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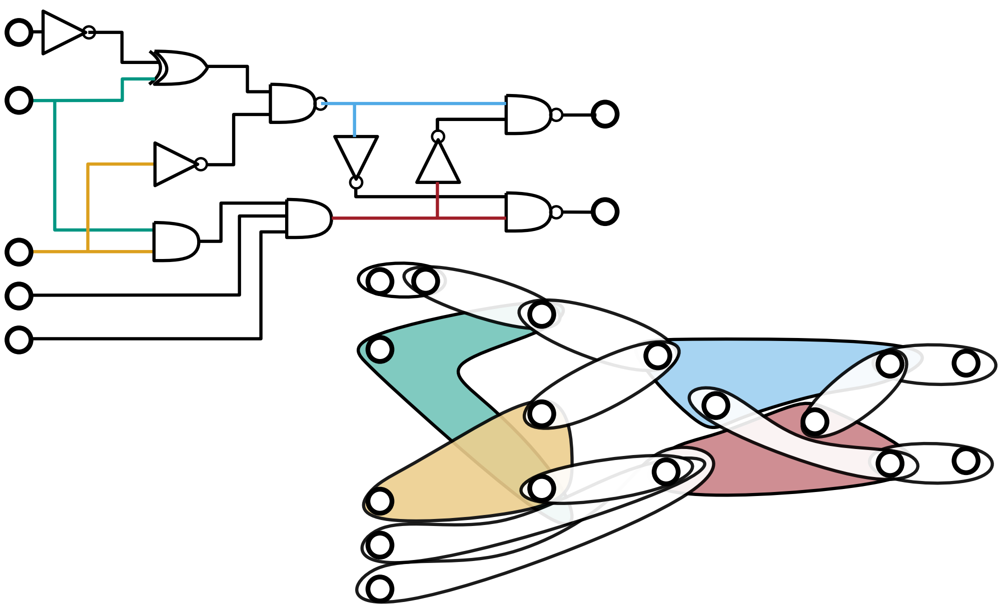
High Quality Hypergraph Partitioning via Max-Flow-Min-Cut Computations

Master Thesis · February 16, 2018

Tobias Heuer

INSTITUTE OF THEORETICAL INFORMATICS · ALGORITHMICS GROUP



Outline

Task

Developing a **local search** algorithm based on **Max-Flow-Min-Cut** computations for the *n*-level hypergraph partitioner **KaHyPar**.

Outline

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Contributions

- Outperforms 5 different systems on 73% of 3216 benchmark instances
- Improve quality of *KaHyPar* by 2.5%, while only incurring a slowdon by a factor of 2
- Comparable running time to *hMetis* and outperforms it on 84% of the instances

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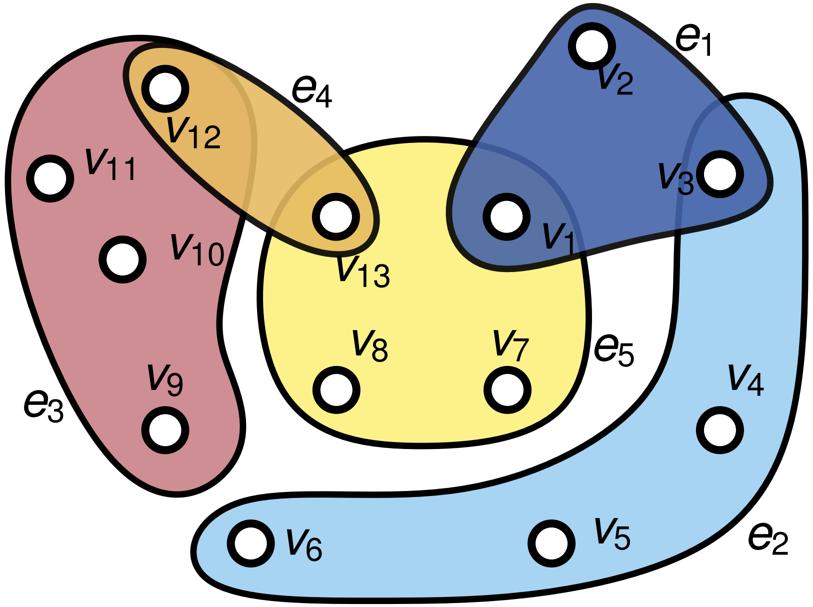
Hypergraphs

[from SEA'17]

- Generalization of graphs
  $\Rightarrow$ hyperedges connect $\geq$ 2 nodes

- Graphs $\Rightarrow$ dyadic ($\mathbf{2}$-**ary**) relationships
- Hypergraphs $\Rightarrow$ ($\mathbf{d}$-**ary**) relationships

- Hypergraph $H = (V, E, c, \omega)$
  - Vertex set $V = \{1, ..., n\}$
  - Edge set $E \subseteq \mathcal{P}(V) \setminus \emptyset$
  - Node weights $c : V \to \mathbb{R}_{\geq 1}$
  - Edge weights $\omega : E \to \mathbb{R}_{\geq 1}$



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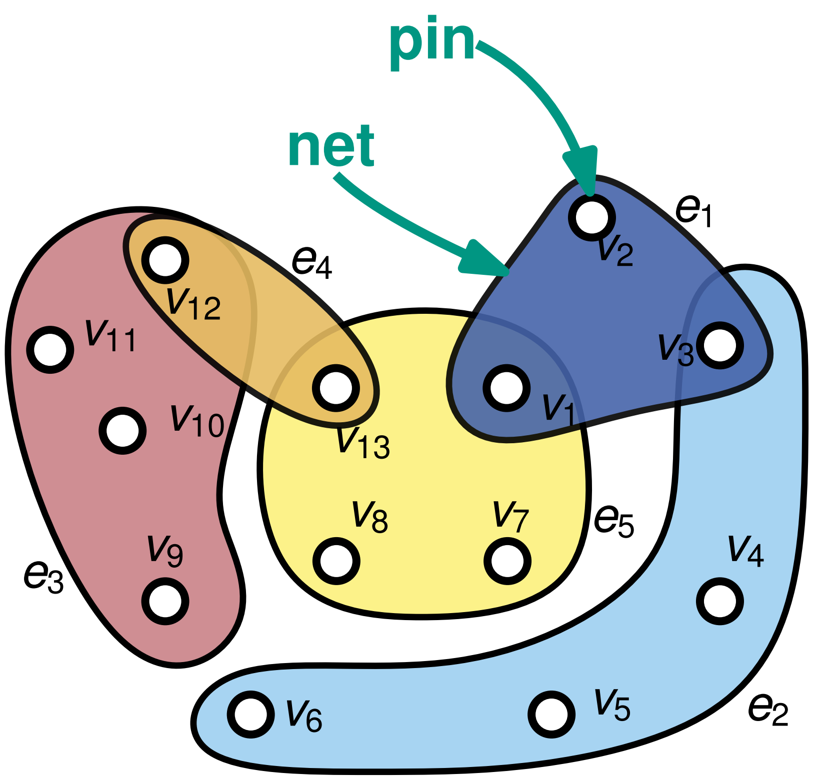
- $|P| = \sum_{e \in E} |e| = \sum_{v \in V} d(v)$



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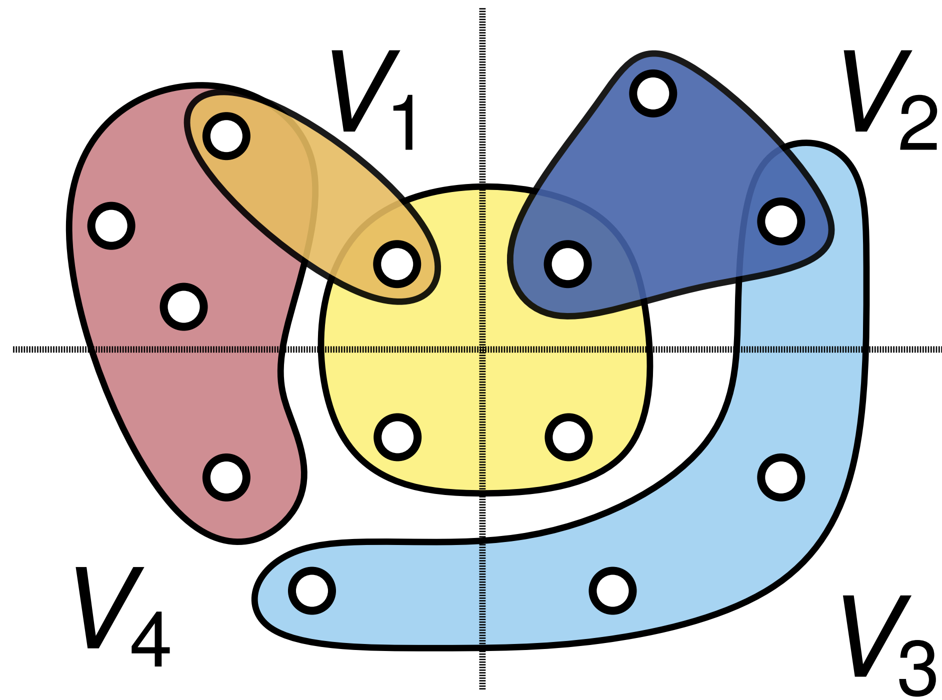
Hypergraph Partitioning Problem

