FRA233 : Control Engineering for Robotics

Lecture 10
State Estimator

FRA233 : Control Engineering for Robotics

Introduction to Control System

LAB 1 : System Identification

State-Space, Linear System

HW1

Feedback Control System I

Feedback Control System II

Feedback Control System Characteristics III

Exam I

The Stability of Linear Feedback Systems

HW2

Controller Design I (Intro)

Controller Design II (PID-Controller & Tuning & Discretization)

Controller Design III (System Requirements)

LAB 2 : Controller Design I

Exam II

YOU are HERE

State Estimator

HW3

Trajectory Generation & Tracking, Control System Architecture

Frequency Response Analysis I

Frequency Response Analysis II

LAB 3 : Controller Design II

Exam III

Quiz

****** เนื้อหาและการบ้านในรายวิชาสามารถปรับเปลี่ยนได้ตามความเหมาะสม

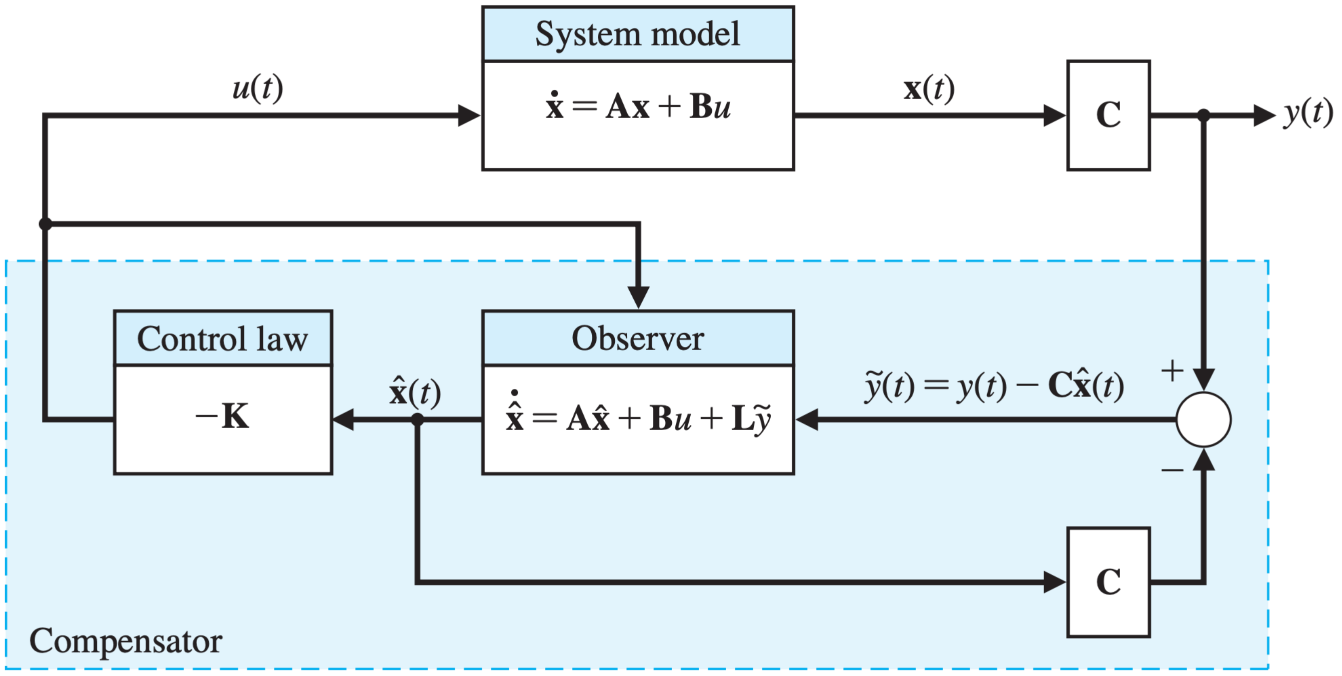
State Estimator

State-Space Systems

- Open-loop Estimators (Observer Theory - no noise)
  - Luenberger
- Closed-loop Estimators (with noise)
  - Kalman

State Estimator

Controllability and Observability



State variable compensator employing full-state feedback in series with a full-state observer.

For the system

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}u(t),$$
$$y(t) = \mathbf{C}\mathbf{x}(t),$$

For a SISO system, the **controllability matrix $\mathbf{P_c}$** is described in tems of **A** and **B** as

$$\boxed{\mathbf{P}_c = [\mathbf{B} \quad \mathbf{AB} \quad \mathbf{A}^2\mathbf{B} \ldots \mathbf{A}^{n-1}\mathbf{B}],}$$

det($\mathbf{P_c}$) is nonzero, the system is controllable.

For a SISO system, the **observability matrix $\mathbf{P_o}$** is described in tems of **A** and **C** as

$$\mathbf{P}_o = \begin{bmatrix} \mathbf{C} \\ \mathbf{CA} \\ \vdots \\ \mathbf{CA}^{n-1} \end{bmatrix},$$

det($\mathbf{P_o}$) is nonzero, the system is observable.

State Estimator

Estimation Schemes

Assume that the system model is the form:

$$\dot{x} = Ax + Bu, \qquad x(0)\ unknow$$
$$y = Cx$$

Where

- $A, B\ and\ C$ are know
- $u(t)$ is know
- Measurable outputs aer $y(t)$ from $C \neq I$

Goal $\longrightarrow$

$$\hat{x}(t) = x(t)$$

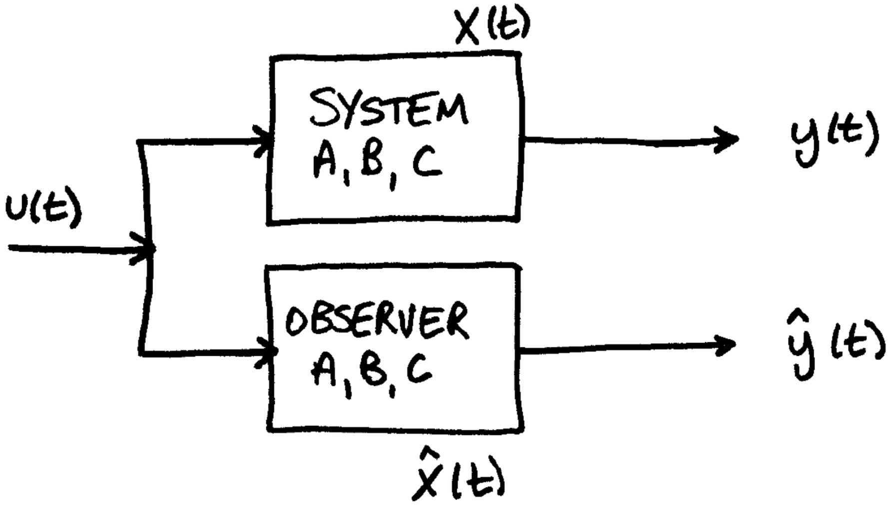
For all time t $\geq$ 0 Two primary approaches:

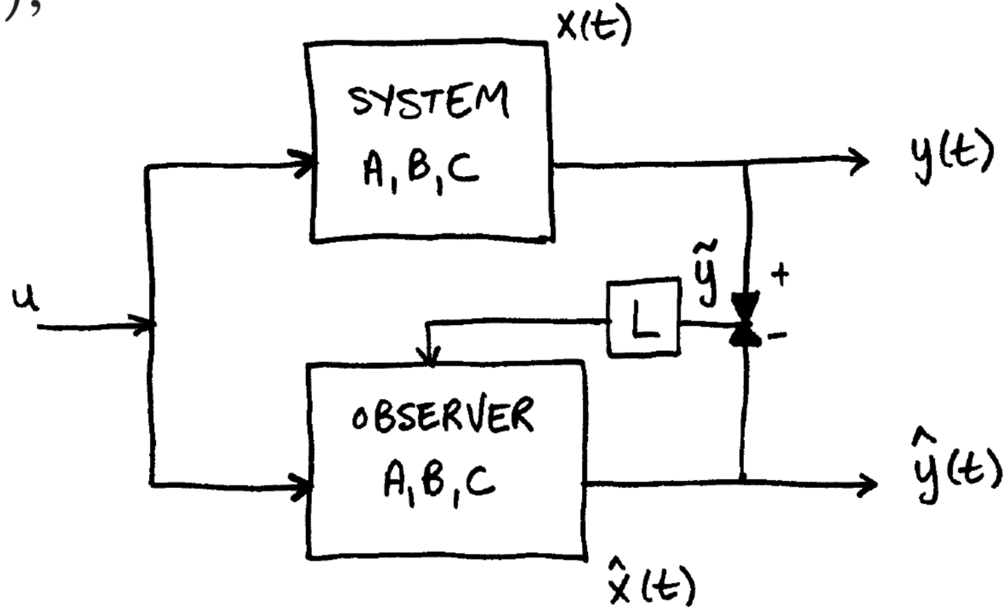
- Open loop
- Closed loop

State Estimator

Estimator : Open loop VS Closed loop

For the system $\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}u(t),\ y(t) = \mathbf{C}\mathbf{x}(t),$

