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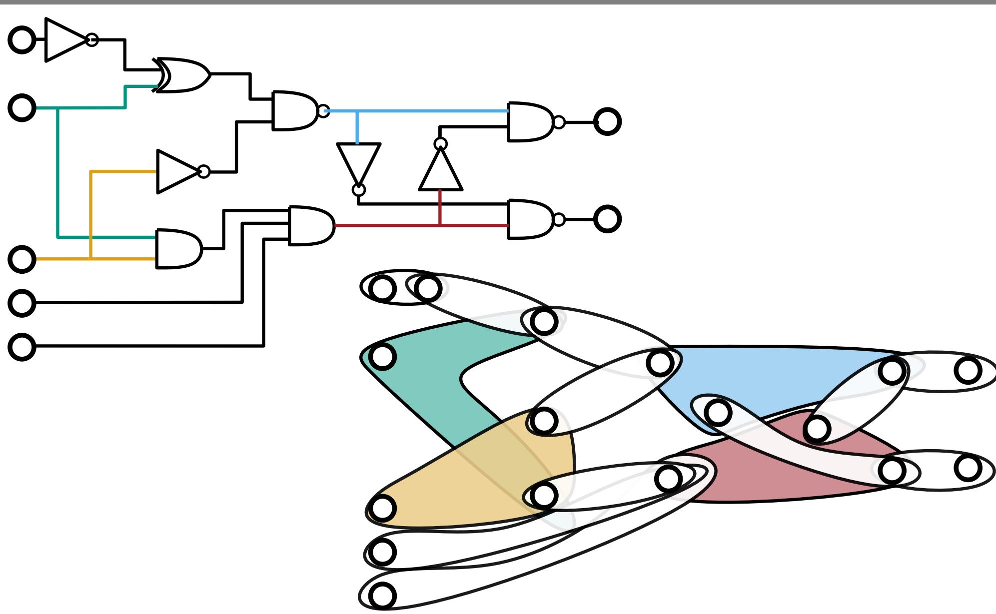
High Quality Hypergraph Partitioning via Max-Flow-Min-Cut Computations

Master Thesis · February 16, 2018

Tobias Heuer

Institute of Theoretical Informatics · Algorithmics Group



Outline

Task

Developing a **refinement** algorithm based on **Max-Flow-Min-Cut** computations for the *n*-level hypergraph partitioner **KaHyPar**.

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Outline

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Contributions

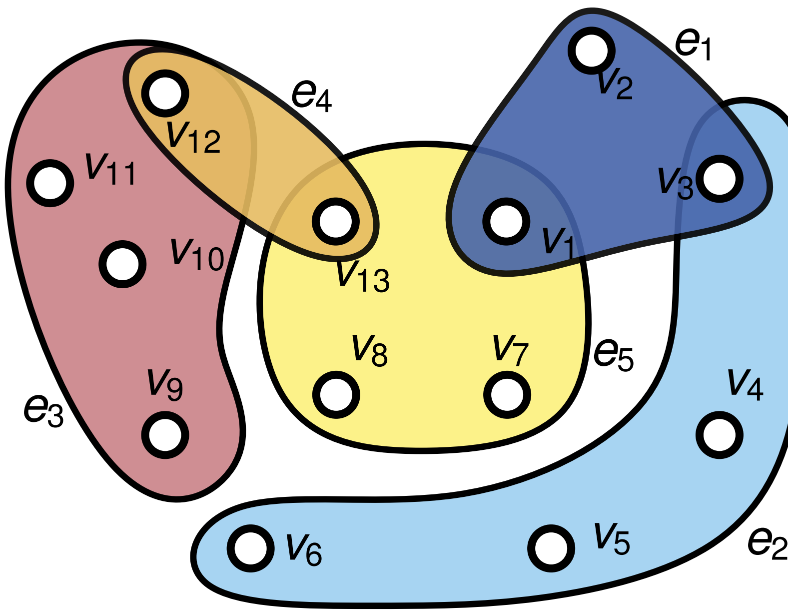
- Outperforms 5 different systems on 73% of 3216 benchmark instances
- Improve quality of *KaHyPar* by 2.5%, while only incurring a slowdon by a factor of 1.8
- Comparable running time to *hMetis* and outperforms it on 84% of the instances

Hypergraphs
[from SEA'17]

- Generalization of graphs
  $\Rightarrow$ hyperedges connect $\geq$ 2 nodes

- Graphs $\Rightarrow$ dyadic (**2-ary**) relationships
- Hypergraphs $\Rightarrow$ (**d-ary**) relationships

- Hypergraph $H = (V, E, c, \omega)$
  - Vertex set $V = \{1, ..., n\}$
  - Edge set $E \subseteq \mathcal{P}(V) \setminus \emptyset$
  - Node weights $c : V \to \mathbb{R}_{\geq 1}$
  - Edge weights $\omega : E \to \mathbb{R}_{\geq 1}$



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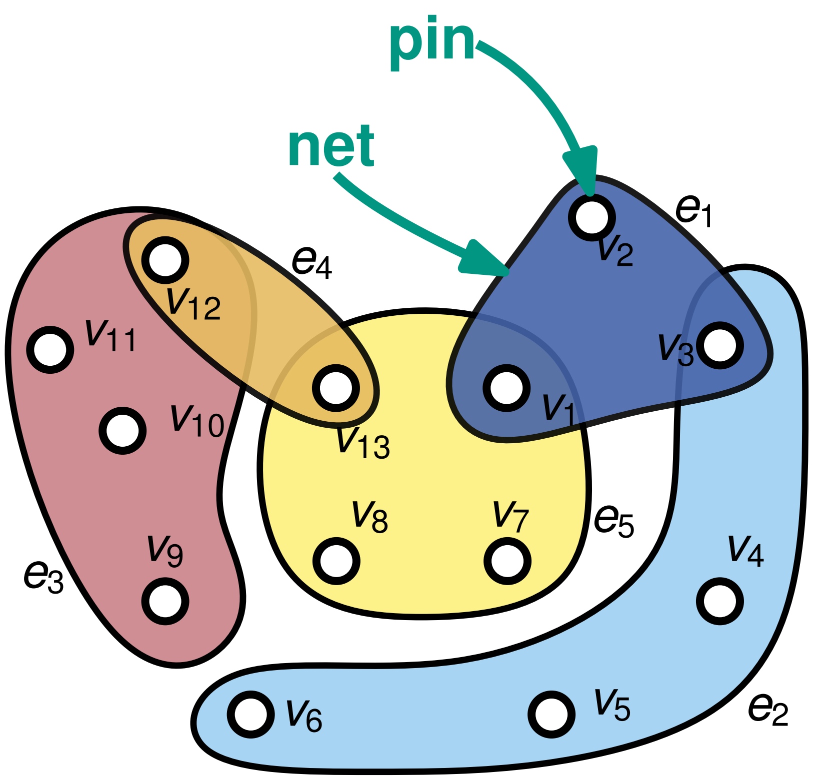
- $|P| = \sum_{e \in E} |e| = \sum_{v \in V} d(v)$



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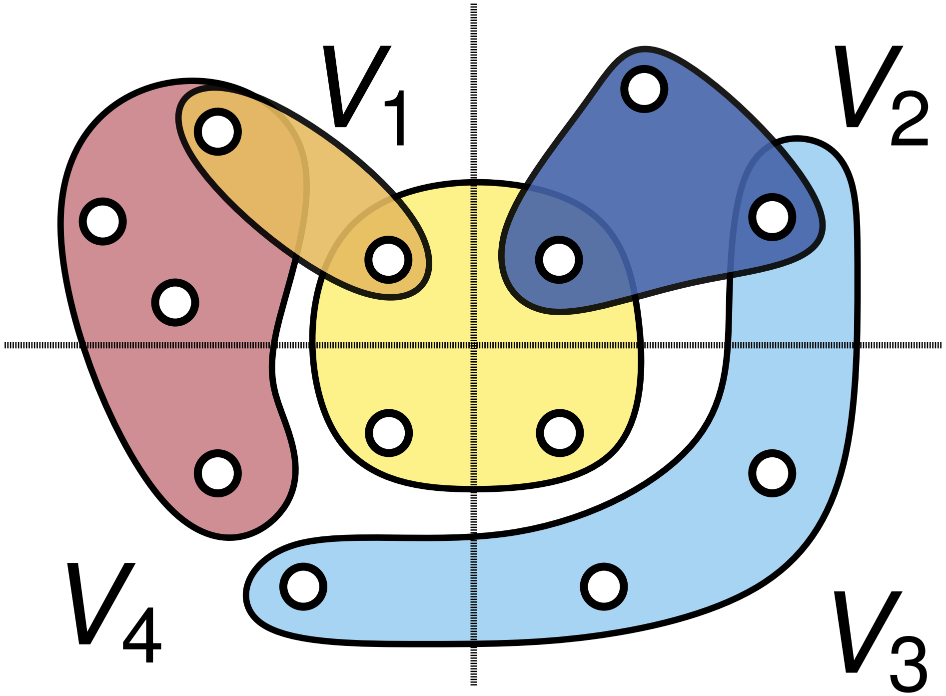
Hypergraph Partitioning Problem
[from SEA'17]

Partition hypergraph $H = (V, E, c, \omega)$ into $\mathbf{k}$ non-empty disjoint blocks $\Pi = \{V_1, \ldots, V_k\}$ such that:

- blocks V_i are **roughly equal-sized**:

$$c(V_i) \leq (1 + \varepsilon) \left\lceil \frac{c(V)}{k} \right\rceil$$



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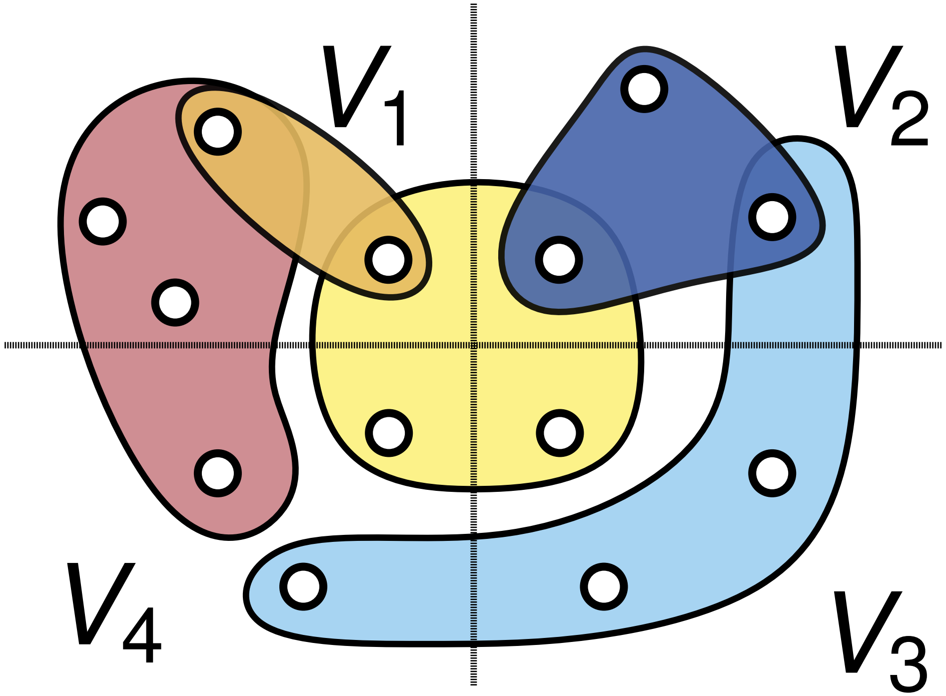

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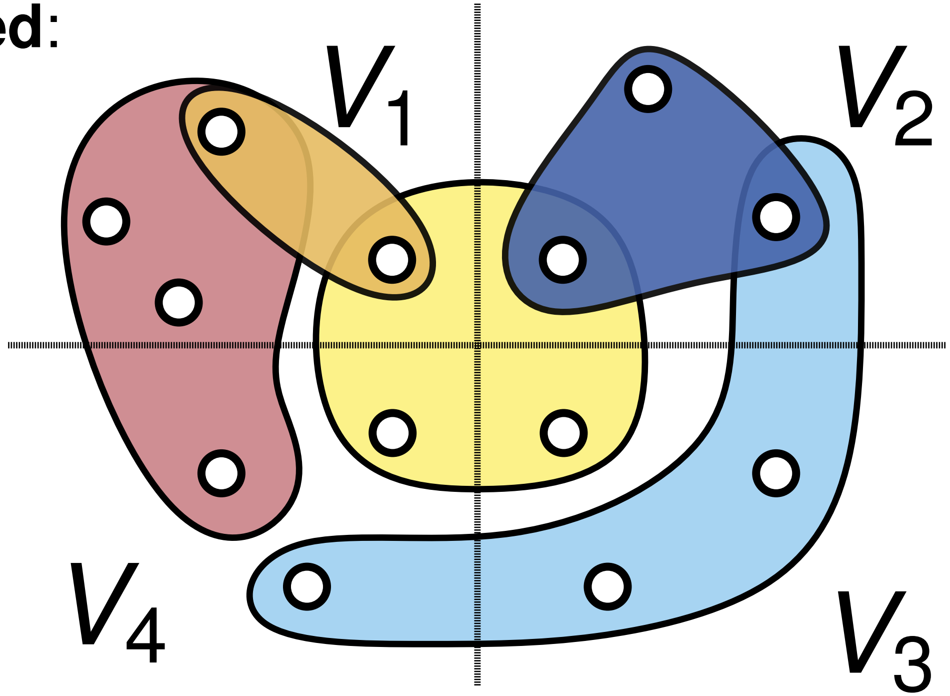
- **connectivity** objective is **minimized**:



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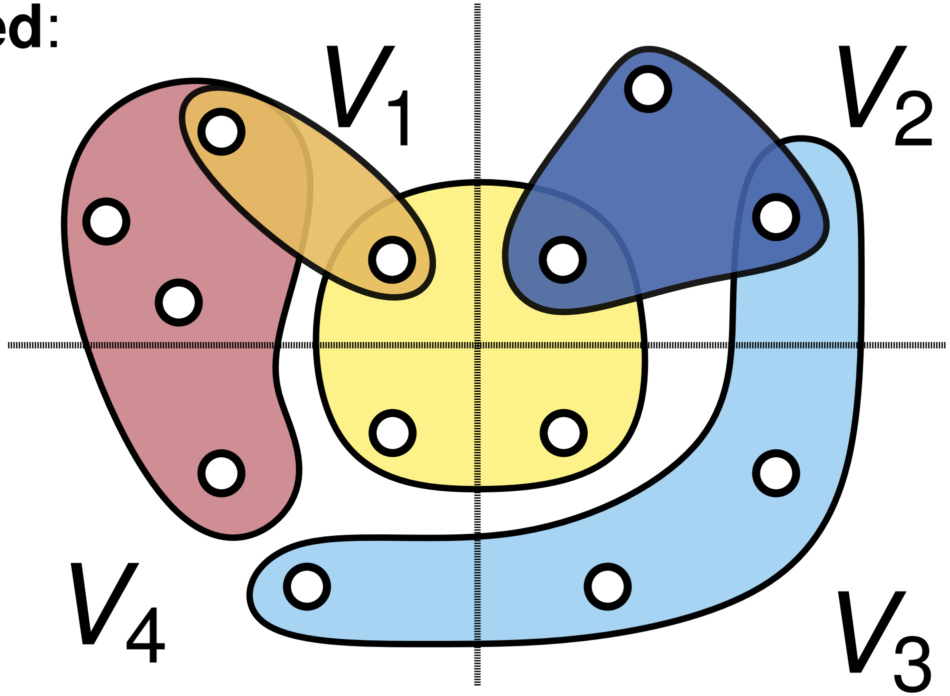
$$\sum_{e \in \mathrm{cut}} (\lambda - 1)\, \omega(e)$$

connectivity:
 # blocks connected by net e



Hypergraph Partitioning Problem

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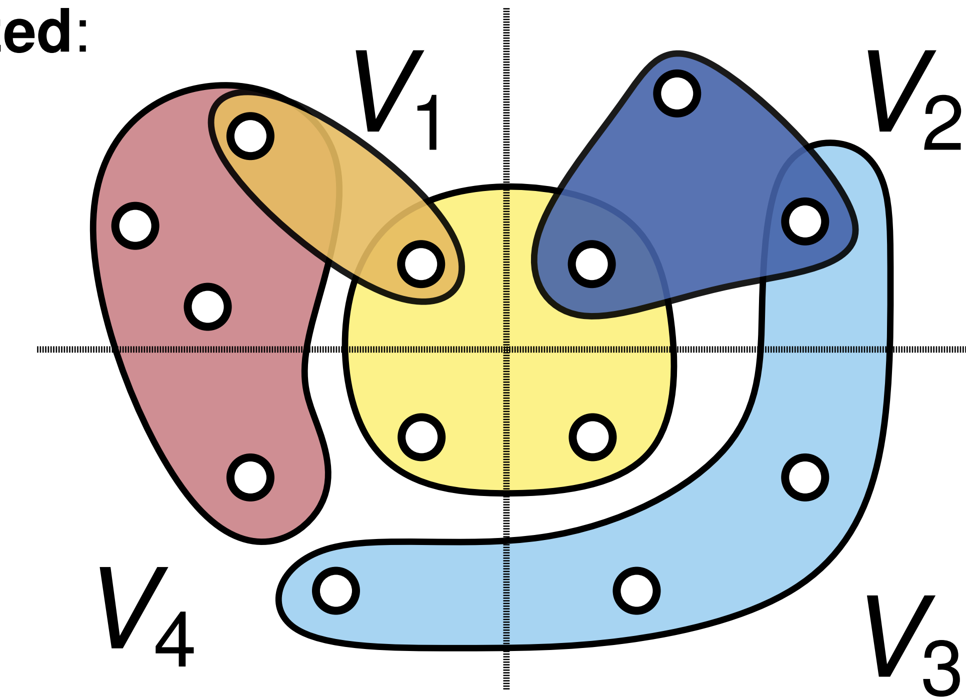
imbalance parameter

$$c(V_i) \leq (1 + \varepsilon) \left\lceil \frac{c(V)}{k} \right\rceil$$

- **connectivity** objective is **minimized**:

$$\sum_{e \in \text{cut}} (\lambda - 1)\, \omega(e) = 6$$

connectivity:
 # blocks connected by net e



Applications
[from SEA'17]

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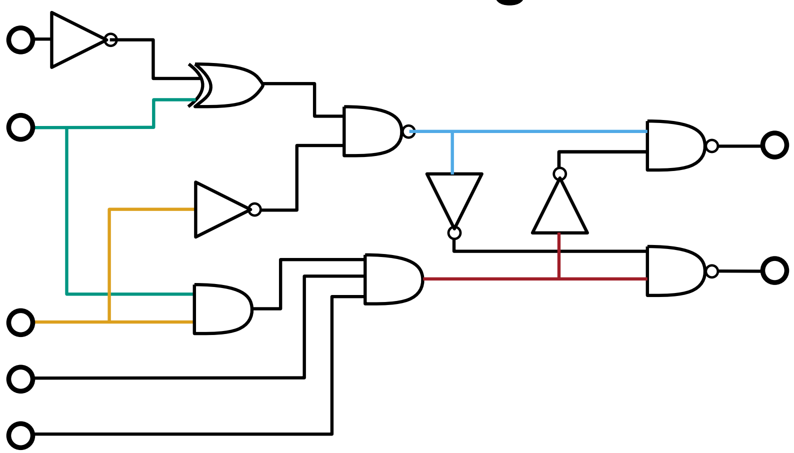
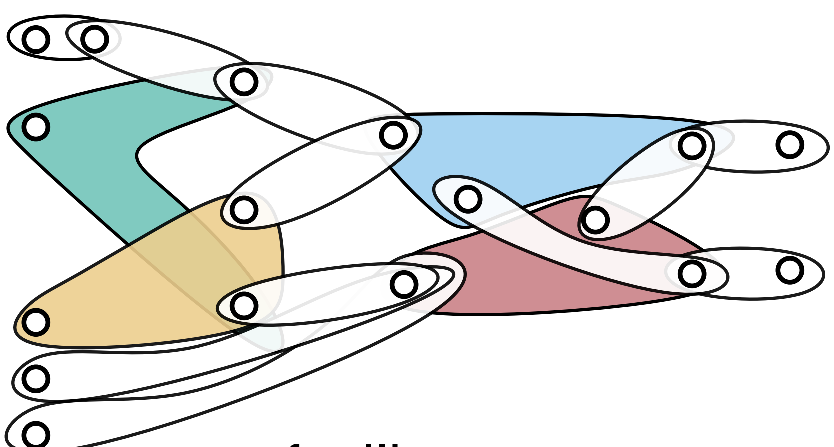
VLSI Design





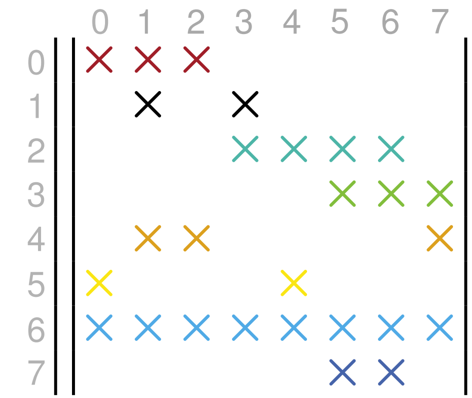
facilitate
floorplanning & placement

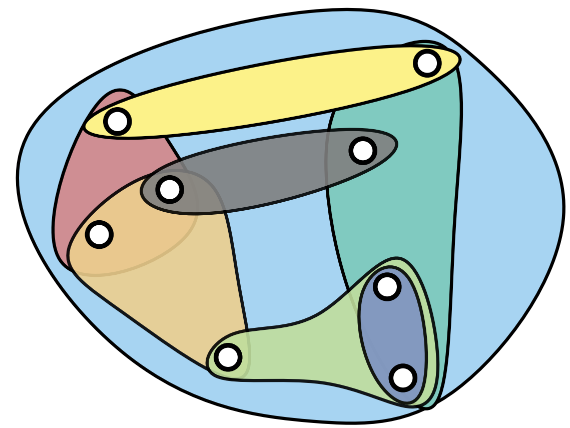
Application Domain

Hypergraph Model

Goal

Scientific Computing

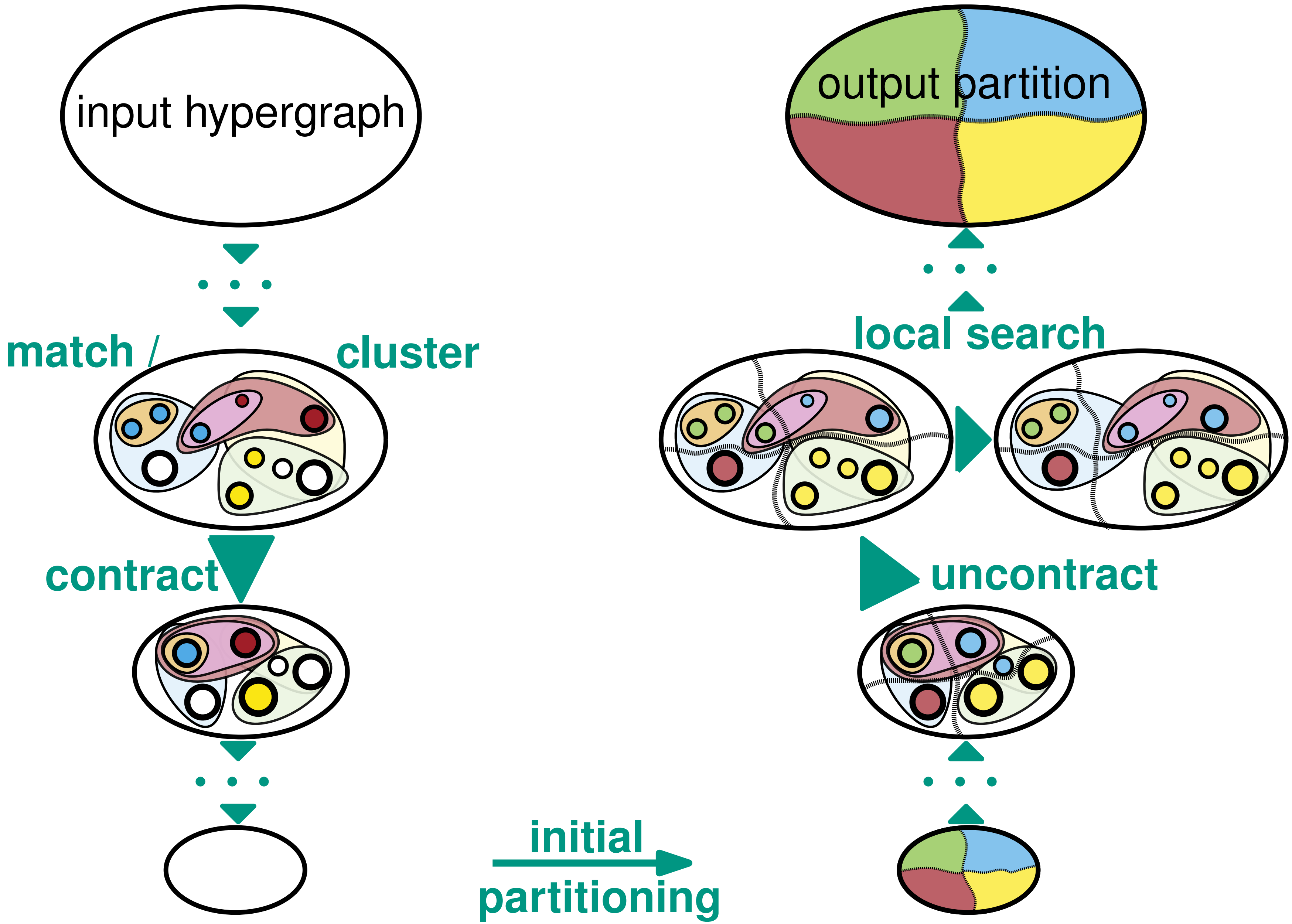




minimize
communication

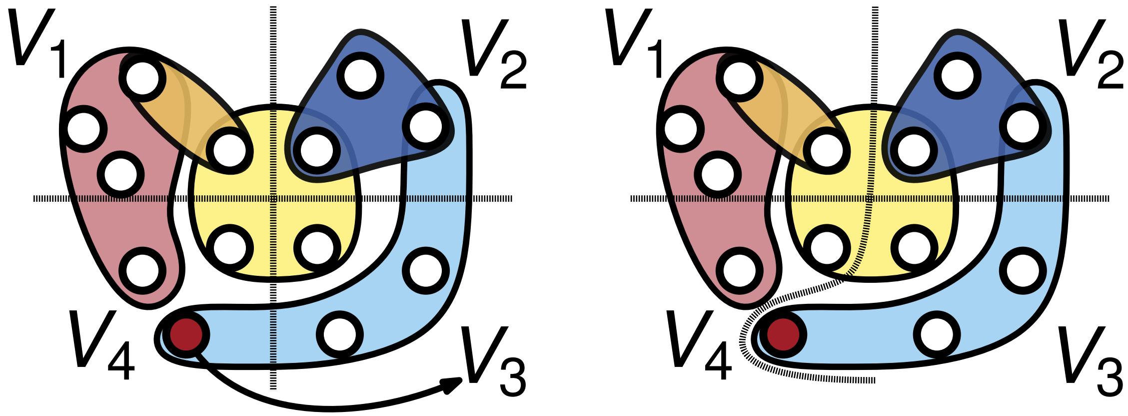
The Multilevel Framework

[from SEA'17]



FM Algorithm

- **Move**-based heuristic that **greedily** move vertices between blocks based on **local** informations of incident nets



Moving  from V_4 to V_3 reduces cut by 1

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FM Algorithm

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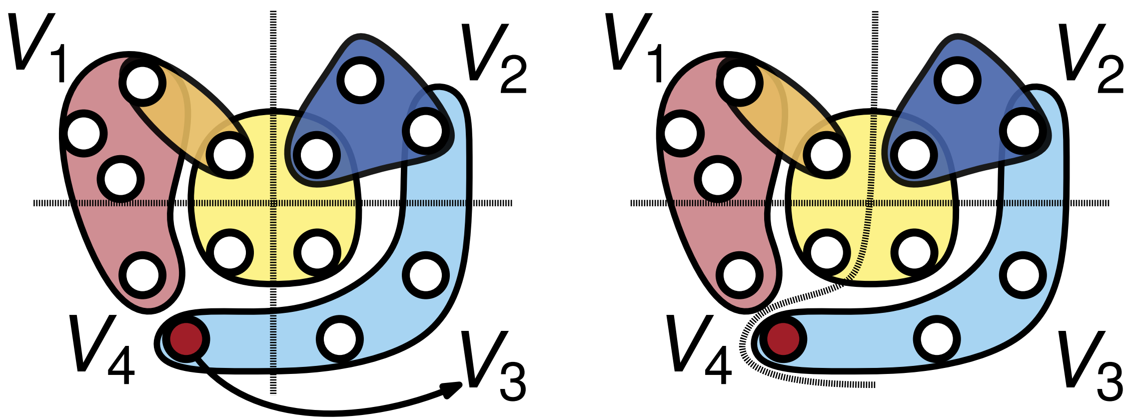

Moving ● from V_4 to V_3 reduces cut by 1

gain

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FM Algorithm

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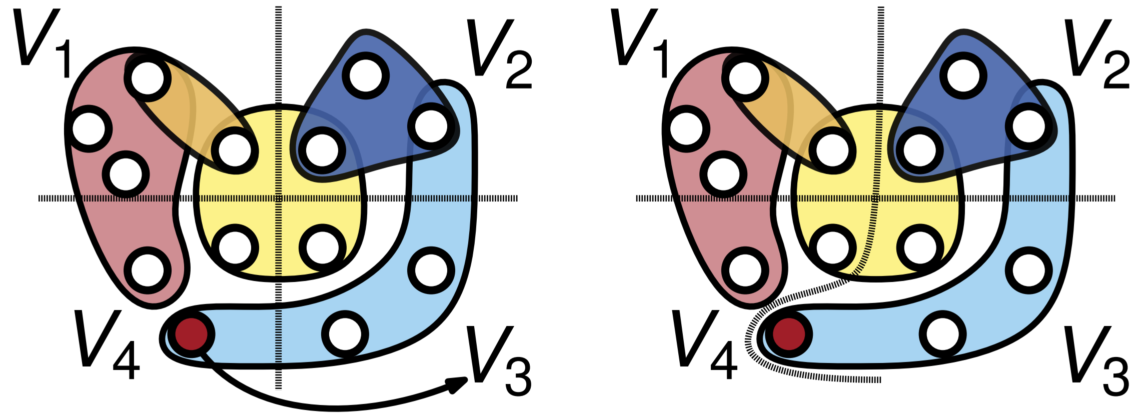


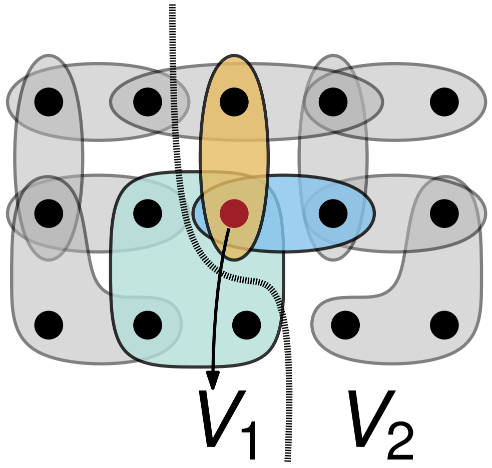
Moving ● from V_4 to V_3 reduces cut by 1

gain

- Performs moves of vertices with **maximum gain** in each step
- All modern hypergraph partitioners implements variations of the *FM* algorithm

FM Algorithm - Disadvantages

- Only incorparates **local** informations about the problem structure
  - Heavily depends on *initial partition*
  - In multilevel context: Depends on quality of *coarsening*



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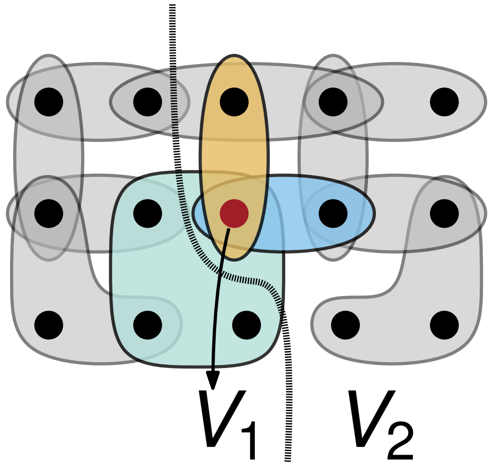
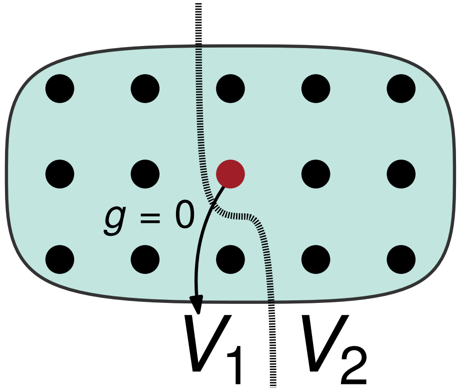


- Large hyperedges induce **Zero-Gain** moves
  - Quality mainly depends on random decisions made within the algorithm



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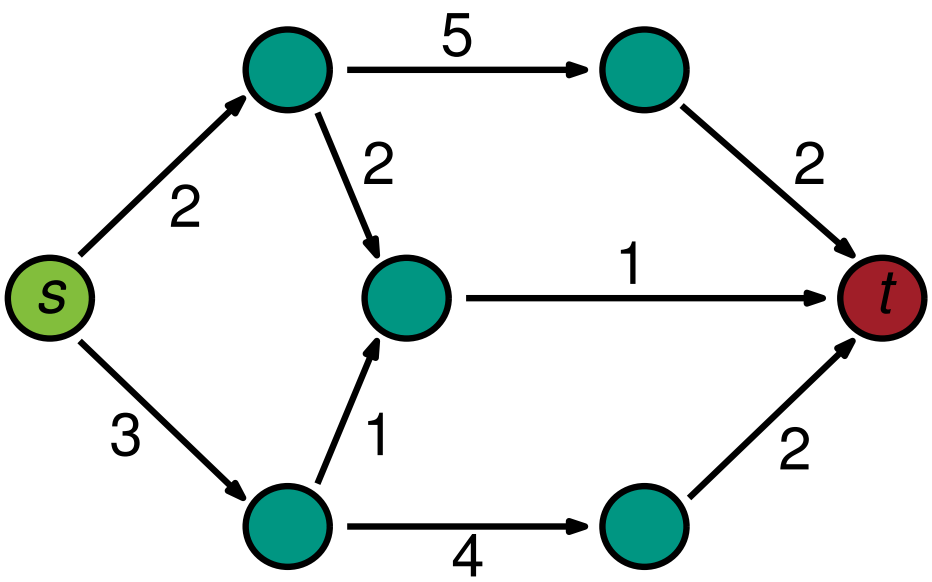
Flows



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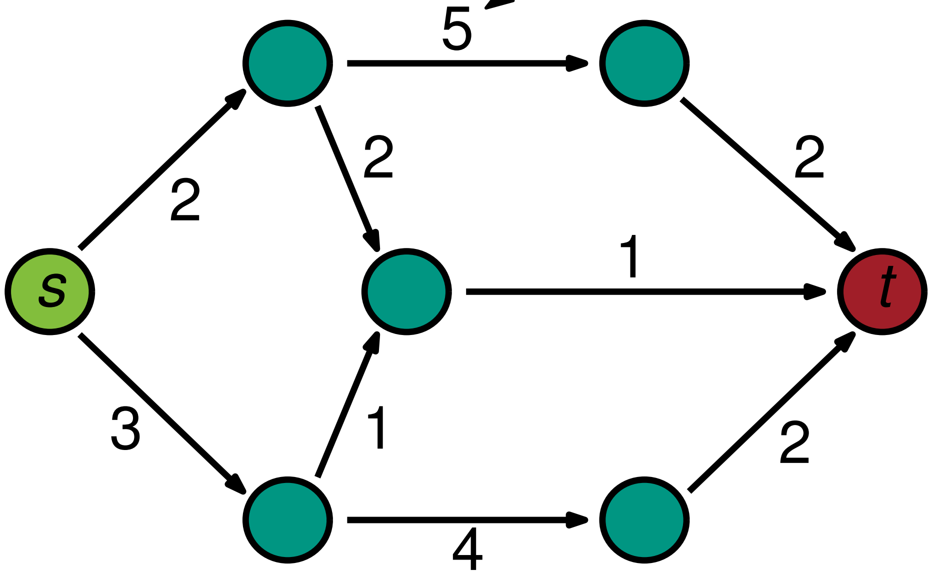
Flows

Capacity function $u : E \to \mathbb{R}_+$



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Flows

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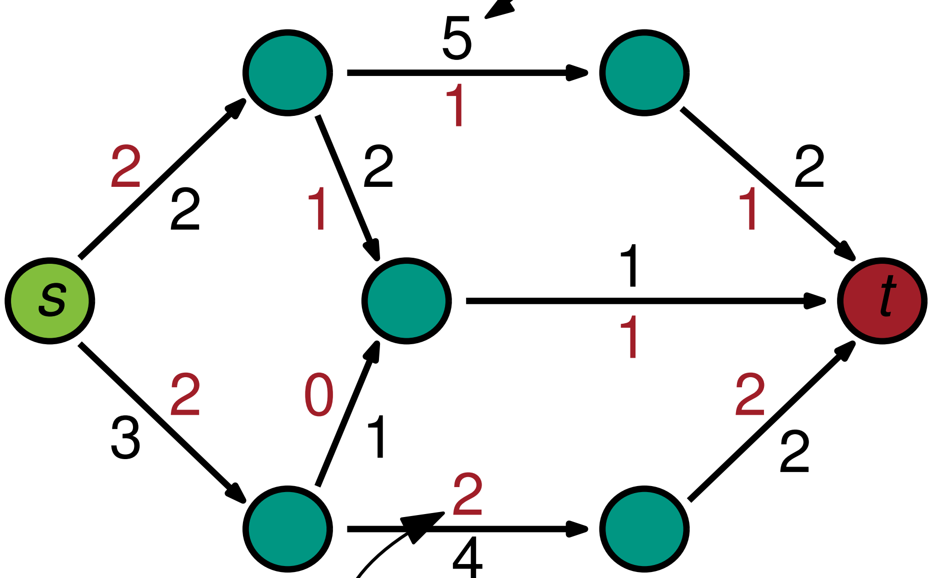

Flow function $f : E \to \mathbb{R}_+$

- $\forall (v, w) \in E : f(v, w) \leq u(v, w)$
- $\forall v \in V \setminus \{s, t\} : \sum_{(w,v) \in E} f(w, v) = \sum_{(v,w) \in E} f(v, w)$

Flows

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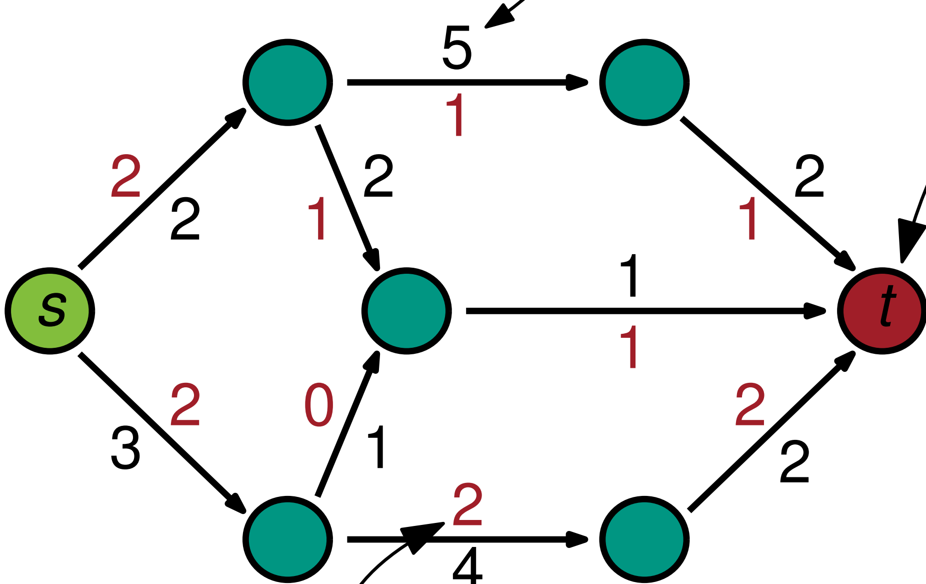
Capacity function $u : E \to \mathbb{R}_+$

Value of flow $|f| = \sum_{(v,t) \in E} f(v,t)$



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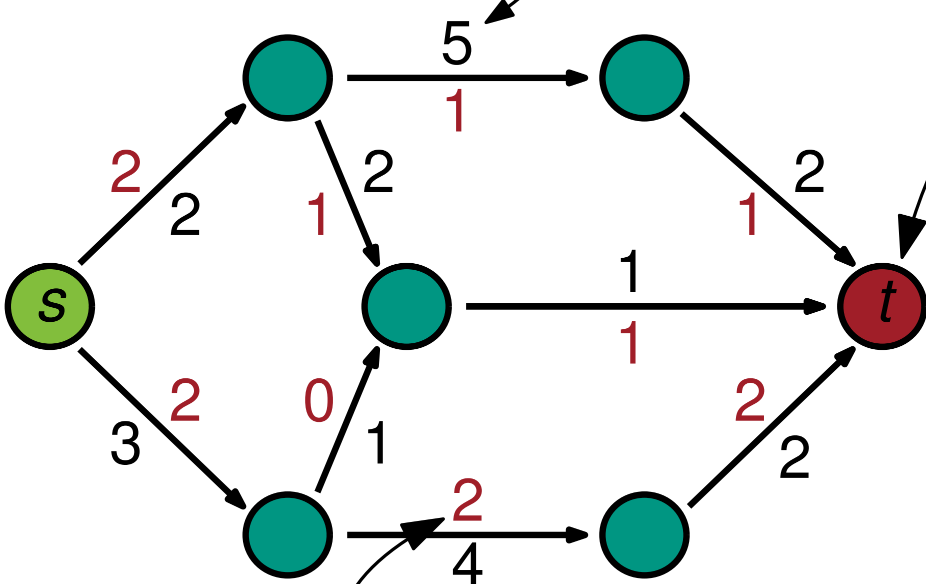