[from SEA'17]

Partition hypergraph $H = (V, E, c, \omega)$ into $\mathbf{k}$ non-empty disjoint blocks $\Pi = \{V_1, \ldots, V_k\}$ such that:

- blocks V_i are **roughly equal-sized**:

$$c(V_i) \leq (1 + \varepsilon) \left\lceil \frac{c(V)}{k} \right\rceil$$



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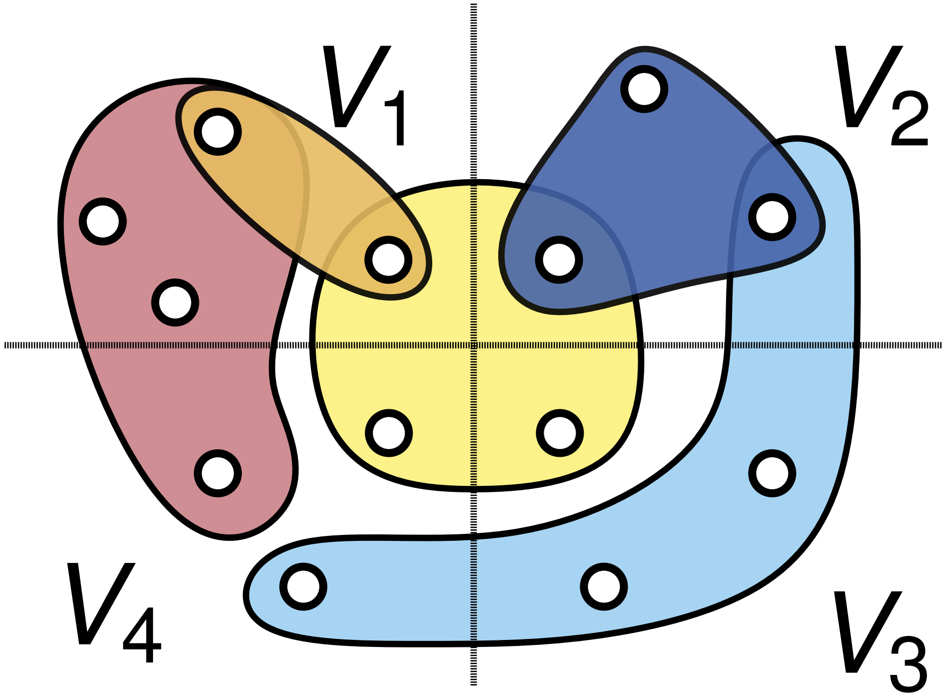

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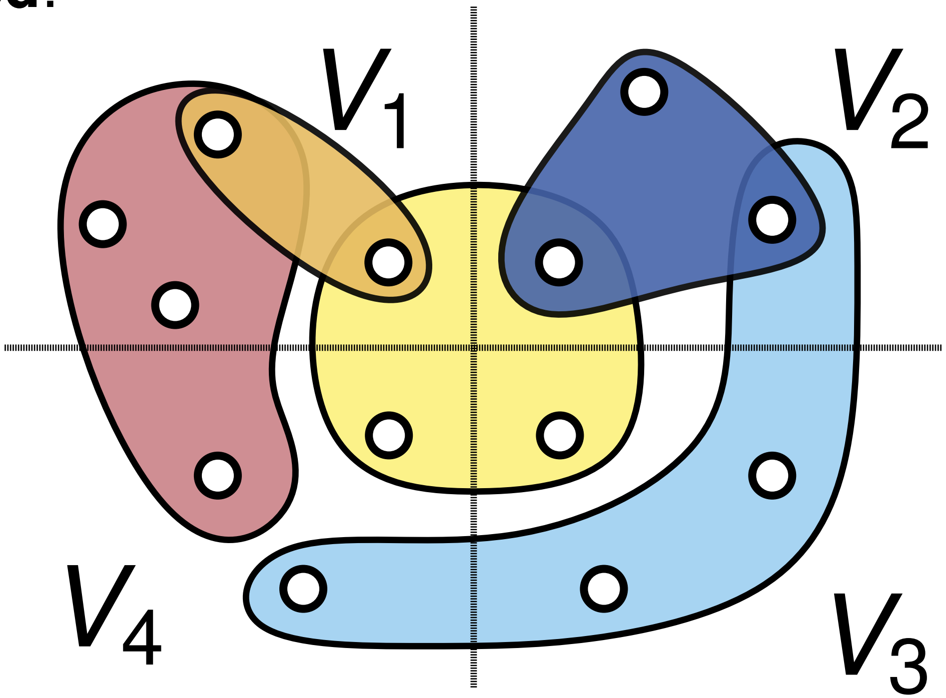

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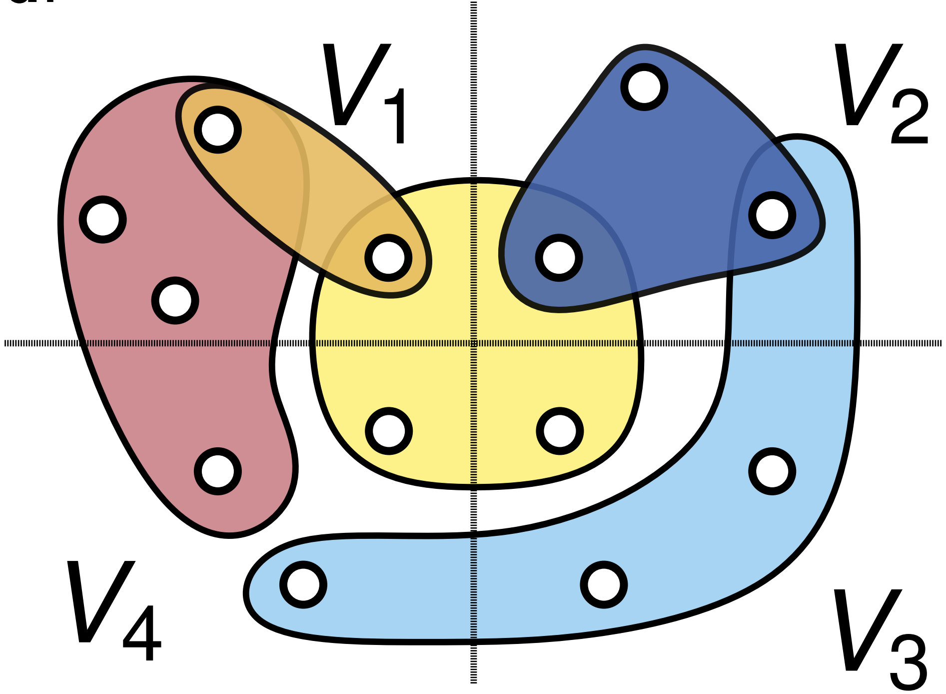
$$\sum_{e \in \text{cut}} (\lambda - 1)\, \omega(e)$$