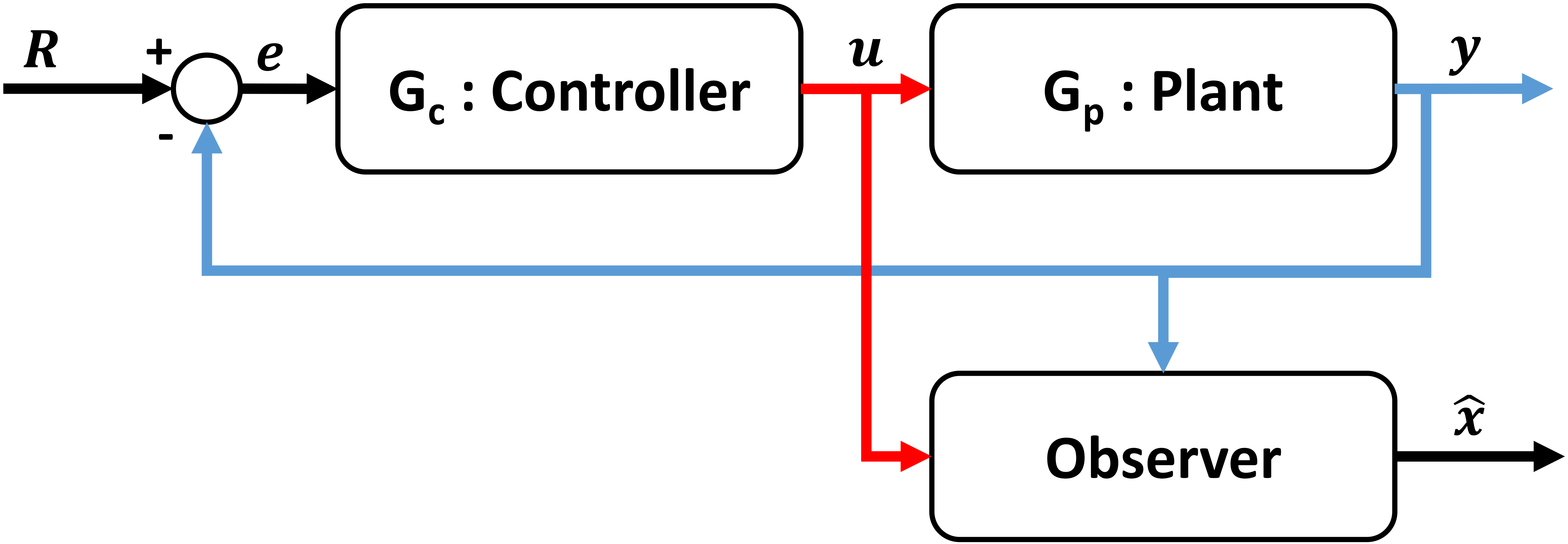




$$\dot{\hat{x}}(t) = A\hat{x} + Bu(t)$$

$$\dot{\hat{x}}(t) = A\hat{x}(t) + Bu(t) + \boxed{L\tilde{y}(t)}$$

State Estimator



Observer Design

The Dynamic System.

$$\dot{x}(t) = Ax(t) + Bu(t)$$
$$y(t) = Cx(t)$$

is give by

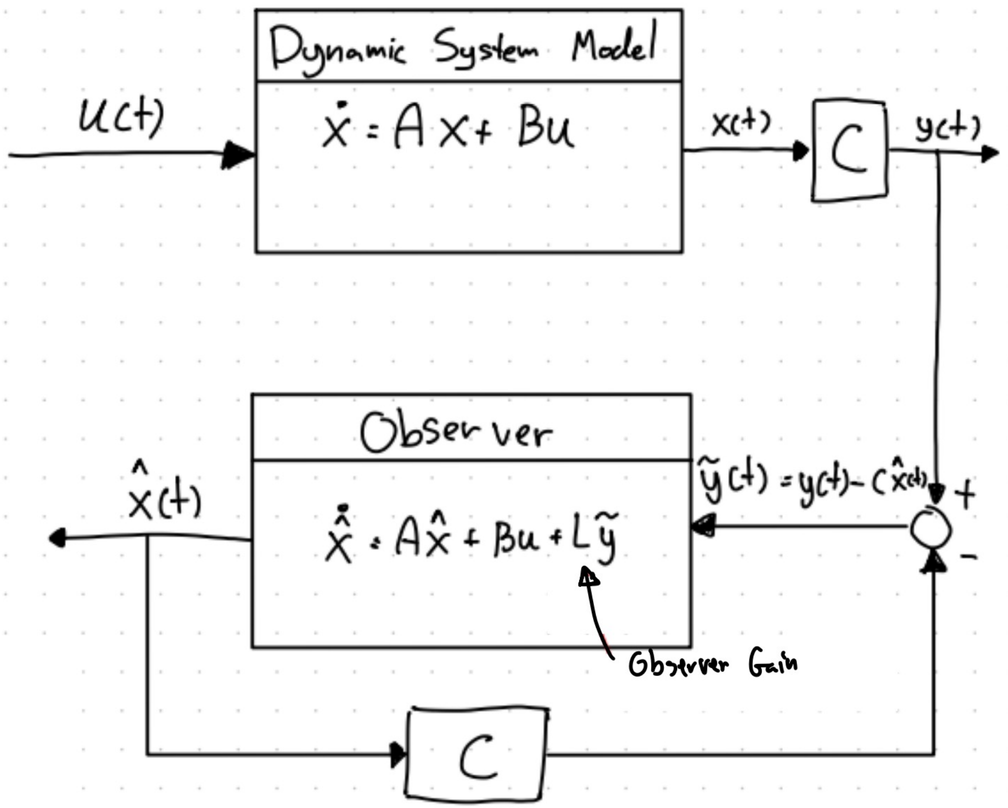
$$\dot{\hat{x}} = A\hat{x}(t) + Bu(t) + L(y(t) - C\hat{x}(t))$$