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Maximum Flow Problem

Find a **maximum flow** f from s to t such that $\forall f' : |f'| < |f|$

Flows

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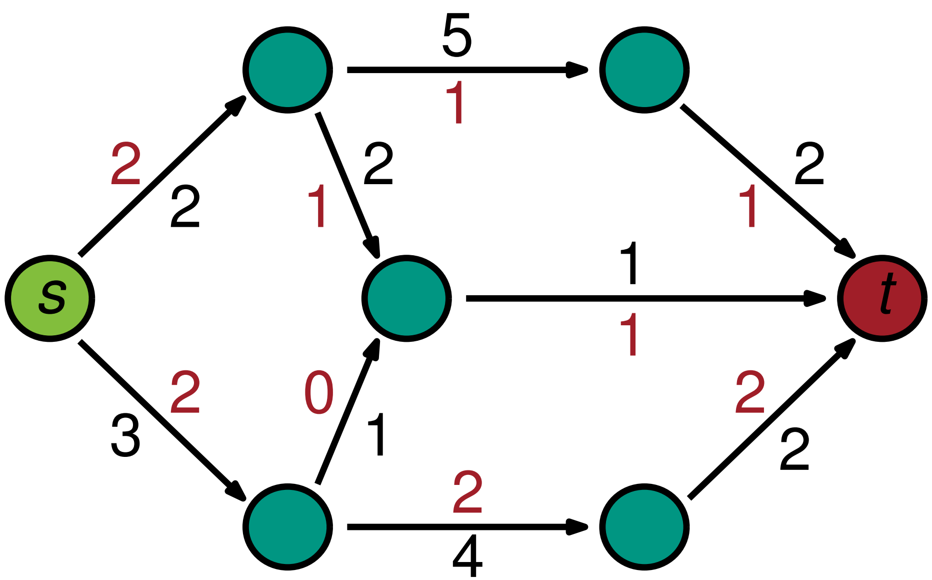

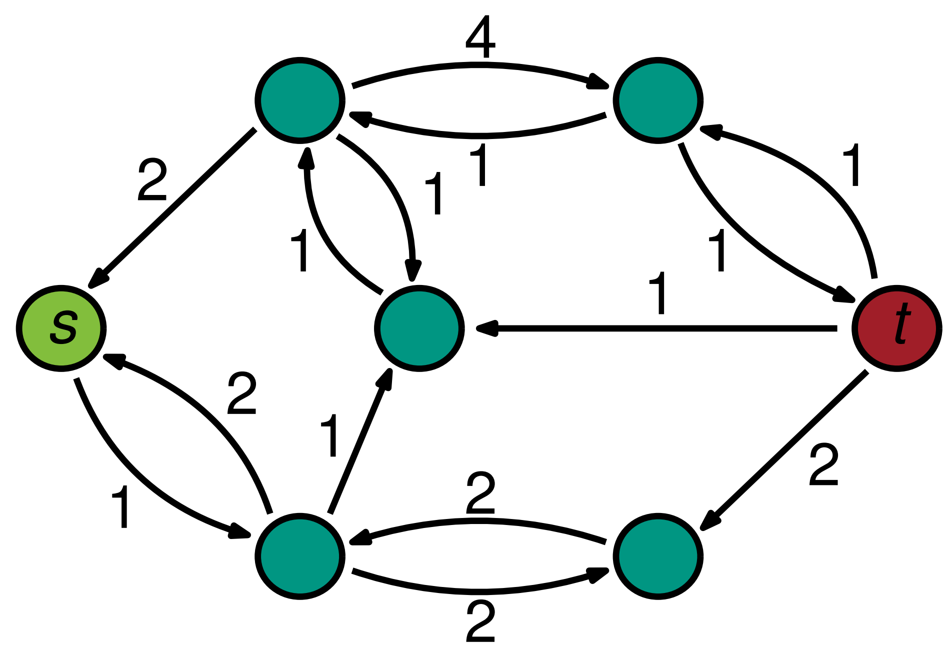

Residual Graph G_f

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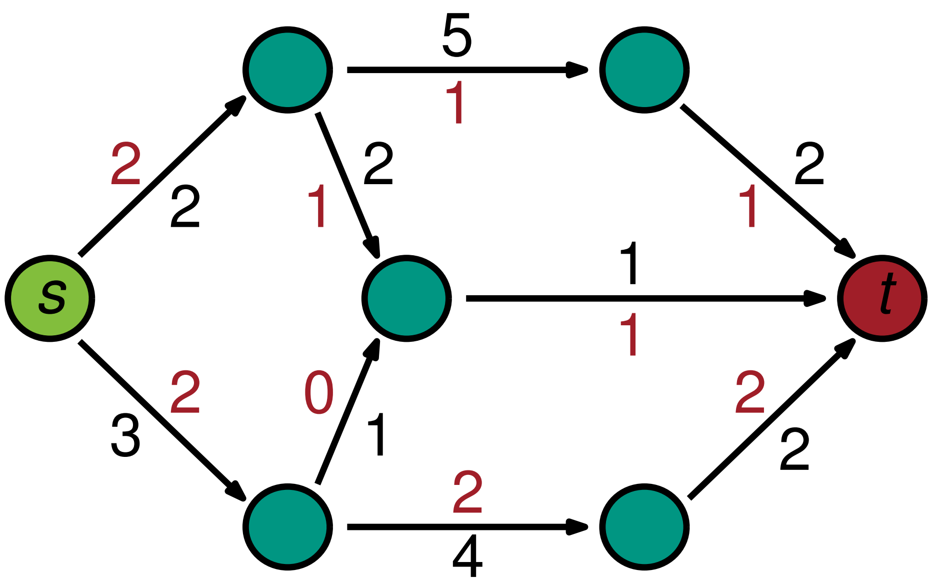
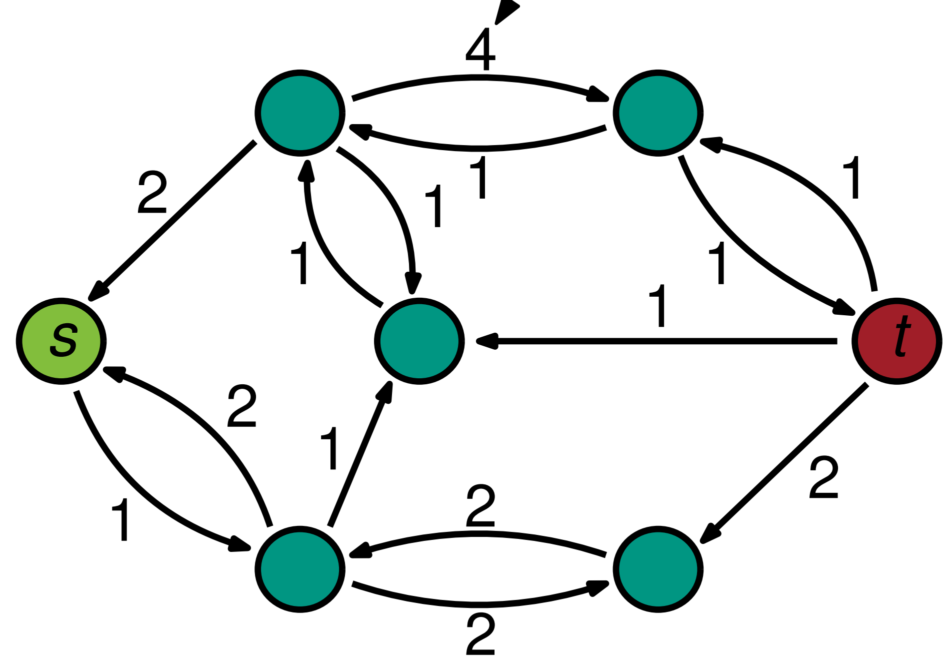
Flows



Residual capacity $r_f : V \times V \rightarrow \mathbb{R}_+$



Residual Graph G_f

Flows

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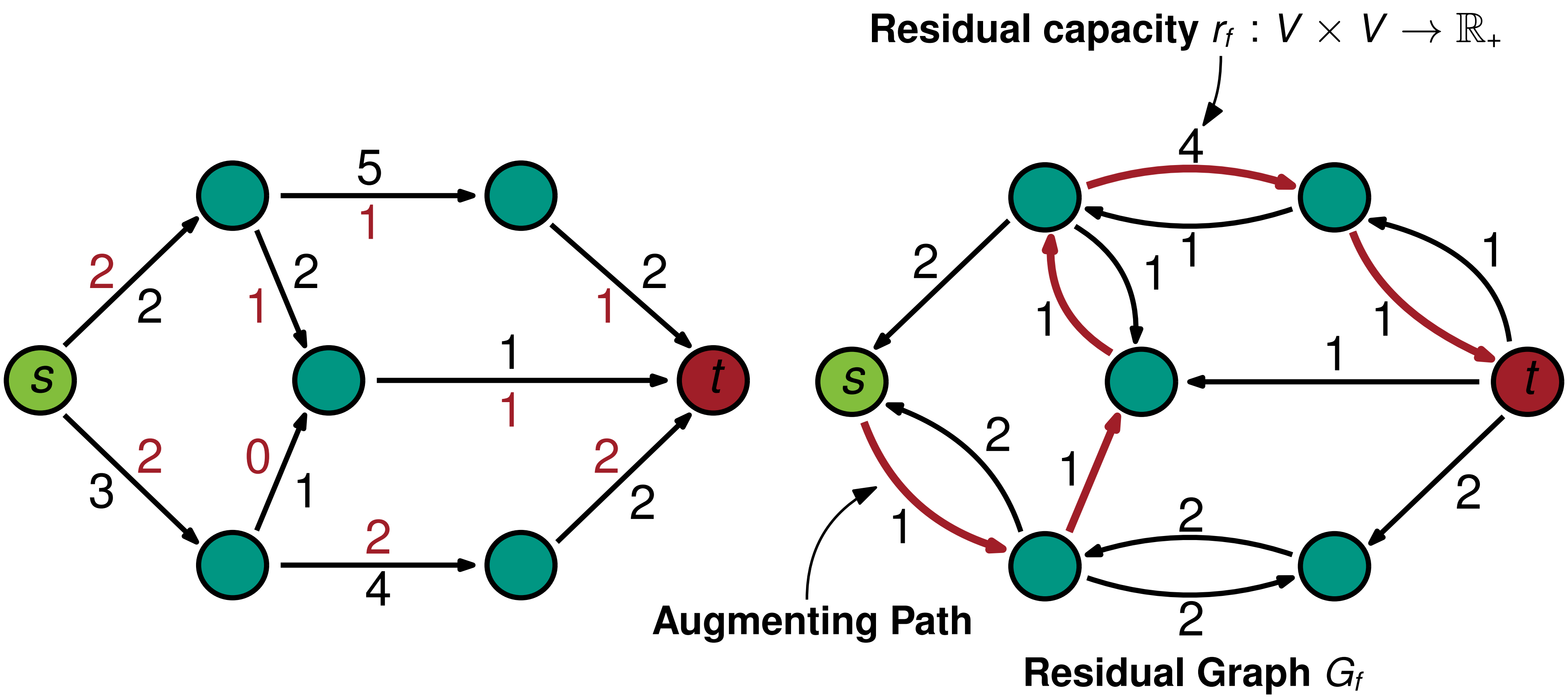
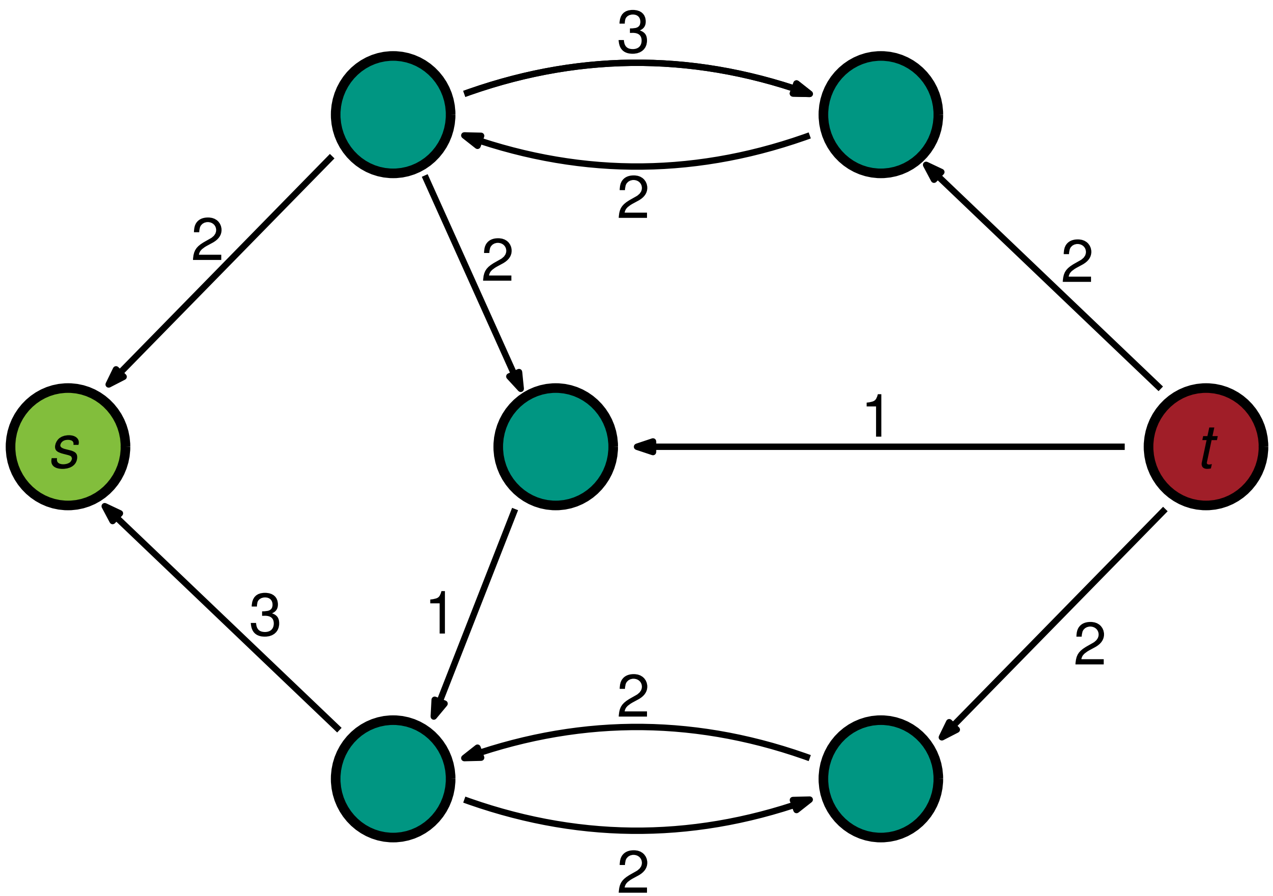


Minimum (s, t)-Bipartition

All nodes *reachable* from s are part of V_1 and $V_2 = V \setminus V_1$



Residual Graph G_f of a maximum flow f

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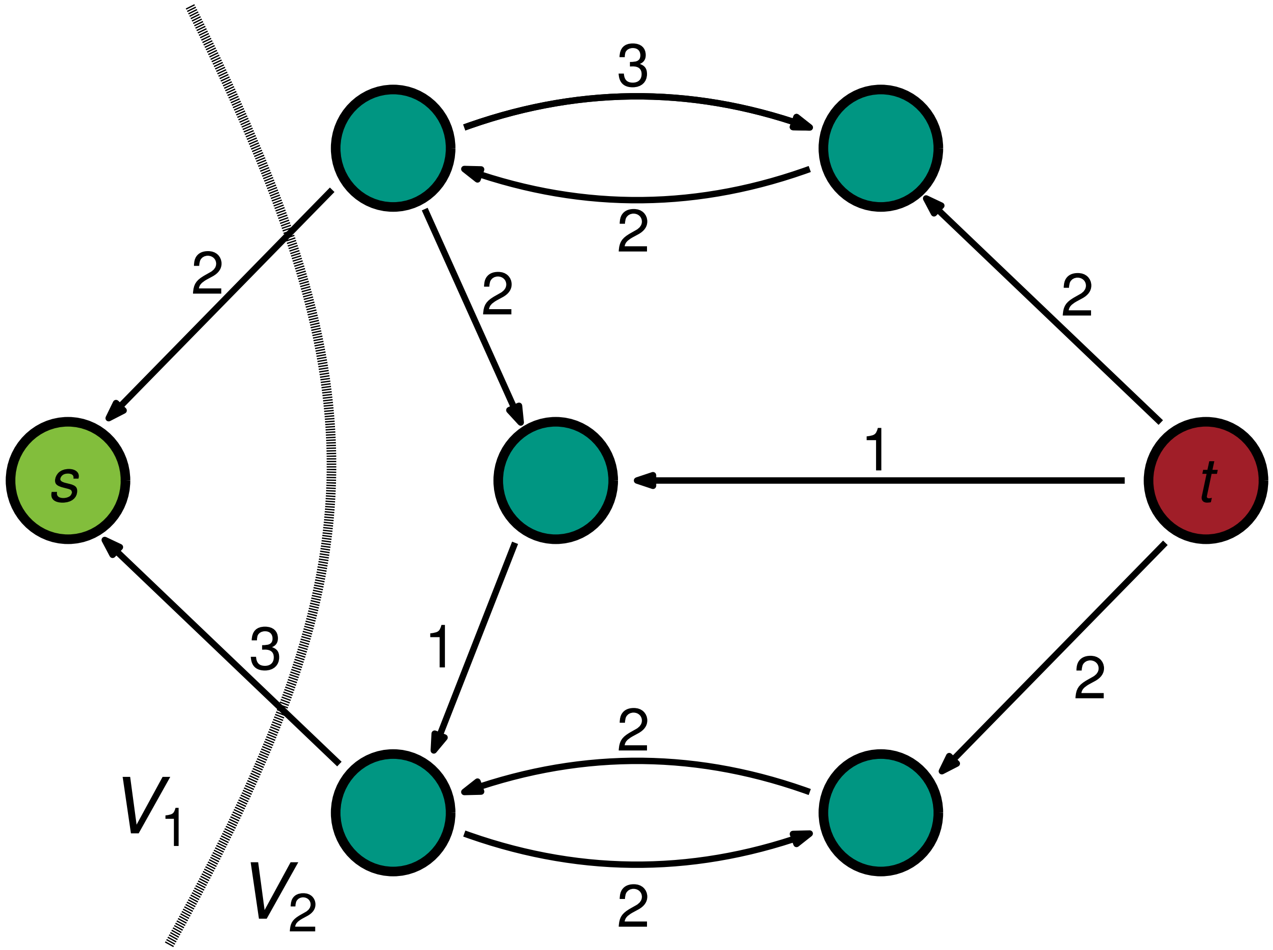

Residual Graph G_f of a maximum flow f

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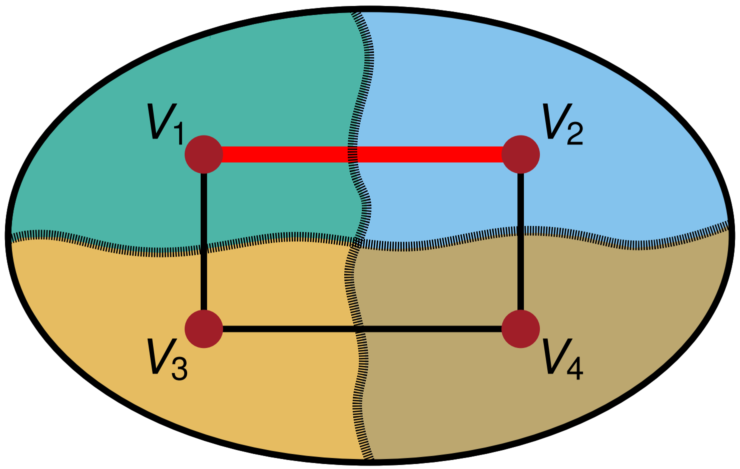
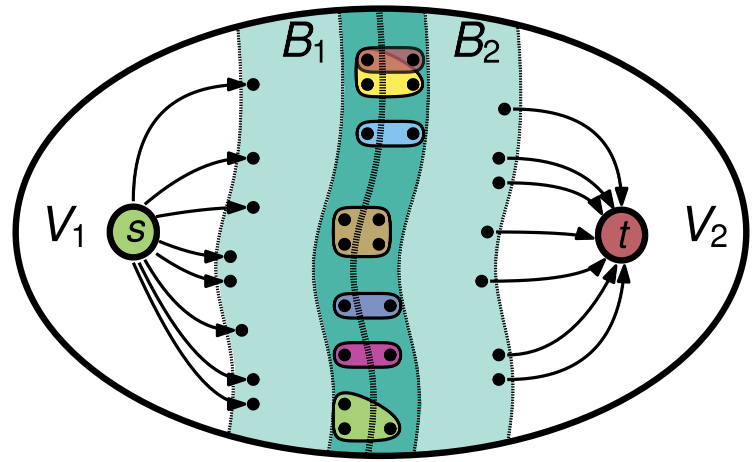
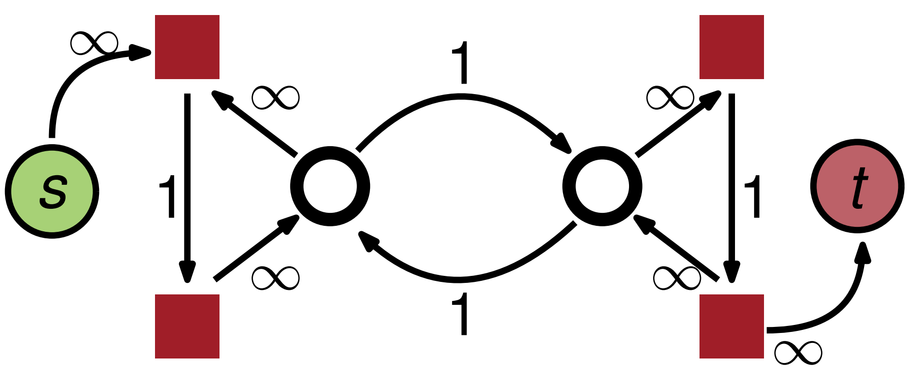
Our Flow-Based Refinement Framework



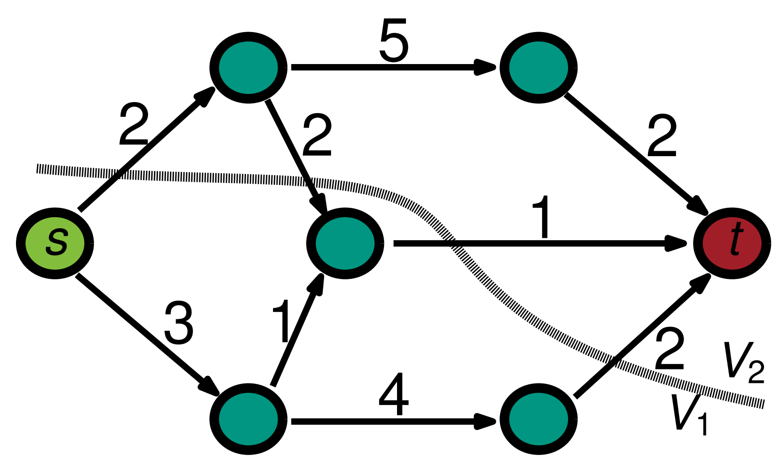
Select two adjacent blocks for refinement



Build Flow Problem

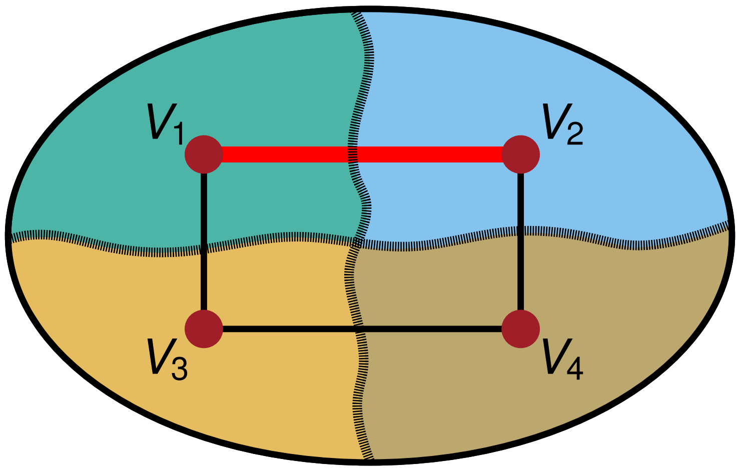


Solve Flow Problem



Find feasible minimum cut

Our Flow-Based Refinement Framework



Select two adjacent blocks for refinement

Build Flow Problem

Solve Flow Problem

Find feasible minimum cut

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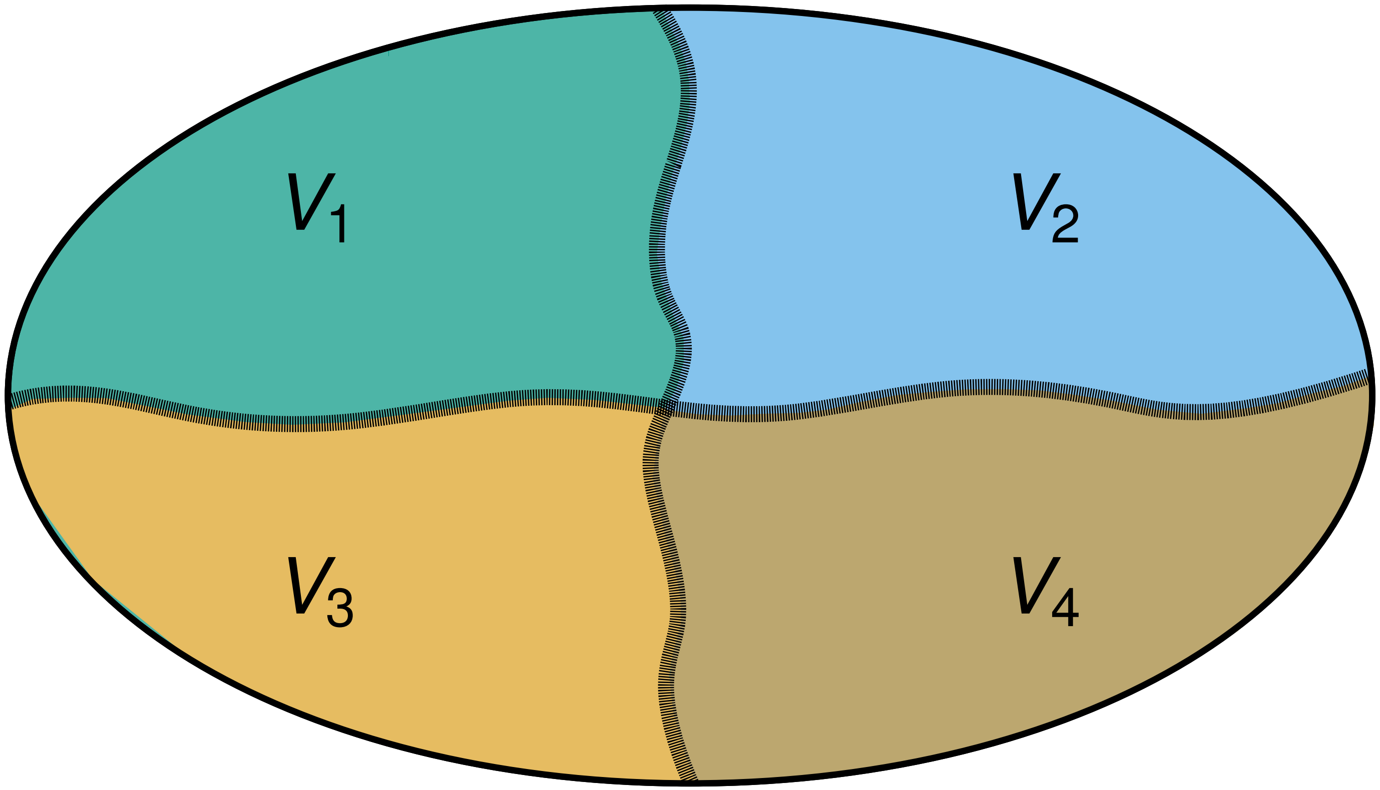
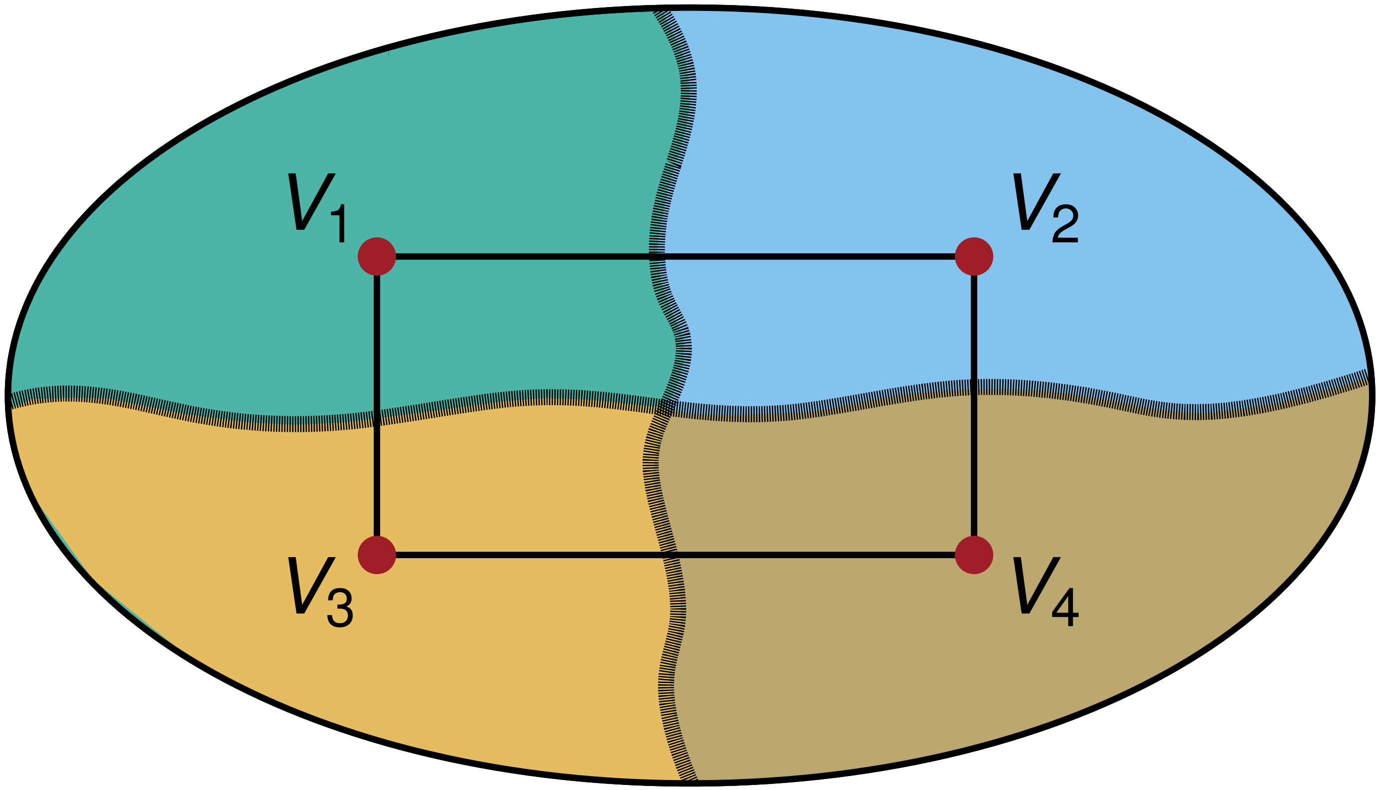
Active Block Scheduling



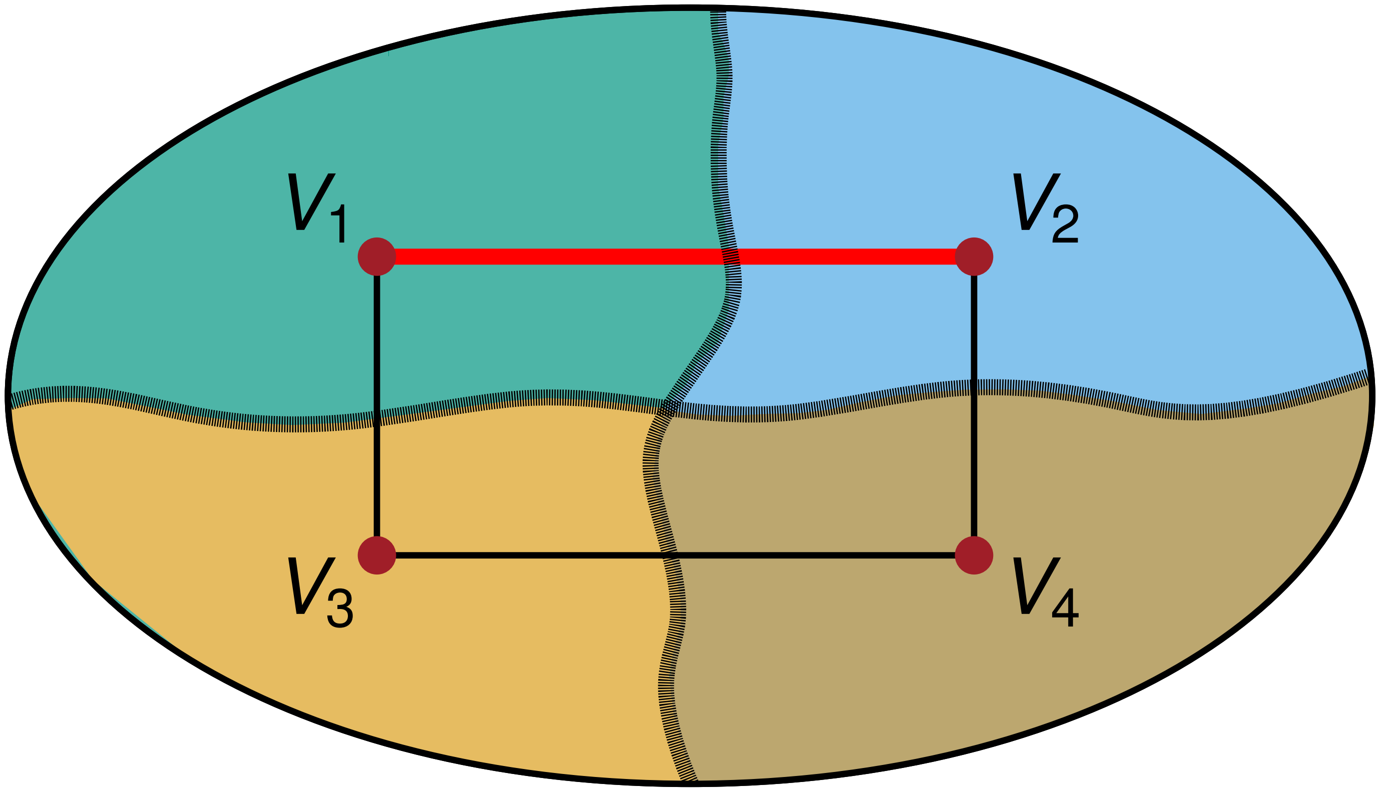
Active Block Scheduling



Build Quotient Graph

Active Block Scheduling

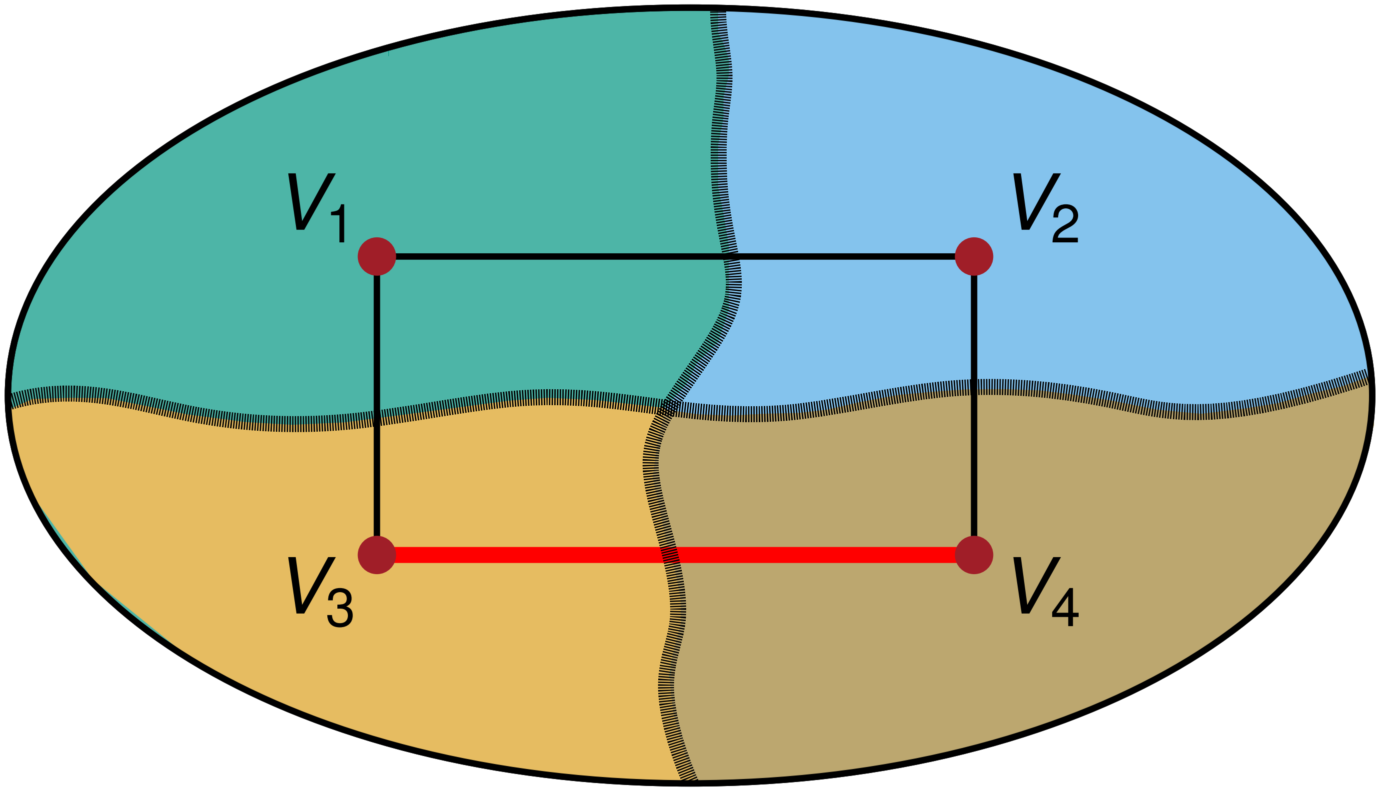
Round 1 *refine*(V_1, V_2) = **Improvement!**



Using 2-way refinement algorithm for **active** blocks of the quotient graph

Active Block Scheduling

Round 1 $\mathit{refine}(V_3, V_4)$ = **No Improvement!**



Using 2-way refinement algorithm for **active** blocks of the quotient graph

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Active Block Scheduling

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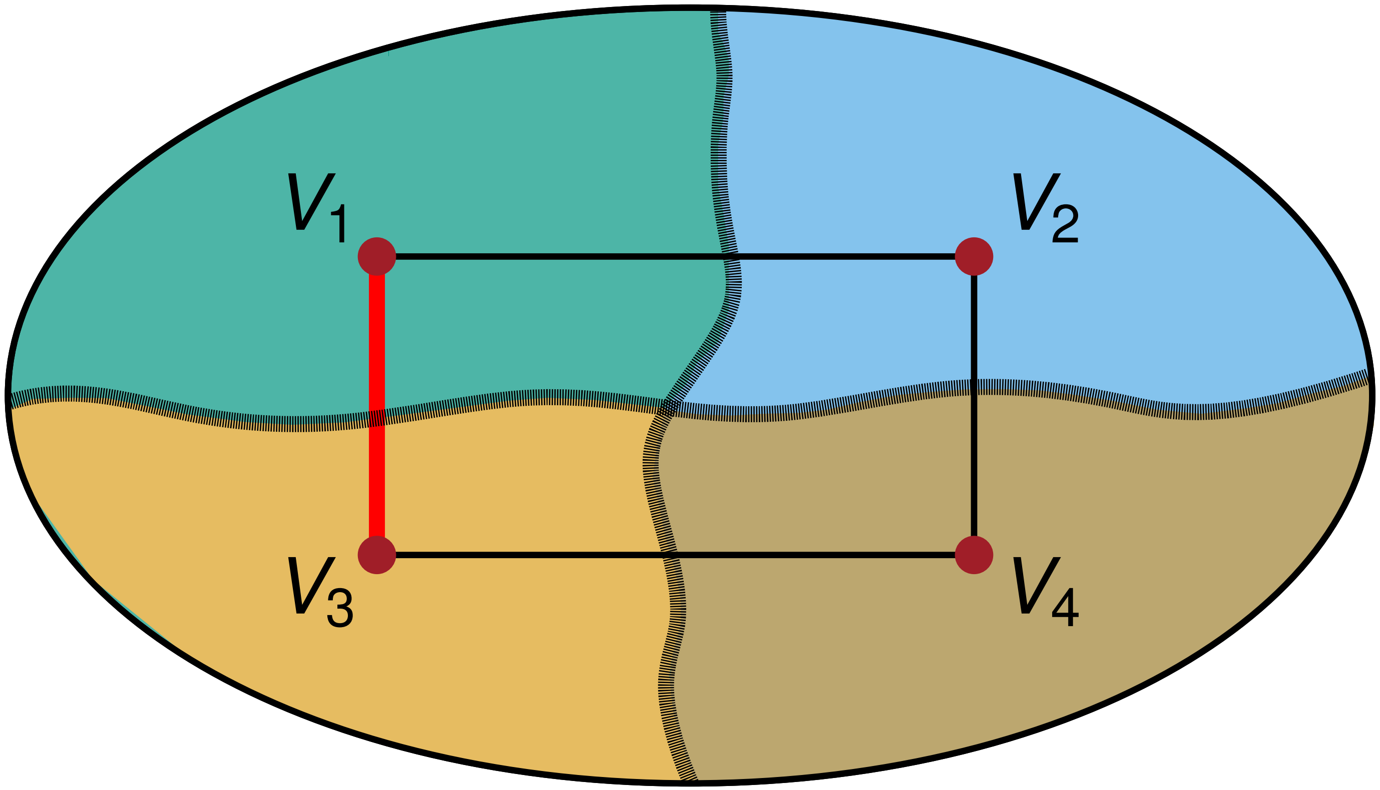
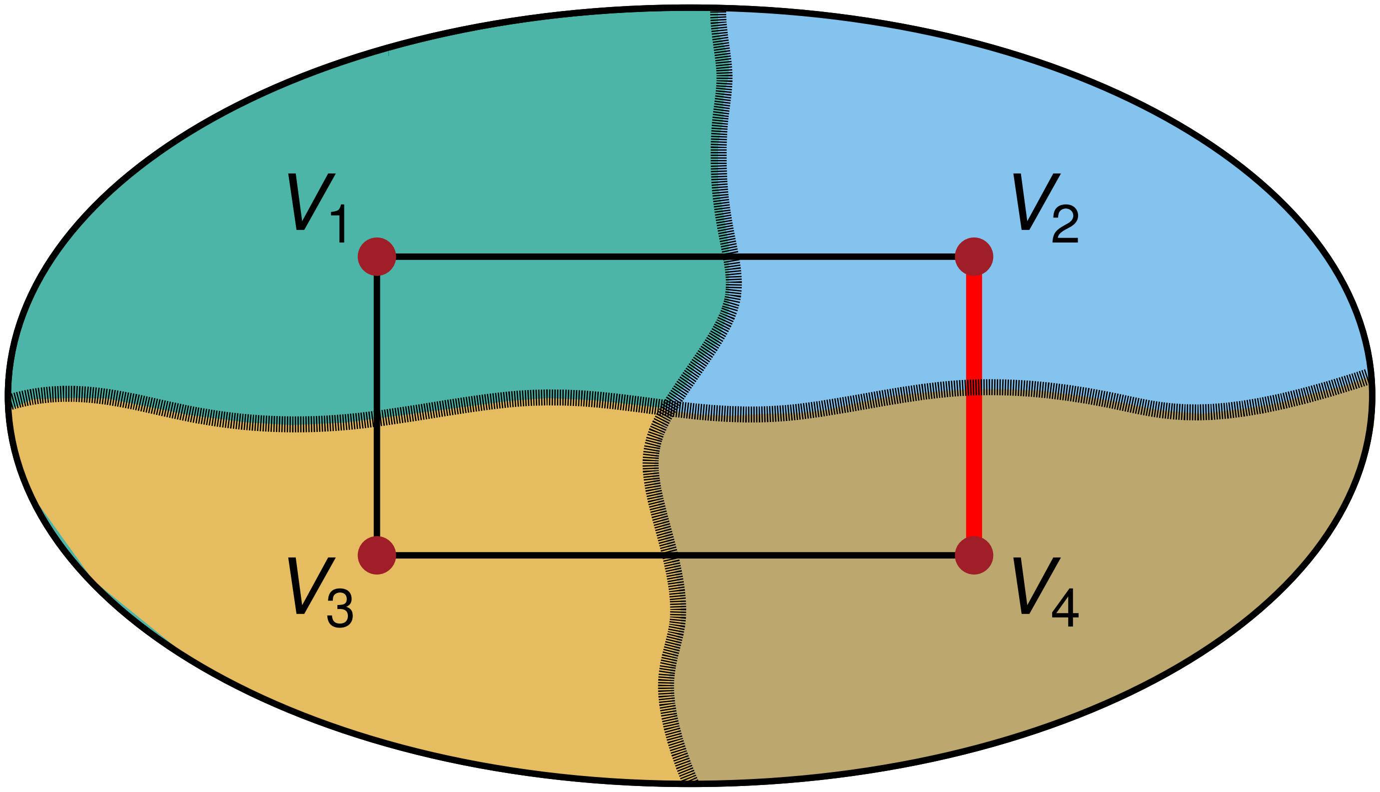