connectivity:
 # blocks connected by net *e*



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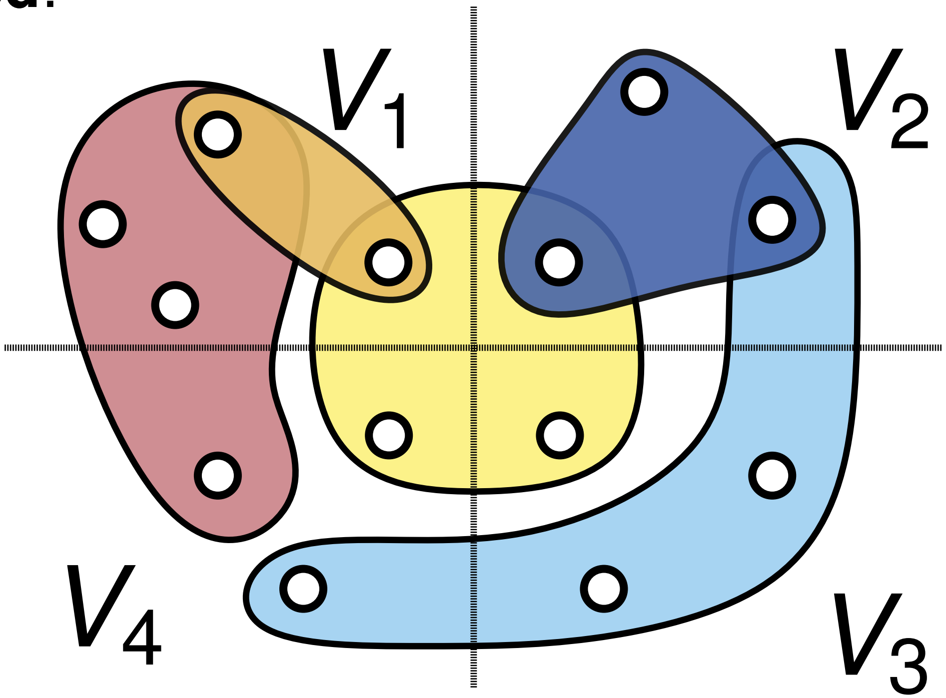
$$c(V_i) \leq (1 + \varepsilon) \left\lceil \frac{c(V)}{k} \right\rceil$$

- **connectivity** objective is **minimized**:

$$\sum_{e \in \text{cut}} (\lambda - 1)\, \omega(e) = 6$$

connectivity:
 # blocks connected by net e



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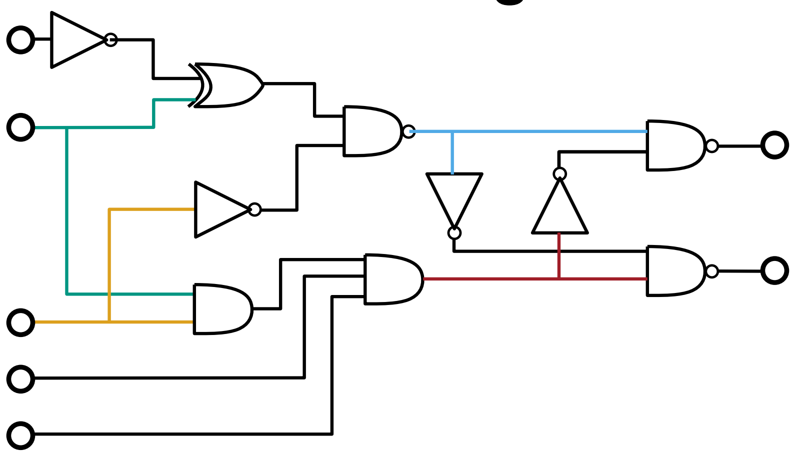
Applications

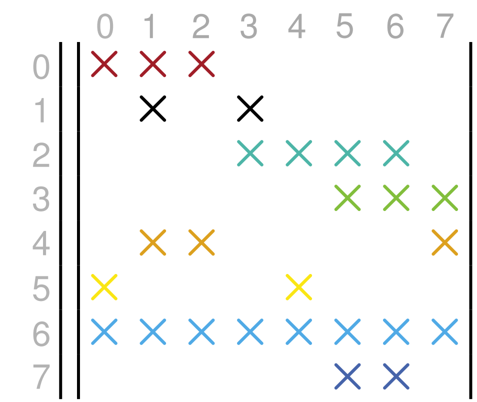
[from SEA'17]

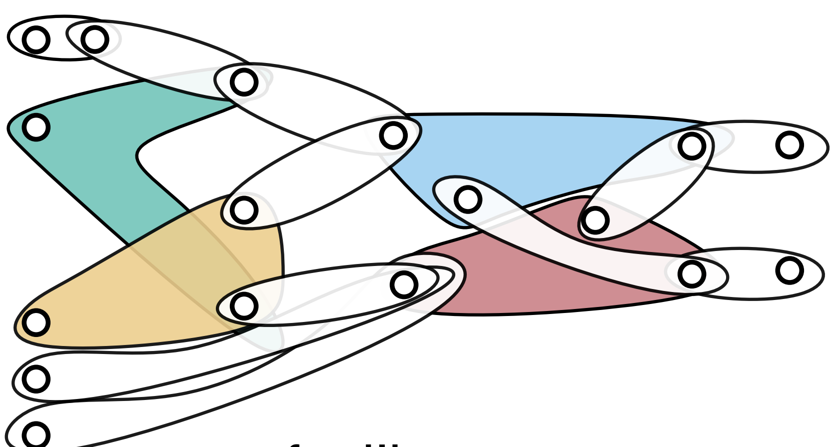
VLSI Design



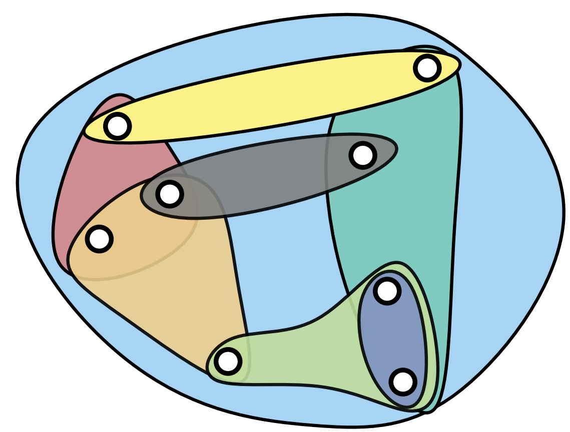
Application Domain

Scientific Computing





Hypergraph Model



facilitate floorplanning & placement

Goal

minimize communication

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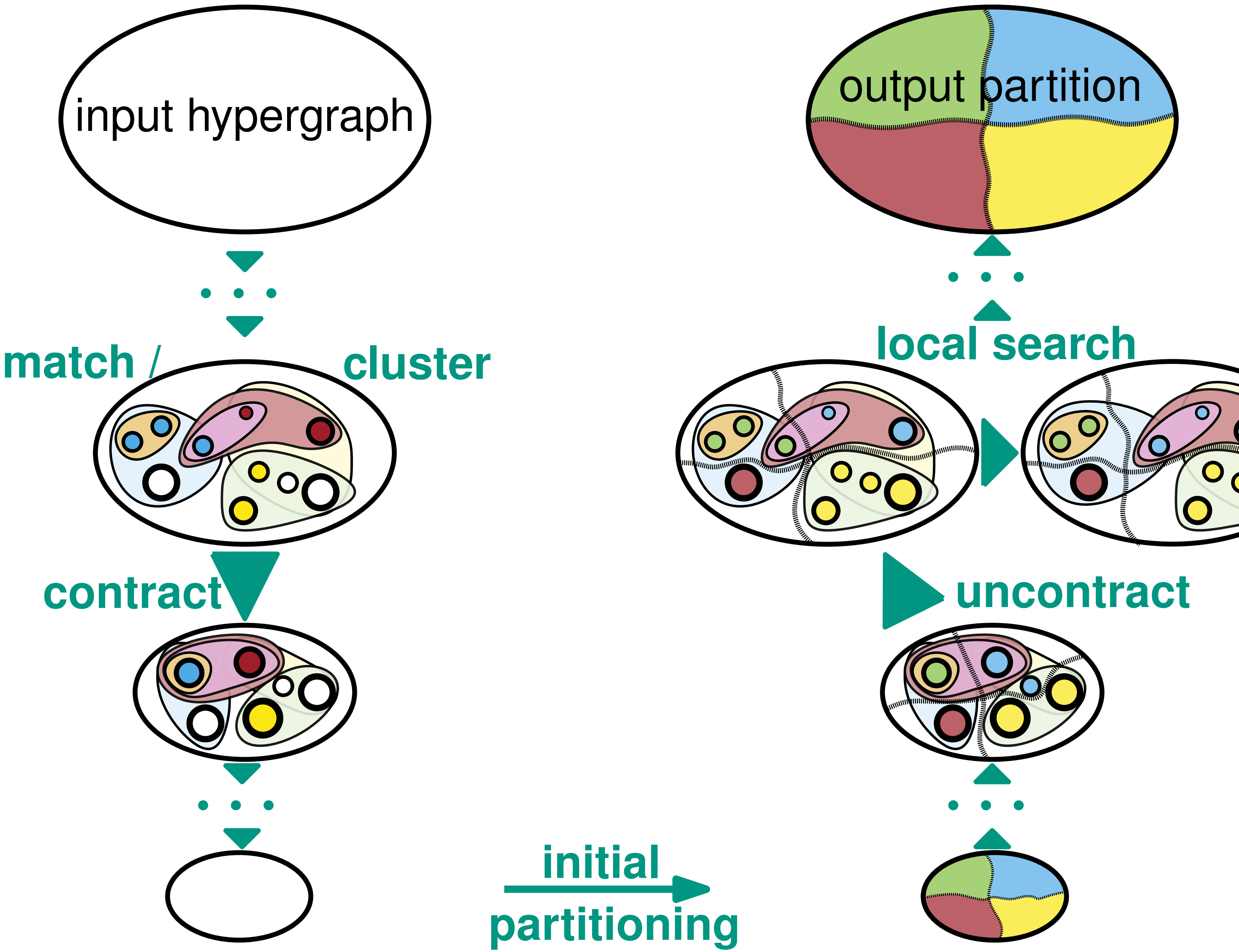
The Multilevel Framework