where $\hat{x}(t)$ estimate of the state $x(t)$

L the observer gain

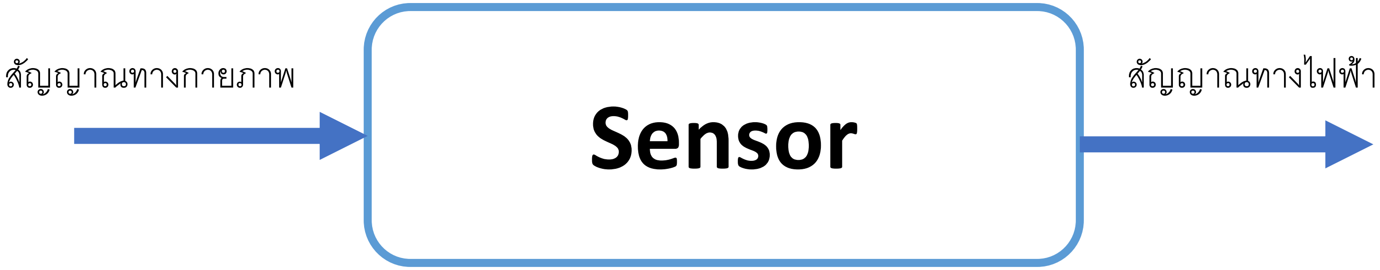
The goal of the observer

$$\hat{x}(t) \longrightarrow x(t) \ @ \ t \rightarrow \infty$$

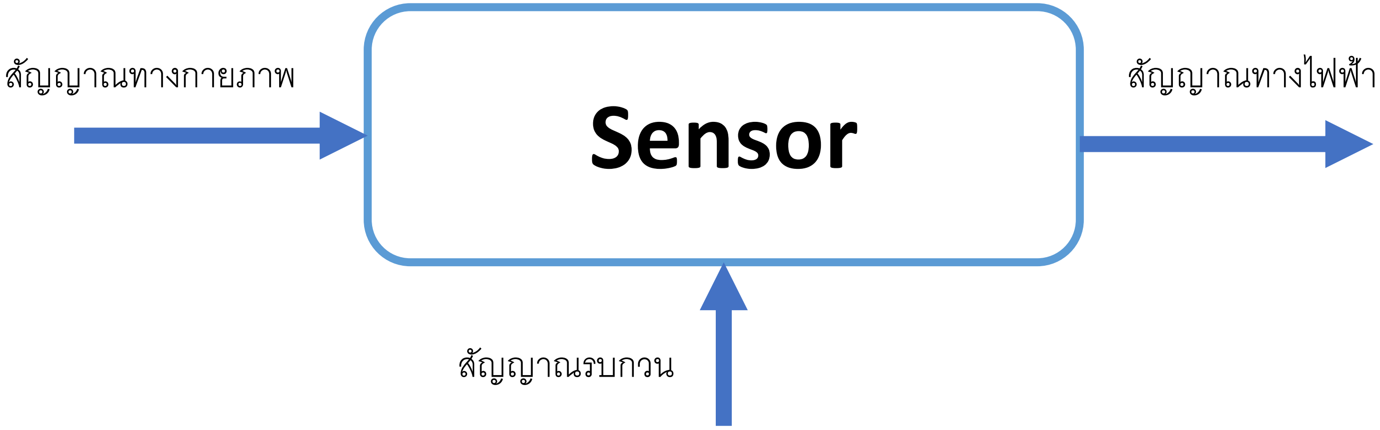


Sensor

Sensor เป็นอุปกรณ์ที่แปลงสัญญาณทางกายภาพให้เป็นสัญญาณทางไฟฟ้า

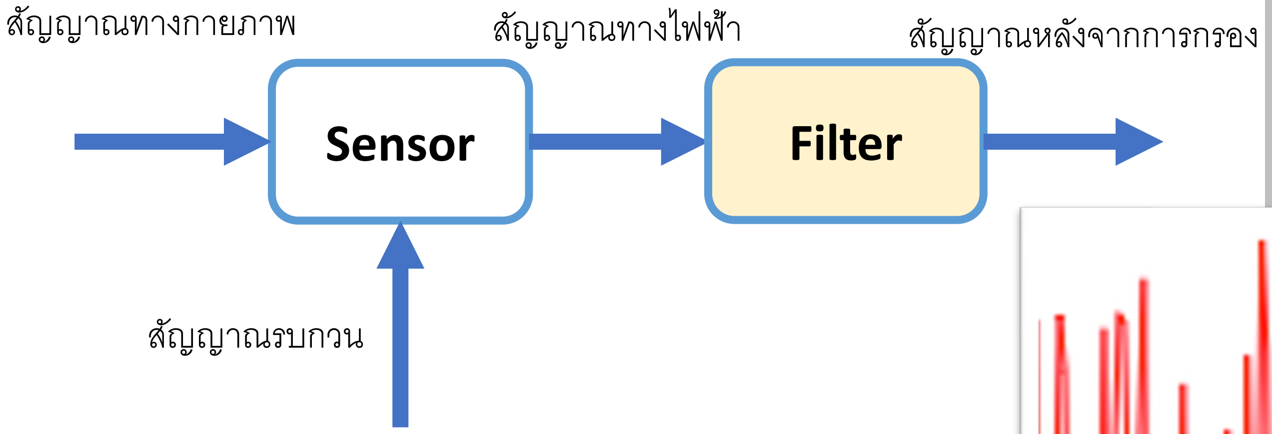


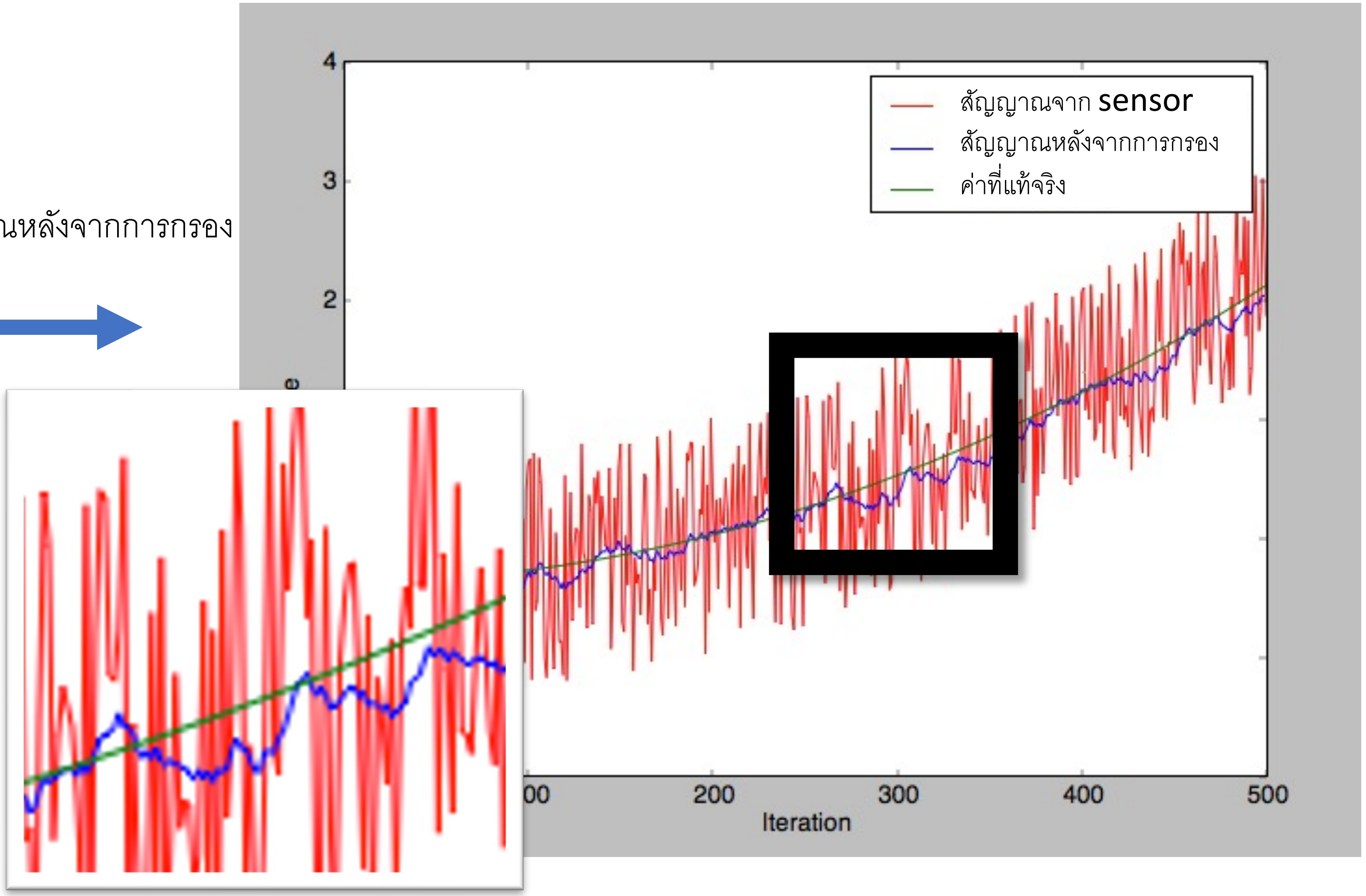
ในความเป็นจริง



Sensor

Sensor เป็นอุปกรณ์ที่แปลงสัญญาณทางกายภาพให้เป็นสัญญาณทางไฟฟ้า เมื่อมีการเปลี่ยนแปลงรูปแบบสัญญาณจะพบว่าสัญญาณหลังจากการเปลี่ยนแปลงจะมีสัญญาณรบกวนเกิดขึ้น





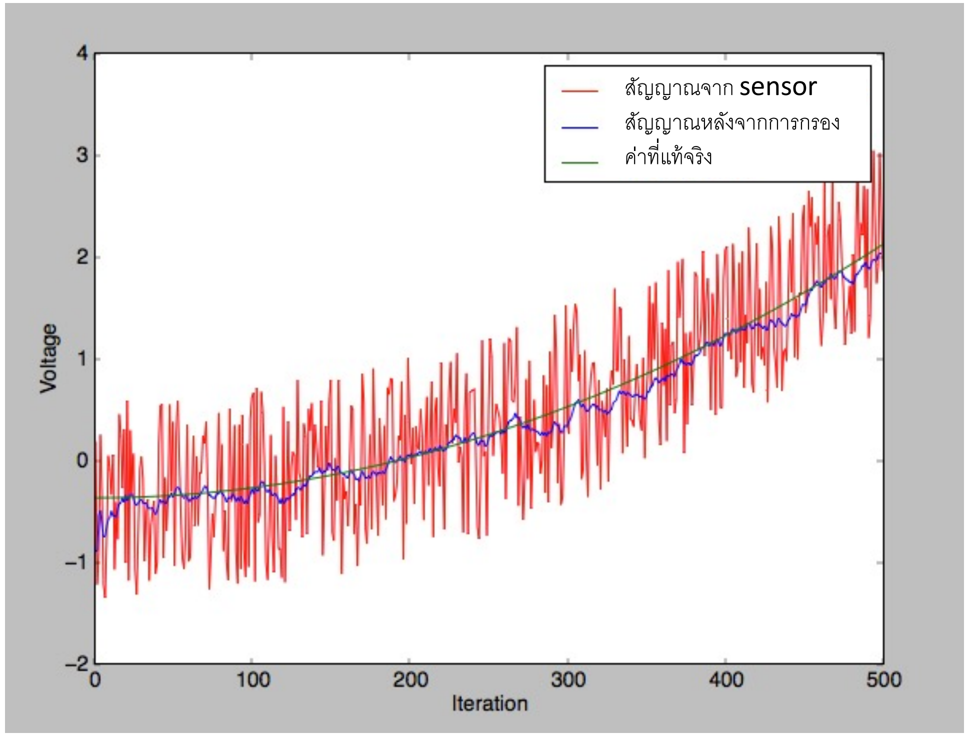
Filter

Type of Filter

- Based on their Construction
  - Passive Filters
  - Active Filters
- Based on their Frequency Response
  - Low pass filter
  - High pass filter
- Based on their model
  - Model → **Kalman Filter**
  - Modeless → …….

Kalman Filter vs Low Pass Filter

If both the Kalman Filter and the Low Pass filter effectively decrease noise, and considering that the Low Pass filter is far less complex than the Kalman Filter, what reasons the existence of the Kalman Filter?