Using 2-way refinement algorithm for **active** blocks of the quotient graph

Active Block Scheduling

Round 1

$refine(V_2, V_4)$ = **No Improvement!**



Using 2-way refinement algorithm for **active** blocks of the quotient graph

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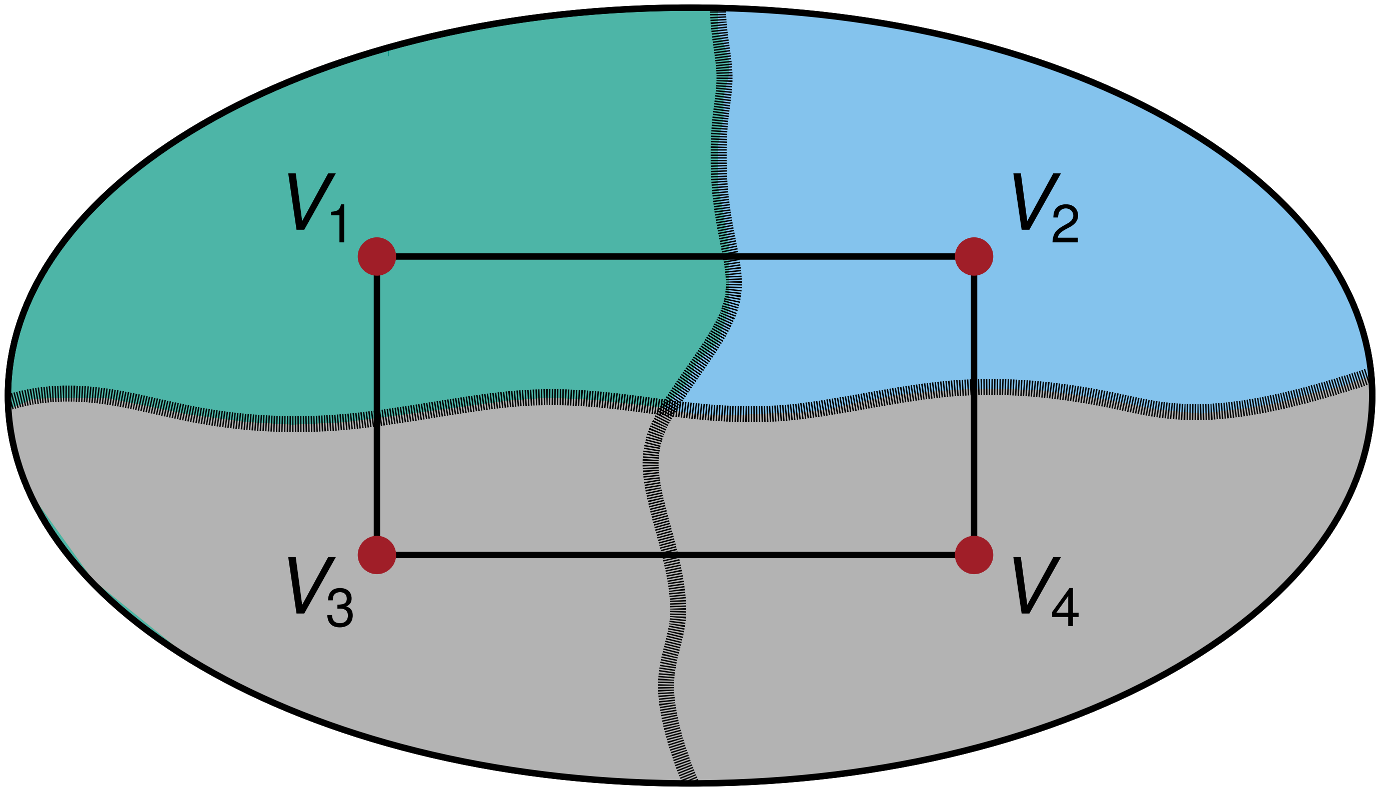
Active Block Scheduling

Round 1 Boundary did not change ⇒ Mark block as **inactive**

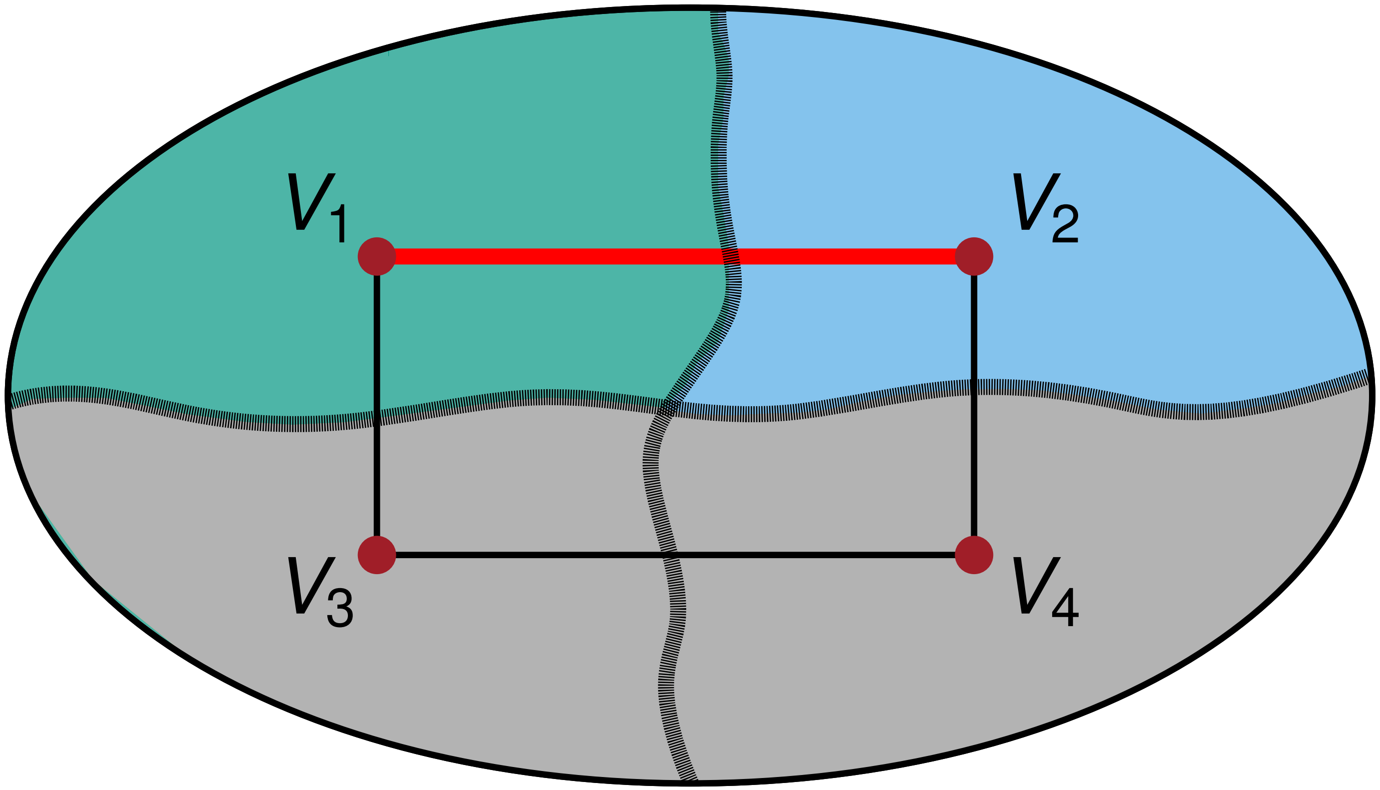


Using 2-way refinement algorithm for **active** blocks of the quotient graph

Active Block Scheduling

Round 2

refine(V_1, V_2) = **No Improvement!**



Using 2-way refinement algorithm for **active** blocks of the quotient graph

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Active Block Scheduling

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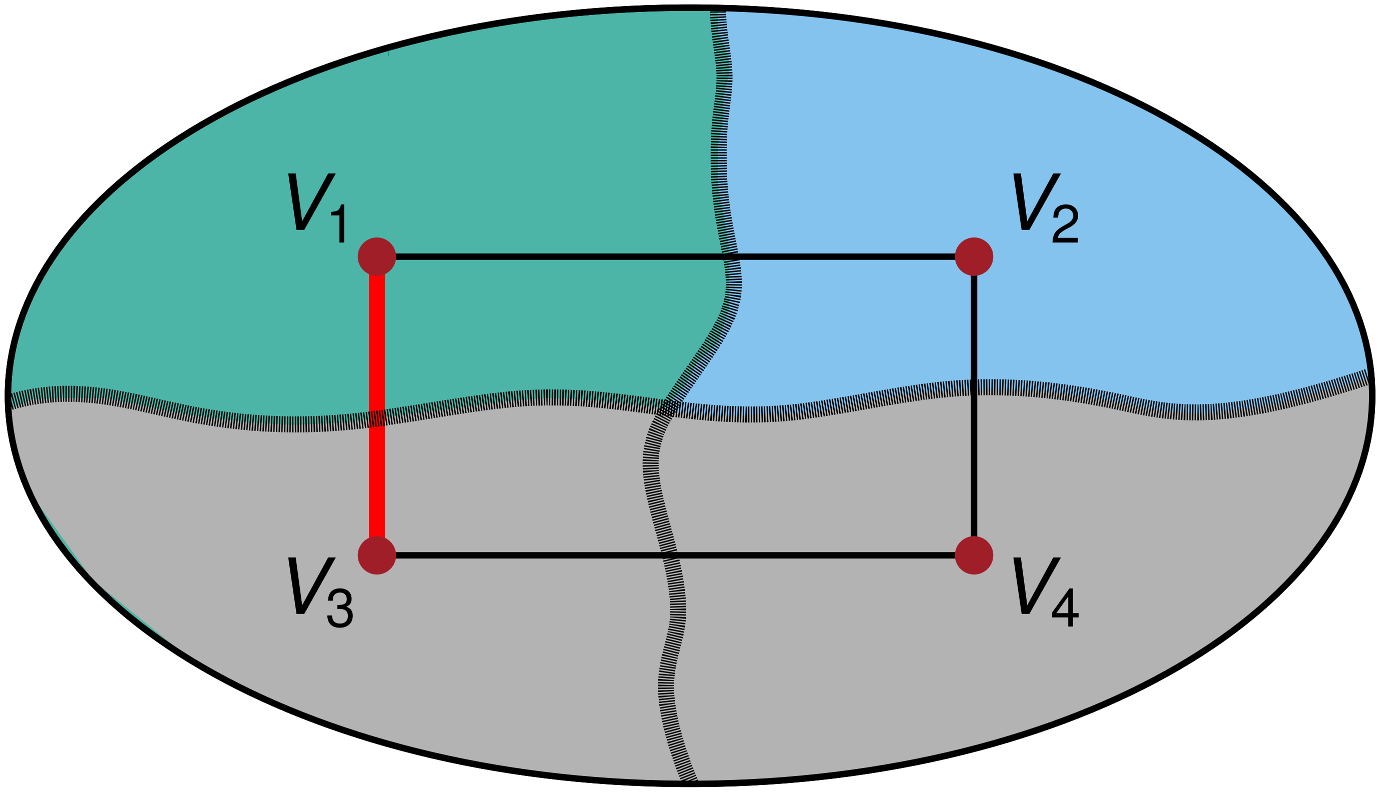

Using 2-way refinement algorithm for **active** blocks of the quotient graph

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Active Block Scheduling

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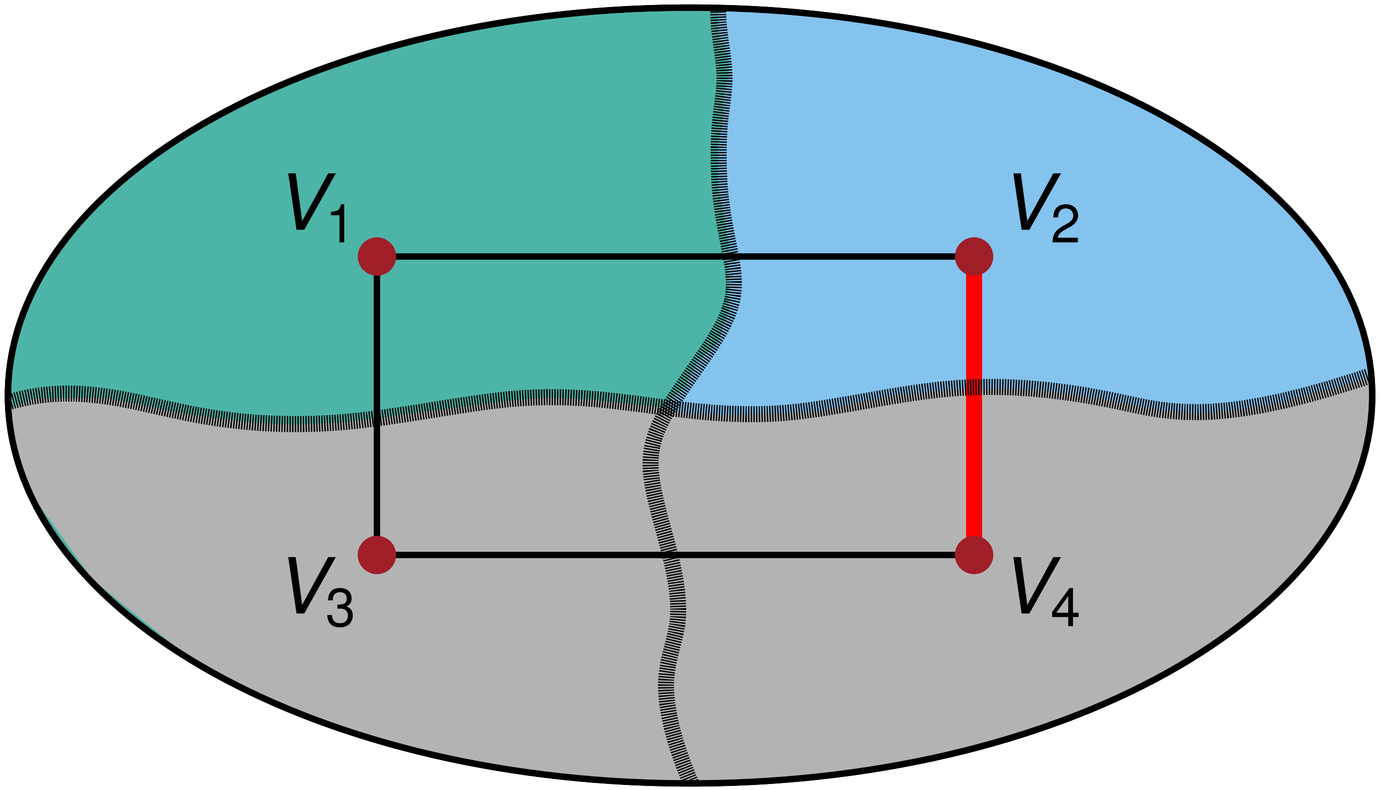
Round 2 *refine*(V_2, V_4) = **No Improvement!**



Using 2-way refinement algorithm for **active** blocks of the quotient graph

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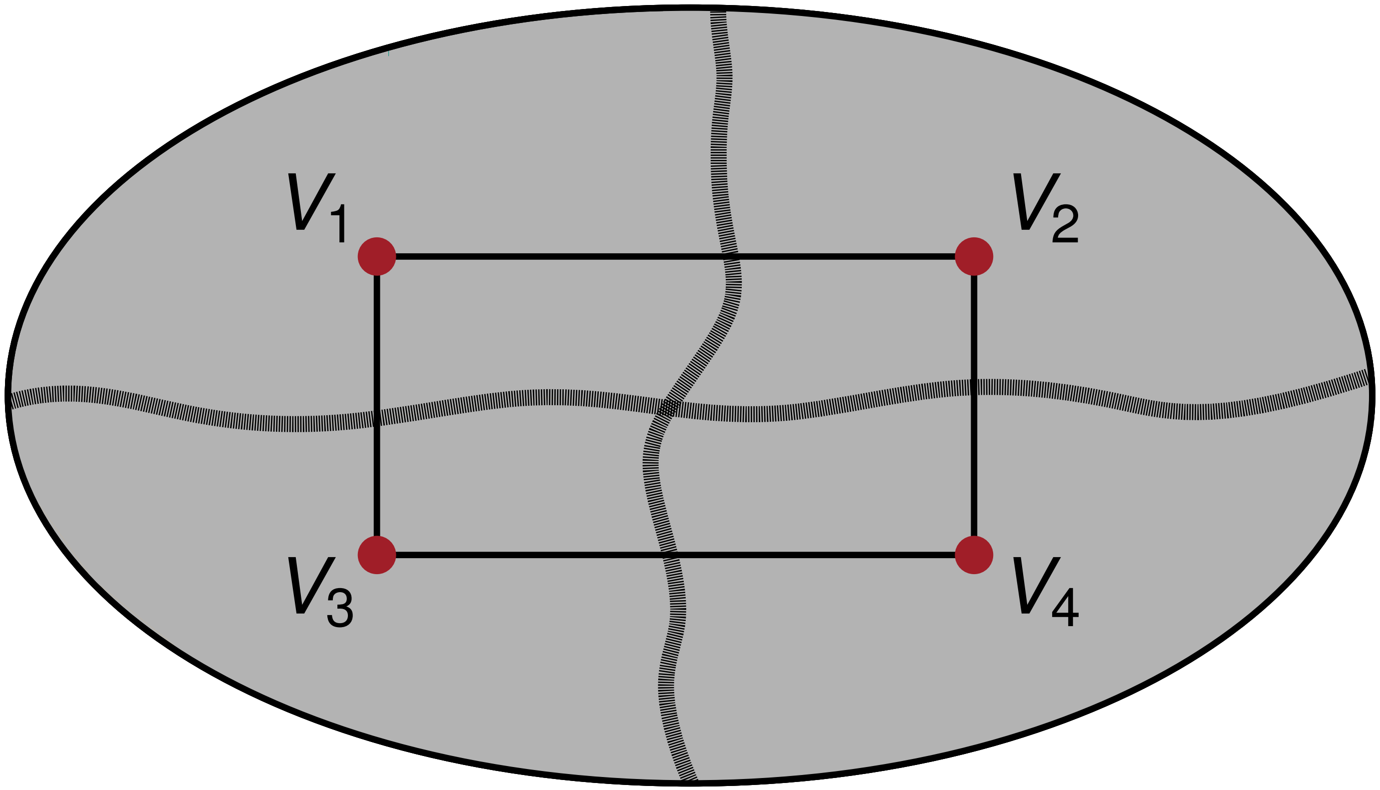
Active Block Scheduling

Round 2 Boundary did not change $\Rightarrow$ Mark block as **inactive**



Using 2-way refinement algorithm for **active** blocks of the quotient graph

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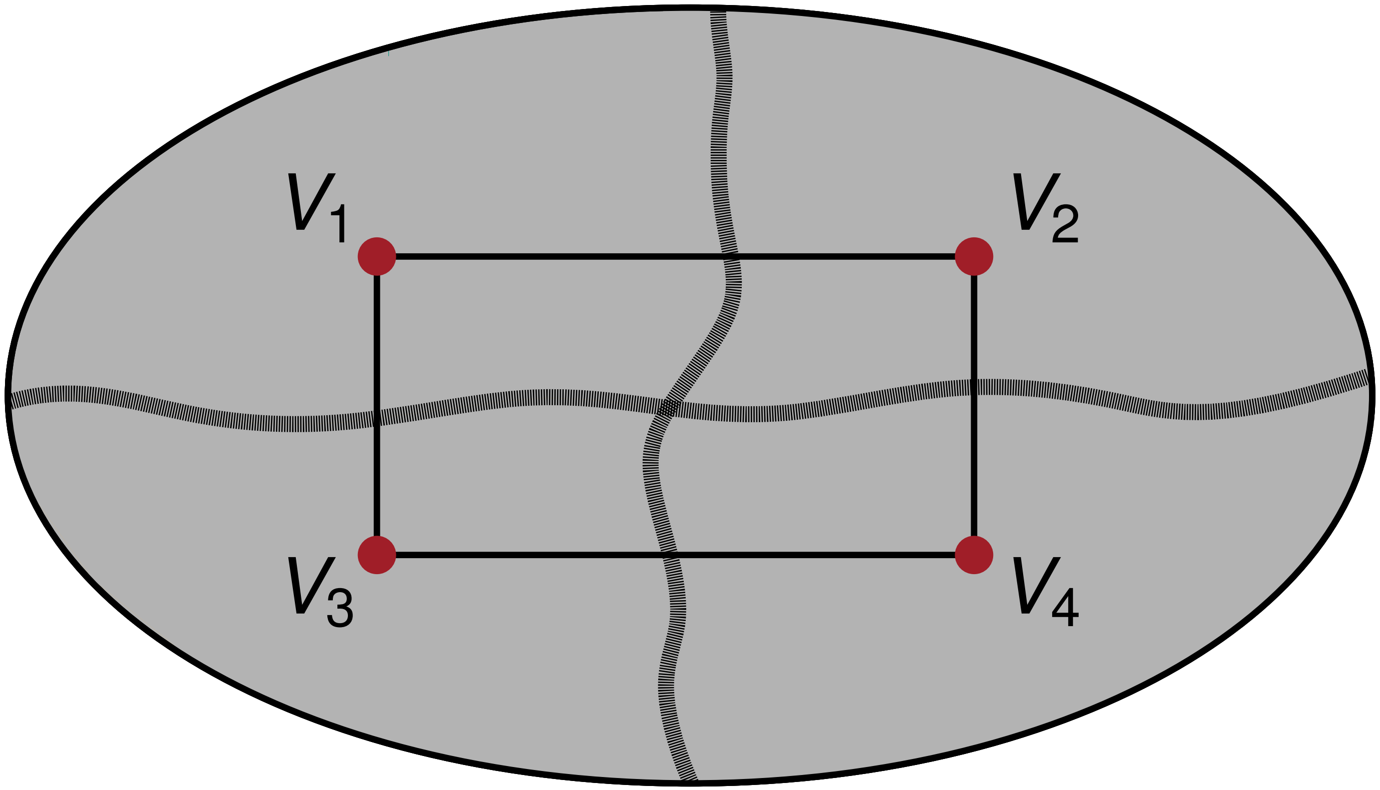
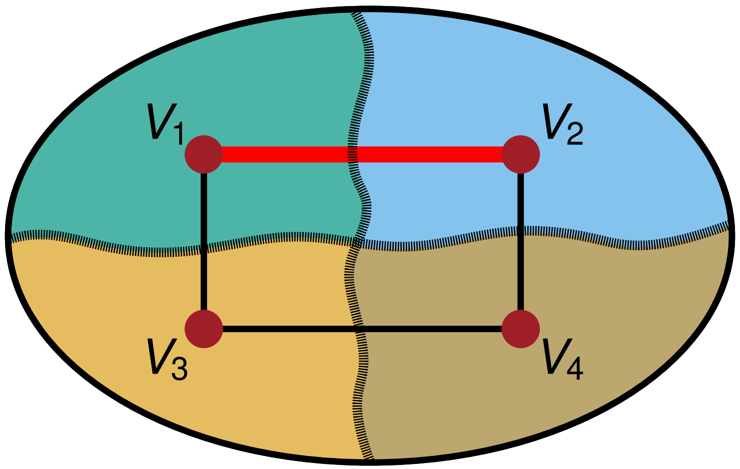
Active Block Scheduling

Round 2 All blocks are **inactive** $\Rightarrow$ Algorithm terminates

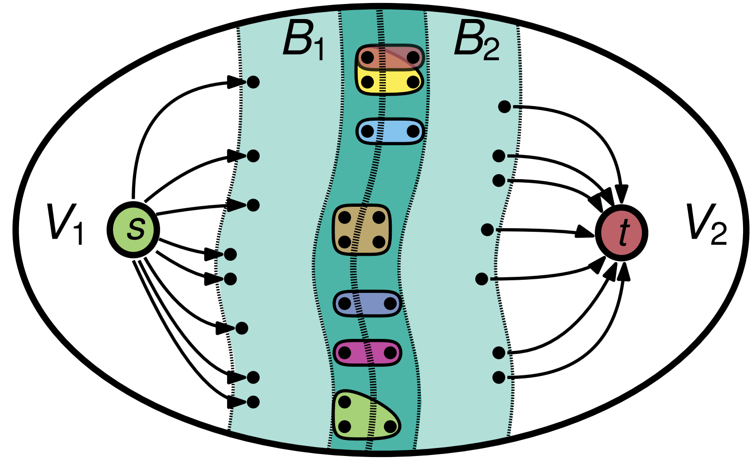


Using 2-way refinement algorithm for **active** blocks of the quotient graph

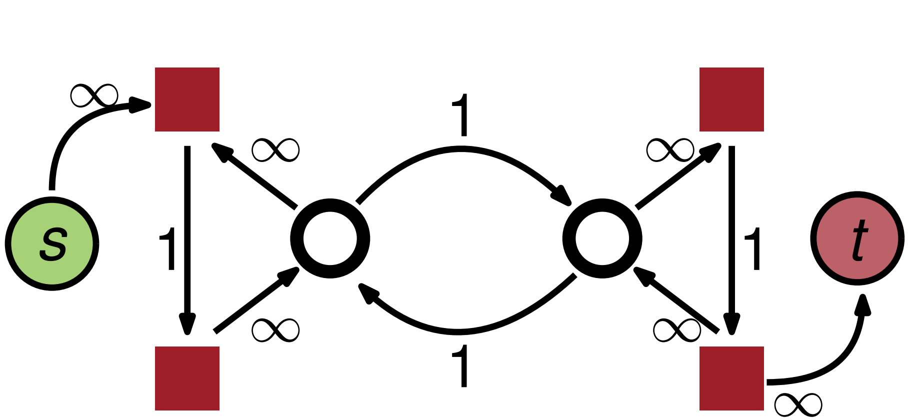
Our Flow-Based Refinement Framework



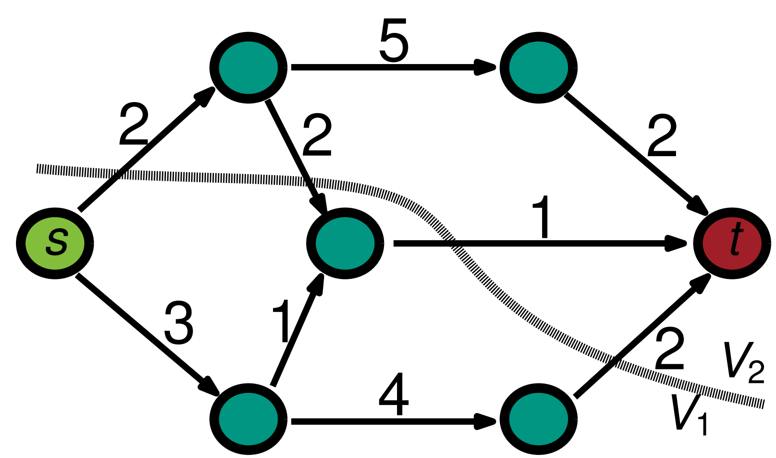
Select two adjacent blocks for refinement



Build Flow Problem



Solve Flow Problem



Find feasible minimum cut

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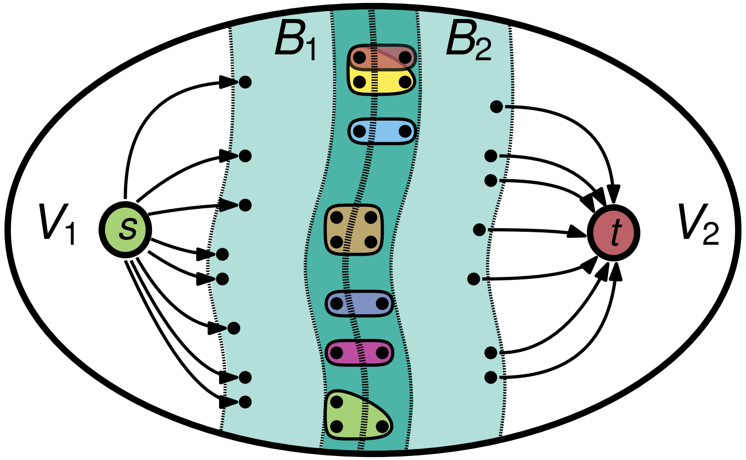
Our Flow-Based Refinement Framework

Select two adjacent blocks for refinement



Build Flow Problem

Solve Flow Problem

Find feasible minimum cut

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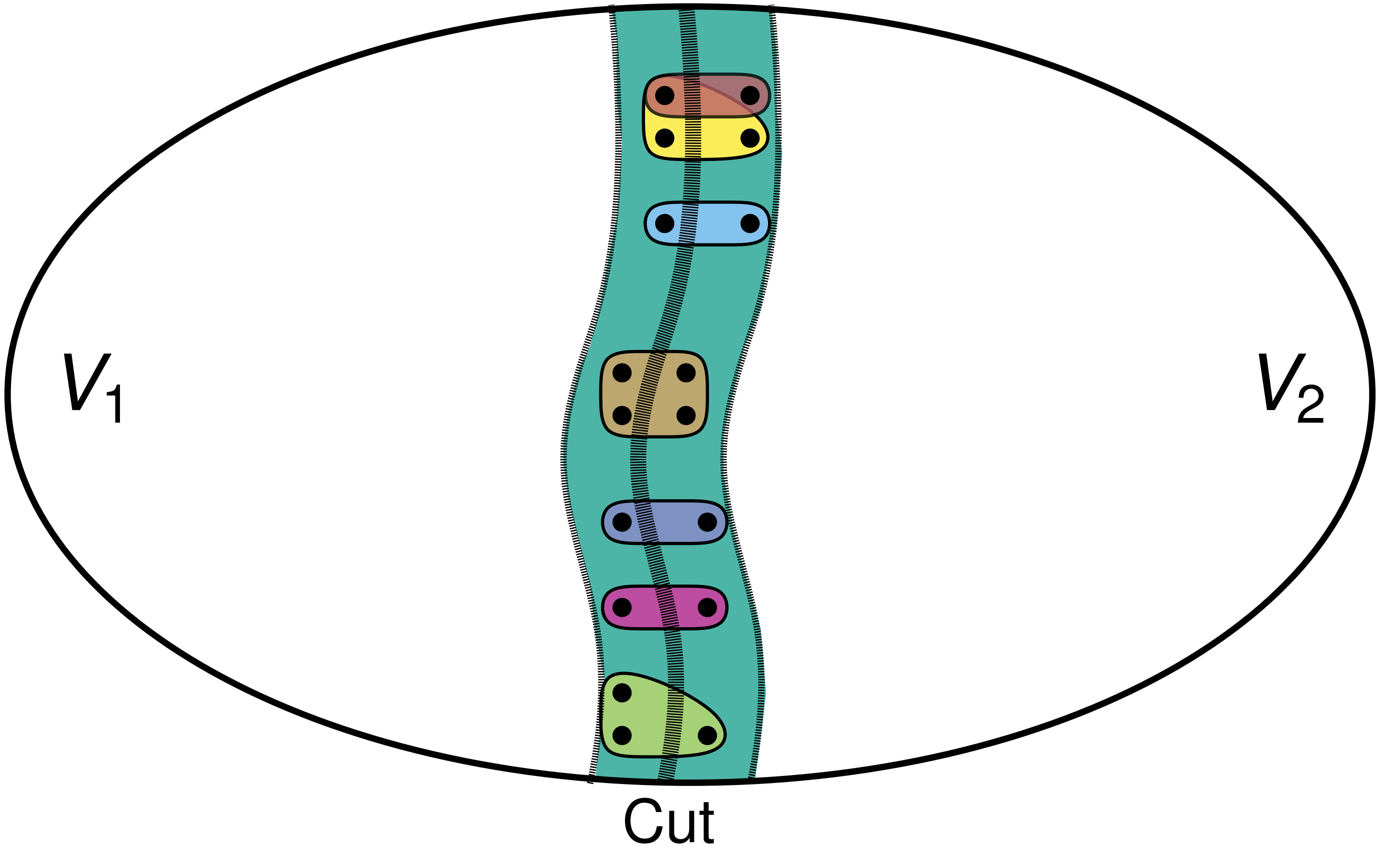
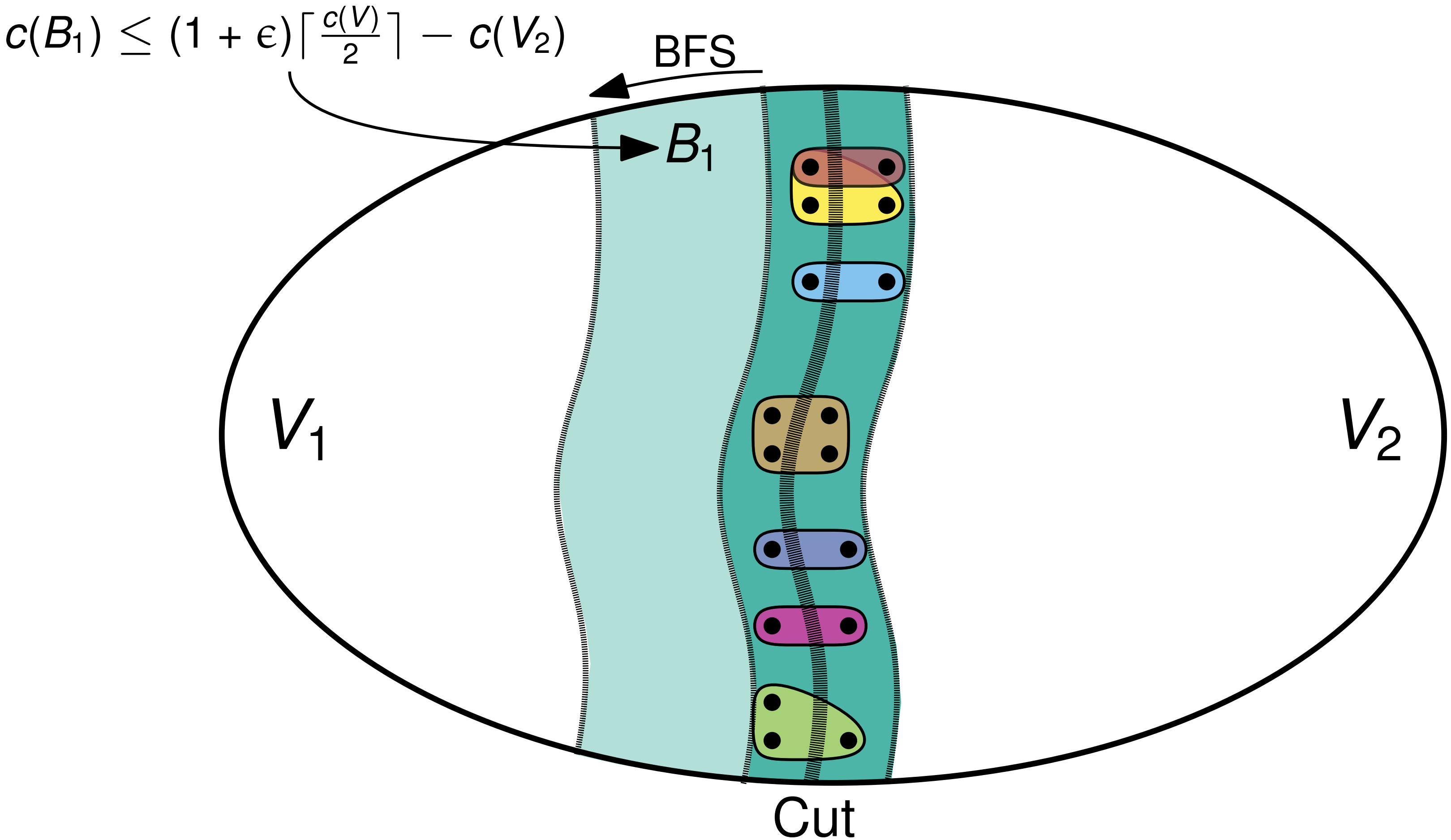
Adaptive Flow Iterations



Adaptive Flow Iterations



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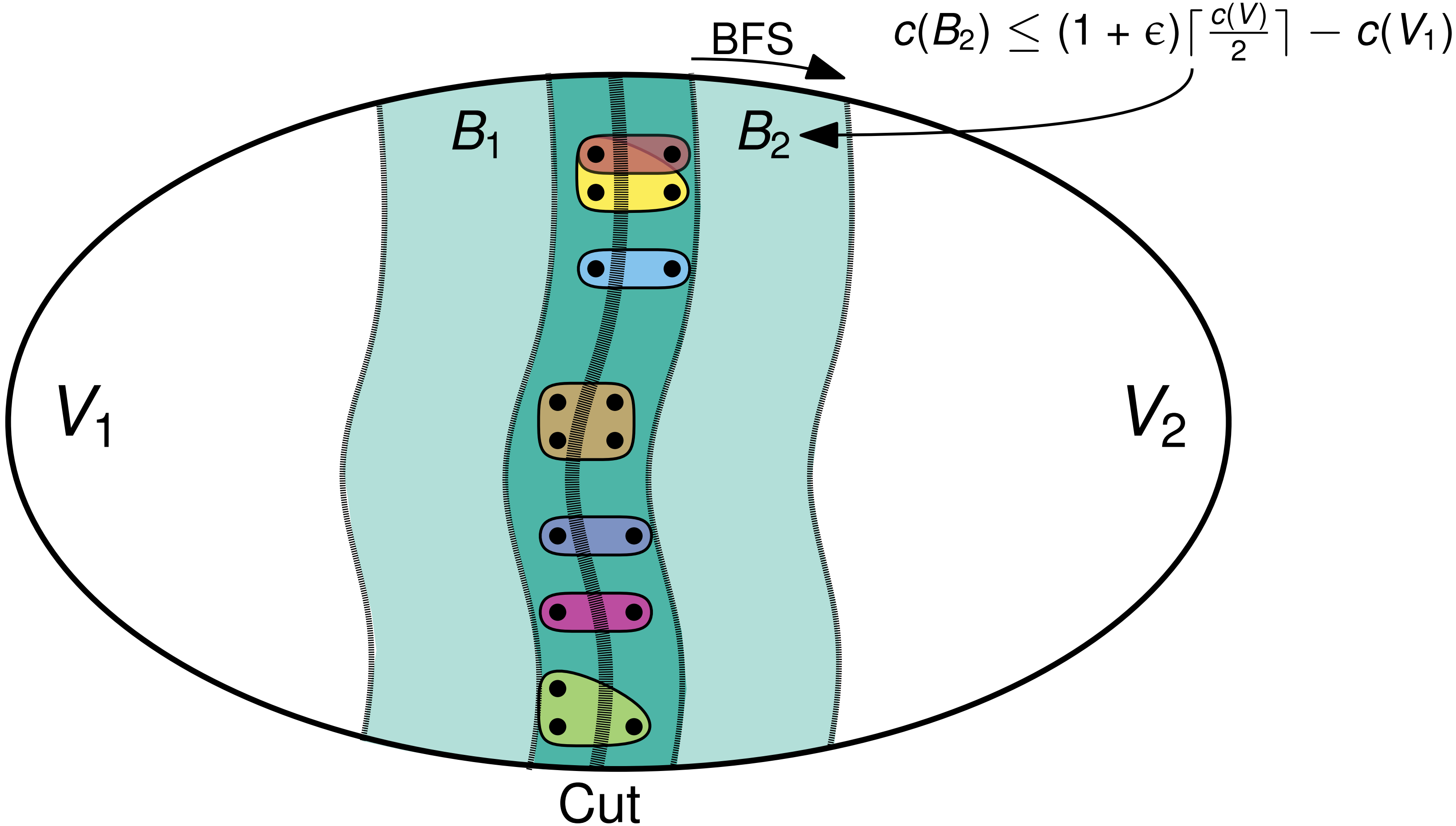
Adaptive Flow Iterations