[from SEA'17]



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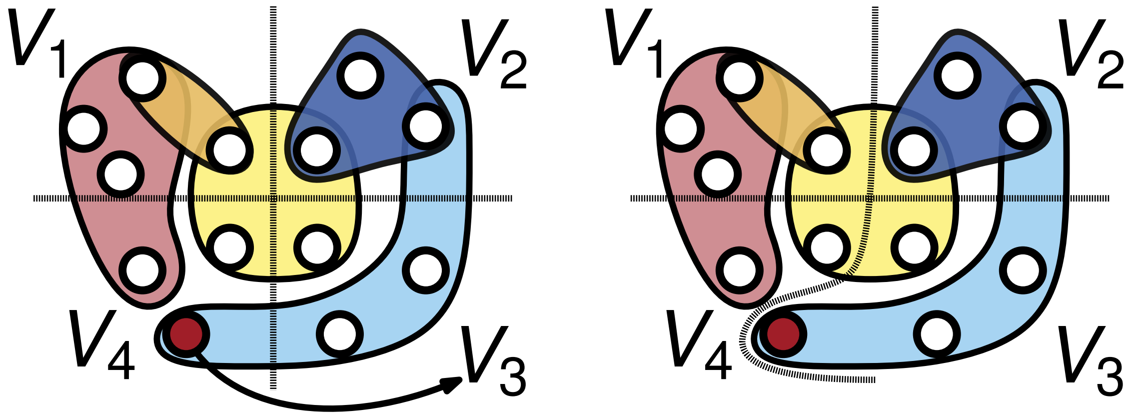
FM Algorithm

- **Move**-based heuristic that **greedily** move vertices between blocks based on **local** informations of incident nets



Moving ● from V_4 to V_3 reduces cut by 1

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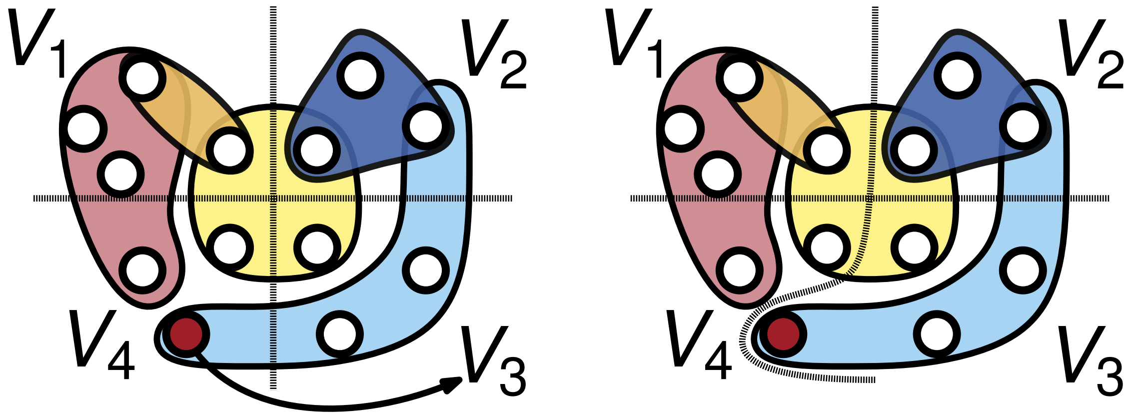

Moving  from V_4 to V_3 reduces cut by 1

gain

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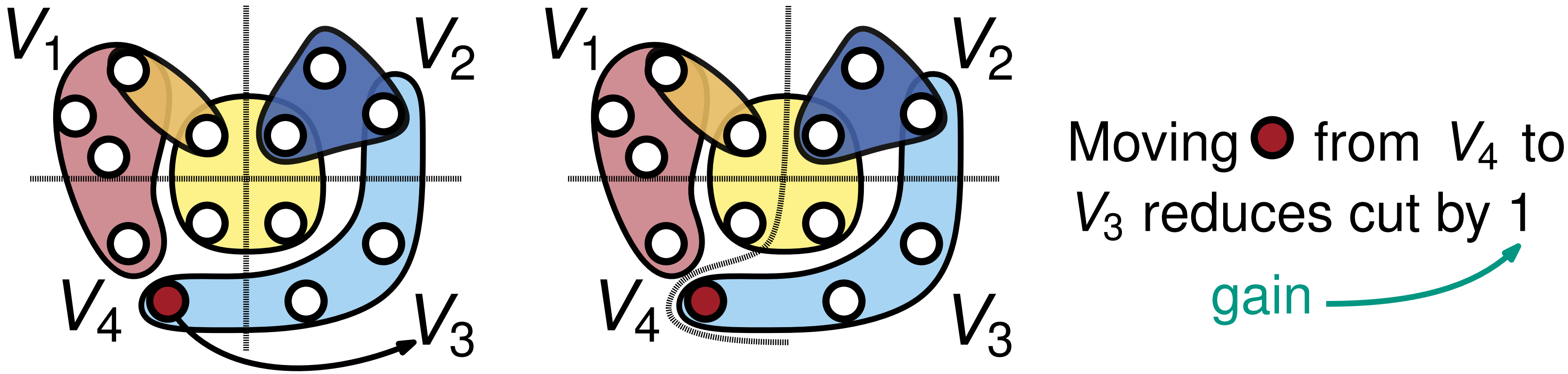

- Performs moves of vertices with **maximum gain** in each step
- All modern hypergraph partitioners implements variations of the *FM* algorithm

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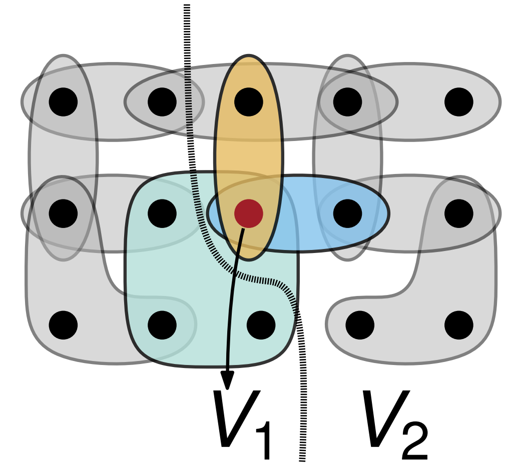
FM Algorithm - Disadvantages

- Only incorparates **local** informations about the problem structure
  - Heavily depends on *initial partition*
  - In multilevel context: Depends on quality of *coarsening*



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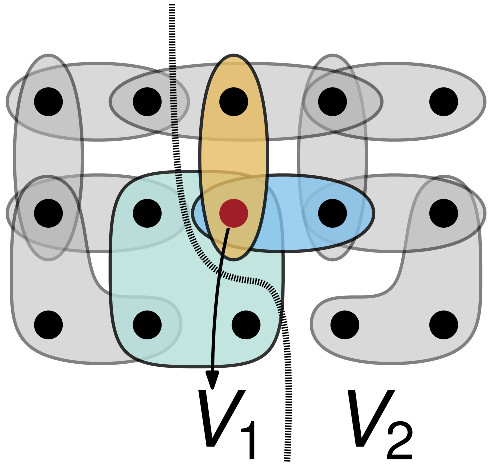
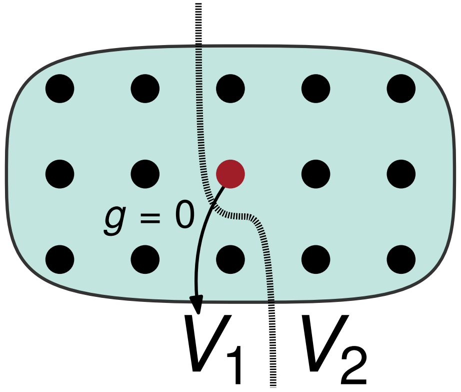