Kalman Filter vs Low Pass Filter

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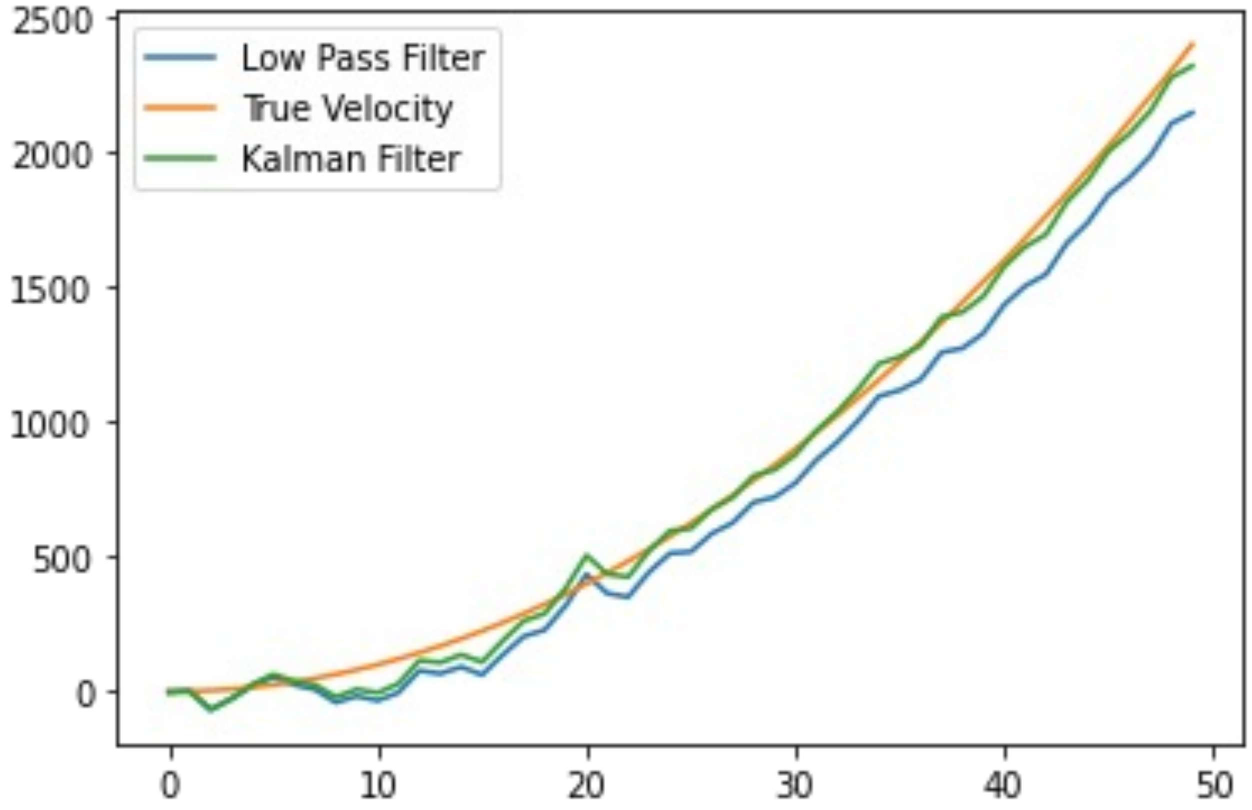
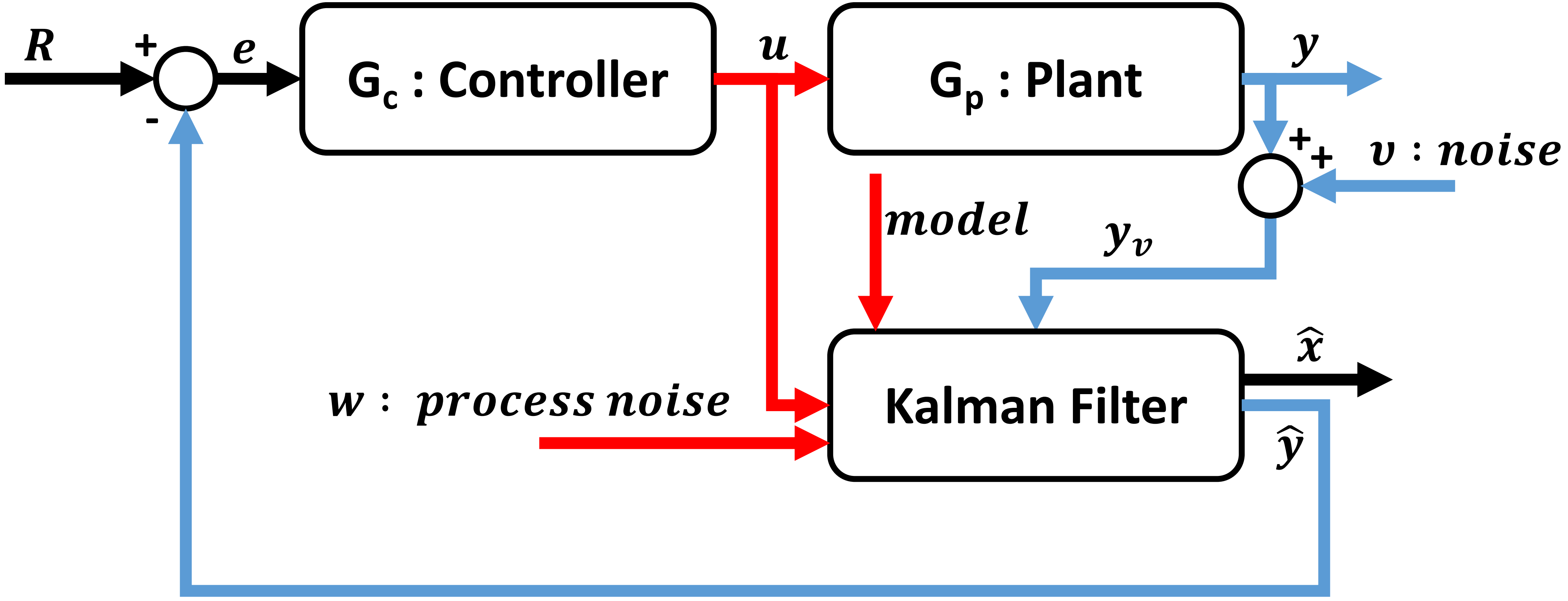
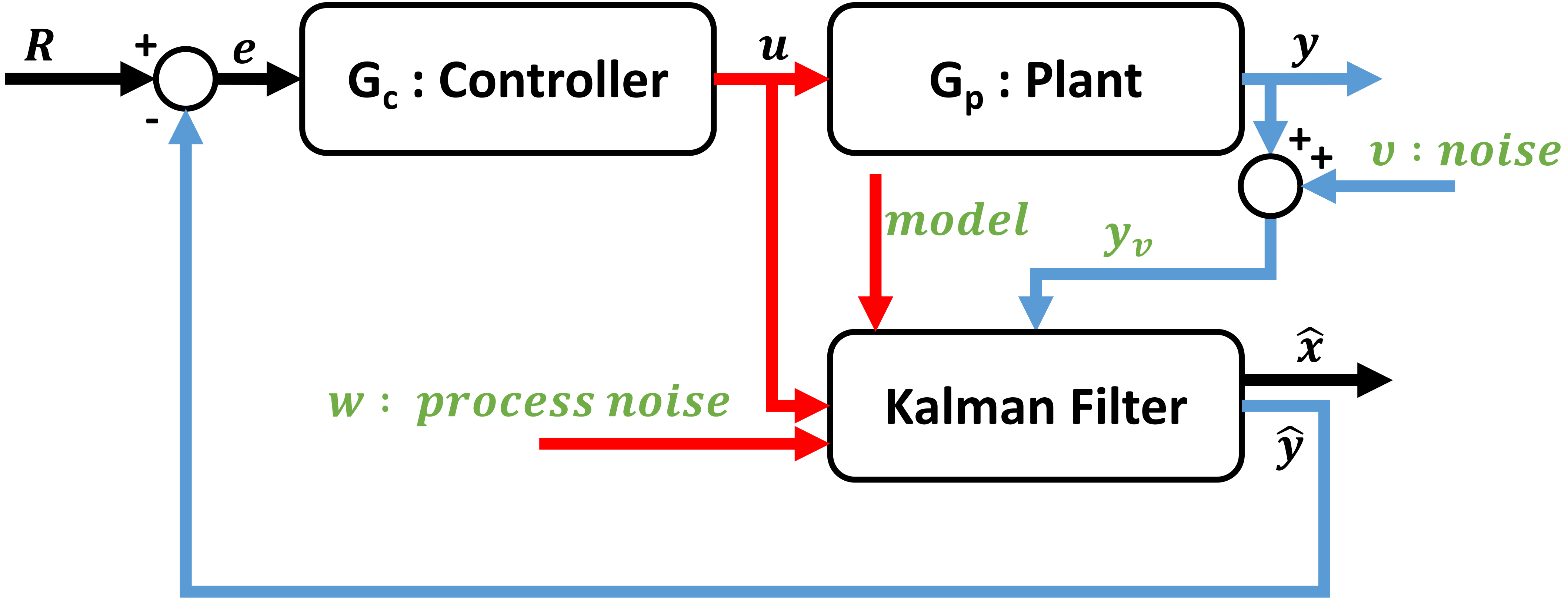