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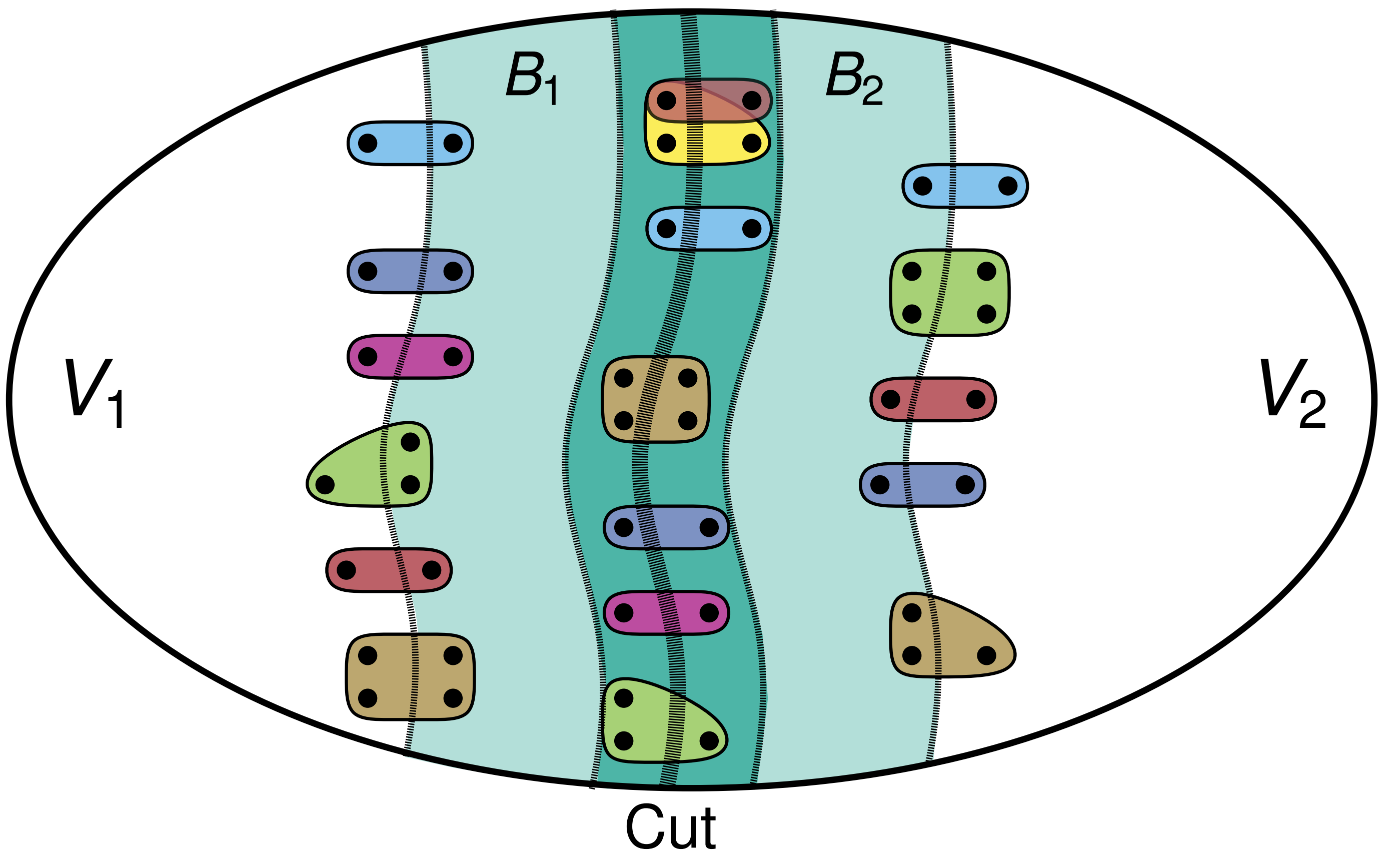
Adaptive Flow Iterations



Adaptive Flow Iterations

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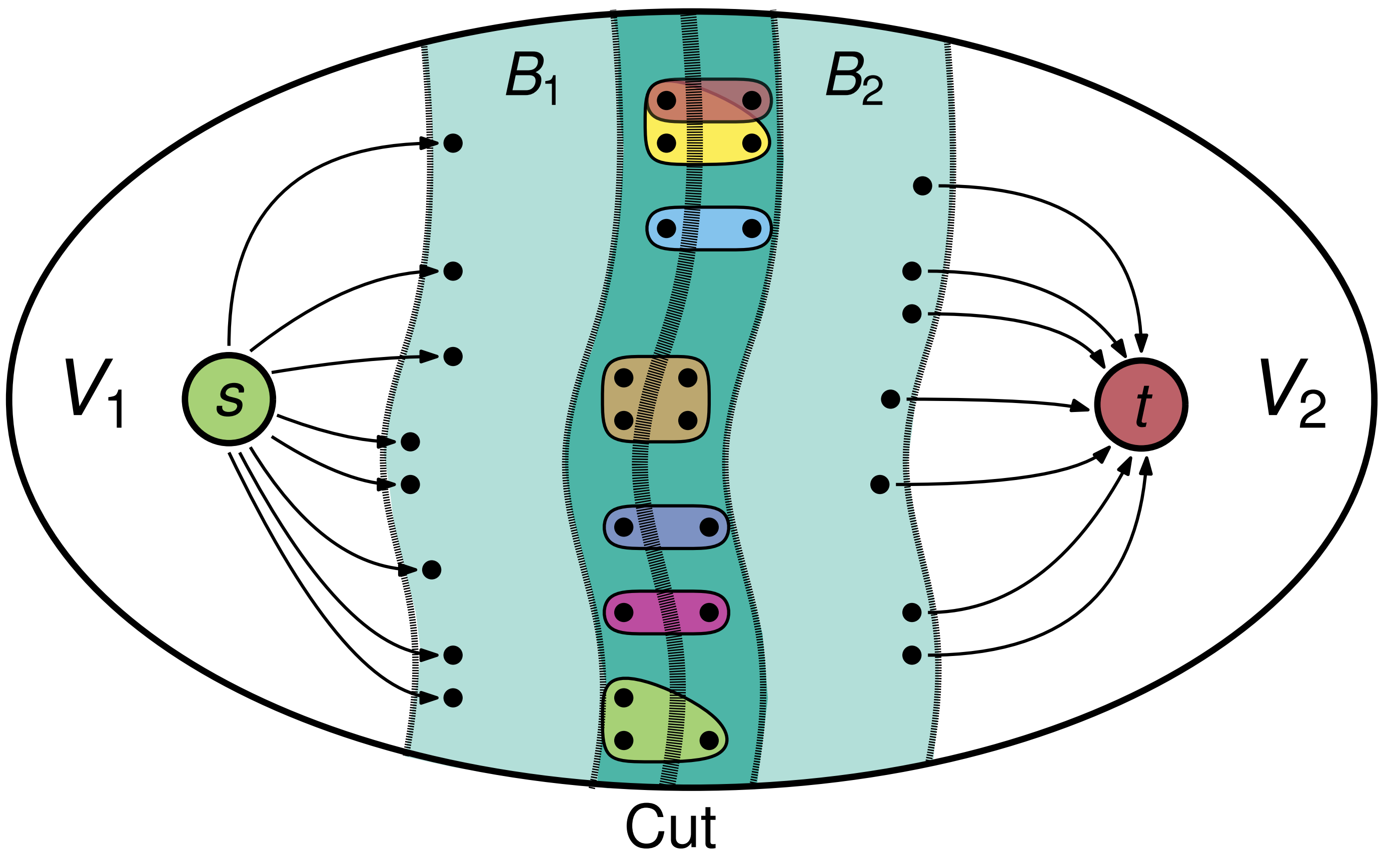
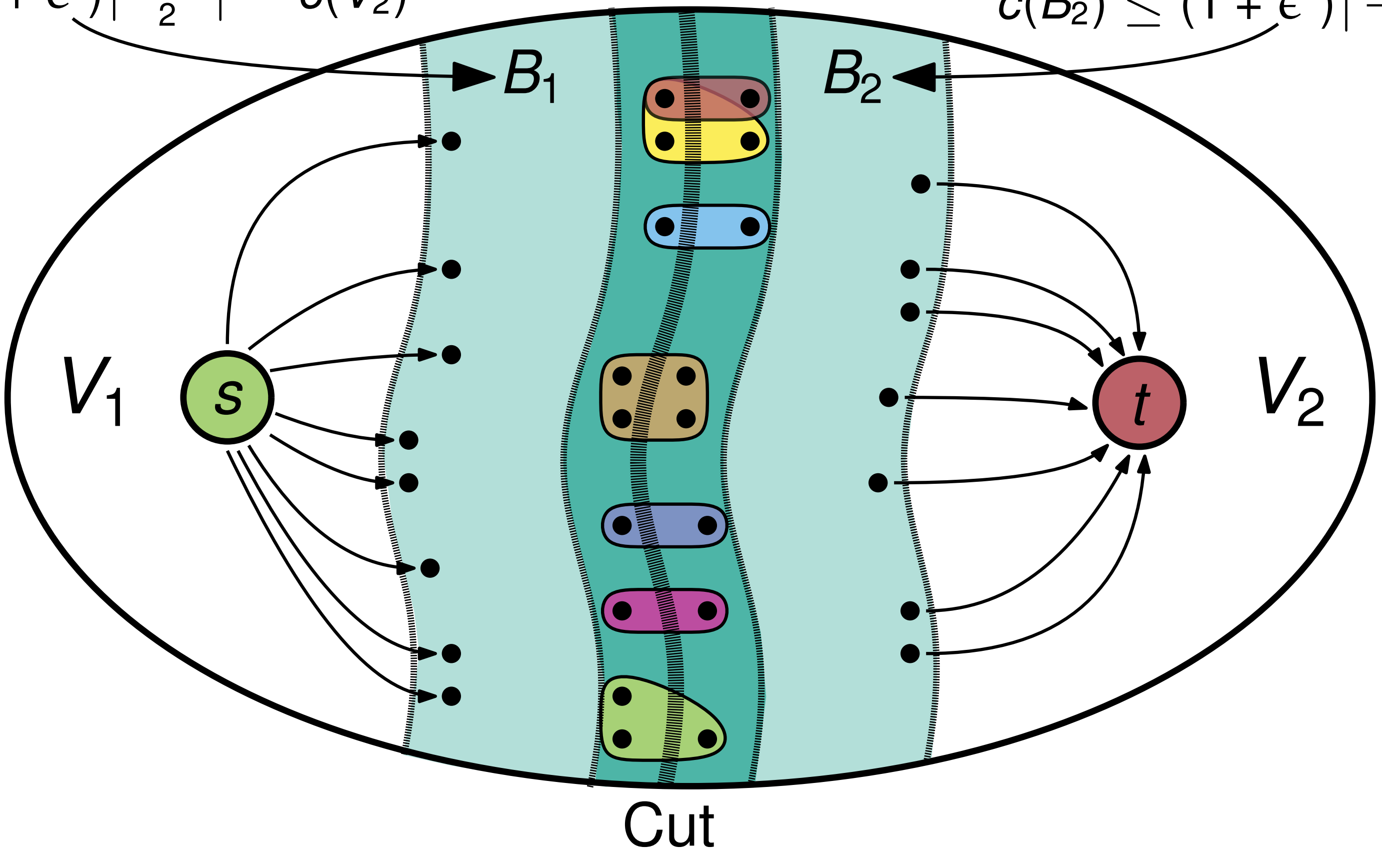


Adaptive Flow Iterations

Use $\epsilon' = \alpha\epsilon$ instead of ϵ

$c(B_1) \leq (1+\epsilon')\lceil \frac{c(V)}{2} \rceil - c(V_2)$

$c(B_2) \leq (1+\epsilon')\lceil \frac{c(V)}{2} \rceil - c(V_1)$



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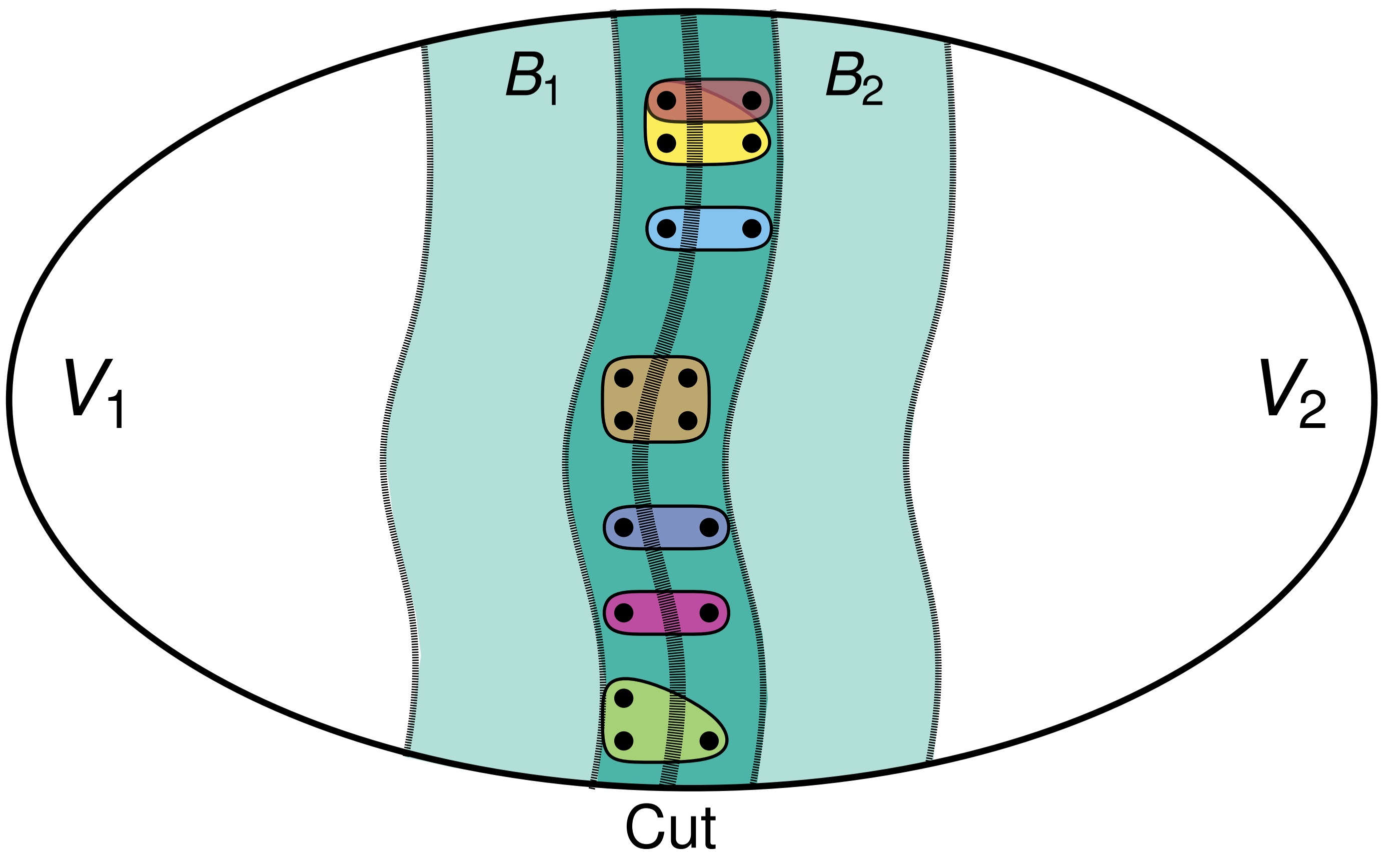
Adaptive Flow Iterations

Use $\epsilon' = \alpha\epsilon$ instead of ϵ

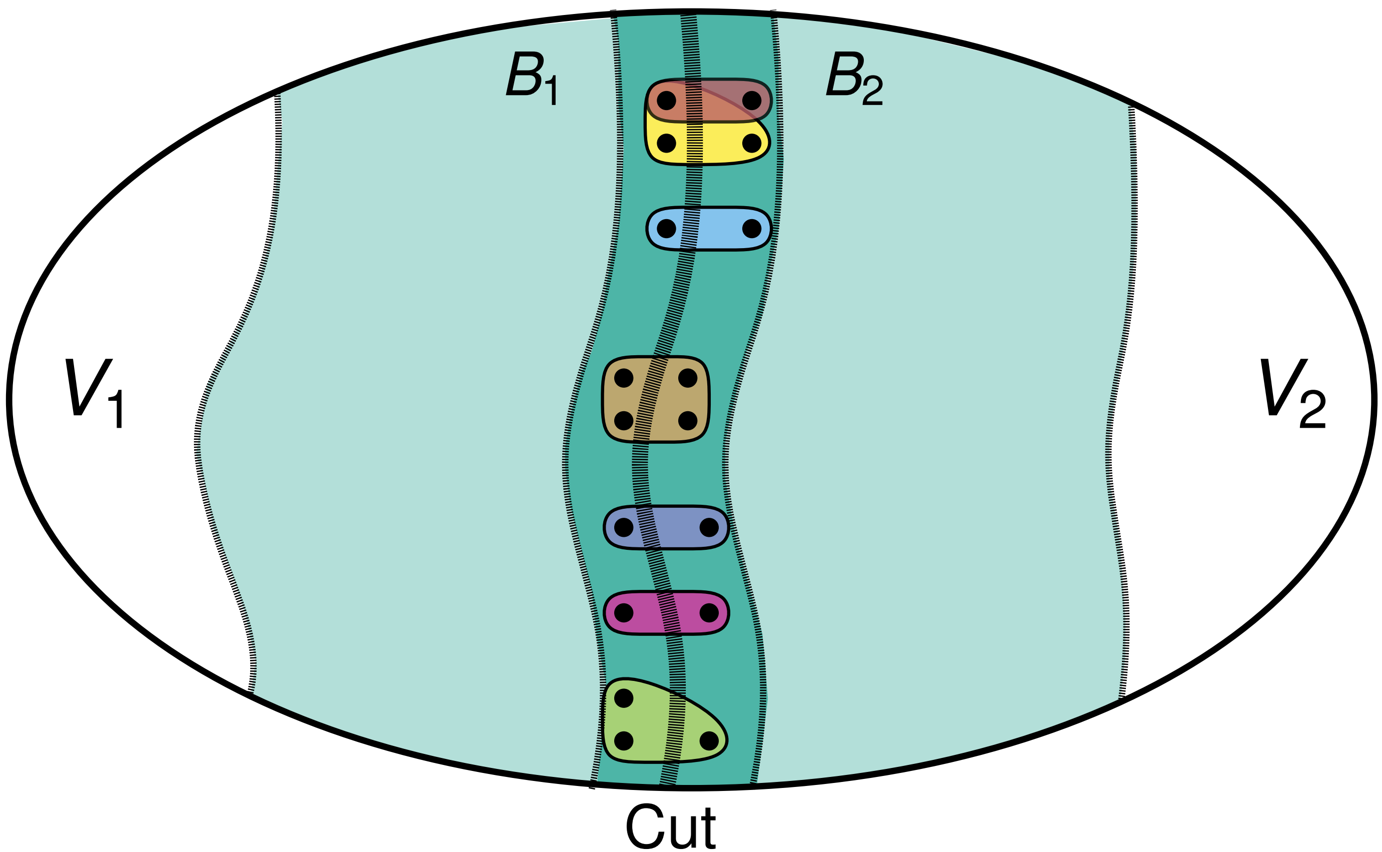
$\alpha = 1 \Rightarrow$ **Improvement Found** $\Rightarrow \alpha = \max(2\alpha, \alpha') = 2$



Adaptive Flow Iterations

Use $\epsilon' = \alpha\epsilon$ instead of ϵ

$\alpha = 2 \Rightarrow$ **No Improvement** $\Rightarrow \alpha = \min(\frac{\alpha}{2}, 1) = 1$



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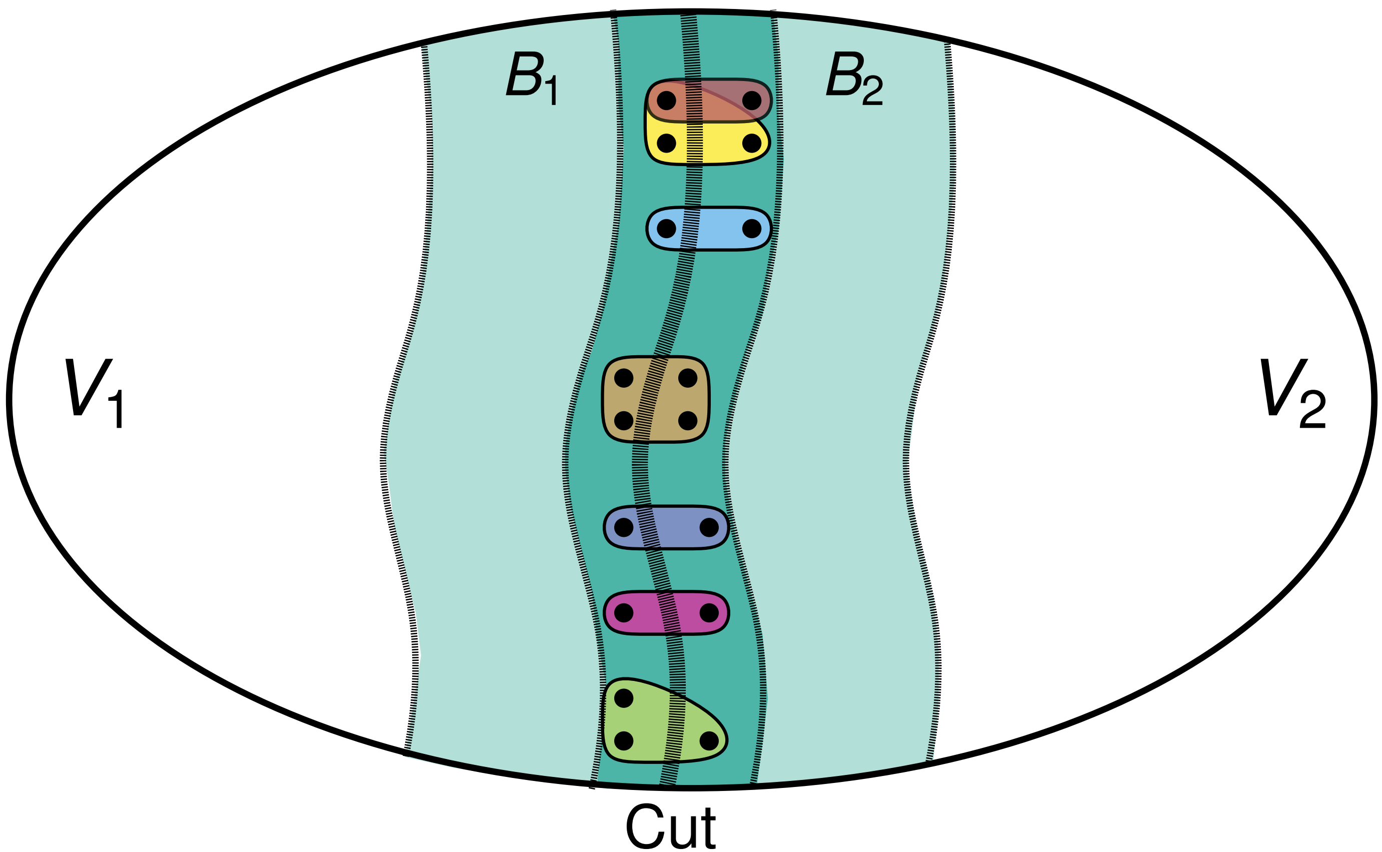
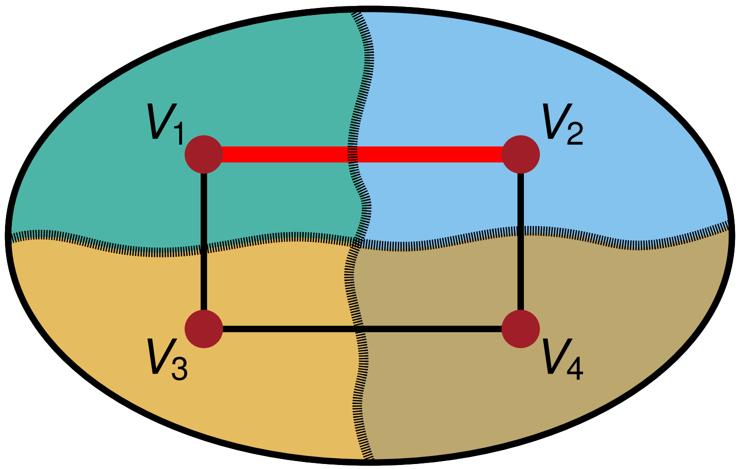
Adaptive Flow Iterations

Use $\epsilon' = \alpha\epsilon$ instead of ϵ

$\alpha = 1 \Rightarrow$ **No Improvement** $\Rightarrow$ **Terminate**



Our Flow-Based Refinement Framework



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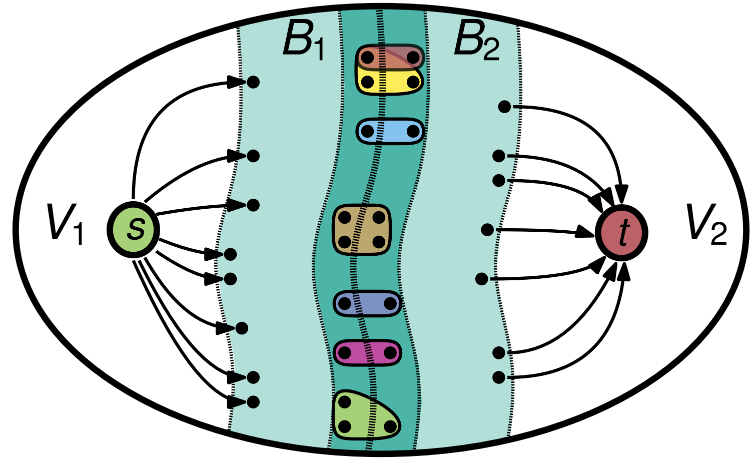
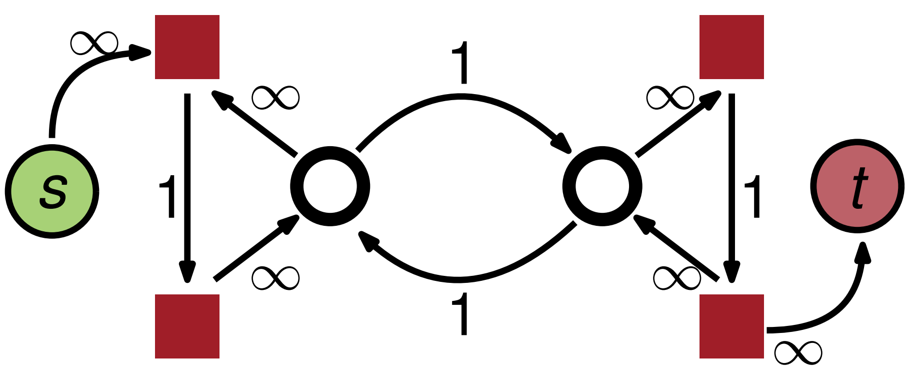
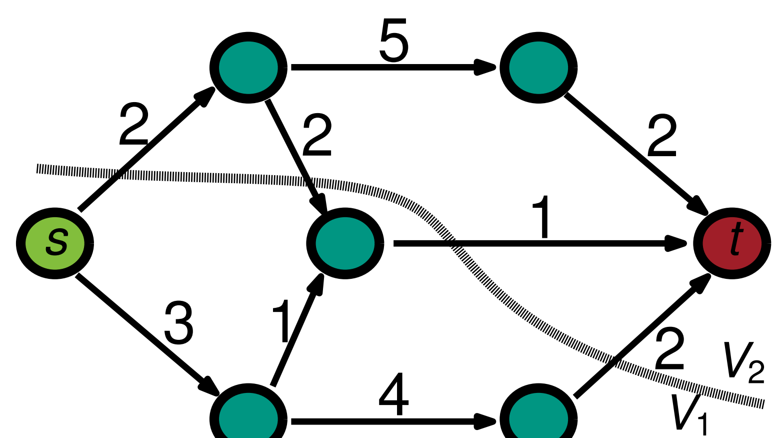


Build Flow Problem



Solve Flow Problem

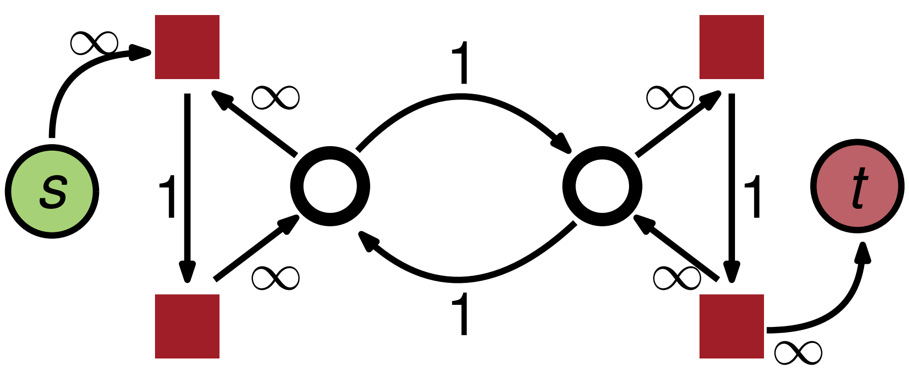


Find feasible minimum cut

Our Flow-Based Refinement Framework

Select two adjacent blocks for refinement

Build Flow Problem



Solve Flow Problem

Find feasible minimum cut

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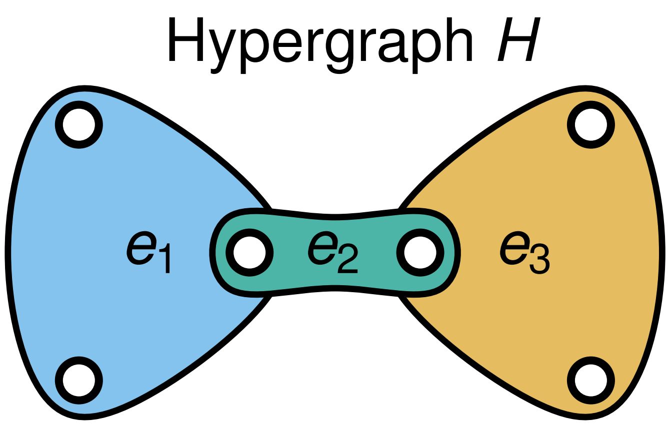
Hypergraph Flow Network



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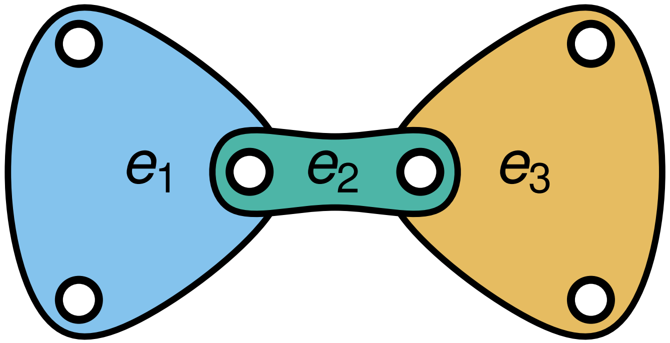
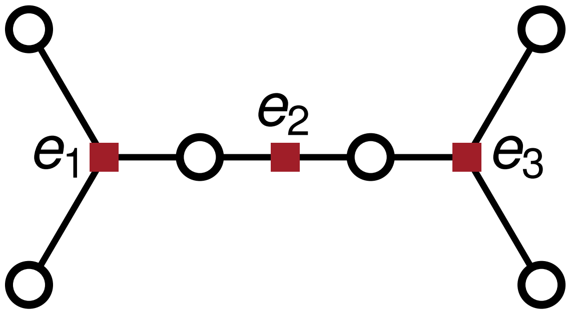
Hypergraph Flow Network

Hypergraph H



Bipartite Graph $G_*(H)$



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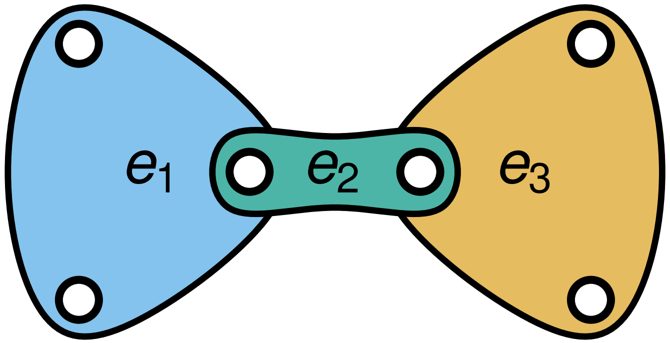
Hypergraph Flow Network

Hypergraph H



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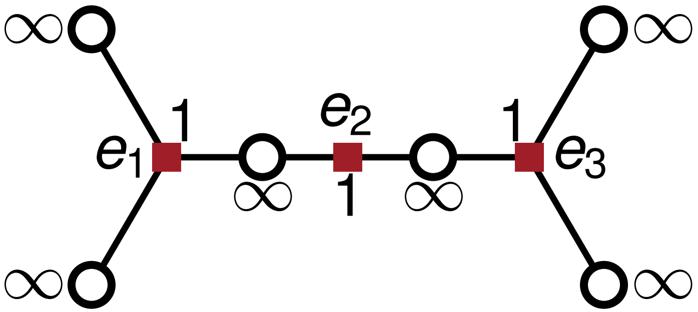

Vertex Separator Problem

Hypergraph Flow Network

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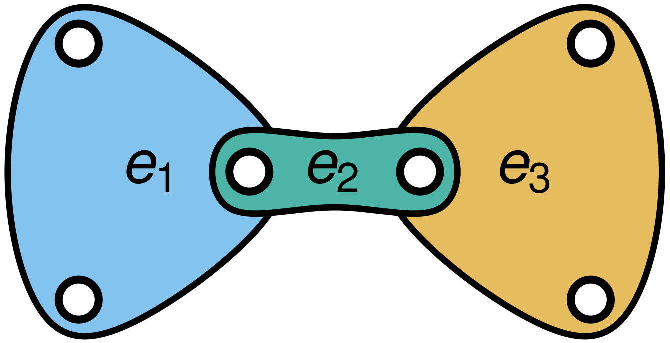
Hypergraph H



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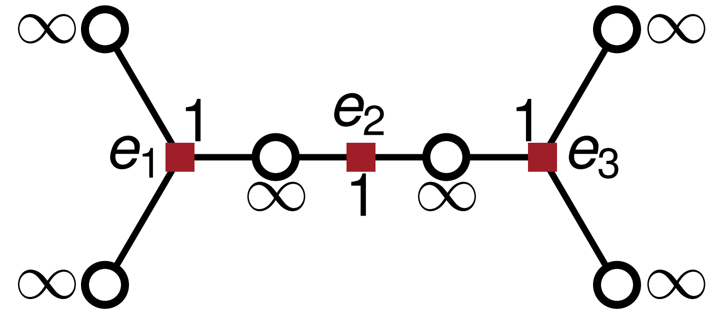
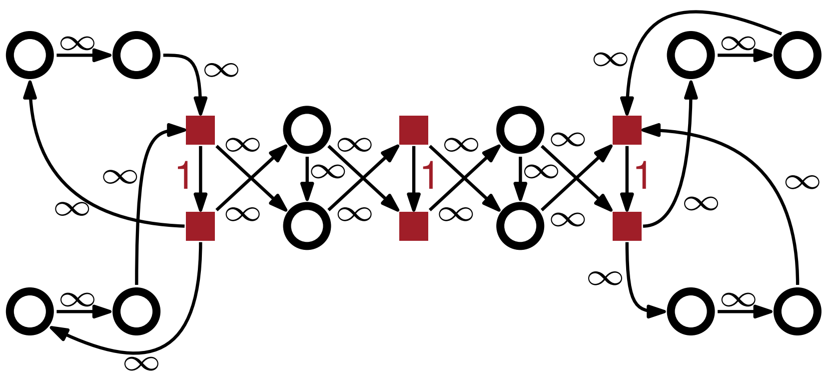


Vertex Separator Problem

Vertex Separator Transformation



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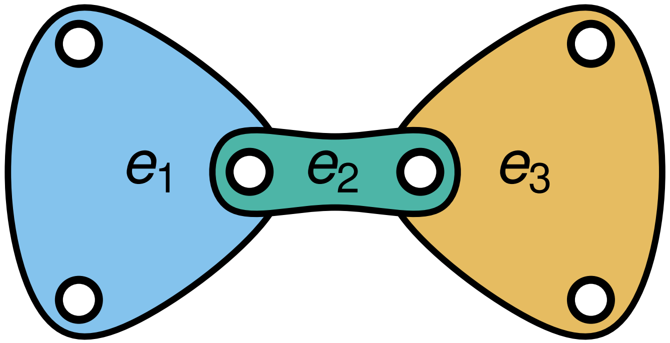
Hypergraph Flow Network

Hypergraph H



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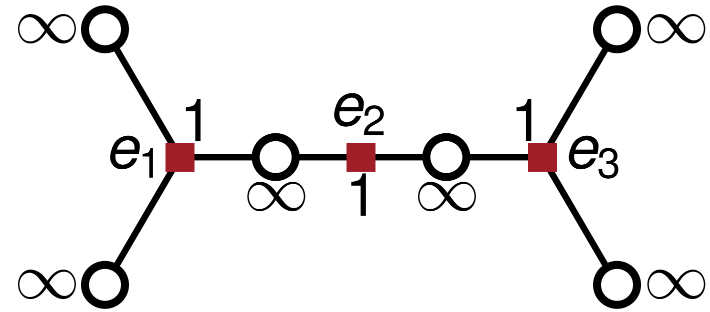

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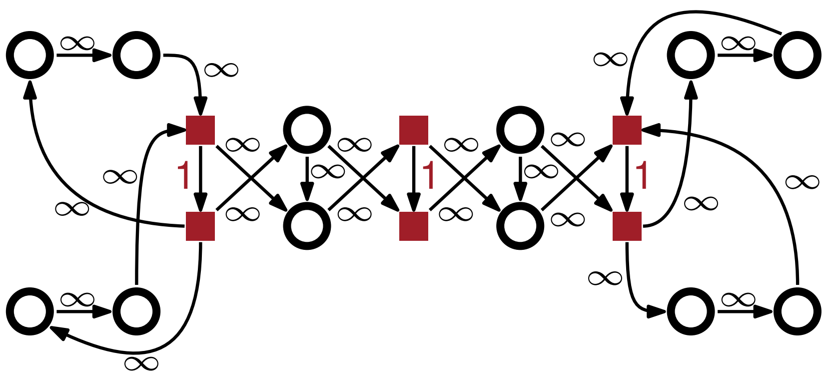
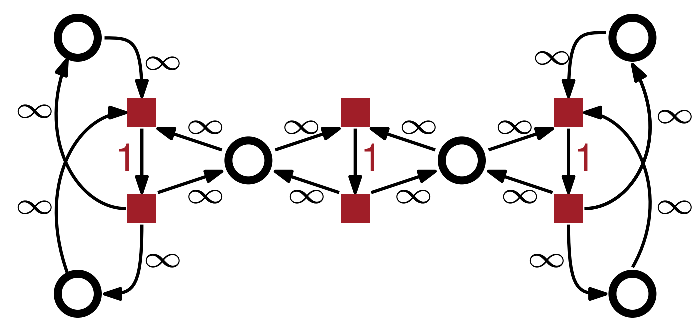
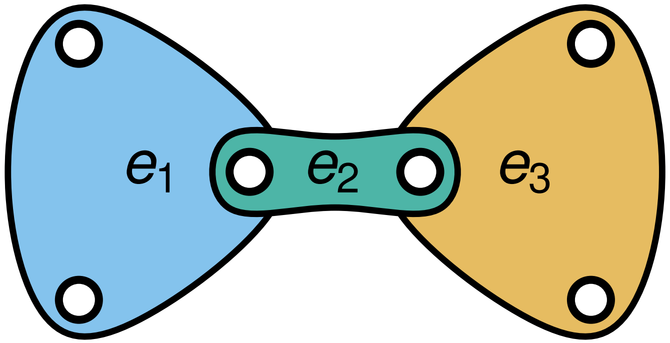


Lawler Network

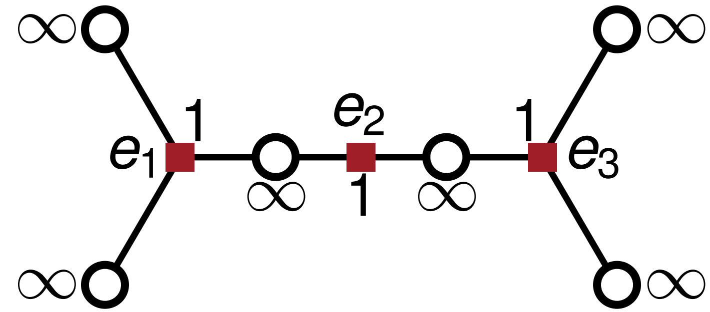


Hypergraph Flow Network

Hypergraph H



Bipartite Graph $G_*(H)$



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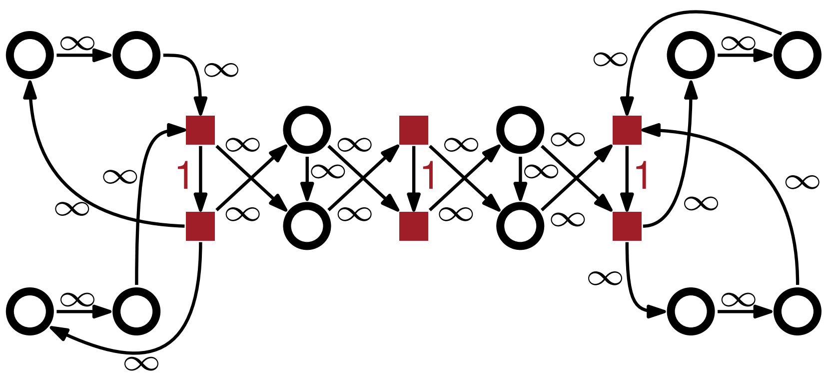
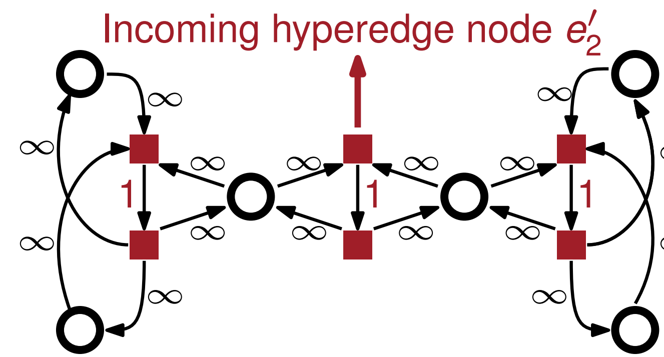