- Large hyperedges induce **Zero-Gain** moves
  - Quality mainly depends on random decisions made within the algorithm



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Flow-based Approaches

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Flows

Given a graph $G = (V, E, u)$ and two nodes $s, t \in V$

- $u : E \to \mathbb{R}_+$ is the **capacity** function
- s and t are called **source** and **sink**

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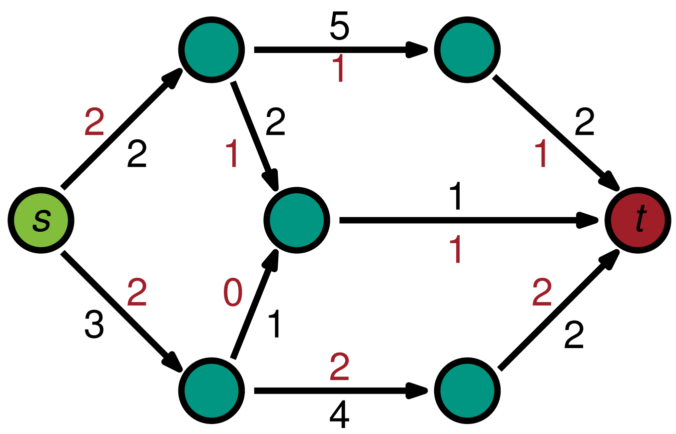
A valid **flow** is a function $f : E \to \mathbb{R}_+$ with the constraints:

- $\forall (v, w) \in E : f(v, w) \leq u(v, w)$
- $\forall v \in V \setminus \{s, t\} : \sum_{(w,v) \in E} f(w, v) = \sum_{(v,w) \in E} f(v, w)$

The value of the flow is $|f| = \sum_{(s,v) \in E} f(s, v)$



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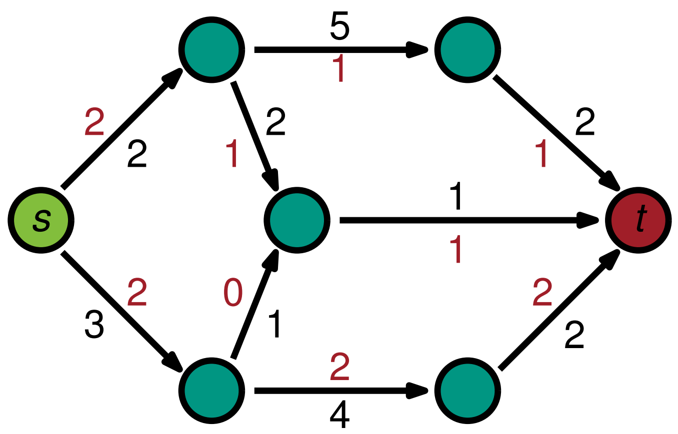
Flows

The **residual capacity** $r_f : V \times V \to \mathbb{R}_+$ is defined as follows:

- $\forall (v, w) \in E : r_f(v, w) = u(v, w) - f(v, w)$
- $\forall (v, w) \in E :$ If $f(v, w) > 0$ and $u(w, v) = 0$, then $r_f(w, v) = f(v, w)$



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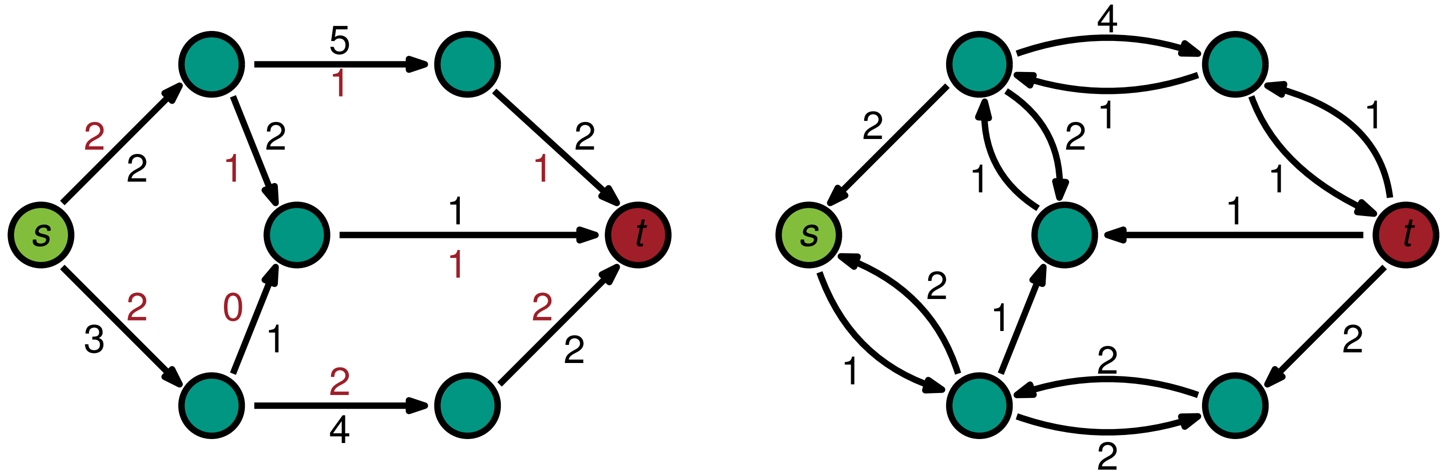
The **residual graph** $G_f = (V, E_f, r_f)$ contains all edges $(v, w) \in V \times V$ with $r_f(v, w) > 0$



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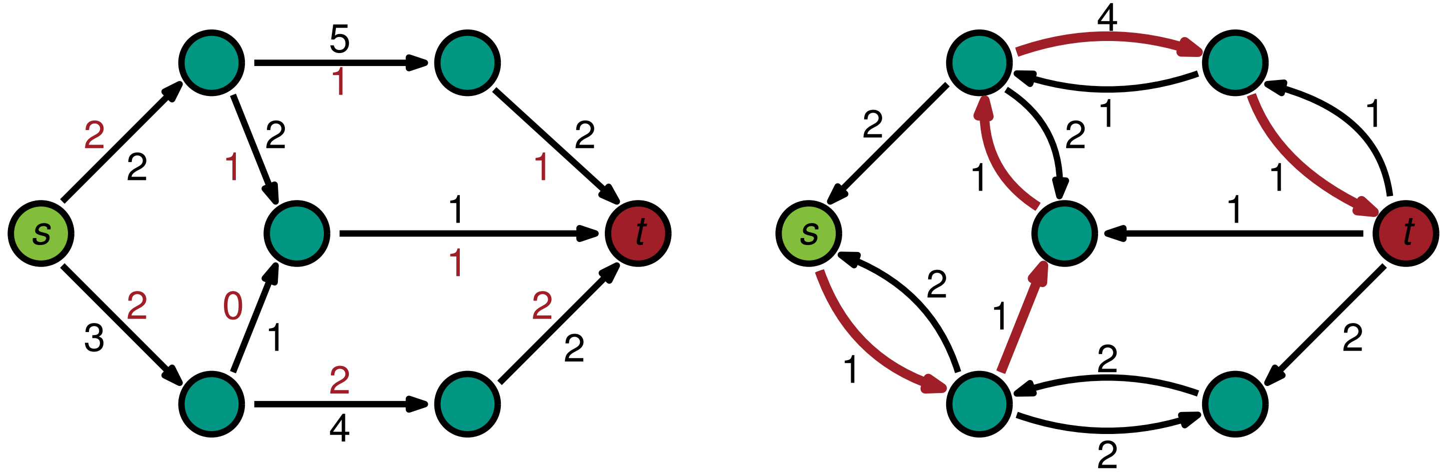
The **residual graph** $G_f = (V, E_f, r_f)$ contains all edges $(v, w) \in V \times V$ with $r_f(v, w) > 0$

- An **augmenting path** is a path in G_f from s to t
- f is a **maximum flow**, if there is no augmenting path from s to t in G_f