The answer: Phase Lag

Kalman Filter

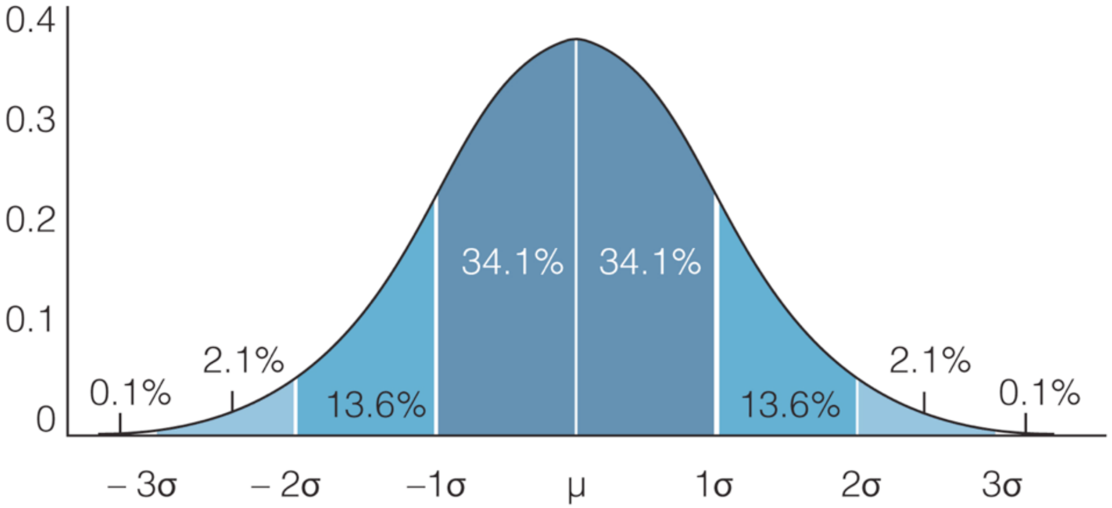


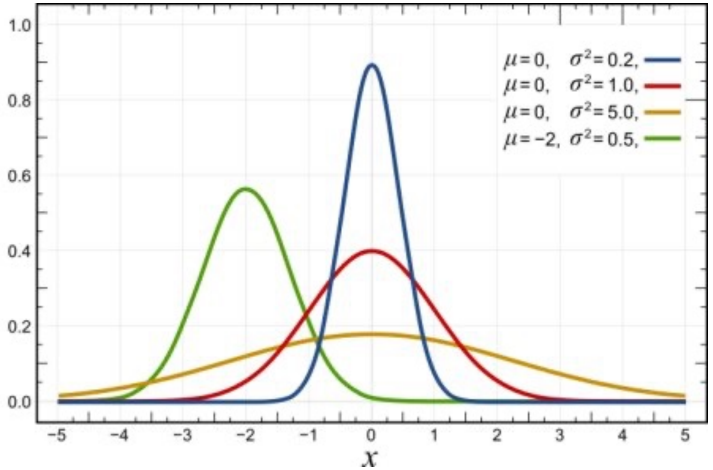
Kalman Filter



Uncertainty and Probability

Gaussian Distribution





- Mean
- Variance
- Standard Deviation

Variance $\sigma^2 = \dfrac{\sum_{i=1}^{N} (x_i - \mu)^2}{N}$

Standard Deviation $\sigma = \sqrt{\dfrac{\sum_{i=1}^{N} (x_i - \mu)^2}{N}}$

Kalman Filter

The Kalman Filter is a state estimator which produces an optimal estimate in the sense that the mean value of the sum (actually of any linear combination) of the estimation errors gets a minimal value.

$$e_x(k) = x_{est}(x) - x(k)$$

This assumes actually that the model is **linear**, so it is not fully correct for **nonlinear** models. It is assumed the that the system for which the states are to be estimated is excited by random ("white") disturbances (or **process noise**) and that the measurements (there must be at least one real measurement in a Kalman Filter) contain random ("white") **measurement noise**

Kalman Filter

The Kalman Filter presented below assumes that the system model consists of this discrete-time (possible nonlinear) state space model:

$$x(k+1) = f[x(k), u(k)] + Gw(k)$$

and this (possible nonlinear) measurement model:

$$y(k) = g[x(k), u(k)] + Hw(k) + v(k)$$

A linear model is just

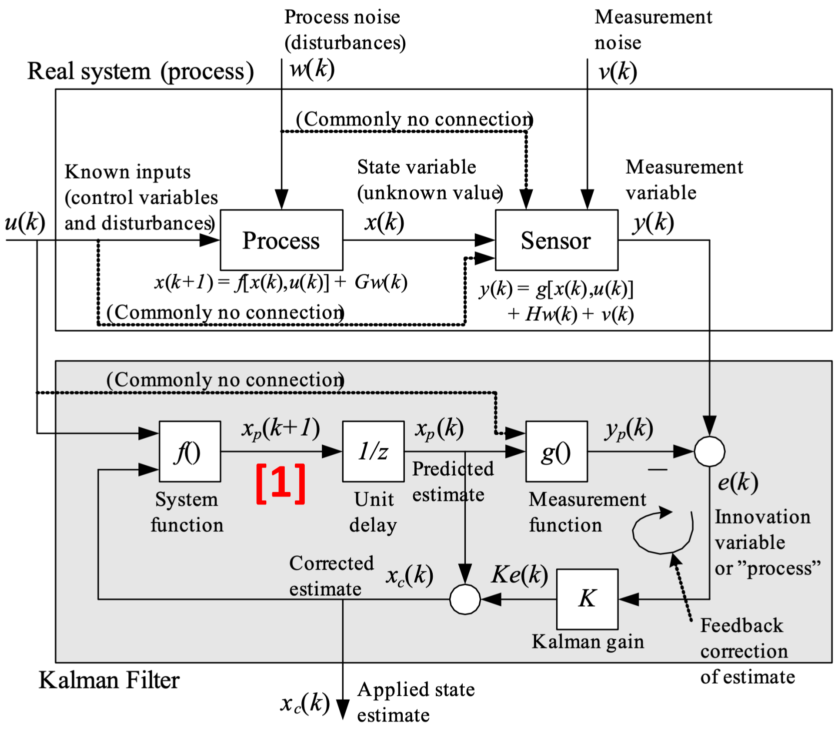
$$x(k+1) = \overbrace{Ax(k) + Bu(k)}^{f[x(k),u(k)]} + Gw(k)$$

$$y(k) = \underbrace{Cx(k) + Du(k)}_{g[x(k),u(k)]} + Hw(k) + v(k)$$

Kalman Filter state estimation

5 Step:

[1] Initial state estimate
[2] Predicted measurement estimate
[3] Innovation variable
[4] Corrected state estimate
[5] Predicted state estimate



This step is the initial step, and the operations here are executed only once. Assume that the initial guess of the state is x_{init}. The initial value $x_p(0)$ of the predicted state estimate x_p (which is calculated continuously as described below) is set equal to this initial value:

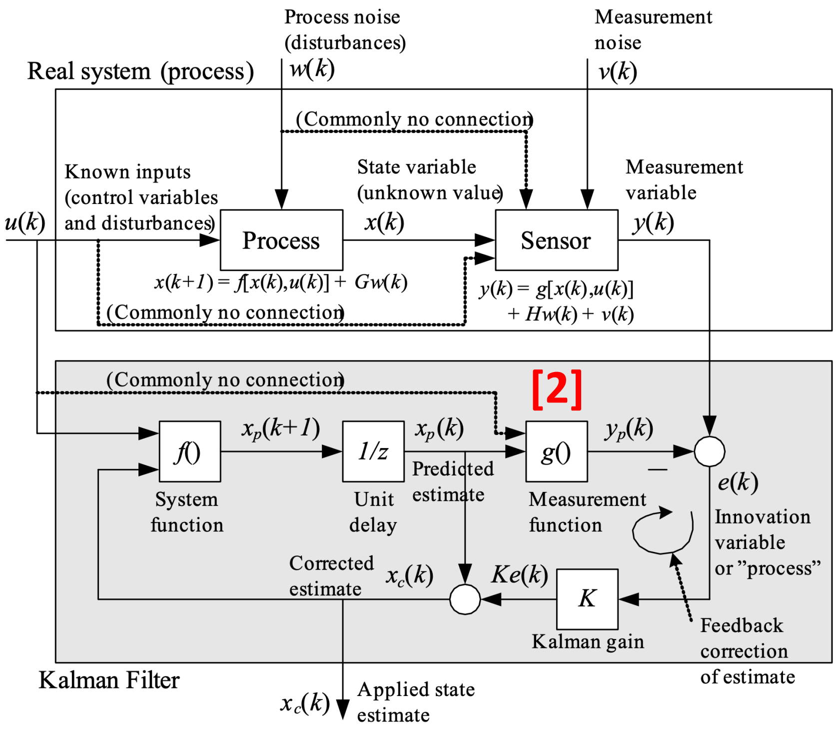
Initial state estimate

$$x_p(0) = x_{init}$$

Kalman Filter state estimation

5 Step:

[1] Initial state estimate
[2] Predicted measurement estimate
[3] Innovation variable
[4] Corrected state estimate
[5] Predicted state estimate



Calculate the predicted measurement estimate from the predicted state estimate:

Predicted measurement estimate

$$y(k) = g[x(k), u(k)] + Hw(k) + v(k)$$

(It is assumed that the noise terms $Hw(k)$ and $v(k)$ are not known or are unpredictable (since they are white noise), so they can not be used in the calculation of the predicted measurement estimate.)