Lawler Network



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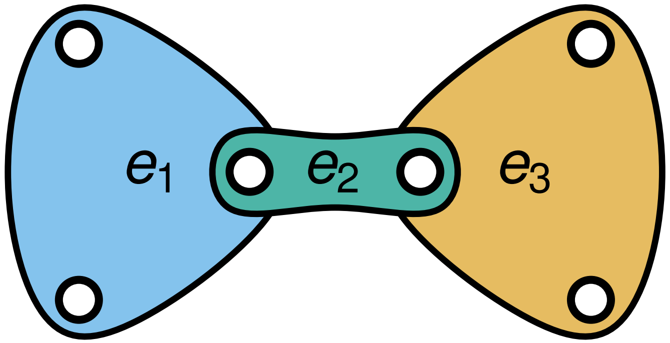
Hypergraph Flow Network

Hypergraph H



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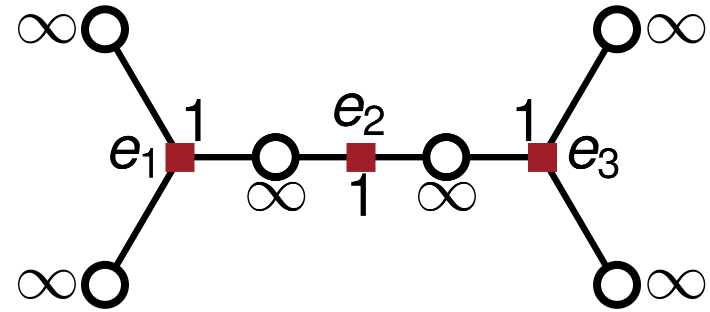

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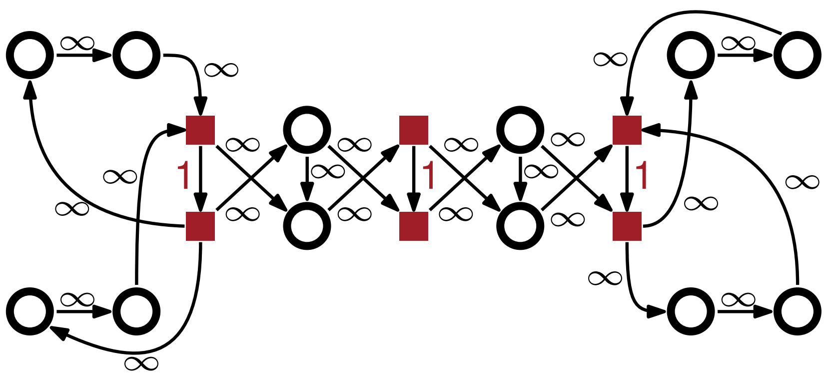
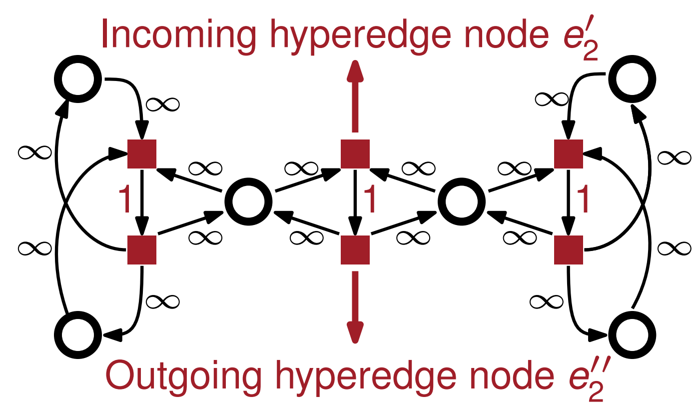
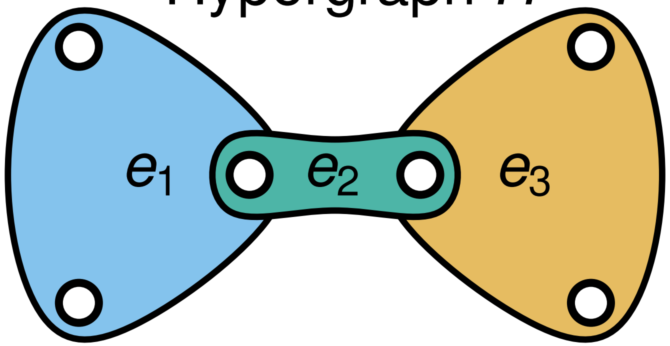


Lawler Network

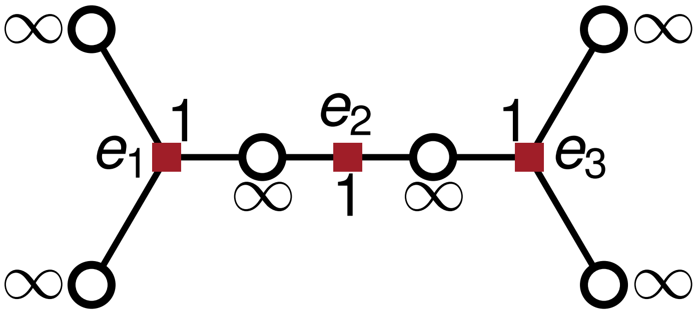


Hypergraph Flow Network - Low Degree Vertices

Hypergraph H

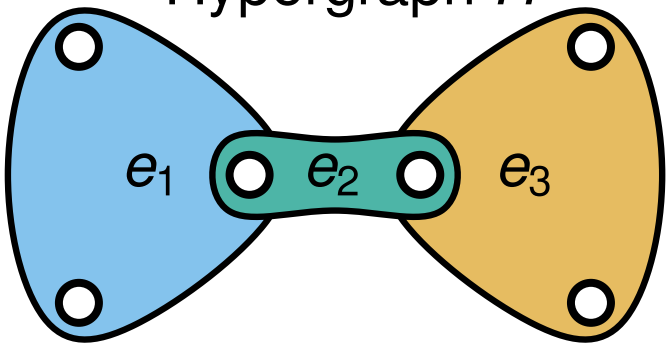


Bipartite Graph $G_*(H)$

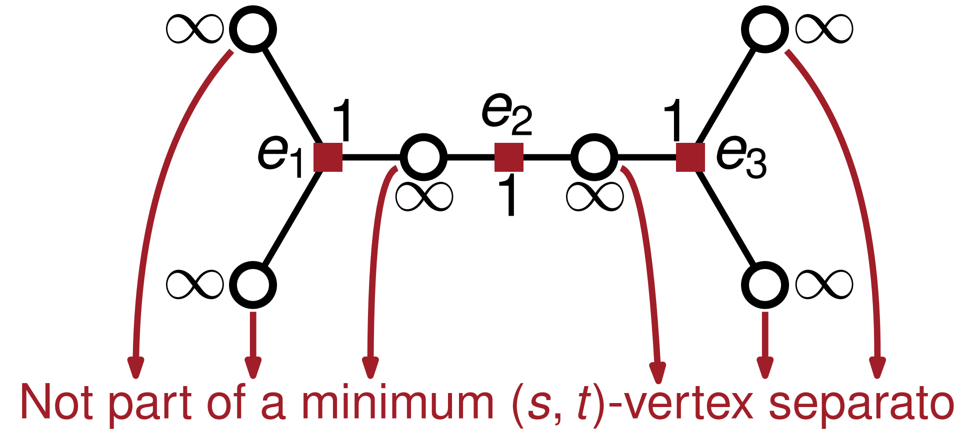


Hypergraph Flow Network - Low Degree Vertices

Hypergraph H



Bipartite Graph $G_*(H)$



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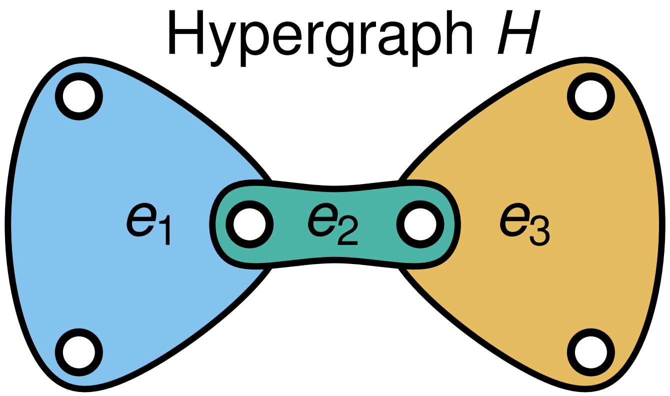
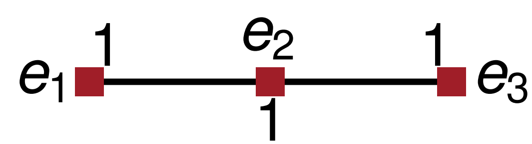
Hypergraph Flow Network - Low Degree Vertices





Remove all vertices by adding a clique

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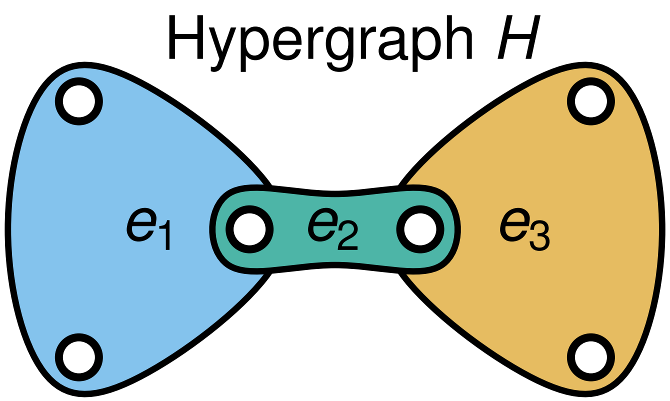
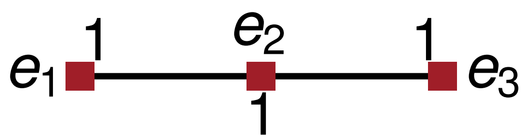
Hypergraph Flow Network - Low Degree Vertices





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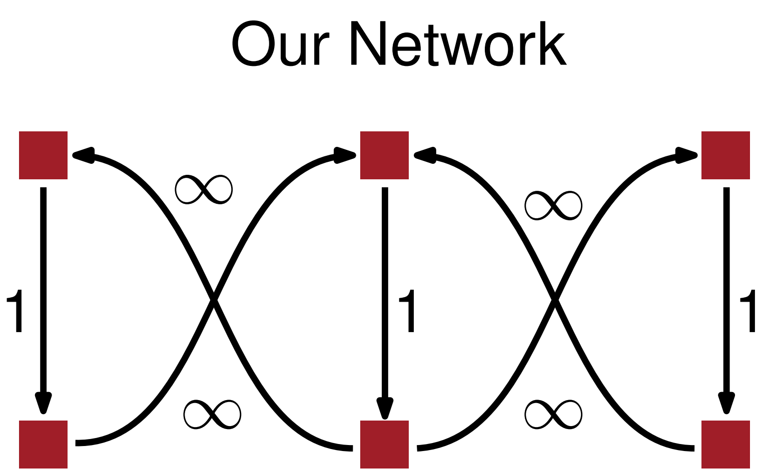

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Hypergraph Flow Network - Low Degree Vertices

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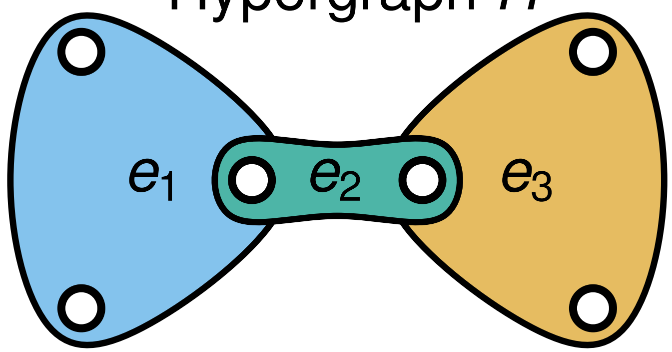
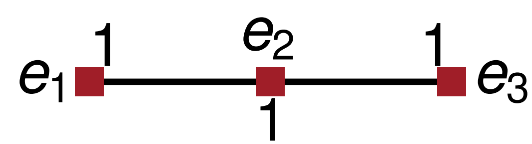
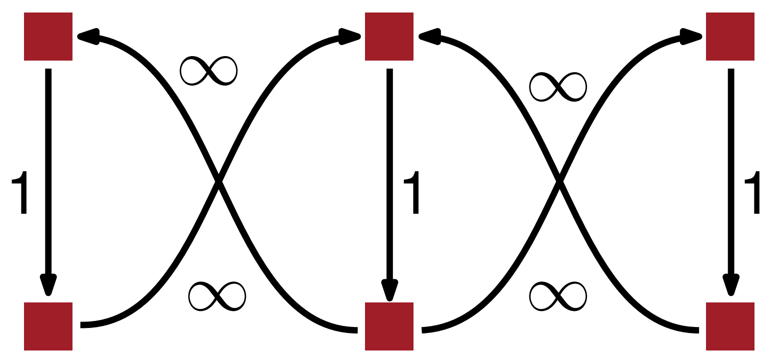
Hypergraph H





Remove all vertices by adding a clique

Our Network



A hypernode v induces ...

- ... $2d(v)$ edges in the Lawler Network
- ... $d(v)(d(v)-1)$ edges in our network

If $d(v) \leq 3$, then $d(v)(d(v)-1) \leq 2d(v)$

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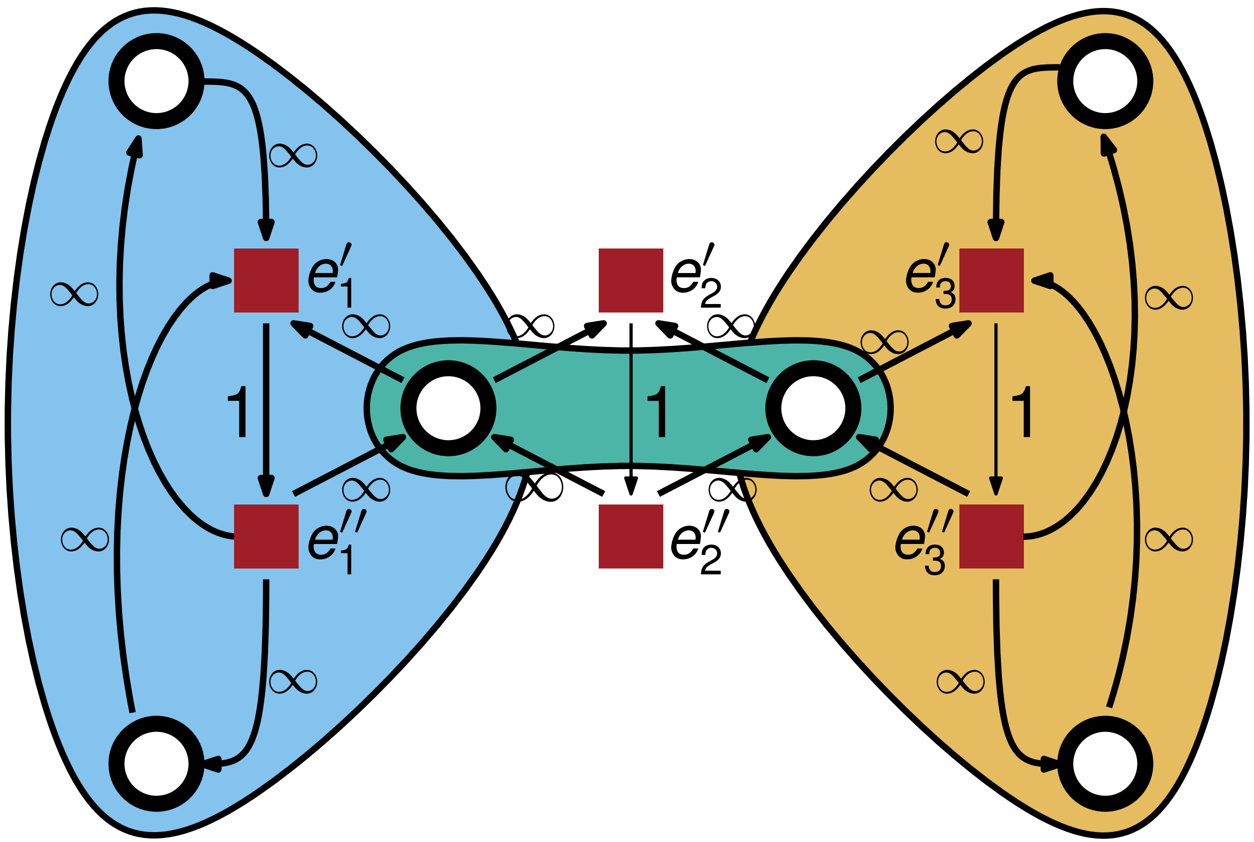
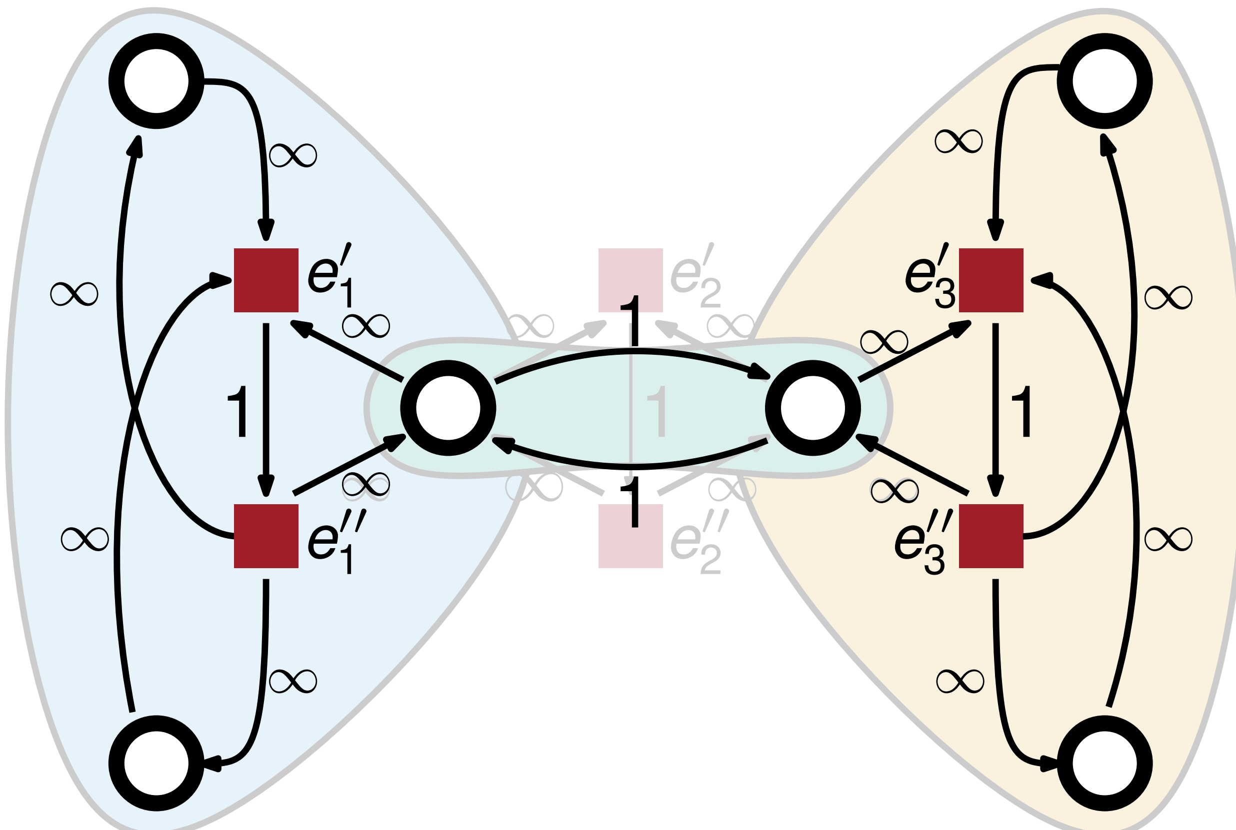
Hypergraph Flow Network - Summary



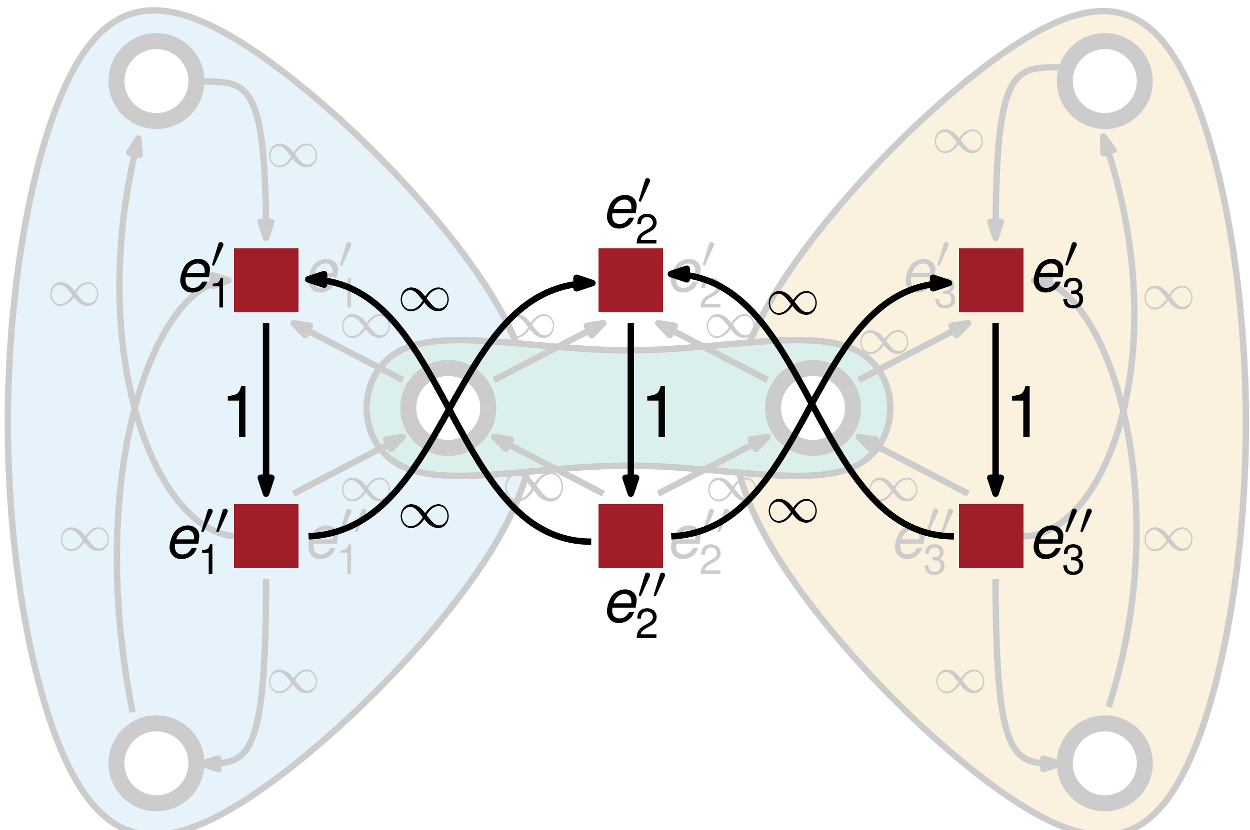
Lawler Network

Hypergraph Flow Network - Summary



Wong Network

Hypergraph Flow Network - Summary



Our Network

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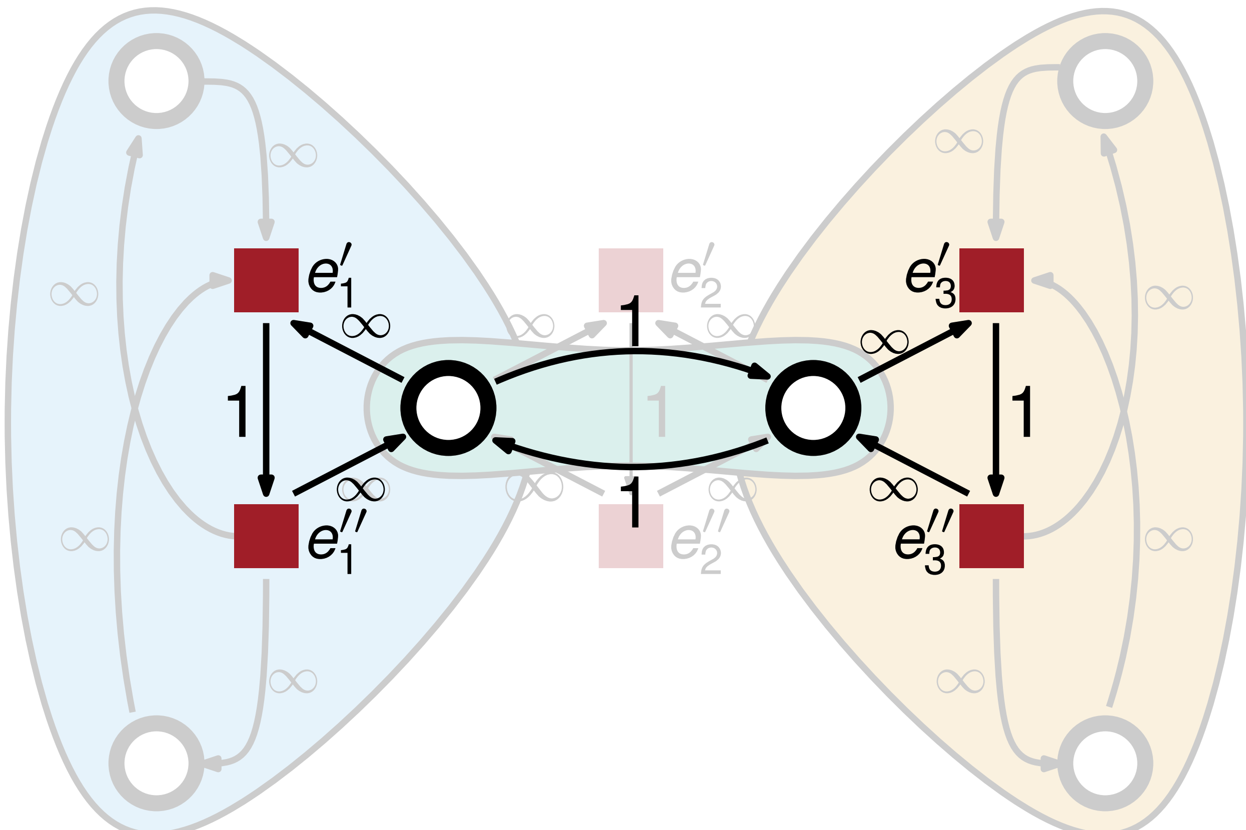
Hypergraph Flow Network - Summary



Hybrid Network

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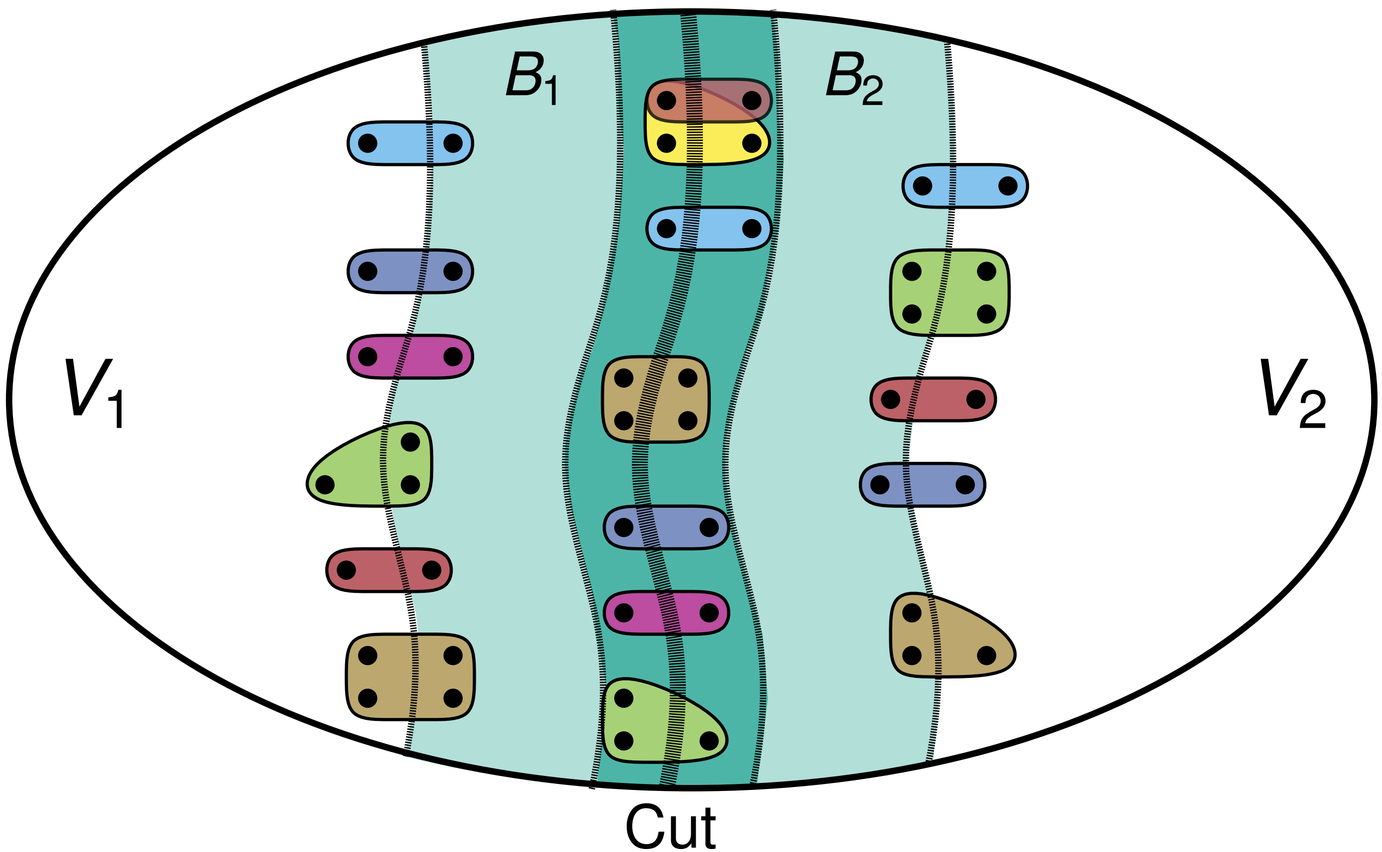
Optimized Flow Problem Modeling Approach

Modeling Approach in *KaFFPa*



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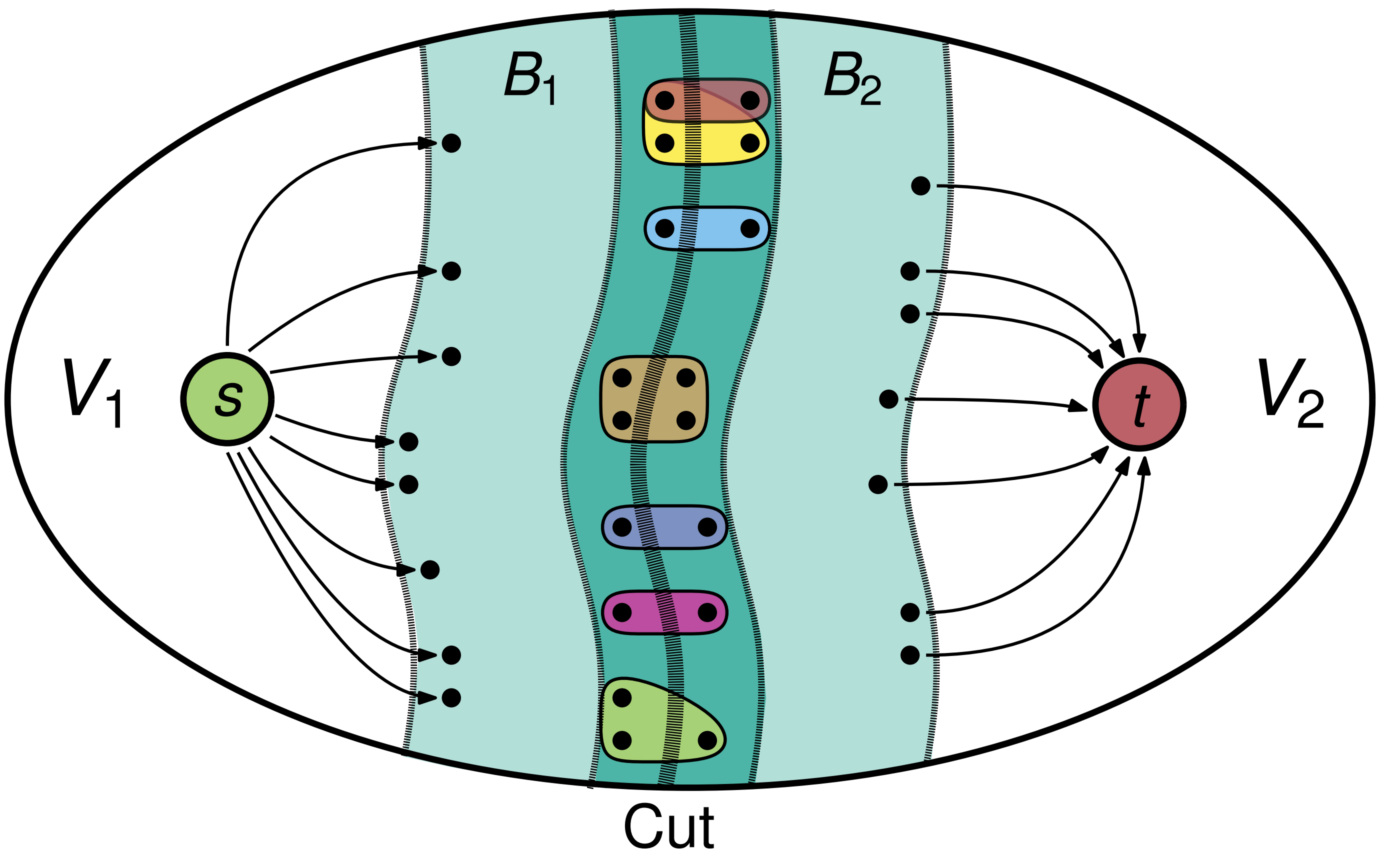
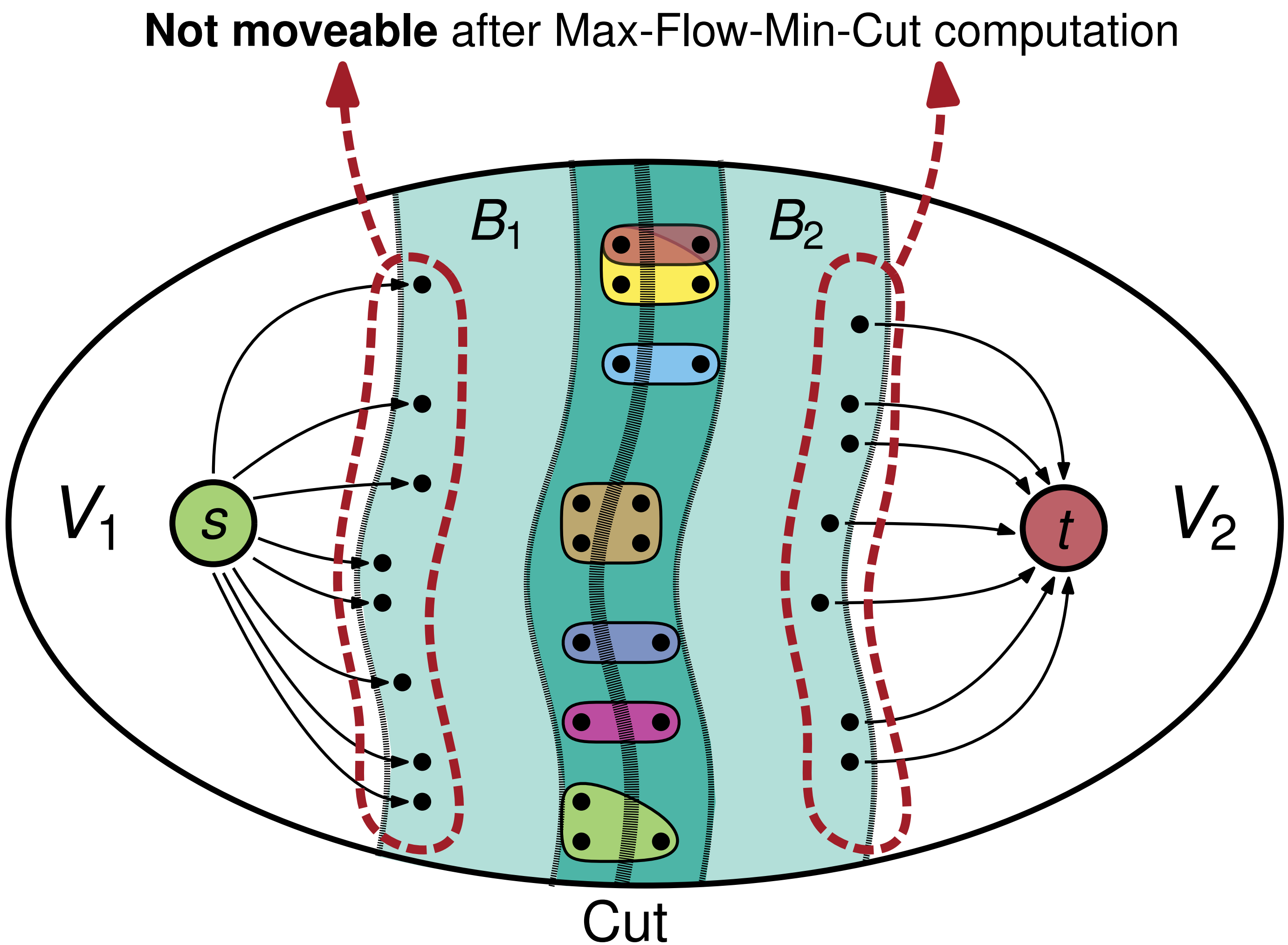
Optimized Flow Problem Modeling Approach

Modeling Approach in *KaFFPa*



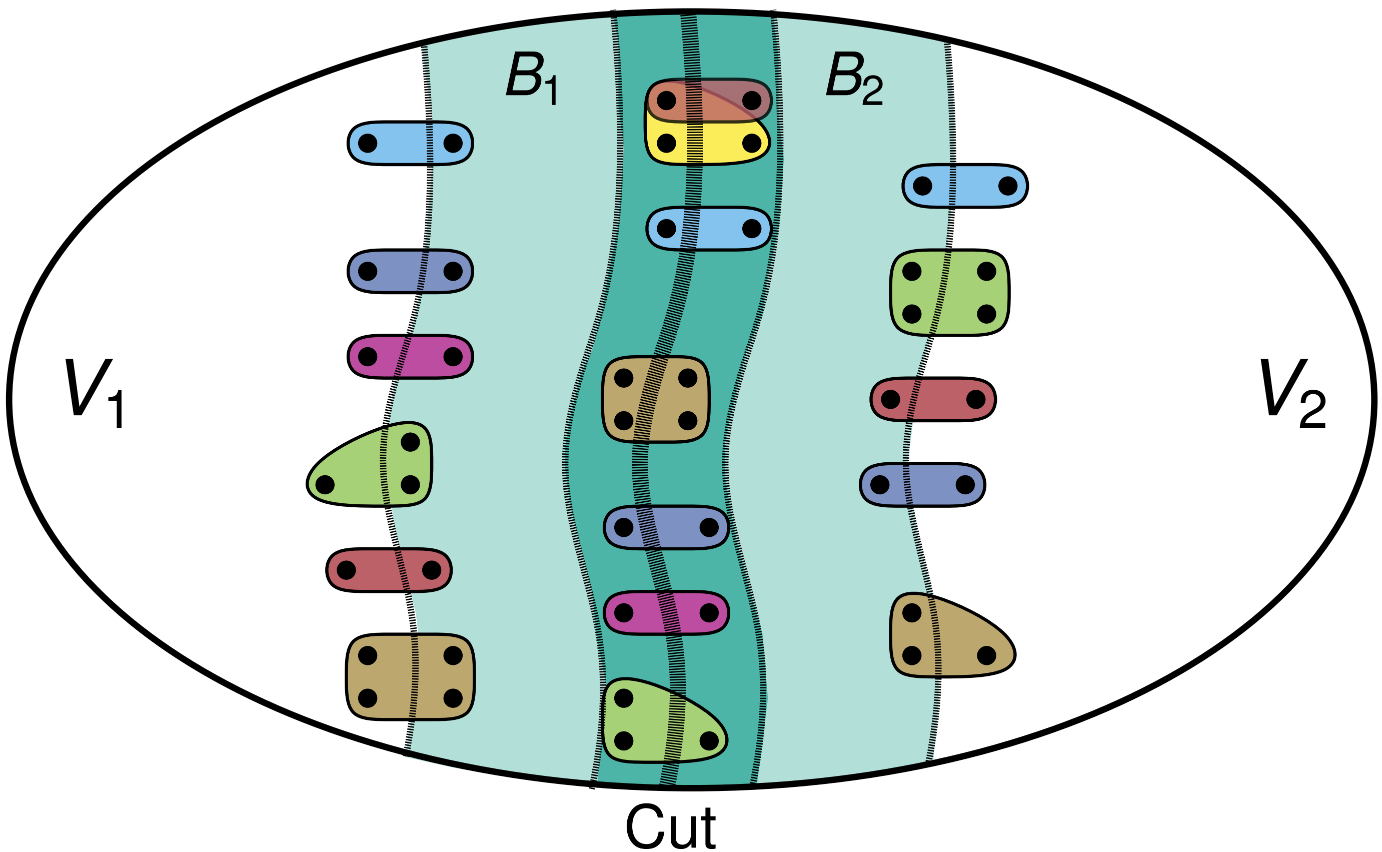
Optimized Flow Problem Modeling Approach