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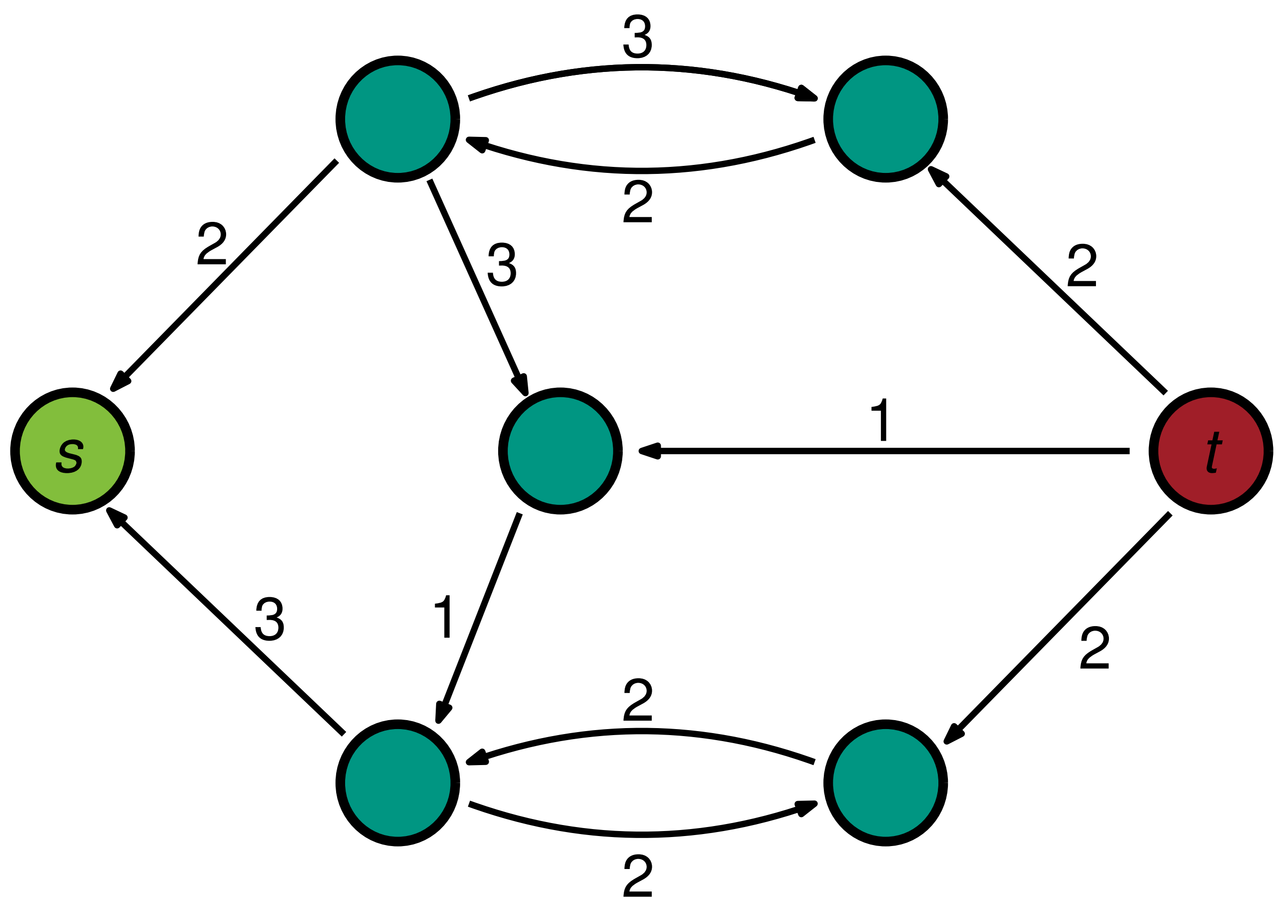
Minimum (s, t)-Bipartition

All nodes *reachable* from s are part of V_1 and $V_2 = V \setminus V_1$



Residual Graph G_f of a maximum flow f

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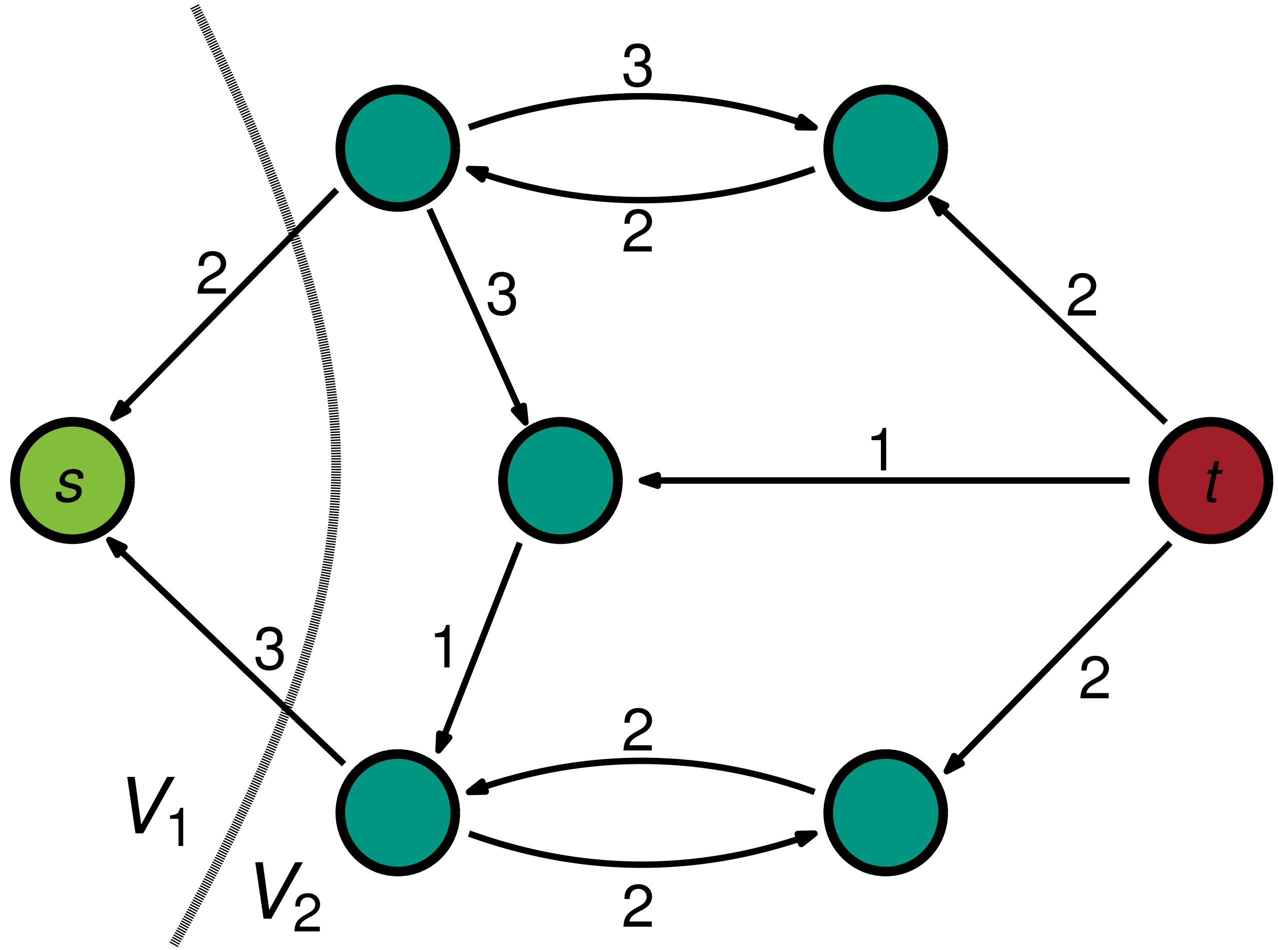
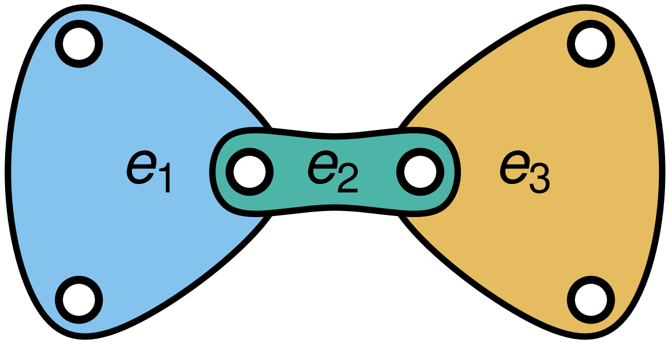