$$y_p(k) = g\left[x_p(k)\right]$$

Kalman Filter state estimation

5 Step:

[1] Initial state estimate
[2] Predicted measurement estimate
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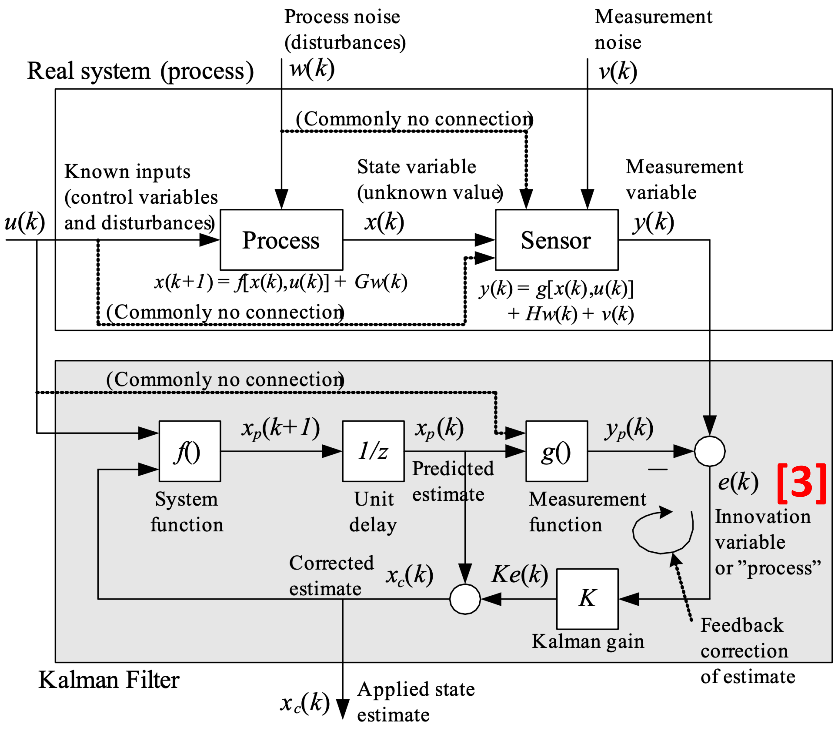


Calculate the so-called innovation process or variable – it is actually the measurement estimate error – as the difference between the measurement $y(k)$ and the predicted measurement $y_p(k)$:

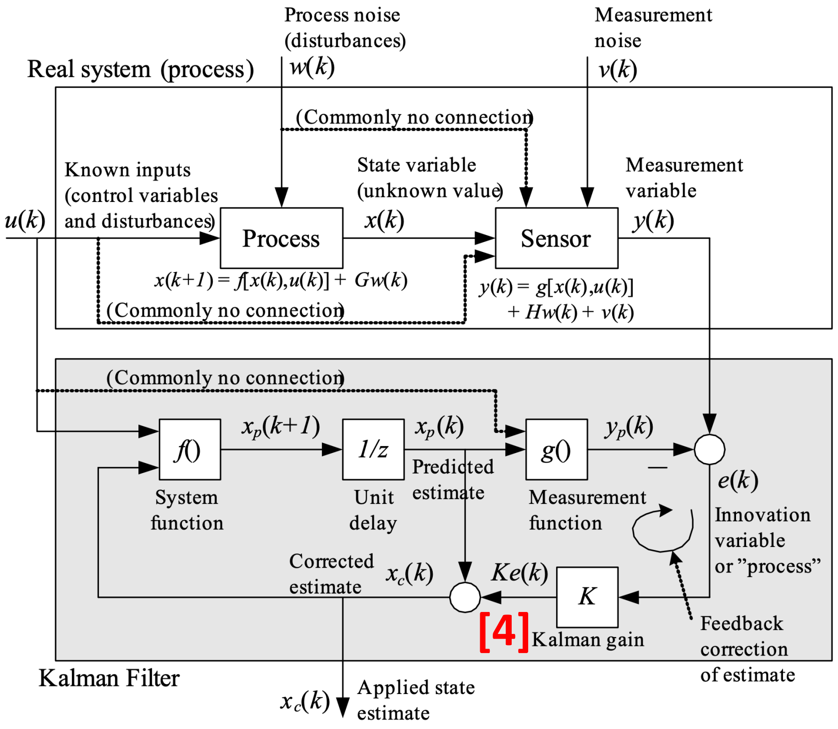
Innovation variable

$$e(k) = y(k) - y_p(k)$$

Kalman Filter state estimation

5 Step:

[1] Initial state estimate
[2] Predicted measurement estimate
[3] Innovation variable
[4] Corrected state estimate
[5] Predicted state estimate



Calculate the corrected state estimate $x_c(k)$ by adding the corrective term $Ke(k)$ to the predicted state estimate $x_p(k)$:

Corrected state estimate

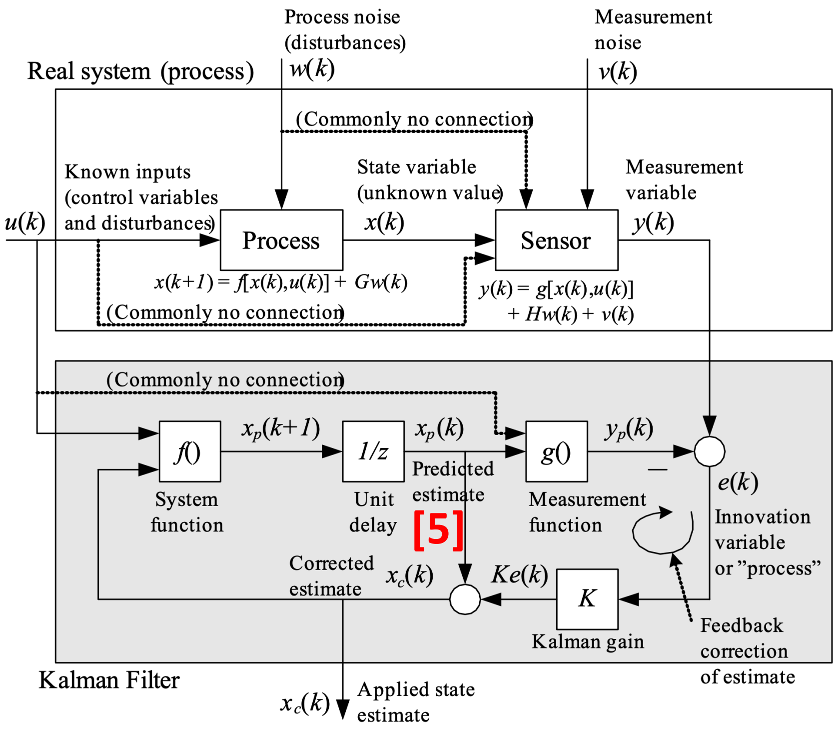
$$x_c(k) = x_p(k) + \boldsymbol{K}e(k)$$

Here, ***K*** is the Kalman Filter gain. The calculation of K is described below

Kalman Filter state estimation

5 Step:

[1] Initial state estimate
[2] Predicted measurement estimate
[3] Innovation variable
[4] Corrected state estimate
[5] Predicted state estimate

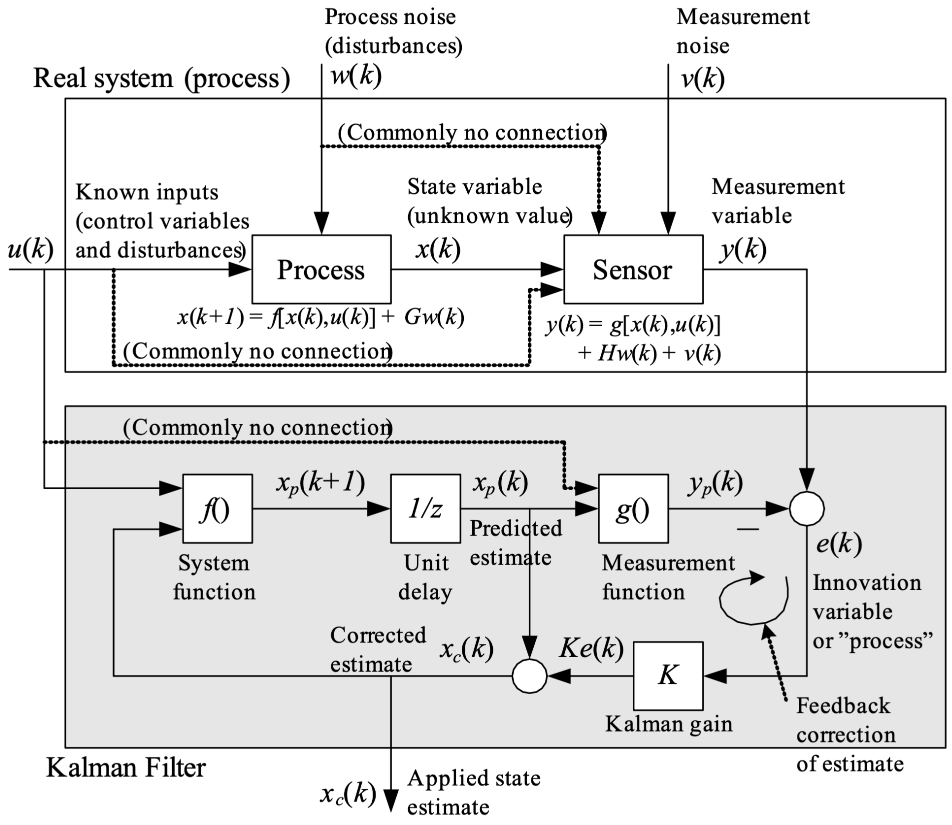


Calculate the corrected state estimate $x_c(k)$ by adding the corrective term $Ke(k)$ to the predicted state estimate $x_p(k)$:

Predicted state estimate

$$x_p(k+1) = f[x_c(k), u(k)]$$

The Kalman Filter algorithm



Prediction Step

Predicted state estimate:

$$x_p(k+1) = f[x_c(k),\, u(k)] \qquad x_{k+1} = Ax_k + Bu_k + Gw_k$$

Auto-covariance of predicted state estimate error:

$$P_p(k+1) = AP_c(k)A^T + GQG^T \qquad P_{k+1} = AP'_{k+1}A^T + GQG^T$$

Measuerment Step

Kalman Filter gain:

$$K(k) = P_p(k)C^T[CP_p(k)C^T + R]^{-1}$$

Predicted measurement estimate:

$$y_p(k) = g\left[x_p(k)\right]$$

Innovation variable:

$$e(k) = y(k) - y_p(k)$$

Corrected state estimate:

$$\underline{x_c(k) = x_p(k) + Ke(k)}$$

Auto-covariance of corrected state estimate error:

$$P_c(k) = [I - K(k)C]\, P_p(k)$$

The Kalman Filter algorithm

ขั้นตอนการ Prediction หรือบางตำราเรียกว่า Propagation โดยที่ขั้นตอนนี้จะประกอบด้วยสมการ 2 สมการด้วยกันได้แก่

$$\hat{x}_p[k] = A\hat{x}[k-1] + B\vec{u}[k-1] + G\vec{w}$$ (Predicted state estimate)

$$\hat{P}_p[k] = A\hat{P}[k-1]A^T + \mathrm{GQG}^T$$ (Predicted error covariance)

ขั้นตอนการ Update หรือบางตำราเรียกว่า Correction โดยที่ในขั้นตอนนี้จะประกอบด้วยUpdate:

$$\tilde{y}[k] = y[k] - C\hat{x}_p[k]$$ (Innovation residual)

$$S[k] = C\hat{P}_p[k]C^T + R$$ (Innovation covariance)

$$K[k] = \hat{P}_p[k]C^T S[k]^{-1}$$ (Optimal Kalman gain)

$$\hat{x}[k] = \hat{x}_p[k] + K[k]\tilde{y}[k]$$ (Corrected state estimate)

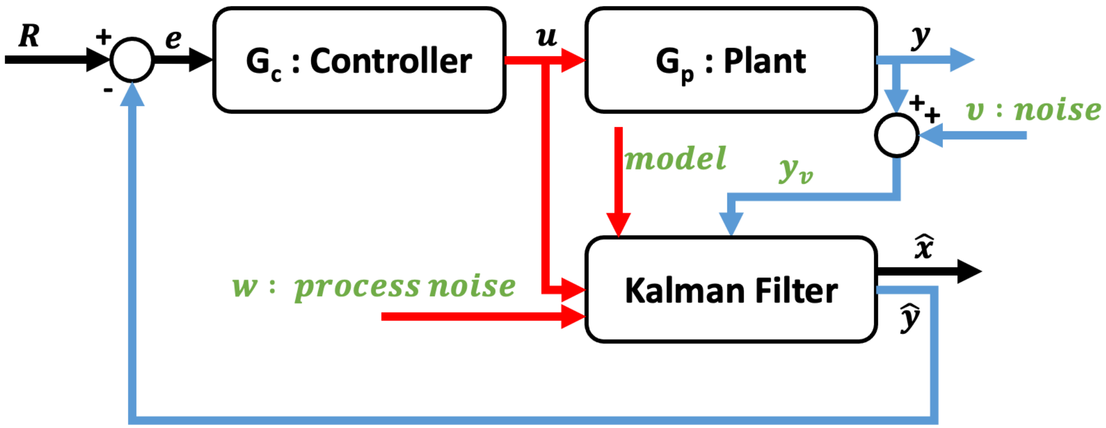
$$\hat{P}[k] = (I - K[k]C)\hat{P}_p[k]$$ (Corrected estimate covariance)

The Kalman Filter algorithm

Implementation

1.) เขียน State Space ของระบบ (Linear System) โดยกำหนด Matrix -> $A, B, H\ and\ G$
2.) กำหนดค่าตัวแปรเริ่มต้น x_0 และ P_0
3.) กำหนดค่า Covariance matrix Q and R
4.) คำนวณหาค่า $\hat{x}_k, P_k$ และ K

The Kalman Filter algorithm



สามารถเขียนสมการได้ดังต่อไปนี้ในรูปของ Discrete Time

$$x_{k+1} = Ax_k + Bu_k + Gw_k$$

$$y_{k+1} = x_{k+1} + v_k$$

w_k, v_k เป็น noise ที่มีการกระจายตัวแบบ normal distribution

โดยมีค่า mean = 0 $\quad p(w) \sim N(0, Q)$

$p(v) \sim N(0, R)$

โดยที่

x คือ ค่าตัวแปรของสถานะ

u คือ ค่า control input

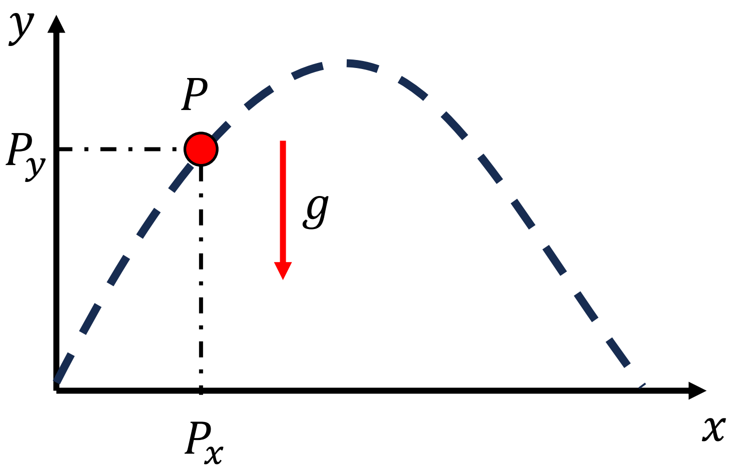
y คือ ค่าตัวแปร Output ที่ได้จากการวัด

w คือ สัญญาณรบกวนของระบบ (Process noise)

v คือ สัญญาณรบกวนที่เกิดจากการวัด (Measurement noise)

The Kalman Filter algorithm

Ballistic Motion with Motion Capture System



ต้องการหาค่าตำแหน่งและความเร็วของระบบ

Position at point P

$$\vec{p} \in \mathcal{R}^2$$

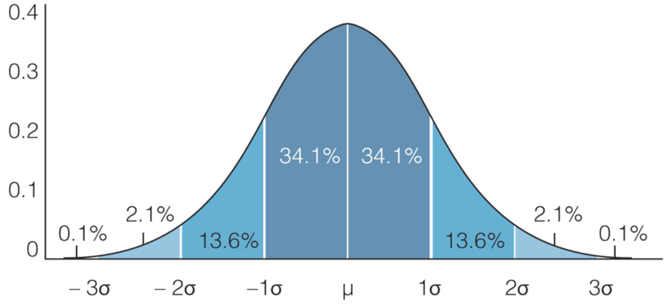
$$\vec{p} = \begin{bmatrix} P_x \\ P_y \end{bmatrix}$$

Measurement

$$\vec{y} = \begin{bmatrix} P_x \\ P_y \end{bmatrix} + \begin{bmatrix} v_x \\ v_y \end{bmatrix}$$

Position at point P (capture system) + Measurement noise

Noise



Determine the kinematics model of ballistic motion

$$\sum \vec{F} = m\vec{a}$$

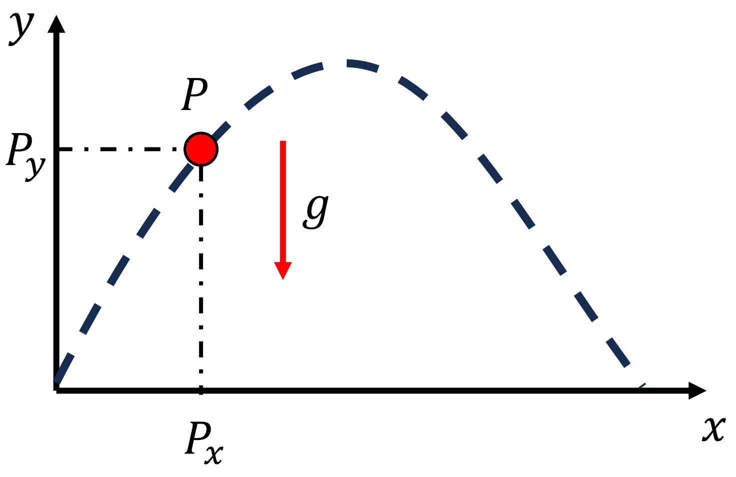
$$m\vec{a} = \begin{bmatrix} F_x \\ F_y \end{bmatrix} + \begin{bmatrix} w_x \\ w_y \end{bmatrix} = \begin{bmatrix} 0 \\ -mg \end{bmatrix} + \begin{bmatrix} w_x \\ w_y \end{bmatrix}$$

$$\vec{a} = \begin{bmatrix} 0 \\ -g \end{bmatrix} + \frac{1}{m}\begin{bmatrix} w_x \\ w_y \end{bmatrix}$$