Modeling Approach in *KaFFPa*



Optimized Flow Problem Modeling Approach

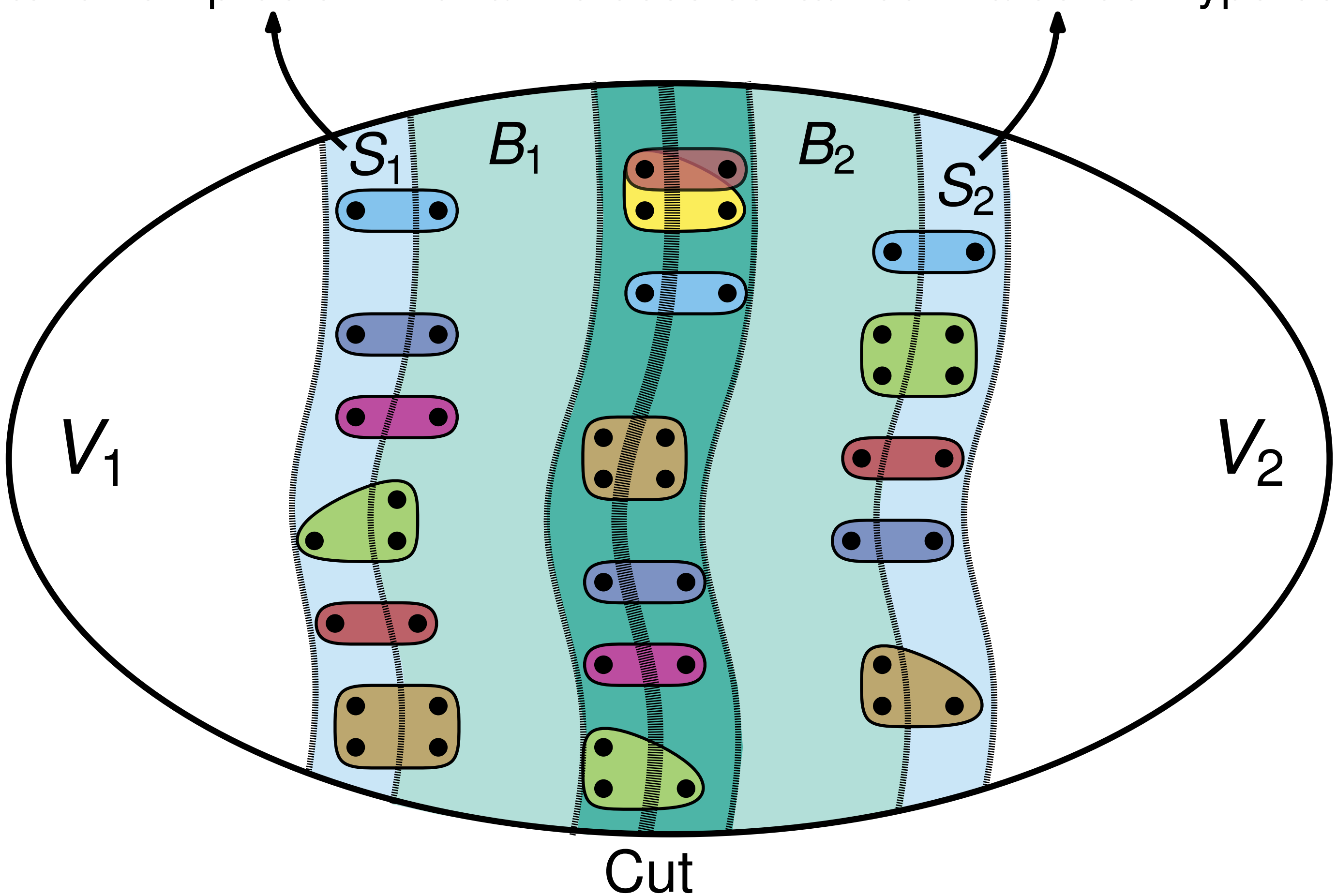
Modeling Approach in *KaHyPar*



Optimized Flow Problem Modeling Approach

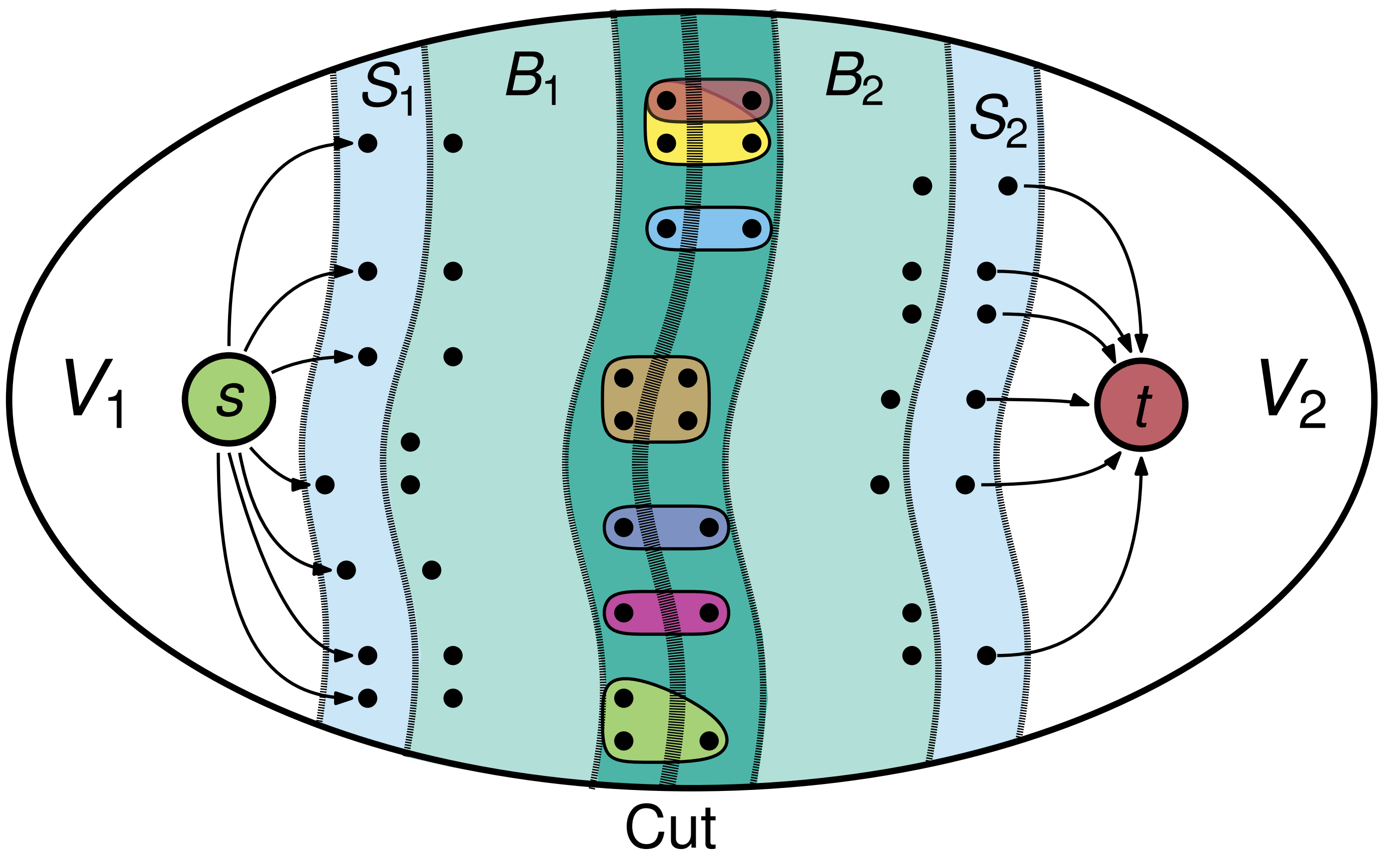
Modeling Approach in *KaHyPar*

Extend flow problem with all vertices contained in a border hyperedge



Optimized Flow Problem Modeling Approach

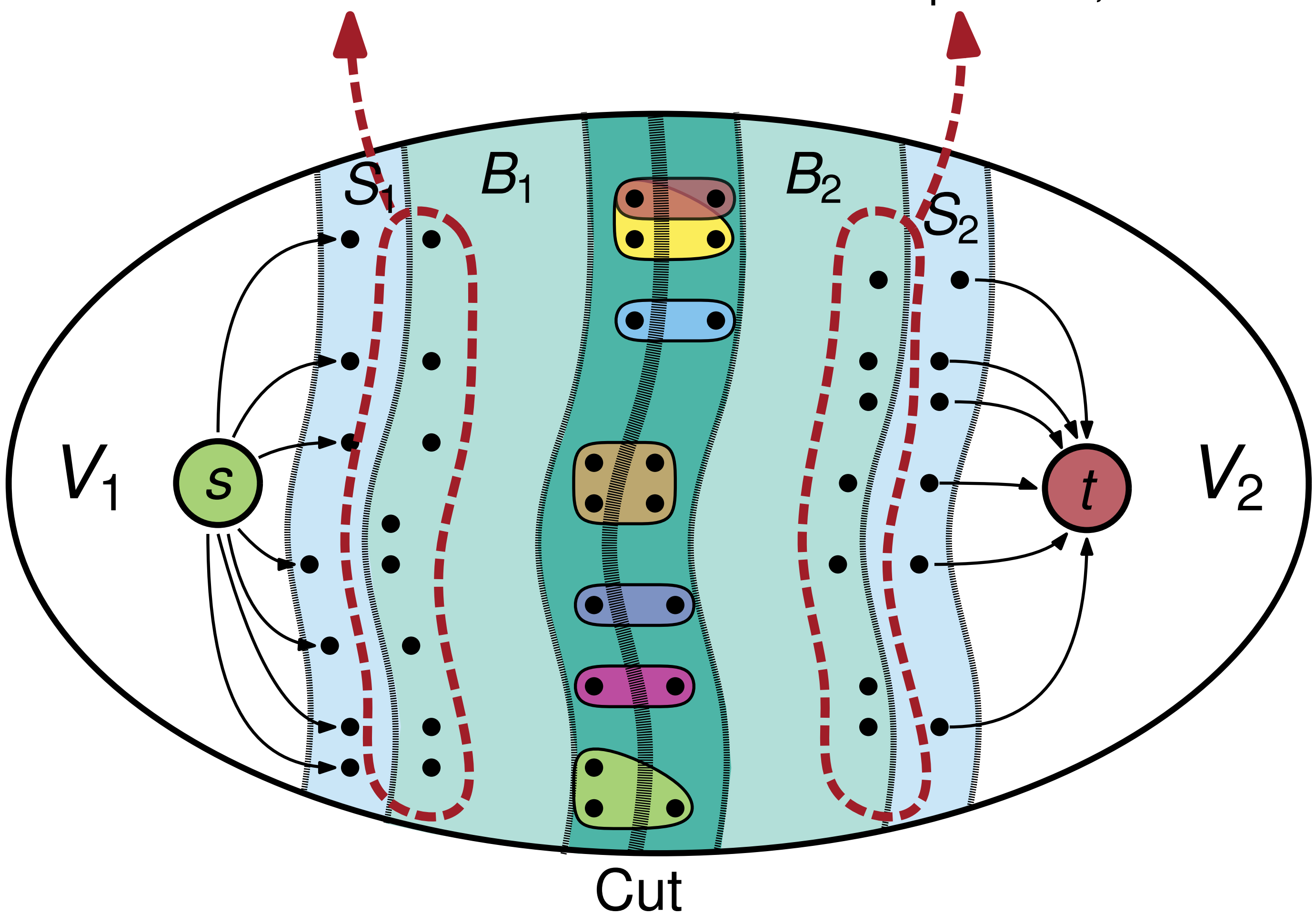
Modeling Approach in *KaHyPar*



Optimized Flow Problem Modeling Approach

Modeling Approach in *KaHyPar*

Moveable after Max-Flow-Min-Cut computation, but . . .



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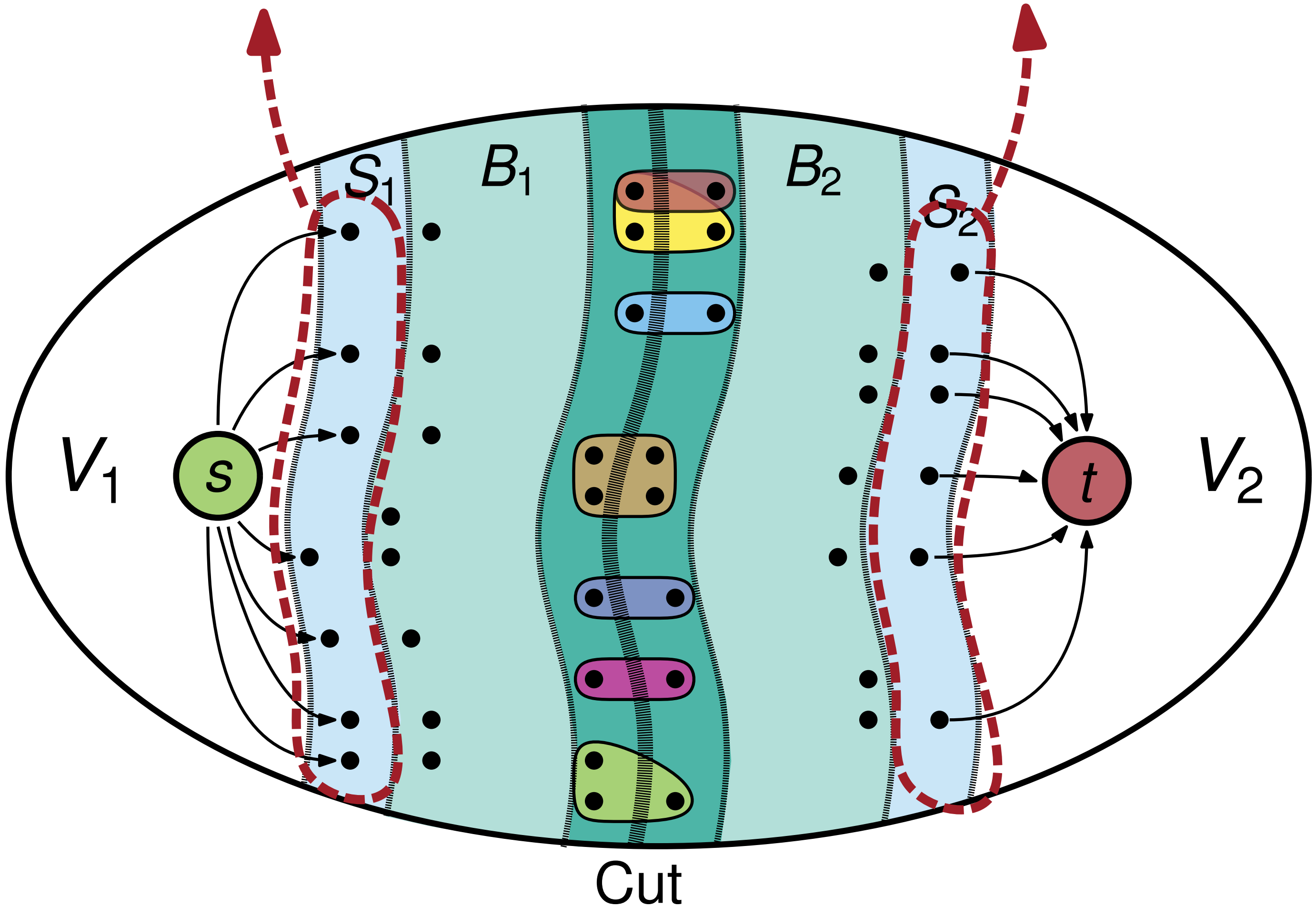
Optimized Flow Problem Modeling Approach

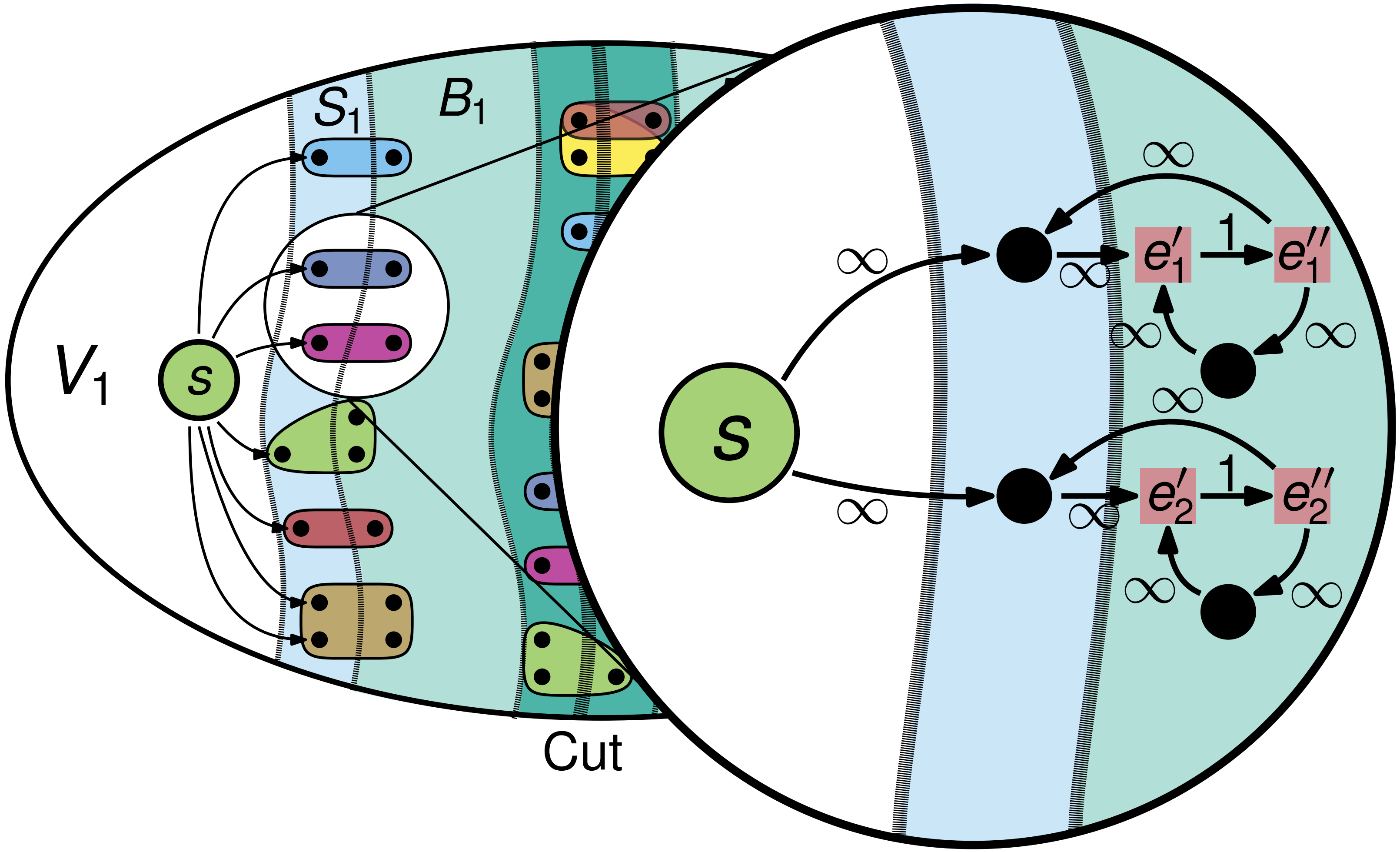
Modeling Approach in *KaHyPar*

. . . flow problem has significantly more nodes and edges.



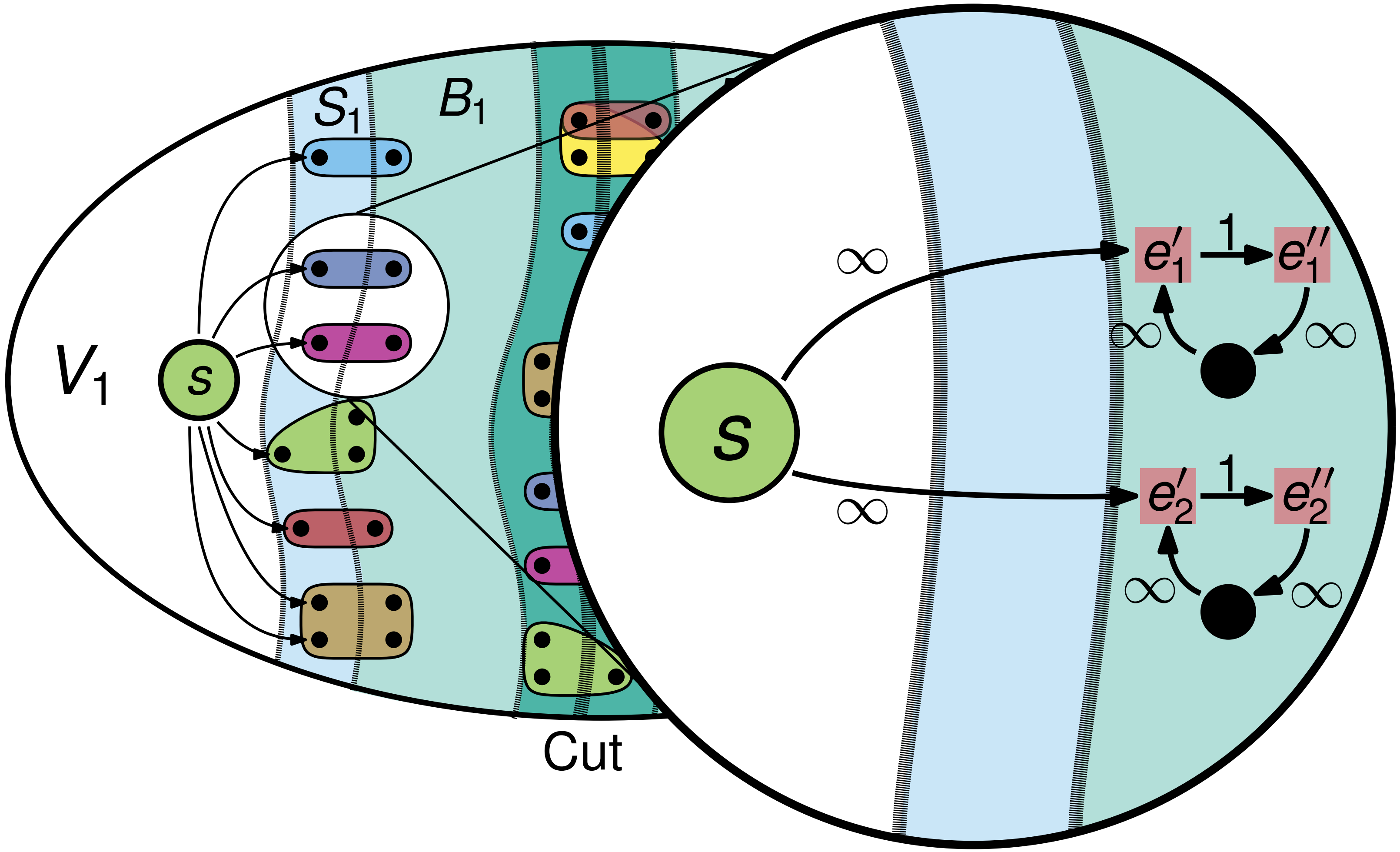
Optimized Flow Problem Modeling Approach

Modeling Approach in *KaHyPar*



Optimized Flow Problem Modeling Approach

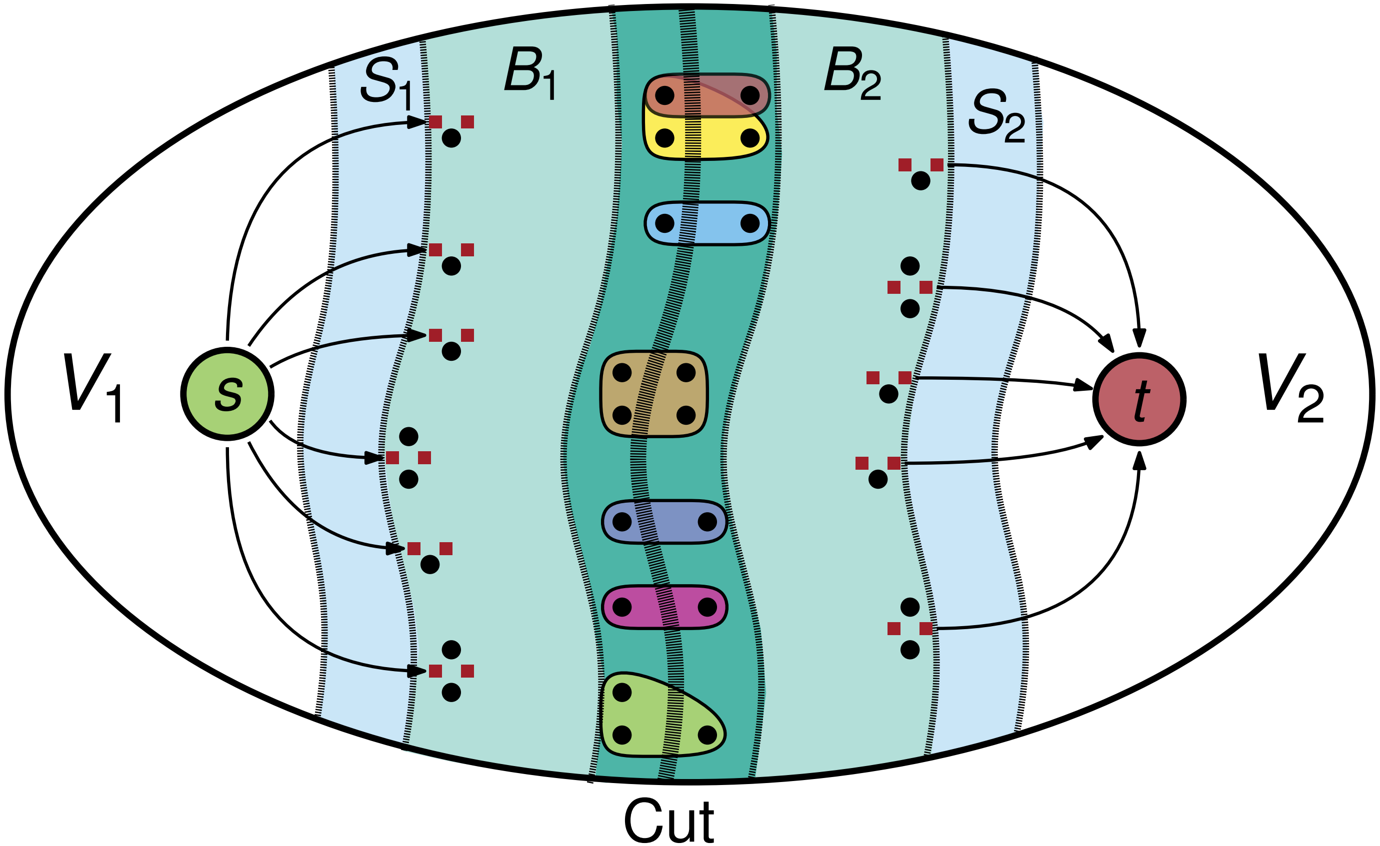
Modeling Approach in *KaHyPar*



Optimized Flow Problem Modeling Approach

Modeling Approach in *KaHyPar*

$$S = \{e' \mid e \in I(S_1)\}$$
$$T = \{e'' \mid e \in I(S_2)\}$$



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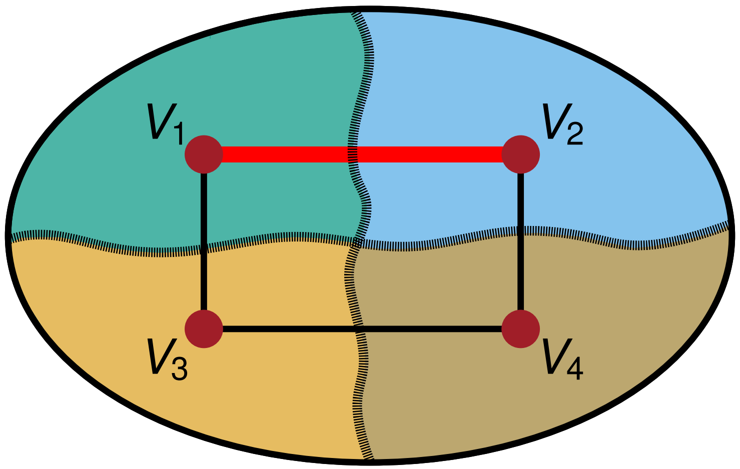
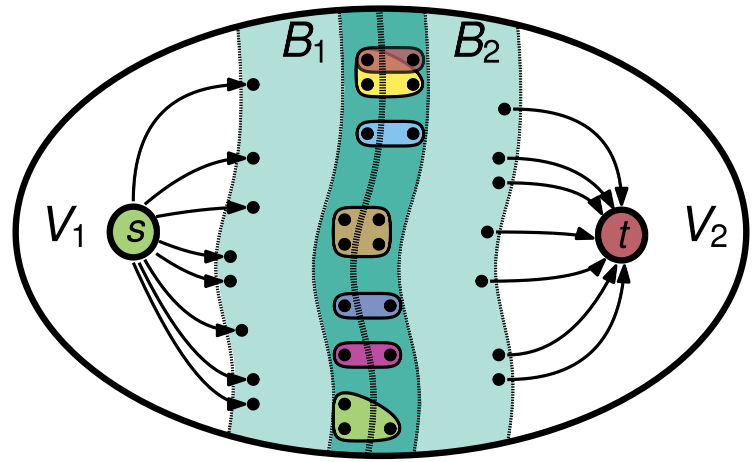
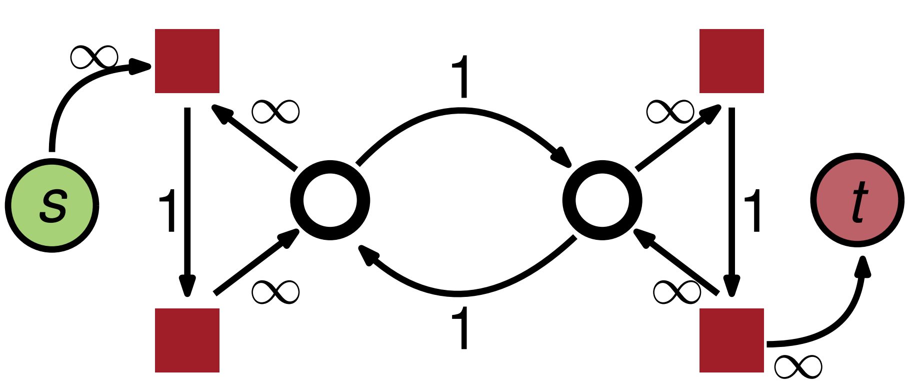
Our Flow-Based Refinement Framework



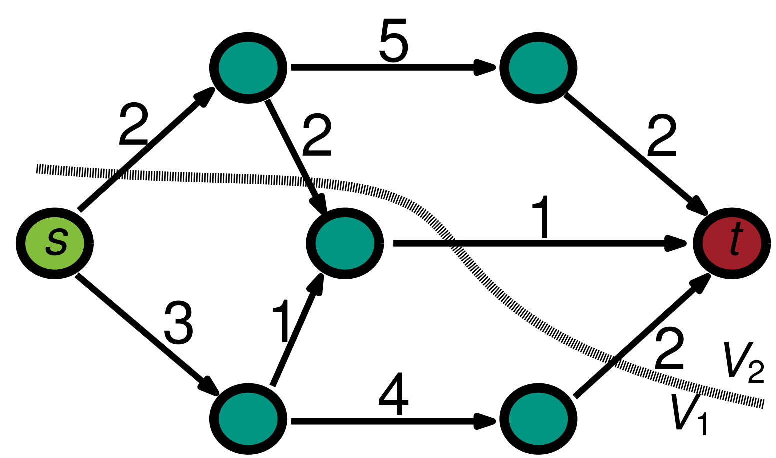
Select two adjacent blocks for refinement



Build Flow Problem



Solve Flow Problem



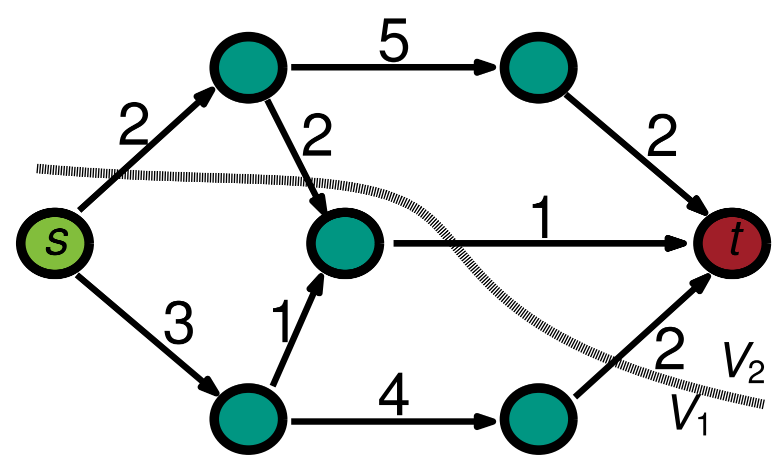
Find feasible minimum cut

Our Flow-Based Refinement Framework

Select two adjacent blocks for refinement

Build Flow Problem

Solve Flow Problem



Find feasible minimum cut

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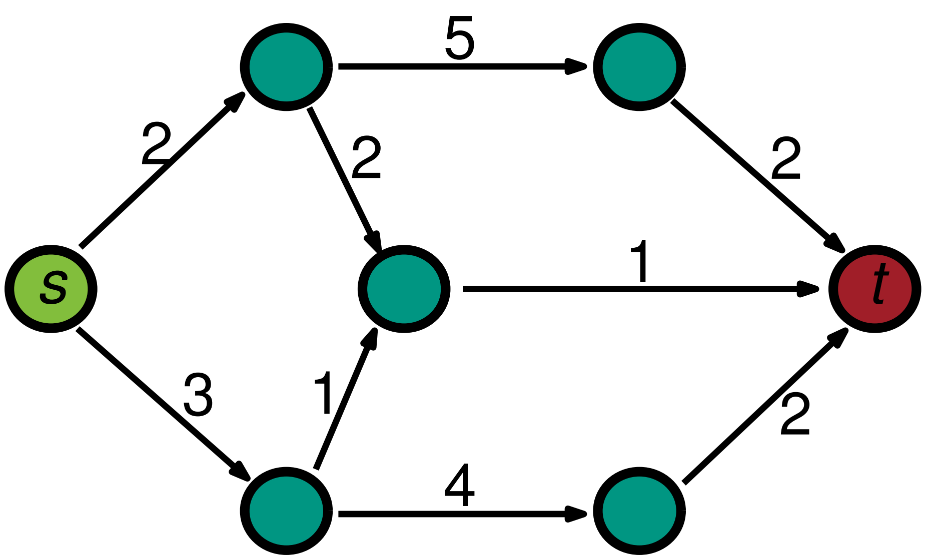
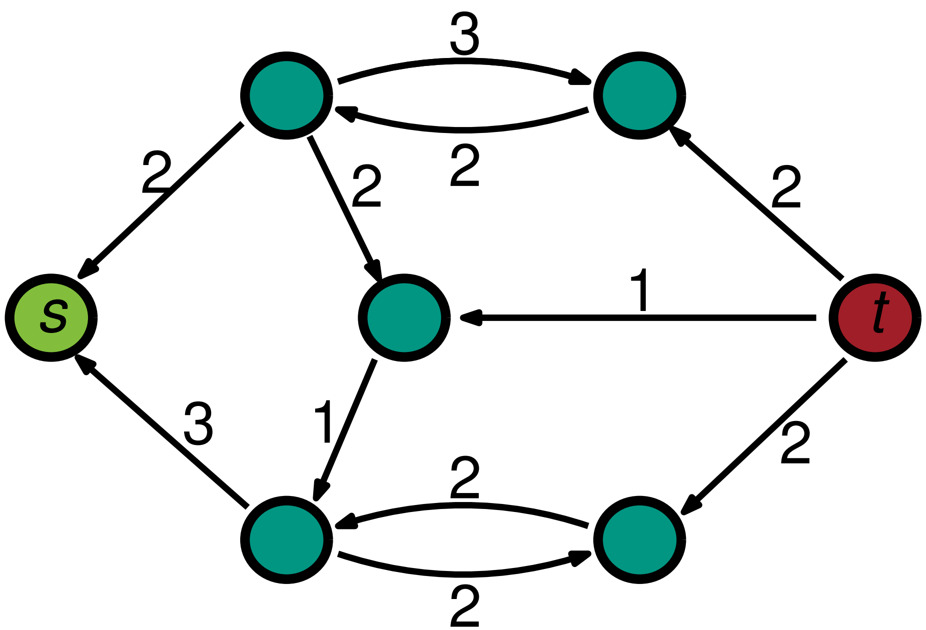
Most Balanced Minimum Cut

One maximum flow f has enough information to enumerate all minimum (s, t)-cuts

Flow Graph



Residual Graph



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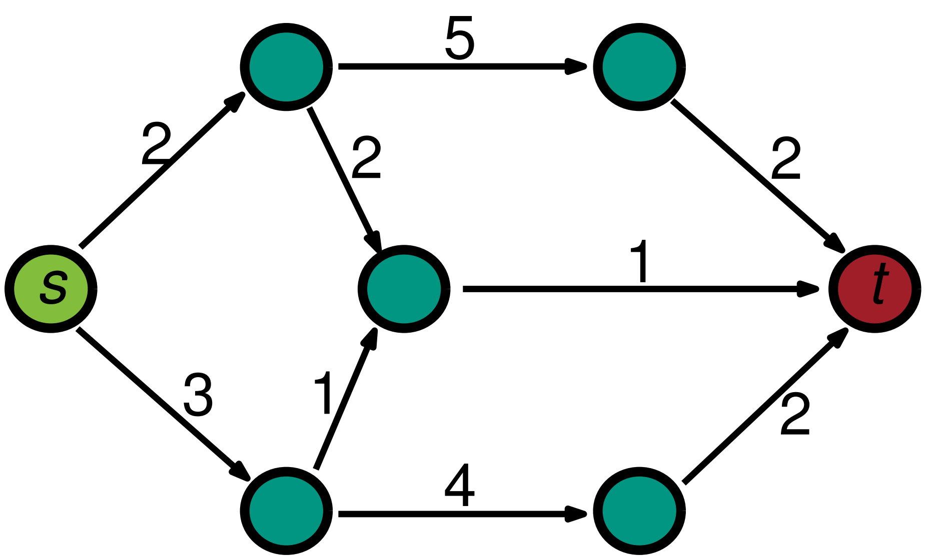
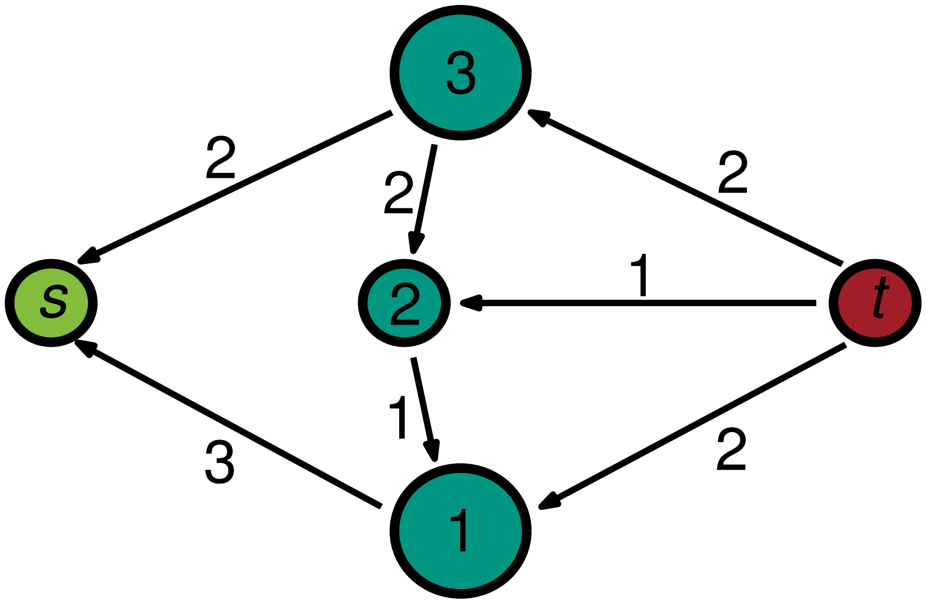
Most Balanced Minimum Cut

One maximum flow *f* has enough information to enumerate all minimum (*s*, *t*)-cuts

Flow Graph



Picard-Queryanne DAC



Contract all *strongly connected components* in the residual graph

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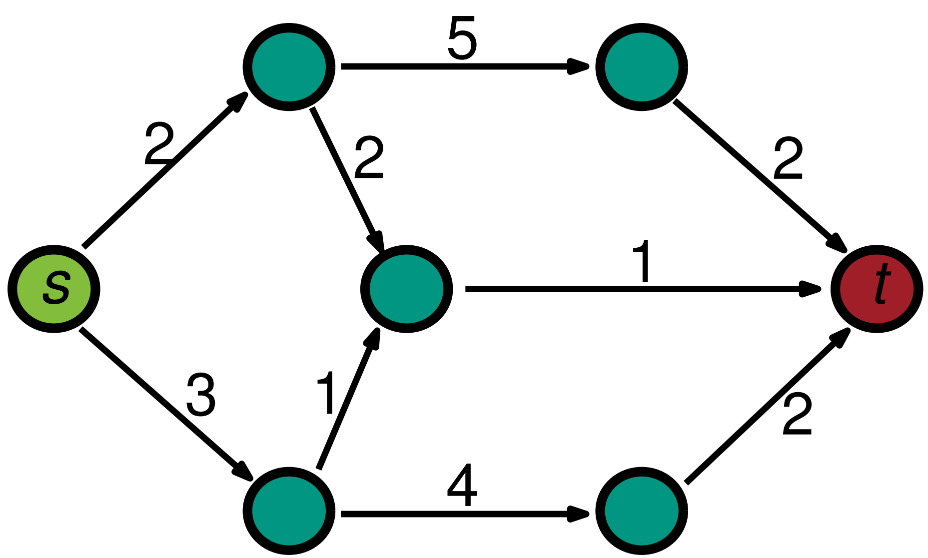
Most Balanced Minimum Cut

One maximum flow f has enough information to enumerate all minimum (s, t)-cuts

Flow Graph



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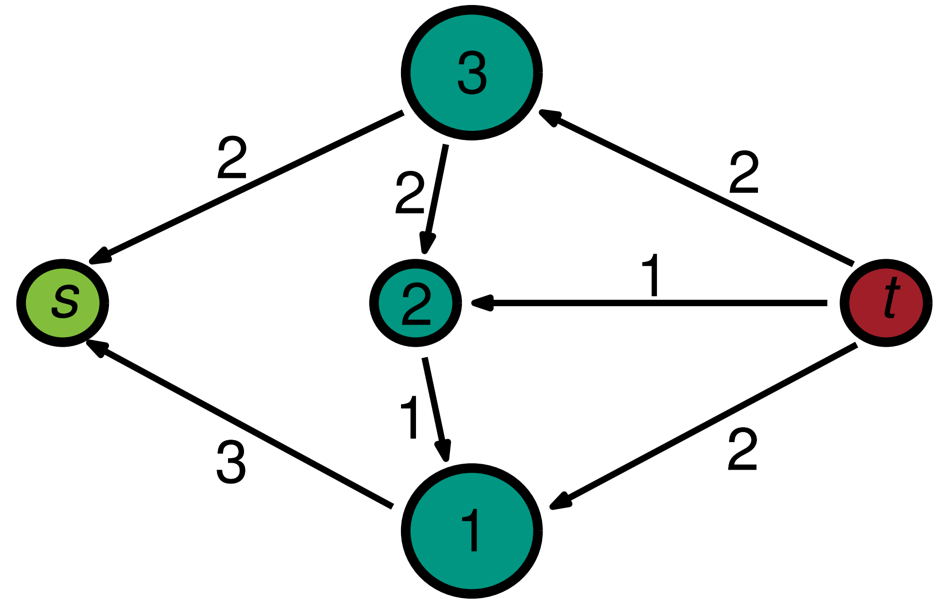