Residual Graph G_f of a maximum flow f

Hypergraph Flow Network

Hypergraph H



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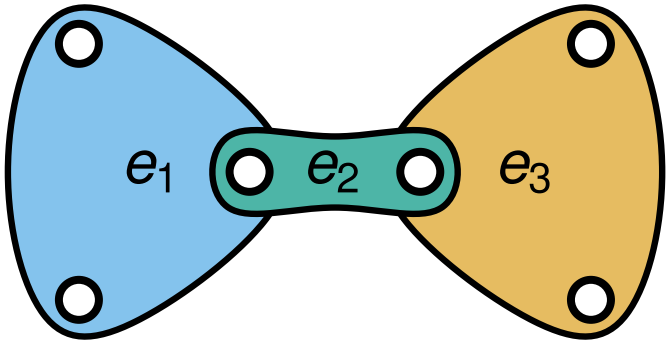
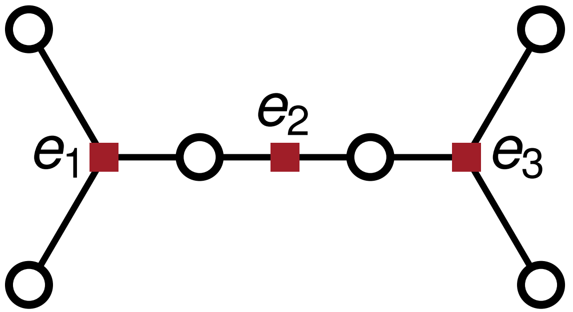


Bipartite Graph $G_*(H)$



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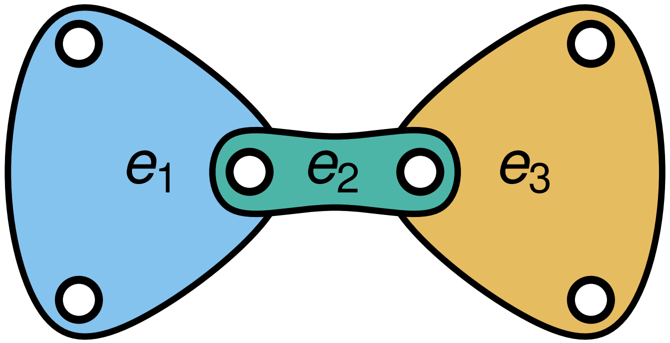

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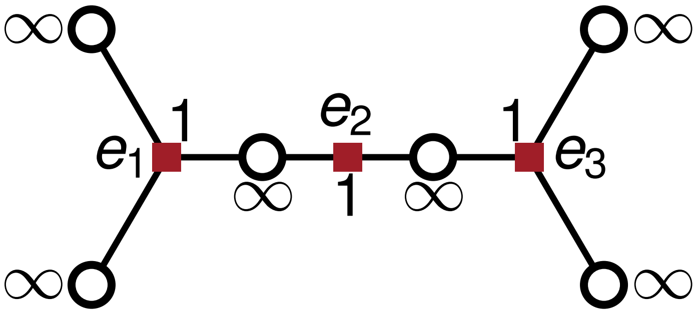

Vertex Separator Problem

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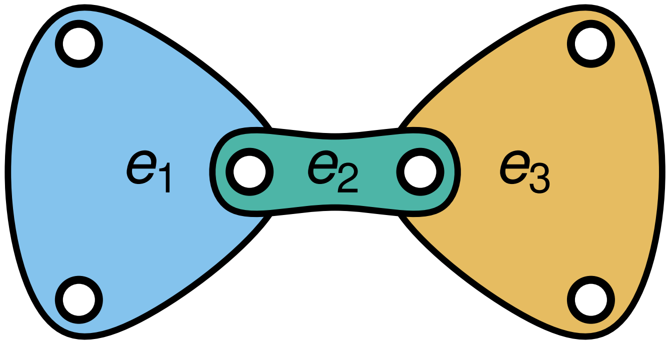

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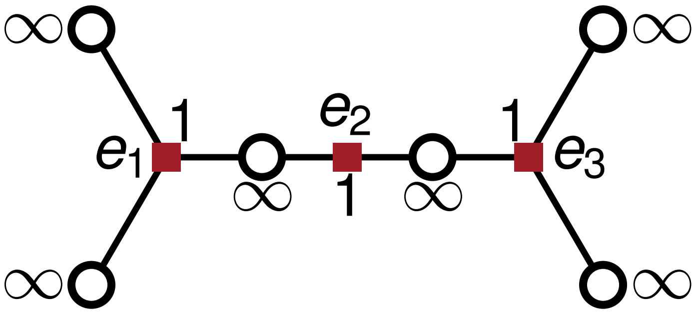
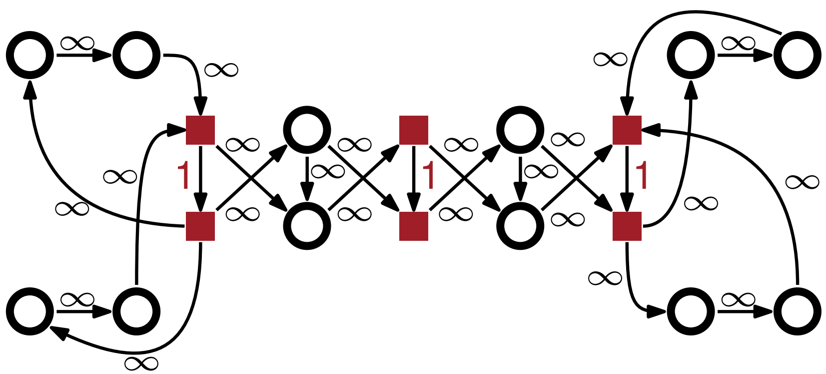


Vertex Separator Problem

Vertex Separator Transformation



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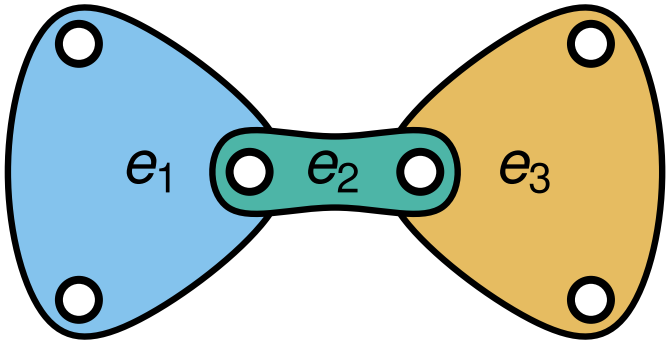

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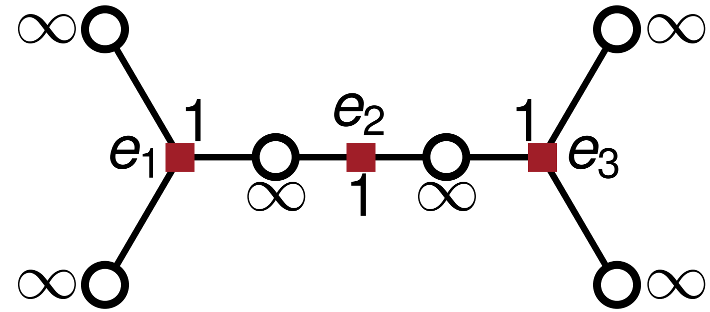

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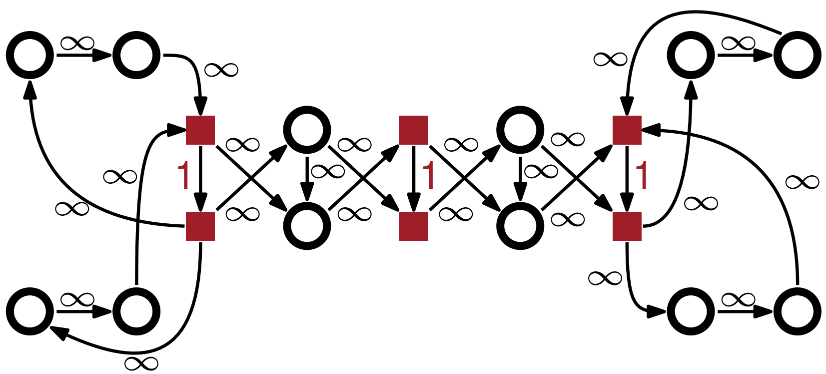
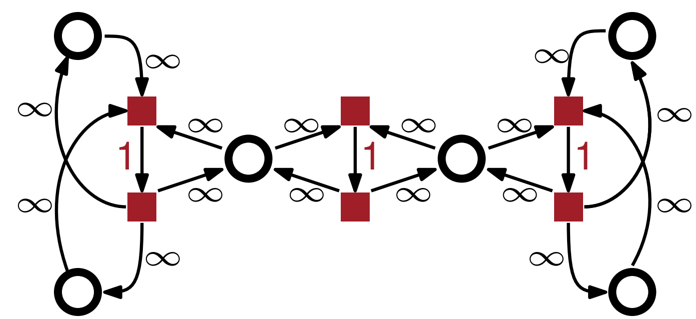