Process Model + Process Noise

The Kalman Filter algorithm

Ballistic Motion with Motion Capture System



Determine the acceleration at point P

$$\vec{a}(t) = \begin{bmatrix} 0 \\ -g \end{bmatrix} + \frac{1}{m}\begin{bmatrix} w_x \\ w_y \end{bmatrix}$$

Determine the velocity at point P

$$\int_{\tau=0}^{\tau=t} \frac{d}{dt}\vec{v}d\tau = \int_{\tau=0}^{\tau=t} \vec{a}d\tau$$

$$\vec{v}(t) - \vec{v}(0) = \vec{a}t - \vec{a}(0)$$

$$\vec{v}(t) = \vec{v}(0) + \vec{a}t$$

Determine the position at point P

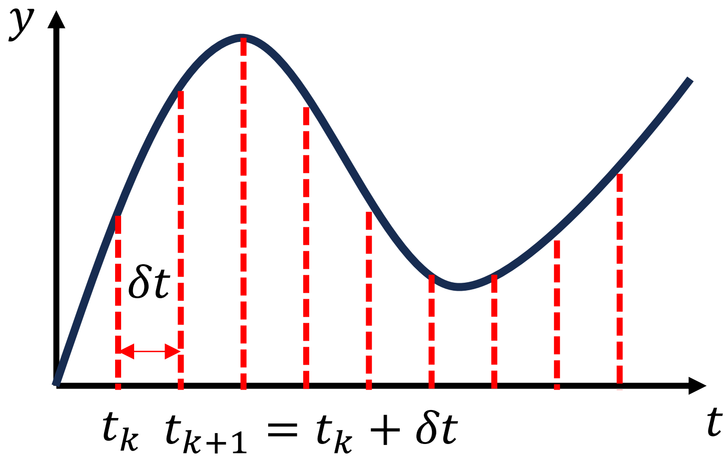
$$\int_{\tau=0}^{\tau=t} \frac{d}{dt}\vec{p}d\tau = \int_{\tau=0}^{\tau=t} \vec{v}(t)d\tau = \int_{\tau=0}^{\tau=t} \vec{v}(0) + \vec{a}td\tau$$

$$\vec{p}(t) - \vec{p}(0) = \vec{v}(0)t + \frac{1}{2}\vec{a}t^2$$

$$\vec{p}(t) = \vec{p}(0) + \vec{v}(0)t + \frac{1}{2}\vec{a}t^2$$

The Kalman Filter algorithm

Discretization Time-Domain to Discrete-Domain



Kinematics model

$$\vec{a}(t) = \begin{bmatrix} 0 \\ -g \end{bmatrix} + \frac{1}{m}\begin{bmatrix} w_x \\ w_y \end{bmatrix}$$

$$\vec{v}(t) = \vec{v}(0) + \vec{a}t$$

$$\vec{p}(t) = \vec{p}(0) + \vec{v}(0)t + \frac{1}{2}\vec{a}t^2$$

Velocity Equation

จากสมการ

$$\vec{v}(t) = \vec{v}(0) + \vec{a}t$$

แปลงเป็นสมการ **Discrete**

$$\vec{v}_k = \vec{v}(0) + \vec{a}t_k$$

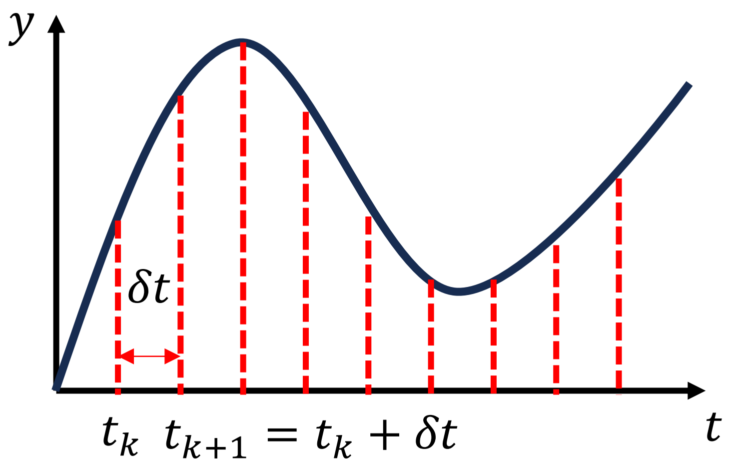
$$\vec{v}_{k+1} = \vec{v}(0) + \vec{a}(t_k + \delta t)$$

$$\vec{v}_{k+1} = \vec{v}(0) + \vec{a}t_k + \vec{a}\delta t$$

$$\vec{v}_{k+1} = \vec{v}_k + \vec{a}\delta t$$

The Kalman Filter algorithm

Discretization Time-Domain to Discrete-Domain



Kinematics model

$$\vec{a}(t) = \begin{bmatrix} 0 \\ -g \end{bmatrix} + \frac{1}{m}\begin{bmatrix} w_x \\ w_y \end{bmatrix}$$

$$\vec{v}(t) = \vec{v}(0) + \vec{a}t$$

$$\vec{p}(t) = \vec{p}(0) + \vec{v}(0)t + \frac{1}{2}\vec{a}t^2$$

Position Equation

จากสมการ

$$\vec{p}(t) = \vec{p}(0) + \vec{v}(0)t + \frac{1}{2}\vec{a}t^2$$

แปลงเป็นสมการ Discrete

$$\vec{p}_k = \vec{p}(0) + \vec{v}(0)t_k + \frac{1}{2}\vec{a}t_k^2$$

$$\vec{p}_{k+1} = \vec{p}(0) + \vec{v}(0)(t_k + \delta t) + \frac{1}{2}\vec{a}(t_k + \delta t)^2$$

$$\vec{p}_{k+1} = \vec{p}(0) + \vec{v}(0)t_k + \vec{v}(0)\delta t + \frac{1}{2}\vec{a}t_k^2 + \vec{a}t_k\delta t + \frac{1}{2}\vec{a}\delta t^2$$

$$\vec{p}_{k+1} = \vec{p}_k + \vec{v}_k\delta t + \frac{1}{2}\vec{a}\delta t^2$$

The Kalman Filter algorithm

จัดสมการเพื่อใช้ในสมการ Kalman Filter

$$\vec{p}_{k+1} = \vec{p}_k + \vec{v}_k \delta t + \frac{1}{2}\vec{a}\delta t^2$$

$$\vec{v}_{k+1} = \vec{v}_k + \vec{a}\delta t$$

แทนค่า $\vec{a}$ ในสมการ

$$\vec{a} = \begin{bmatrix} 0 \\ -g \end{bmatrix} + \frac{1}{m}\begin{bmatrix} w_x \\ w_y \end{bmatrix}$$

$$\vec{p}_{k+1} = \vec{p}_k + \vec{v}_k \delta t + \frac{1}{2}\begin{bmatrix} 0 \\ -g \end{bmatrix}\delta t^2 + \frac{1}{2m}\begin{bmatrix} w_x \\ w_y \end{bmatrix}\delta t^2$$

$$\vec{v}_{k+1} = \vec{v}_k + \begin{bmatrix} 0 \\ -g \end{bmatrix}\delta t + \frac{1}{m}\begin{bmatrix} w_x \\ w_y \end{bmatrix}\delta t$$

จัดรูปสมการ Process model

$$\begin{bmatrix} \vec{p}_{k+1} \\ \vec{v}_{k+1} \end{bmatrix} = \begin{bmatrix} \mathbb{I}_2 & \delta t \mathbb{I}_2 \\ o_{2x2} & \mathbb{I}_2 \end{bmatrix}\begin{bmatrix} \vec{p}_k \\ \vec{v}_k \end{bmatrix} + \begin{bmatrix} \frac{1}{2}\delta t^2 \mathbb{I}_2 \\ \delta t \mathbb{I}_2 \end{bmatrix}\begin{bmatrix} 0 \\ -g \end{bmatrix} + \begin{bmatrix} \frac{1}{2m}\delta t^2 \mathbb{I}_2 \\ \frac{1}{m}\delta t \mathbb{I}_2 \end{bmatrix}\vec{w}_k$$

$$\vec{x}_{k+1} = A\vec{x}_k + B\vec{u}_k + G\vec{w}_k$$

จัดรูปสมการ Measuement model

$$\vec{y}_k = C\vec{x}_k + D\vec{u}_k + v_k$$

$$\vec{y}_k = \begin{bmatrix} \mathbb{I}_2 & 0_{2x2} \end{bmatrix}\begin{bmatrix} \vec{p}_k \\ \vec{v}_k \end{bmatrix} + \begin{bmatrix} 0_{2x2} \end{bmatrix}\begin{bmatrix} 0 \\ -g \end{bmatrix} + v_k$$

Kalman filter Matrix

$$A = \begin{bmatrix} 1 & 0 & \delta t & 0 \\ 0 & 1 & 0 & \delta t \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$B = \begin{bmatrix} \frac{1}{2}\delta t^2 & 0 \\ 0 & \frac{1}{2}\delta t^2 \\ \delta t & 0 \\ 0 & \delta t \end{bmatrix}$$

$$G = B\frac{1}{m}$$

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

$$Q = \vec{w}$$

$$R = \vec{v}$$