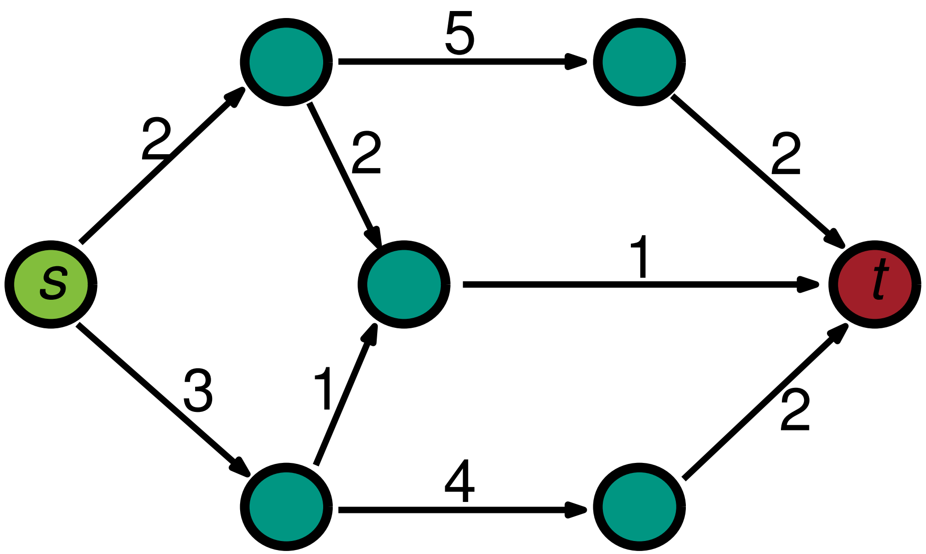
$\langle t,\quad 3,\quad 2,\quad 1,\quad s \rangle$

Find *topological order*

Most Balanced Minimum Cut

One maximum flow f has enough information to enumerate all minimum (s, t)-cuts

Flow Graph



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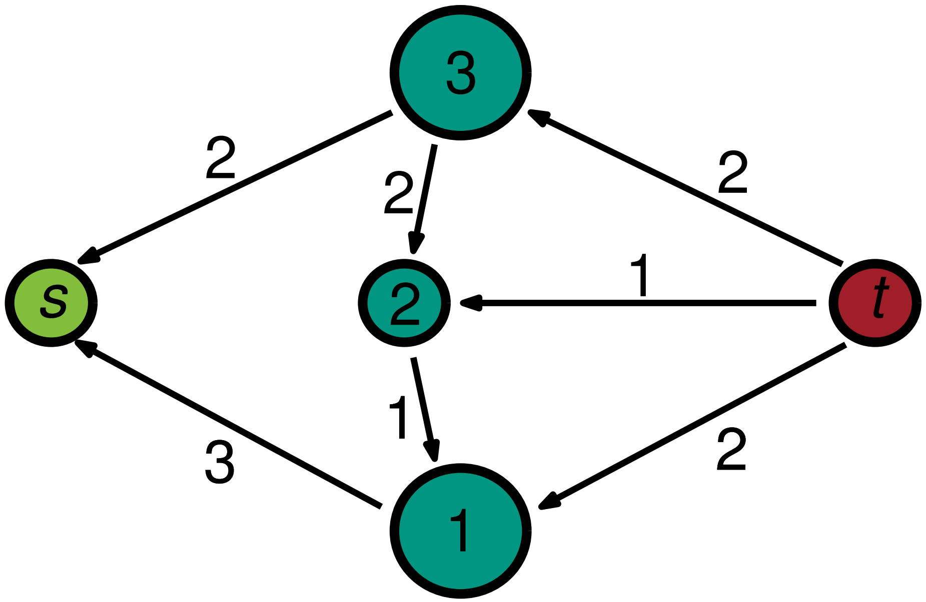


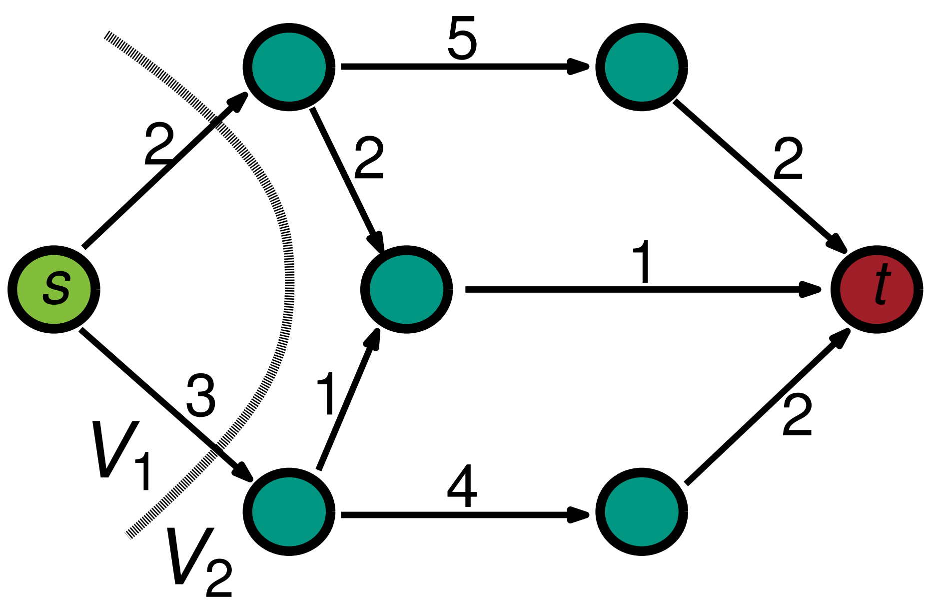
$\langle t, \quad 3, \quad 2, \quad 1, \quad s \rangle$

Sweep through **reverse** topological order

Most Balanced Minimum Cut

One maximum flow f has enough information to enumerate all minimum (s, t)-cuts

Flow Graph



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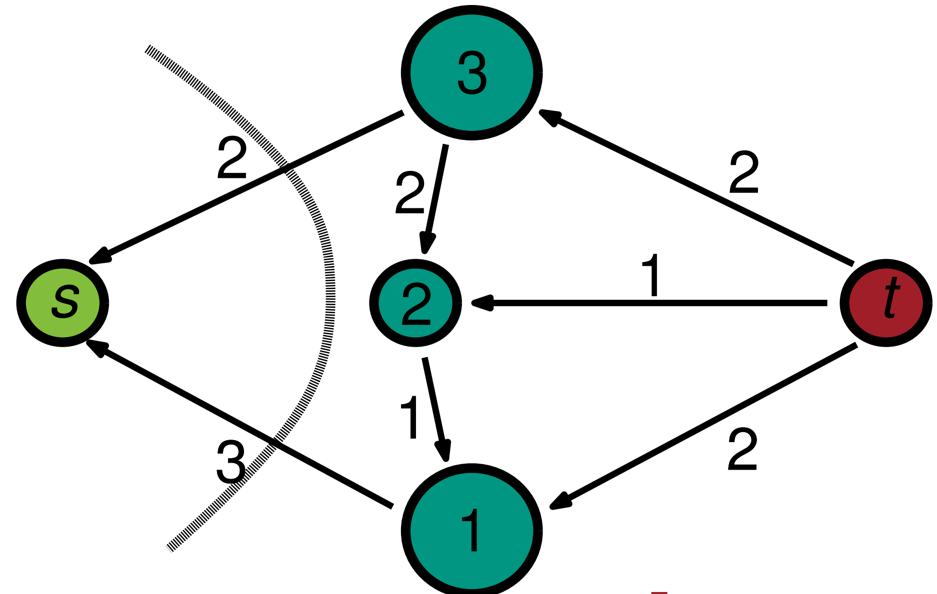
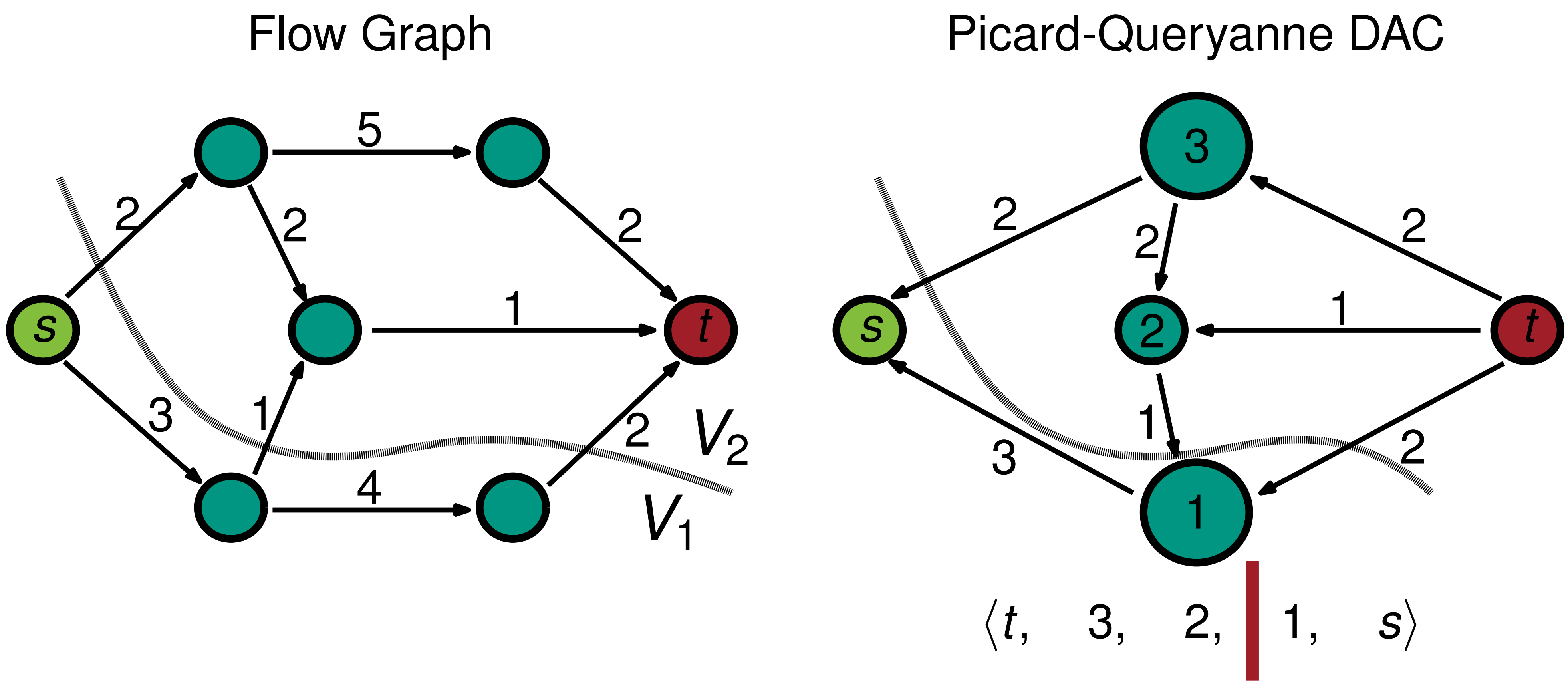


$\langle t,\quad 3,\quad 2,\quad 1, \mid s \rangle$

Sweep through **reverse** topological order

Most Balanced Minimum Cut

One maximum flow f has enough information to enumerate all minimum (s, t)-cuts



Sweep through **reverse** topological order

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Most Balanced Minimum Cut

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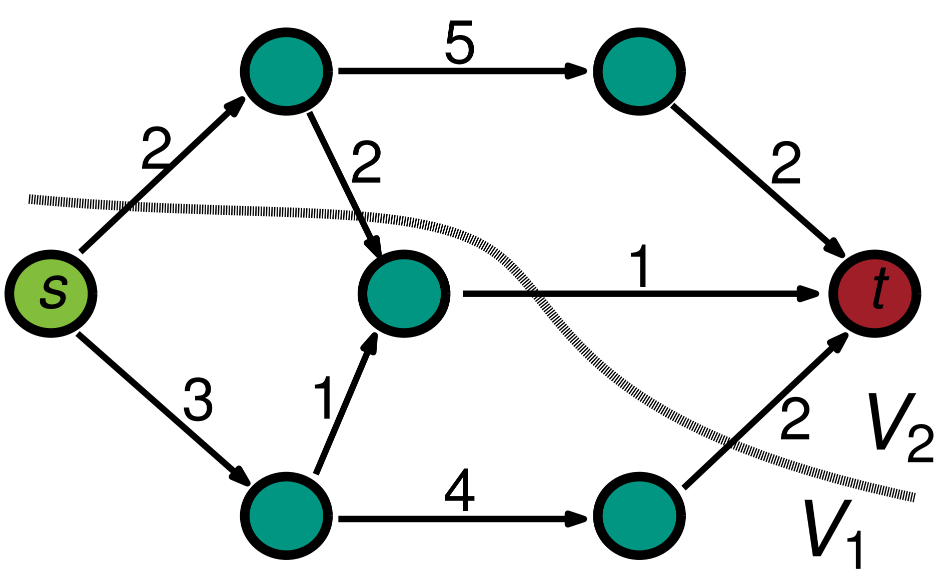
Flow Graph



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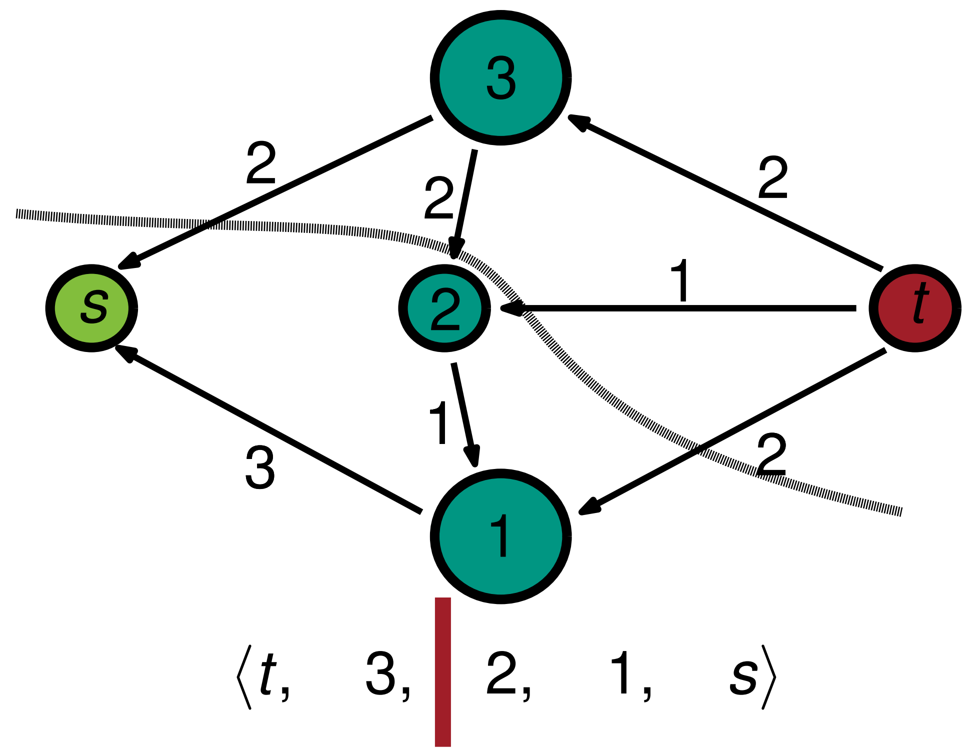

Sweep through **reverse** topological order

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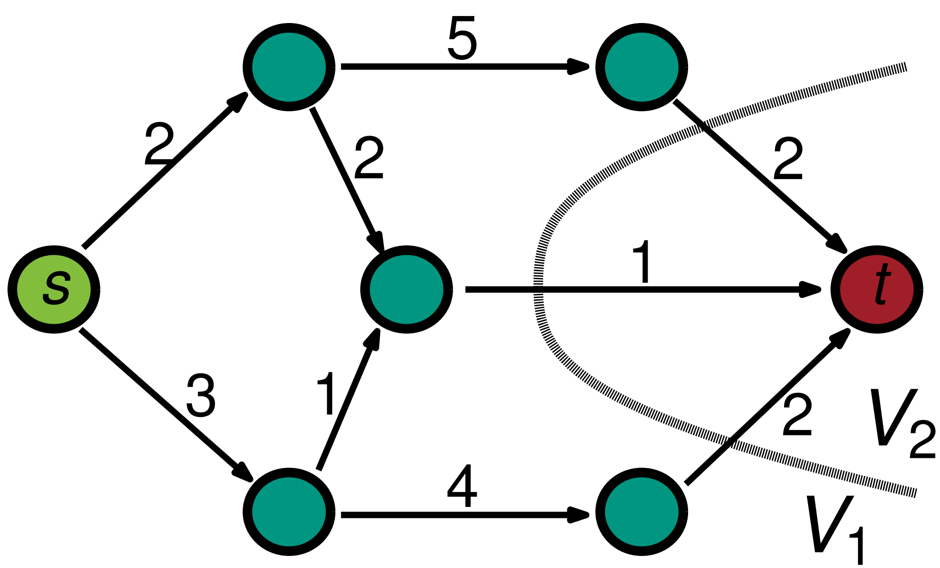
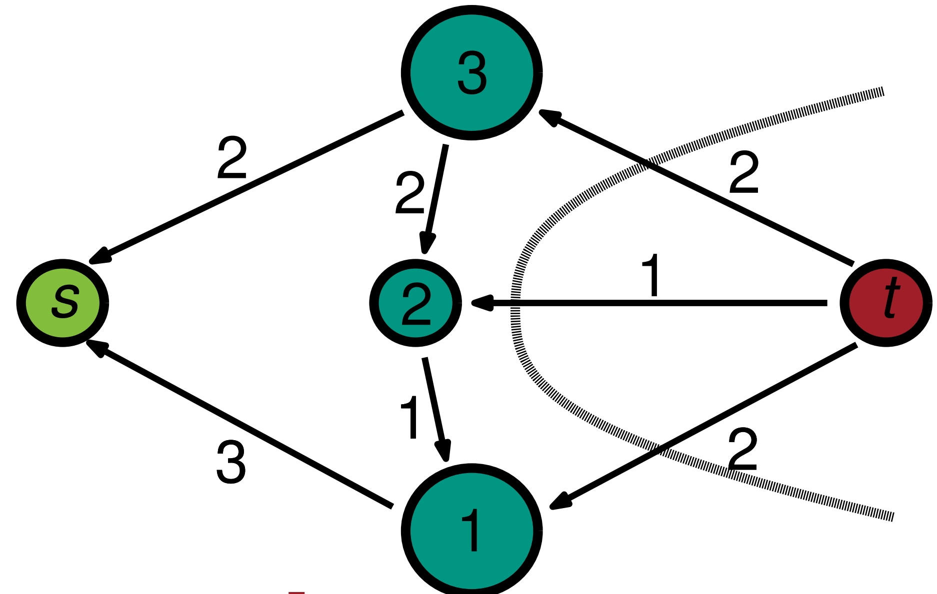
Flow Graph



Picard-Queryanne DAC



$\langle t, \mid 3, \quad 2, \quad 1, \quad s \rangle$

Sweep through **reverse** topological order

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Integration into KaHyPar

Karlsruhe Institute of Technology

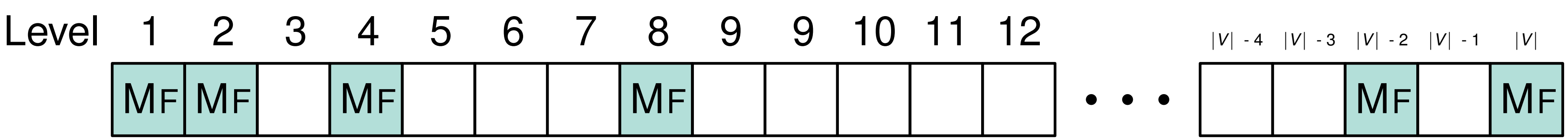
- KaHyPar is a *n*-level hypergraph partitioner

Integration into KaHyPar

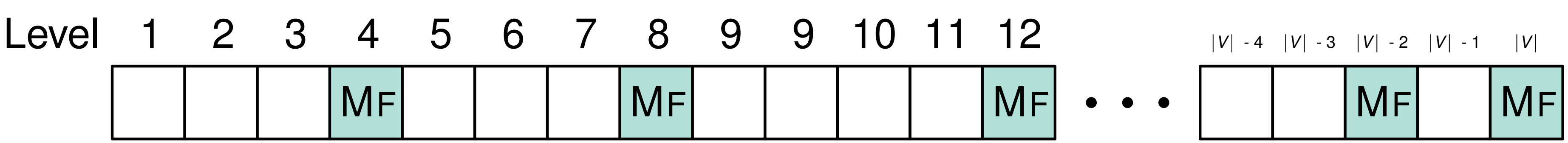
- KaHyPar is a <u>*n*-level</u> hypergraph partitioner

Flow Execution Policies

Exponential: Execute *Max-Flow-Min-Cut* computations (MF) on each level i with $i = 2^j$



Constant: Execute *Max-Flow-Min-Cut* computations (MF) on each level i with $i = \beta \cdot j$



Integration into KaHyPar

Karlsruhe Institute of Technology

- KaHyPar is a *n-level* hypergraph partitioner

Flow Execution Policies

Exponential: Execute *Max-Flow-Min-Cut* computations (MF) on each level i with $i = 2^j$

Level	1	2	3	4	5	6	7	8	9	9	10	11	12		$\|V\| - 4$	$\|V\| - 3$	$\|V\| - 2$	$\|V\| - 1$	$\|V\|$
	MF	MF		MF				MF						• • •			MF		MF

Constant: Execute *Max-Flow-Min-Cut* computations (MF) on each level i with $i = \beta \cdot j$

Level	1	2	3	4	5	6	7	8	9	9	10	11	12		$\|V\| - 4$	$\|V\| - 3$	$\|V\| - 2$	$\|V\| - 1$	$\|V\|$
				MF				MF					MF	• • •			MF		MF

Note, each policy uses *flow*-based refinement on the **last level**

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Integration into KaHyPar

- KaHyPar is a *n*-level hypergraph partitioner

Flow Execution Policies

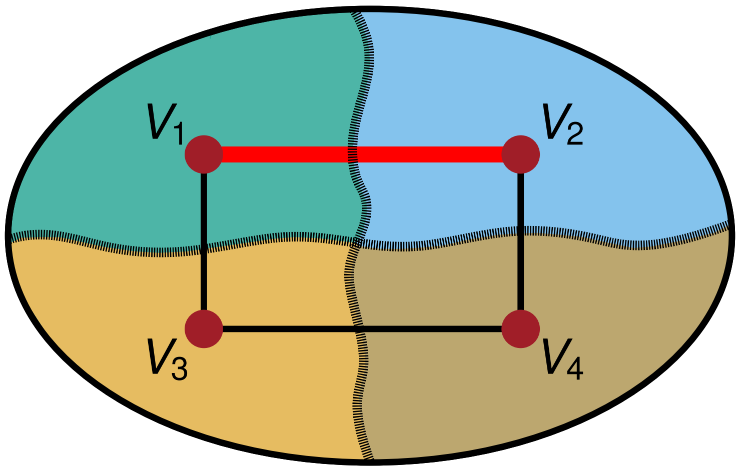
Exponential: Execute *Max-Flow-Min-Cut* computations (MF) on each level i with $i = 2^j$

Level	1	2	3	4	5	6	7	8	9	9	10	11	12		$\|V\|-4$	$\|V\|-3$	$\|V\|-2$	$\|V\|-1$	$\|V\|$
	MF	MF	FM	MF	FM	FM	FM	MF	FM	FM	FM	FM	FM	• • •	FM	FM	MF	FM	MF

Constant: Execute *Max-Flow-Min-Cut* computations (MF) on each level i with $i = \beta \cdot j$

Level	1	2	3	4	5	6	7	8	9	9	10	11	12		$\|V\|-4$	$\|V\|-3$	$\|V\|-2$	$\|V\|-1$	$\|V\|$
	FM	FM	FM	MF	FM	FM	FM	MF	FM	FM	FM	FM	MF	• • •	FM	FM	MF	FM	MF

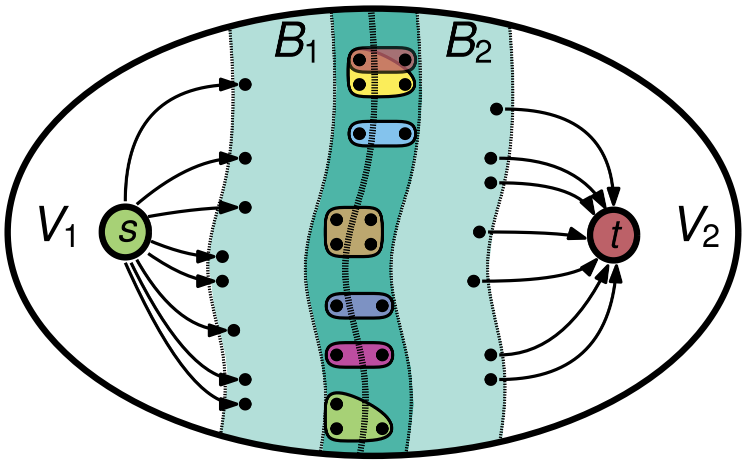
Speed-Up Heuristics



Active Block Scheduling

(R1) If cut between two blocks is small (e.g. $\leq$ 10) skip flow-based refinement, except on the last level

(R2) Only execute flow-based refinement if previous computations lead to an improvement (except in first round)



Adaptive Flow Iterations

(R3) If no hypernode change its block after *Max-Flow-Min-Cut* computation, then break

Experimental Setup

Karlsruhe Institute of Technology

System

- Intel Xeon E5-2670 Octa-Core (2.6 GHz)
- 64 GB Main Memory
- 20 MB L3-Cache, 8 $\times$ 256 KB L2-Cache
- **Compiler**: g++-5.2
- **Flags**: *-O3 -mtune=nactive -march=native*

Flow Algorithms

- EdmondKarp
- GoldbergTarjan
- BoykovKolmogorov
- IBFS (fastest in our experiments)

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Flow Algorithms

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Own Implementations

- BOYKOVKOLMOGOROV
- IBFS (fastest in our experiments)

Third-Party Implementations

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Experimental Setup

Karlsruhe Institute of Technology

Benchmarks

- Parameter Tuning Benchmark Set (25 Hypergraphs)
- Benchmark Subset (165 Hypergraphs)
- Full Benchmark Set (488 Hypergraphs)

Benchmark Type

- DAC (10 Hypergraphs)
- ISPD98 (18 Hypergraphs)
- PRIMAL (92 Hypergraphs)
- LITERAL (92 Hypergraphs)
- DUAL (92 Hypergraphs)
- SPM (184 Hypergraphs)

Experimental Setup

Benchmarks

- Parameter Tuning Benchmark Set (25 Hypergraphs)
- Benchmark Subset (165 Hypergraphs)
- Full Benchmark Set (488 Hypergraphs)

Benchmark Type

VLSI Design
- DAC (10 Hypergraphs)
- ISPD98 (18 Hypergraphs)

SAT Formulas
- PRIMAL (92 Hypergraphs)
- LITERAL (92 Hypergraphs)
- DUAL (92 Hypergraphs)

Sparse Matrices
- SPM (184 Hypergraphs)

Experimental Setup

Benchmarks

- Parameter Tuning Benchmark Set (25 Hypergraphs)
- Benchmark Subset (165 Hypergraphs)
- Full Benchmark Set (488 Hypergraphs)

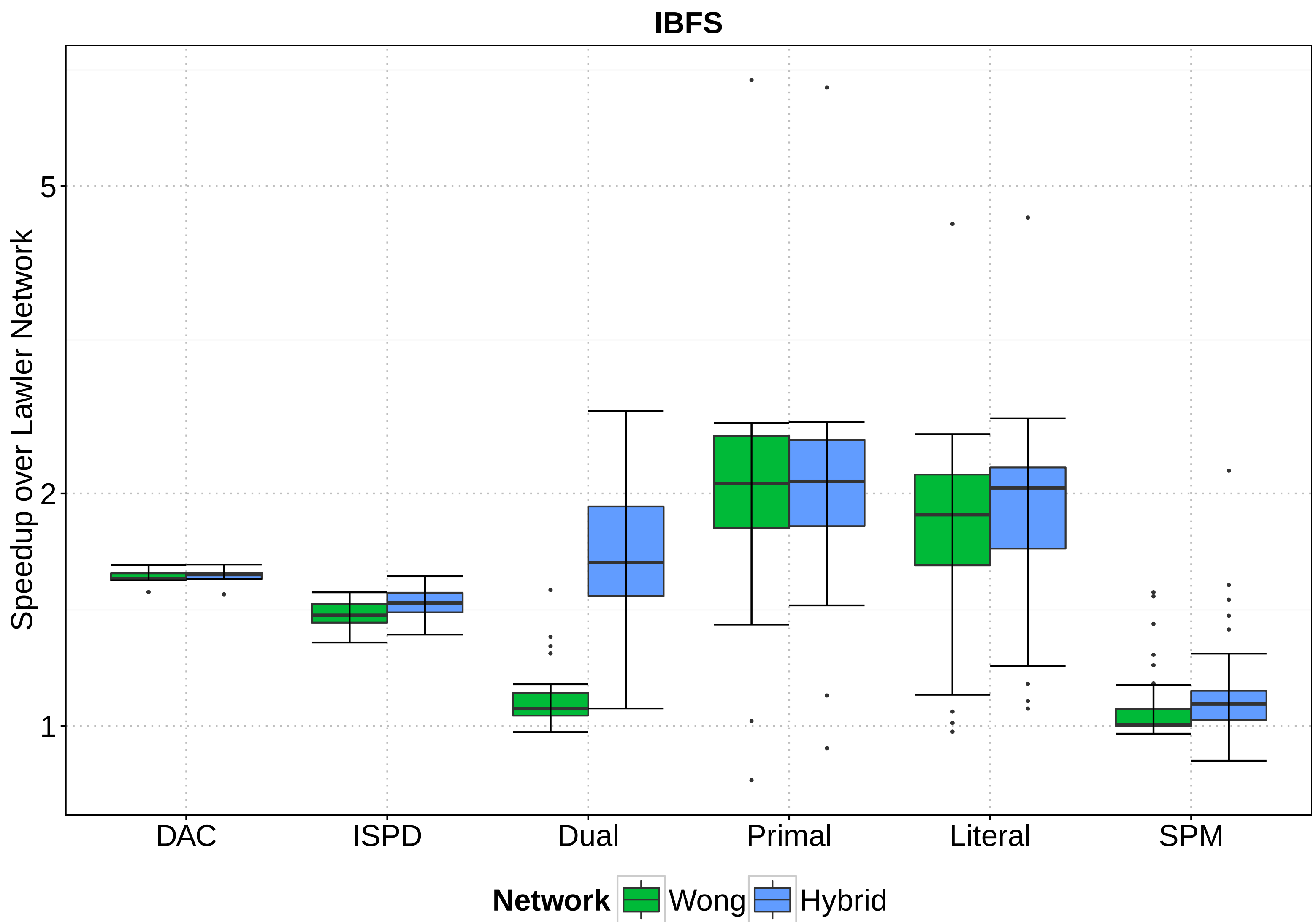
Benchmark Type

- DAC (10 Hypergraphs)
- ISPD98 (18 Hypergraphs)
- PRIMAL (92 Hypergraphs)
- LITERAL (92 Hypergraphs)
- DUAL (92 Hypergraphs)
- SPM (184 Hypergraphs)

Methodology

- $\varepsilon = 3\%$
- $k \in \{2, 4, 8, 16, 32, 64, 128\}$
- 10 seeds

Flow Networks



Flow Configuration

- $+/-$ F = Enabled/Disabled Flow-based refinement
- $+/-$ M = Enabled/Disabled Most Balanced Minimum Cut
- $+/-$ FM = Enabled/Disabled FM Heuristic
- CONSTANT128 = (+F,+M,+FM) with **constant** flow execution policy and $\beta = 128$

Config.	(+F,-M,-FM)		(+F,+M,-FM)		(+F,+M,+FM)		CONSTANT128	
α'	Avg [%]	$t[s]$	Avg [%]	$t[s]$	Avg [%]	$t[s]$	Avg [%]	$t[s]$
1	−6.09	12.91	−5.60	13.40	0.23	15.37	0.53	55.75
2	−3.19	15.75	−2.07	16.74	0.74	18.06	1.09	87.93
4	−1.82	20.37	−0.19	21.88	1.21	22.49	1.61	144.42
8	−0.85	28.49	0.98	30.67	1.71	30.23	2.16	257.41
16	−0.19	43.32	1.75	46.66	2.21	43.53	2.69	498.29
Ref.	KaHyPar-CA		6373.88	13.73				

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Flow Configuration

Improvement of **MBMC** more significantly for large α':

- $\alpha' \in \{1, 2\}$: Improvement 0.5% to 1.12%
- $\alpha' \in \{4, 8, 16\}$: Improvement 1.63% to 1.94%

Config.	(+F,-M,-FM)		(+F,+M,-FM)		(+F,+M,+FM)		CONSTANT128	
α'	Avg [%]	$t[s]$	Avg [%]	$t[s]$	Avg [%]	$t[s]$	Avg [%]	$t[s]$
1	−6.09	12.91	−5.60	13.40	0.23	15.37	0.53	55.75
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Flow Configuration

Flows with FM are faster than flows on its own for large α'

Config.	(+F,-M,-FM)		(+F,+M,-FM)		(+F,+M,+FM)		CONSTANT128	
α'	Avg [%]	$t[s]$	Avg [%]	$t[s]$	Avg [%]	$t[s]$	Avg [%]	$t[s]$
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Flow Configuration

Quality improvement around 0.3% to 0.5%, but running time is impractical

Config.	(+F,-M,-FM)		(+F,+M,-FM)		(+F,+M,+FM)		CONSTANT128	
α'	Avg [%]	$t[s]$	Avg [%]	$t[s]$	Avg [%]	$t[s]$	Avg [%]	$t[s]$
1	−6.09	12.91	−5.60	13.40	0.23	15.37	0.53	55.75
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Flow Configuration

Config.	(+F,-M,-FM)		(+F,+M,-FM)		(+F,+M,+FM)		Constant128	
α'	Avg [%]	$t[s]$	Avg [%]	$t[s]$	Avg [%]	$t[s]$	Avg [%]	$t[s]$
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Ref.	KaHyPar-CA		6373.88	13.73				

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Flow Configuration

Config. / α'	(+F,-M,-FM)		(+F,+M,-FM)	
	KaFFPa	Our	KaFFPa	Our
1	-15.48	-6.10	-15.26	-5.62
2	-10.50	-3.20	-10.12	-2.08
4	-5.98	-1.82	-5.08	-0.20
8	-3.22	-0.85	-1.64	0.98
16	-1.52	-0.20	0.51	1.75
Ref.	KaHyPar-CA		6373.88	

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Flow Configuration

Quality improvement around 5% to 10% for small α'

Config. / α'	(+F,-M,-FM) KaFFPa	(+F,-M,-FM) Our	(+F,+M,-FM) KaFFPa	(+F,+M,-FM) Our
1	-15.48	-6.10	-15.26	-5.62
2	-10.50	-3.20	-10.12	-2.08
4	-5.98	-1.82	-5.08	-0.20
8	-3.22	-0.85	-1.64	0.98
16	-1.52	-0.20	0.51	1.75
Ref.	KaHyPar-CA		6373.88	

Flow Configuration

Quality improvement around 1.5% to 2.5% for large α'

α' \ Config.	(+F,-M,-FM)		(+F,+M,-FM)	
	KaFFPa	Our	KaFFPa	Our
1	-15.48	-6.10	-15.26	-5.62
2	-10.50	-3.20	-10.12	-2.08
4	-5.98	-1.82	-5.08	-0.20
8	-3.22	-0.85	-1.64	0.98
16	-1.52	-0.20	0.51	1.75
Ref.	KaHyPar-CA		6373.88	

Speed-Up Heuristics

Algorithm	Avg [%]	Min [%]	$t_{\text{flow}}[s]$	$t[s]$
KaHyPar-CA	7077.20	6820.17	-	29.26
KaHyPar-MF	-2.47	-2.12	43.04	72.30
KaHyPar-MF$_{(R1)}$	-2.41	-2.06	33.89	63.15
KaHyPar-MF$_{(R1,R2)}$	-2.40	-2.05	28.52	57.78
KaHyPar-MF$_{(R1,R2,R3)}$	-2.41	-2.06	21.23	50.49

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Speed-Up Heuristics

Algorithm	Avg [%]	Min [%]	$t_{flow}[s]$	$t[s]$
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KaHyPar-MF$_{(R1,R2)}$	-2.40	-2.05	28.52	57.78
KaHyPar-MF$_{(R1,R2,R3)}$	-2.41	-2.06	21.23	50.49

Comparable Quality

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Speed-Up Heuristics

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KaHyPar-MF$_{(R1,R2,R3)}$	-2.41	-2.06	21.23	50.49

Speed-up by a factor of 2

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Speed-Up Heuristics

Algorithm	Avg [%]	Min [%]	$t_{\text{flow}}[s]$	$t[s]$
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KaHyPar-MF$_{(R1)}$	-2.41	-2.06	33.89	63.15
KaHyPar-MF$_{(R1,R2)}$	-2.40	-2.05	28.52	57.78
KaHyPar-MF$_{(R1,R2,R3)}$	-2.41	-2.06	21.23	50.49

Slow-down compared to KaHyPar-CA by factor of 1.72

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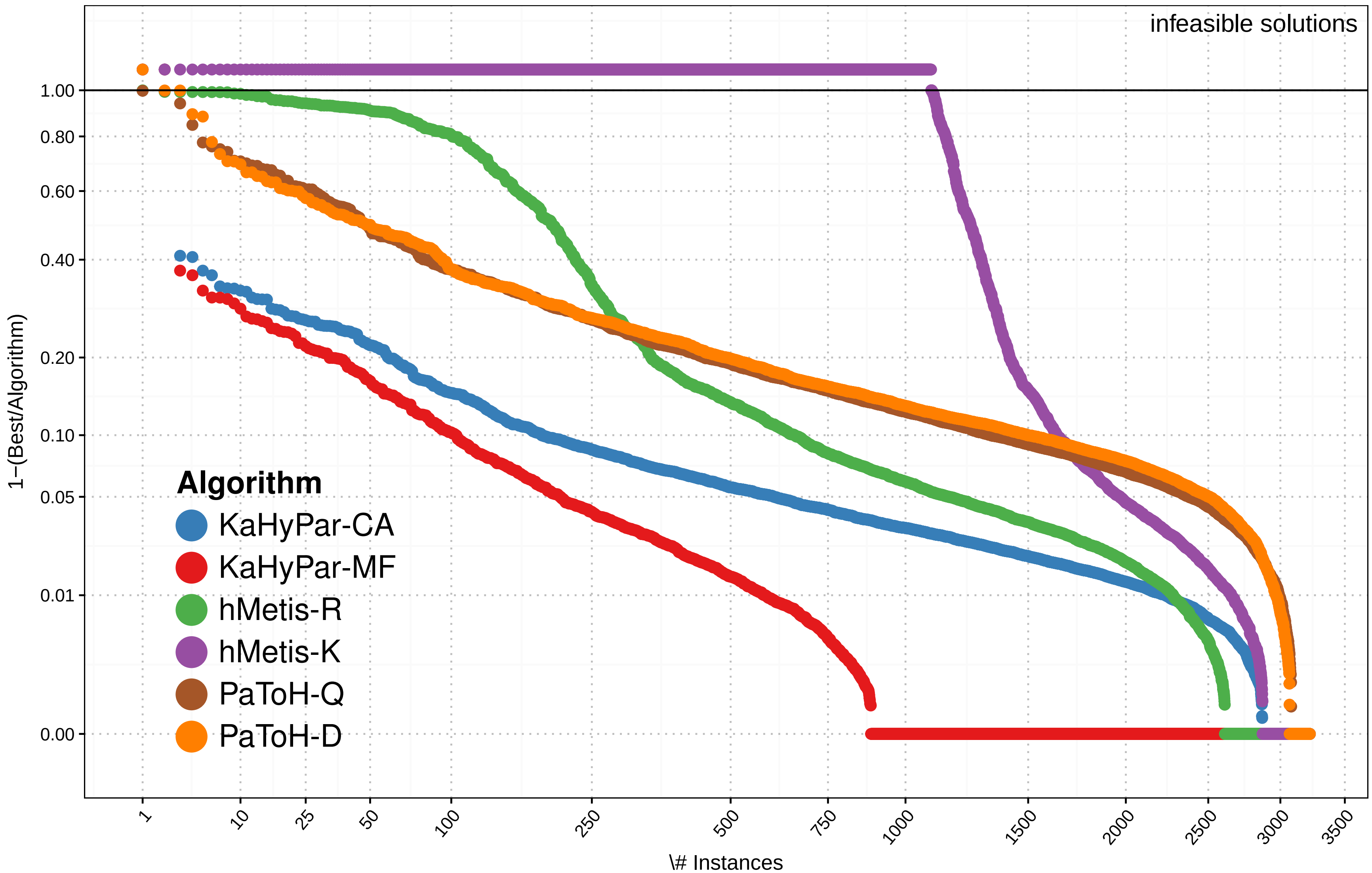
Quality - Full Benchmark Set



Running Time - Full Benchmark Set

Algorithm	Running Time $t[s]$						
	ALL	DAC	ISPD98	PRIMAL	LITERAL	DUAL	SPM
KaHyPar-MF	55.67	504.27	20.83	61.78	119.51	97.22	27.40
KaHyPar-CA	31.05	368.97	12.35	32.91	64.65	68.27	13.91
hMetis-R	79.23	446.36	29.03	66.25	142.12	200.36	41.79
hMetis-K	57.86	240.92	23.18	44.23	94.89	125.55	35.95
PaToH-Q	5.89	28.34	1.89	6.90	9.24	10.57	3.42
PaToH-D	1.22	6.45	0.35	1.12	1.58	2.87	0.77

Running Time - Full Benchmark Set

Overall Running Time

Algorithm	Running Time $t[s]$						
	ALL	DAC	ISPD98	PRIMAL	LITERAL	DUAL	SPM
KaHyPar-MF	55.67	504.27	20.83	61.78	119.51	97.22	27.40
KaHyPar-CA	31.05	368.97	12.35	32.91	64.65	68.27	13.91
hMetis-R	79.23	446.36	29.03	66.25	142.12	200.36	41.79
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PaToH-Q	5.89	28.34	1.89	6.90	9.24	10.57	3.42
PaToH-D	1.22	6.45	0.35	1.12	1.58	2.87	0.77

Slow-down by a factor of 1.8

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Running Time - Full Benchmark Set

Overall Running Time

Algorithm	Running Time $t[s]$						
	ALL	DAC	ISPD98	PRIMAL	LITERAL	DUAL	SPM
KaHyPar-MF	55.67	504.27	20.83	61.78	119.51	97.22	27.40
KaHyPar-CA	31.05	368.97	12.35	32.91	64.65	68.27	13.91
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PaToH-Q	5.89	28.34	1.89	6.90	9.24	10.57	3.42
PaToH-D	1.22	6.45	0.35	1.12	1.58	2.87	0.77

Comparable running time to hMetis-K

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Running Time - Full Benchmark Set

Small Flow Network Instances

Algorithm	Running Time $t[s]$						
	ALL	DAC	ISPD98	PRIMAL	LITERAL	DUAL	SPM
KaHyPar-MF	55.67	504.27	20.83	61.78	119.51	97.22	27.40
KaHyPar-CA	31.05	368.97	12.35	32.91	64.65	68.27	13.91
hMetis-R	79.23	446.36	29.03	66.25	142.12	200.36	41.79
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PaToH-Q	5.89	28.34	1.89	6.90	9.24	10.57	3.42
PaToH-D	1.22	6.45	0.35	1.12	1.58	2.87	0.77

Slow-down by a factor of 1.4

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Running Time - Full Benchmark Set

Large Flow Network Instances

Algorithm	Running Time $t[s]$						
	ALL	DAC	ISPD98	PRIMAL	LITERAL	DUAL	SPM
KaHyPar-MF	55.67	504.27	20.83	61.78	119.51	97.22	27.40
KaHyPar-CA	31.05	368.97	12.35	32.91	64.65	68.27	13.91
hMetis-R	79.23	446.36	29.03	66.25	142.12	200.36	41.79
hMetis-K	57.86	240.92	23.18	44.23	94.89	125.55	35.95
PaToH-Q	5.89	28.34	1.39	6.90	9.24	10.57	3.42
PaToH-D	1.22	6.45	0.35	1.12	1.58	2.87	0.77

Slow-down by a factor of 1.85

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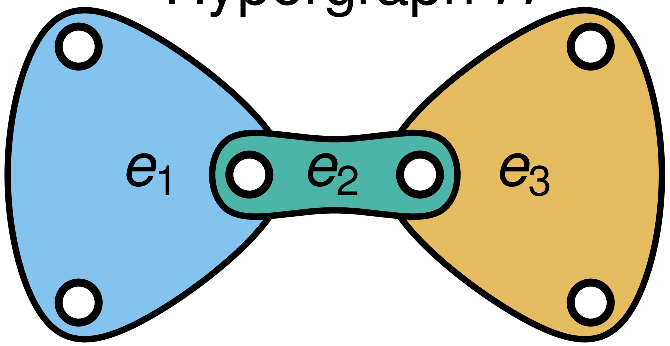
Conclusion

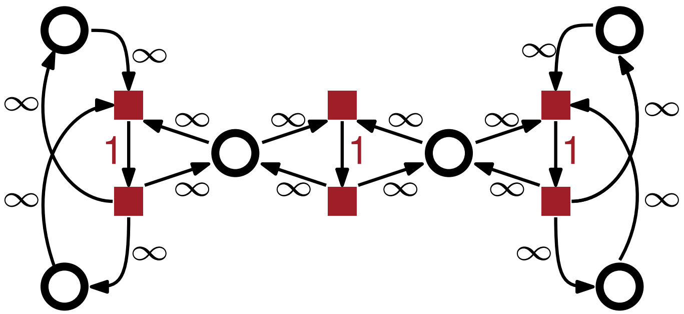
Appendix

Hypergraph Flow Network - Graph Edges

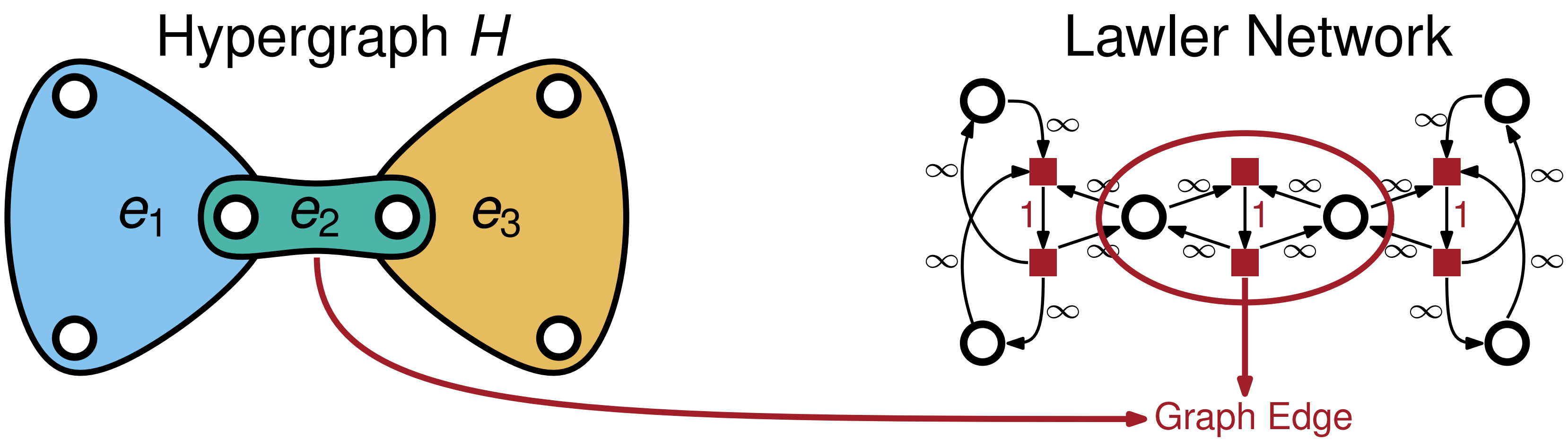
Hypergraph H



Lawler Network



Hypergraph Flow Network - Graph Edges



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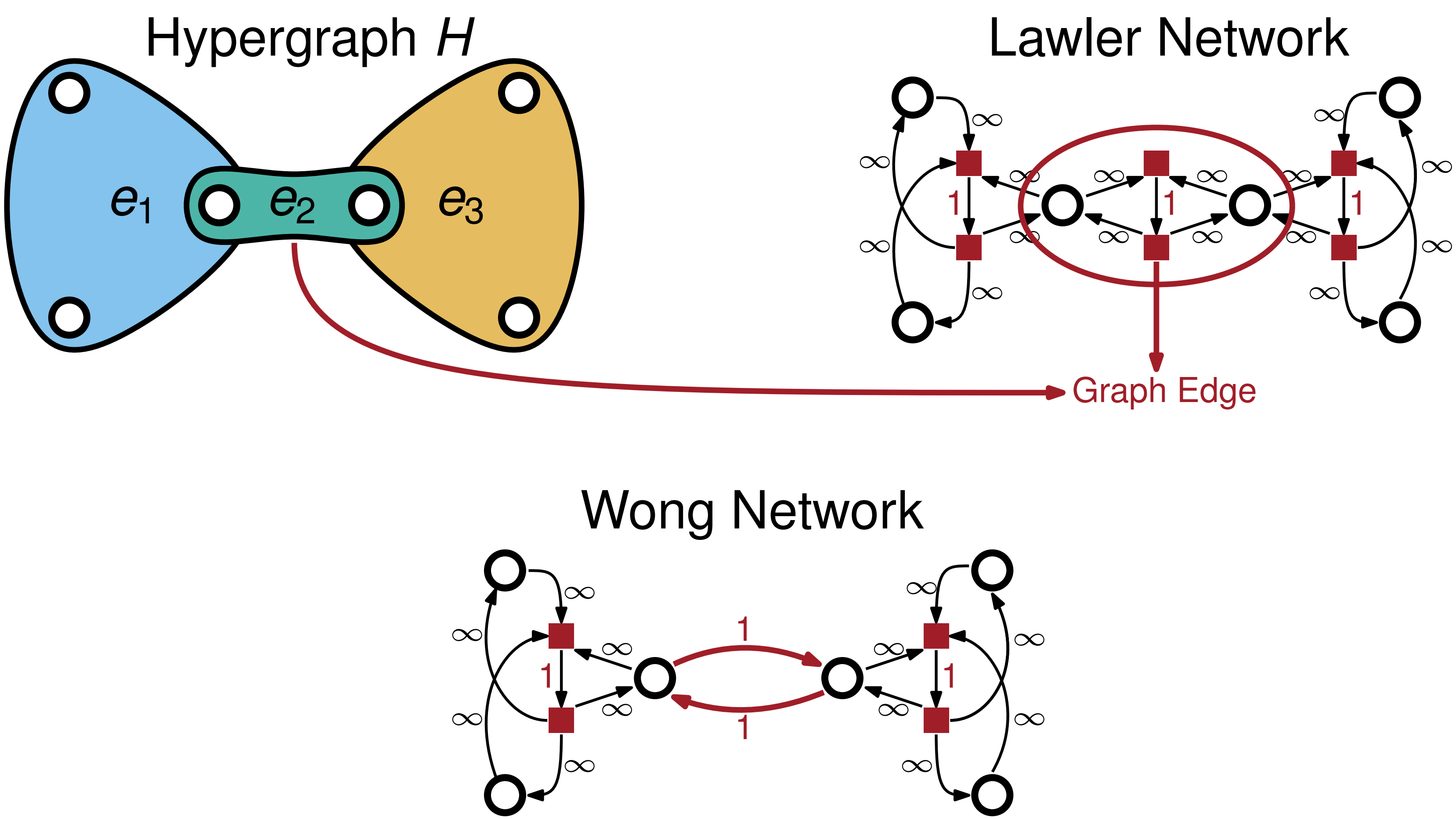
Hypergraph Flow Network - Graph Edges



Removing Source and Sink Vertices

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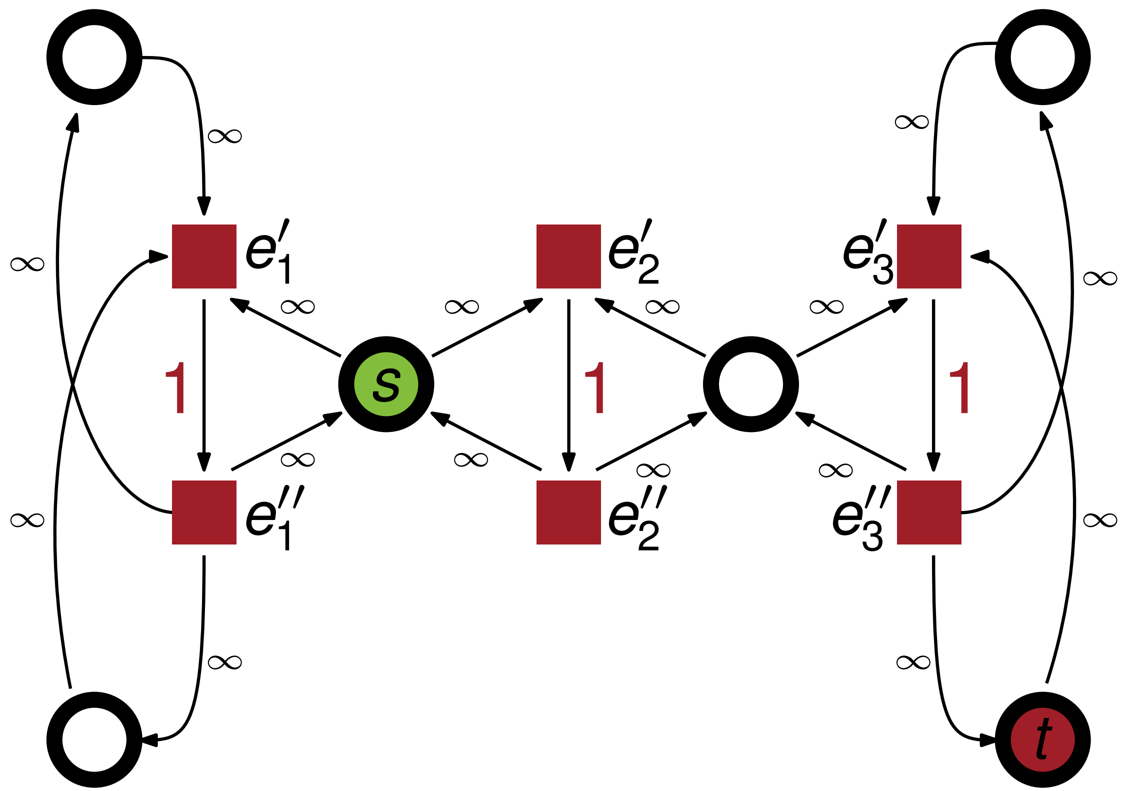
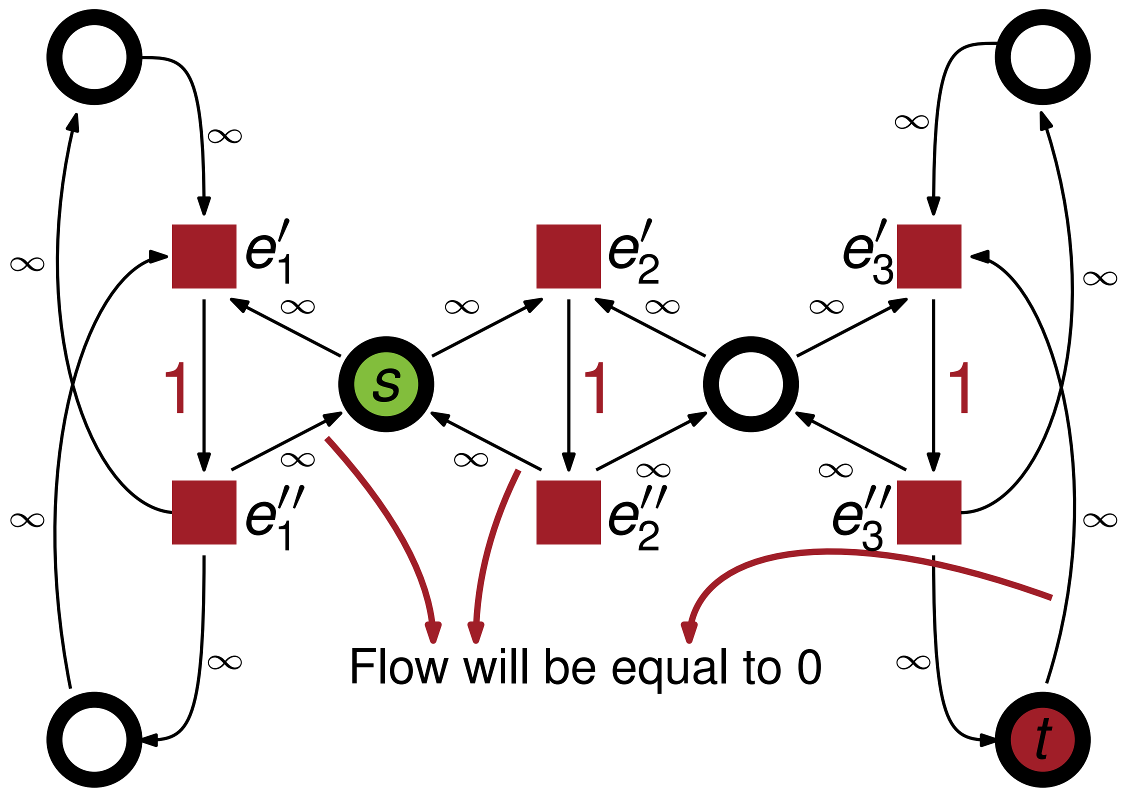


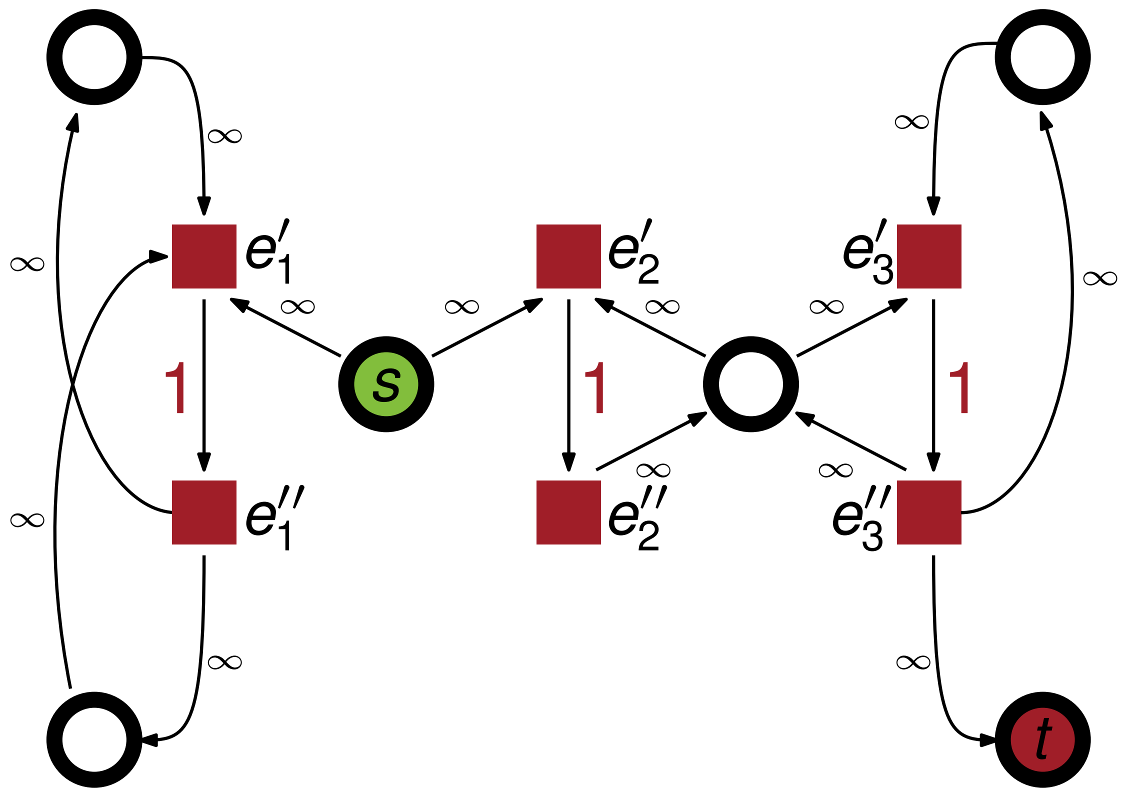
Lawler Network

Removing Source and Sink Vertices



Lawler Network

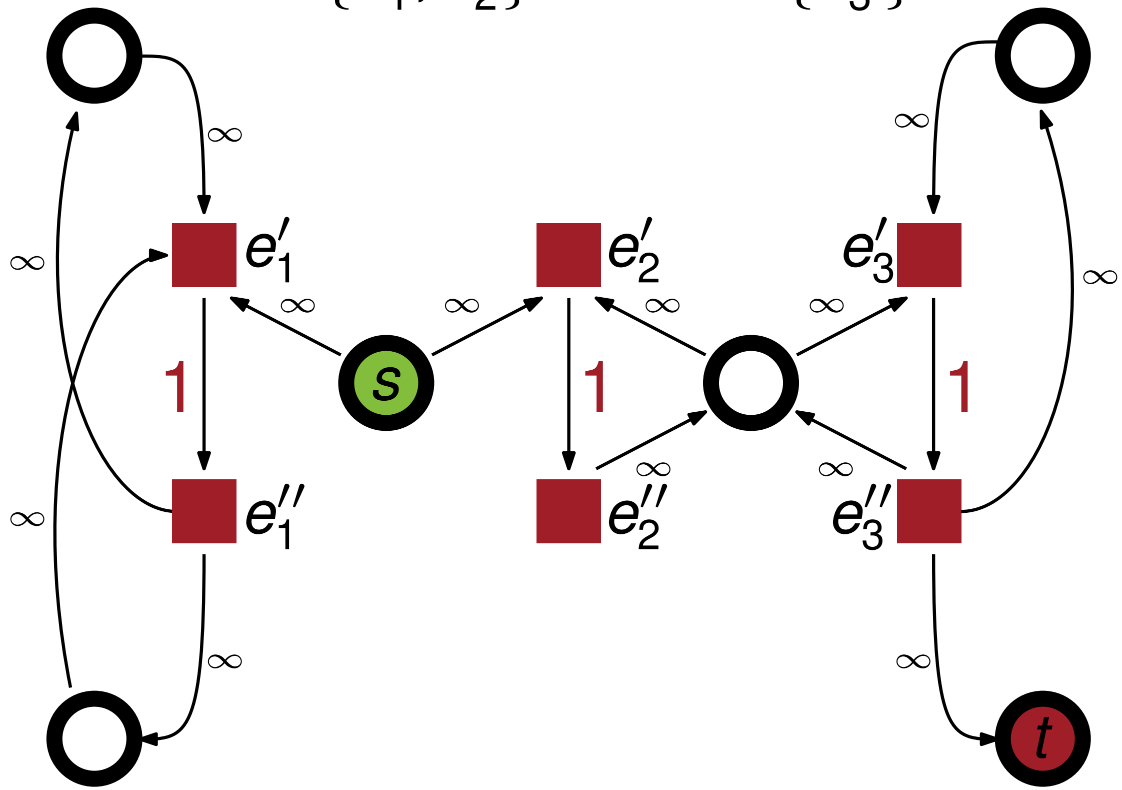
Removing Source and Sink Vertices



Lawler Network

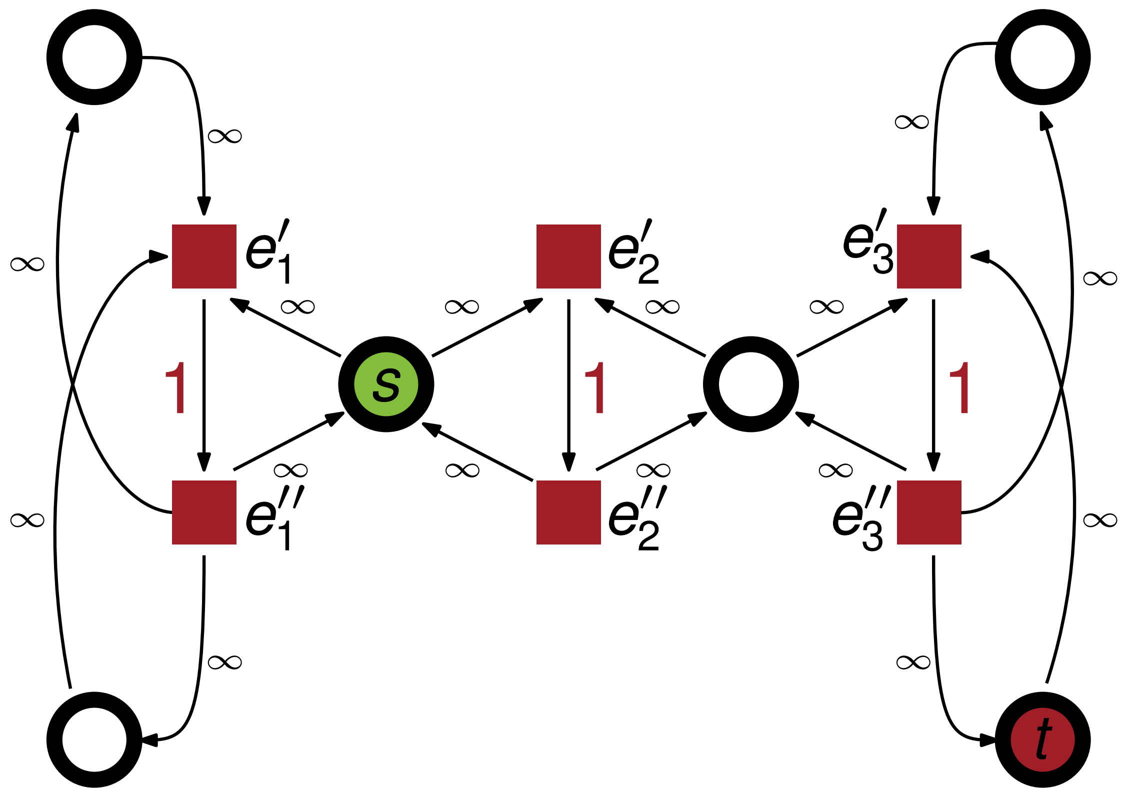
Removing Source and Sink Vertices

Corresponds to *Multi-Source Multi-Sink* problem with
$S = \{e'_1, e'_2\}$ and $T = \{e''_3\}$

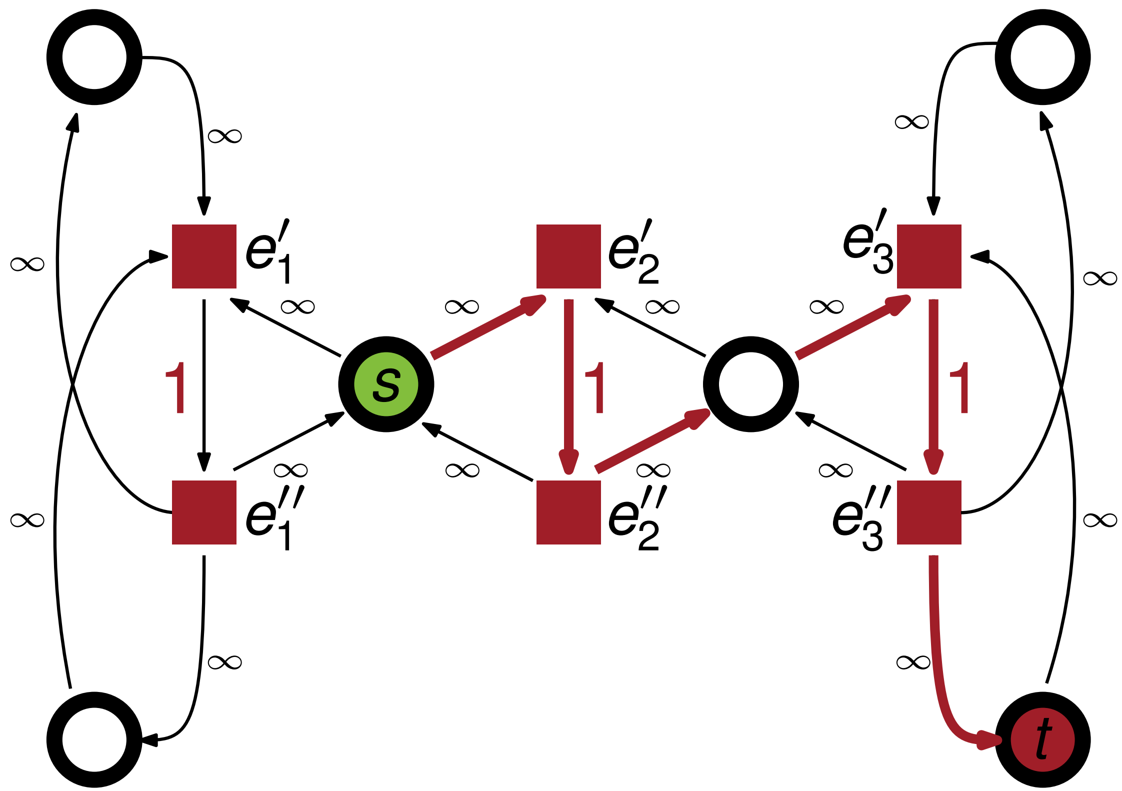


Lawler Network

Minimum (s, t)-Bipartition



Minimum (s, t)-Bipartition



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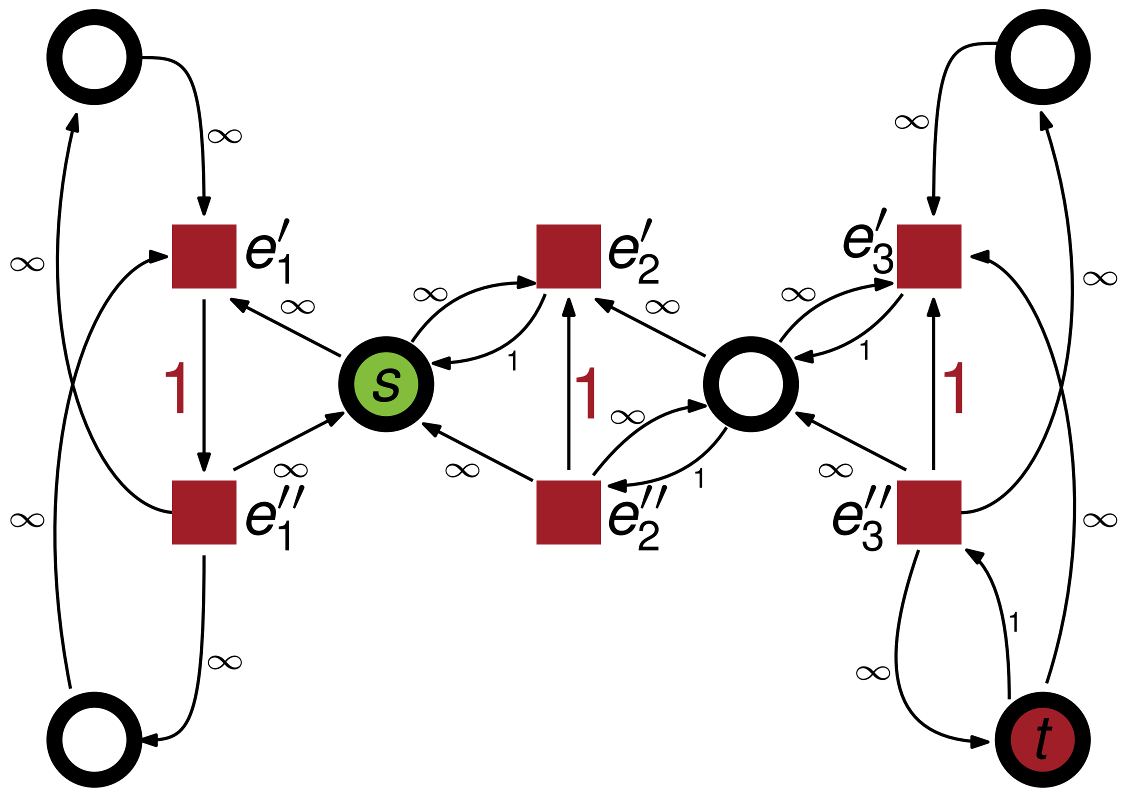
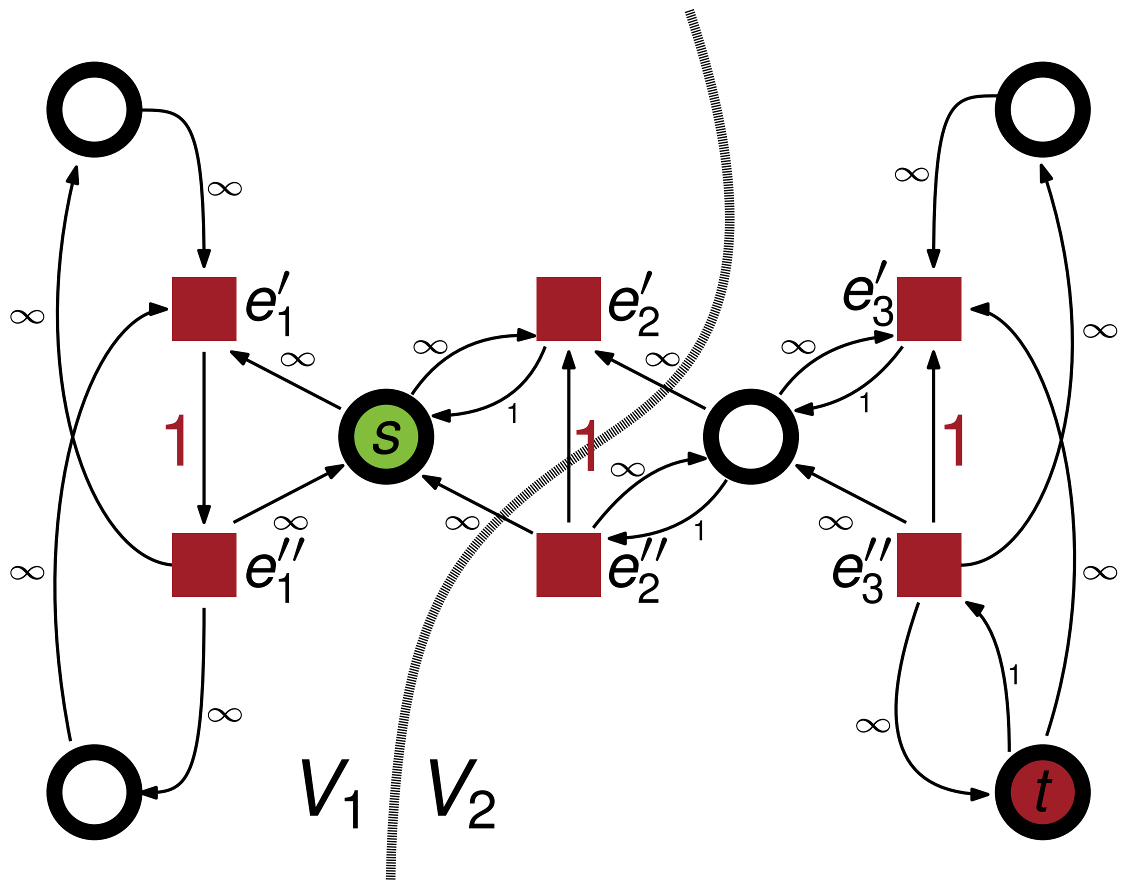
Minimum (s, t)-Bipartition



Minimum (s, t)-Bipartition



All nodes *reachable* from s are part of V_1 and $V_2 = V \setminus V_1$

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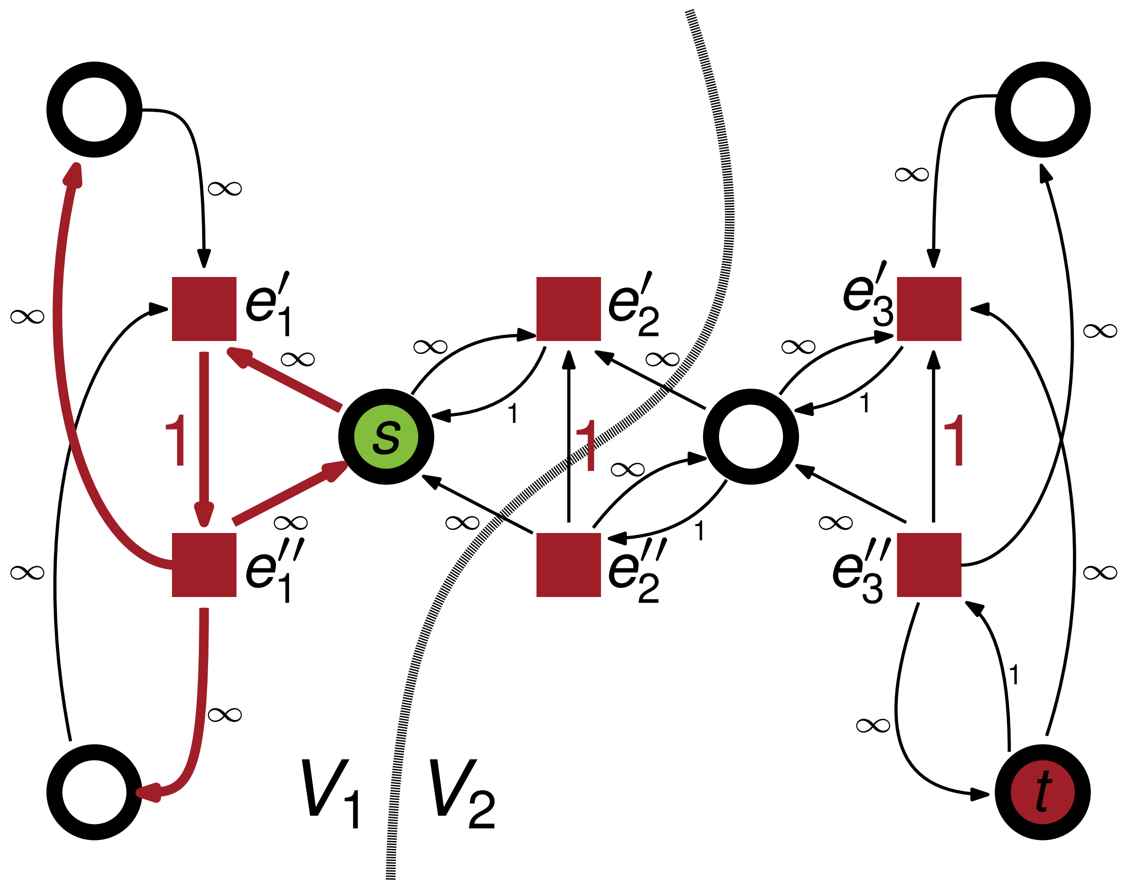
Minimum (s, t)-Bipartition

For each hypernode $v \in V_1$, there exists at least one $e \in I(v)$ with $e'' \in V_1$



All nodes *reachable* from s are part of V_1 and $V_2 = V \setminus V_1$

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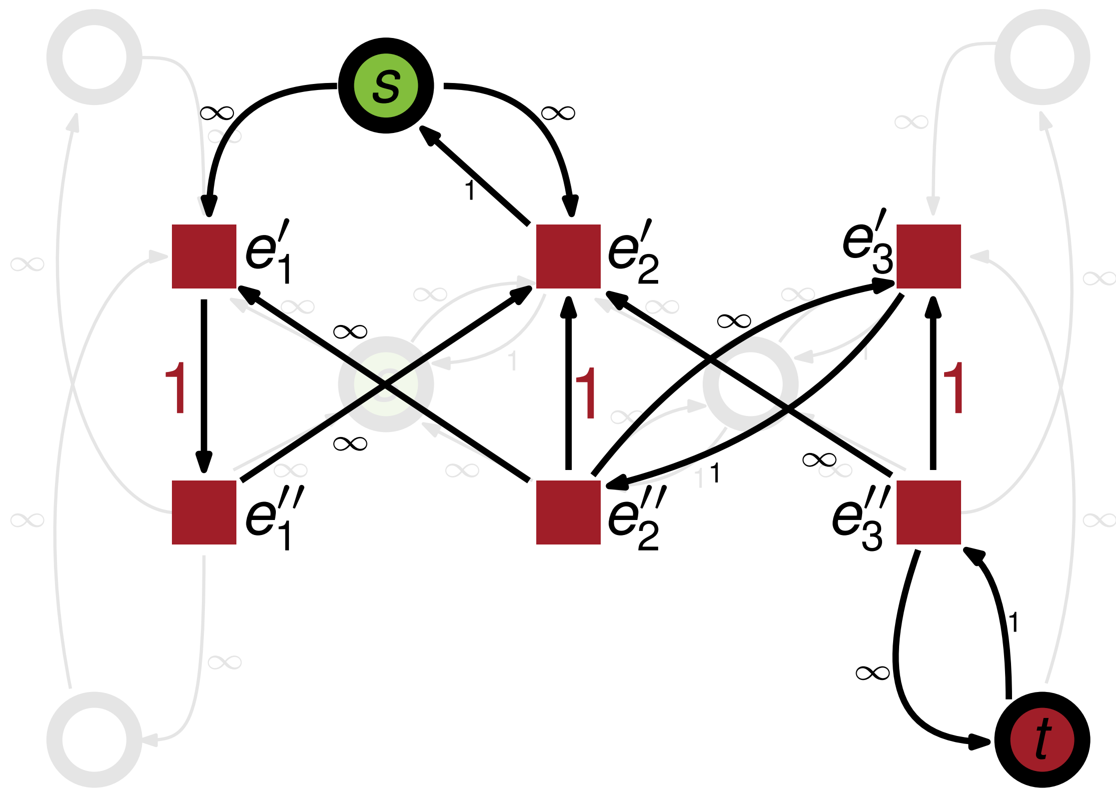
Minimum (s, t)-Bipartition



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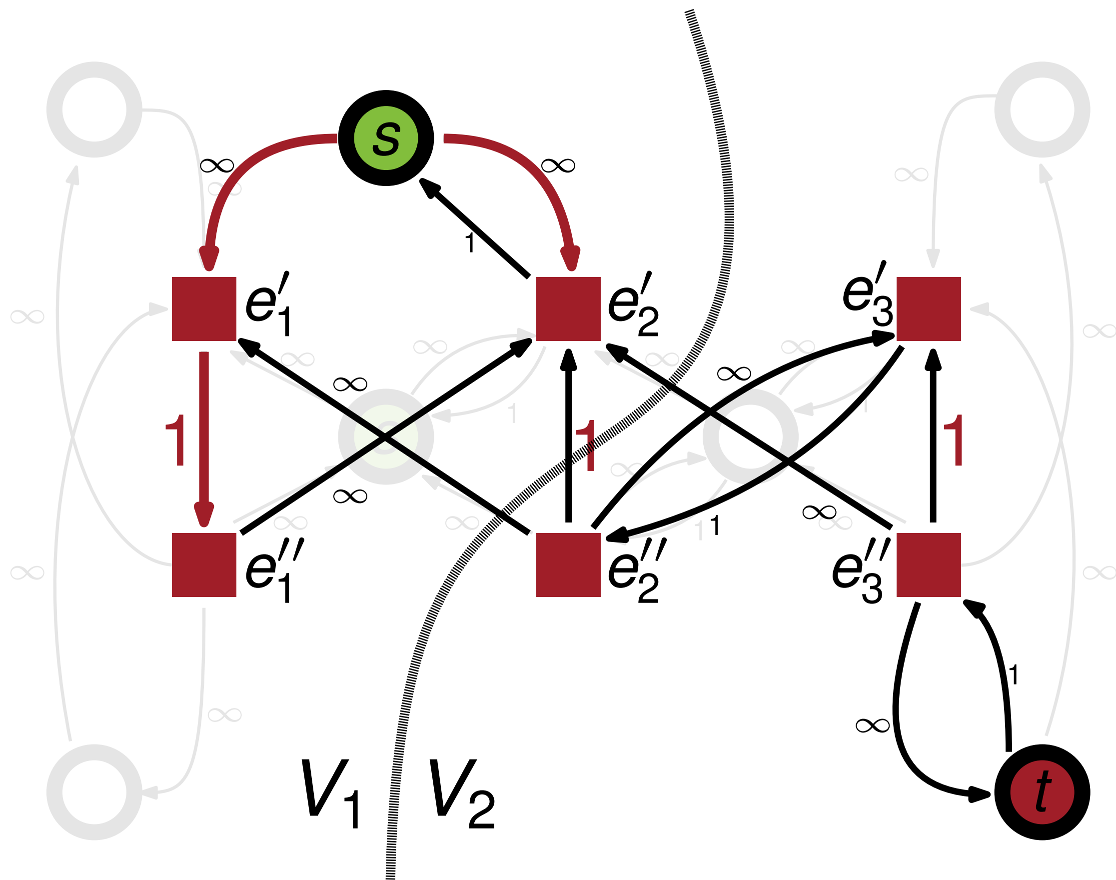
Minimum (s, t)-Bipartition



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Minimum (s, t)-Bipartition