Lawler Network



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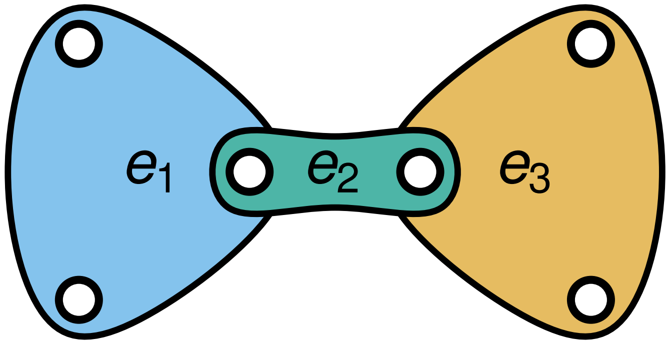

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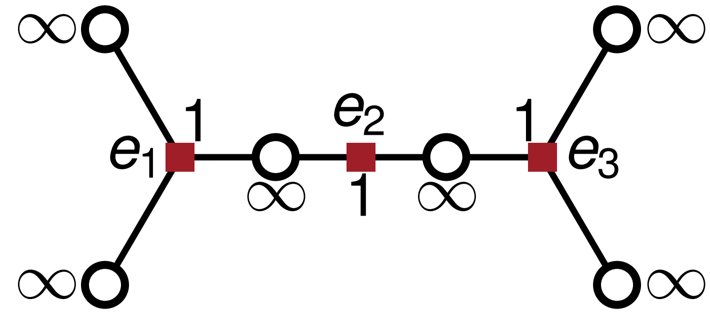

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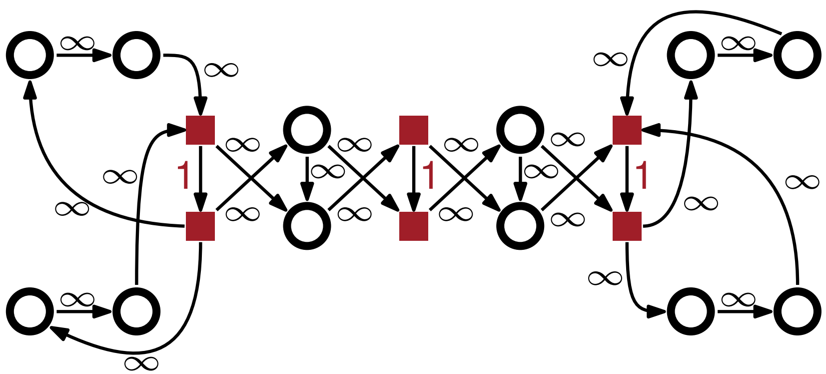

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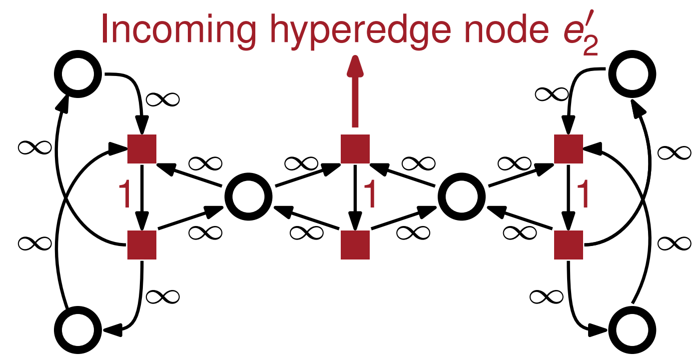

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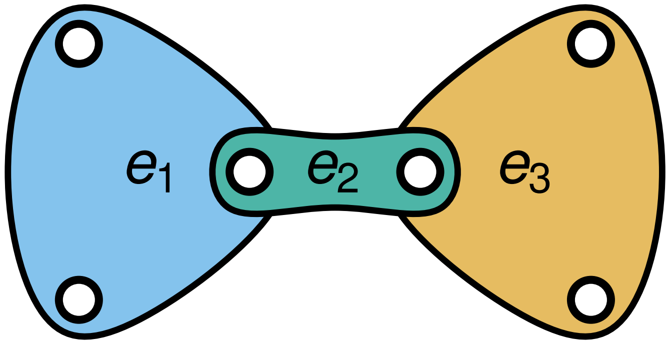

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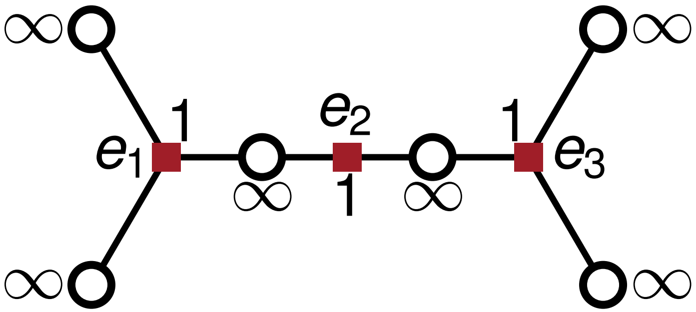

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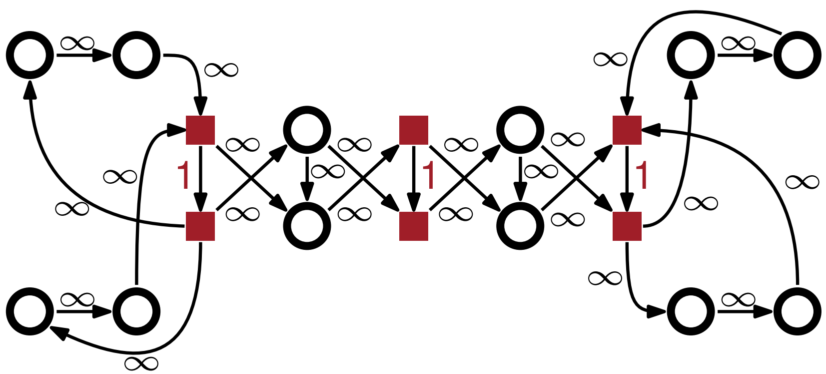
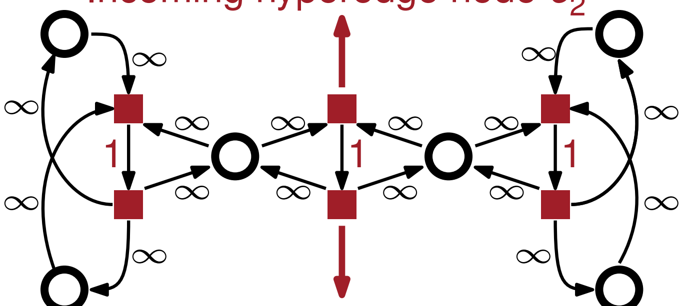


Lawler Network

Incoming hyperedge node e_2'



Outgoing hyperedge node e_2''

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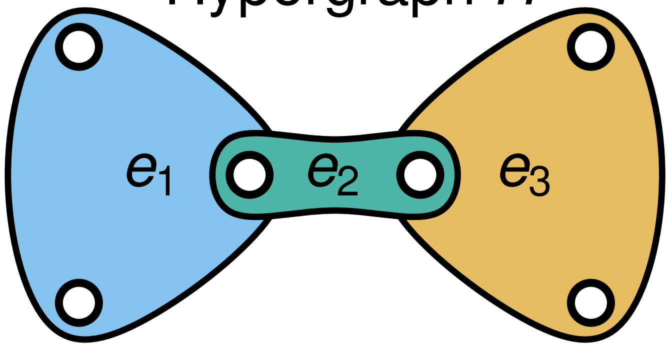
Hypergraph Flow Network - Graph Edges

Hypergraph H



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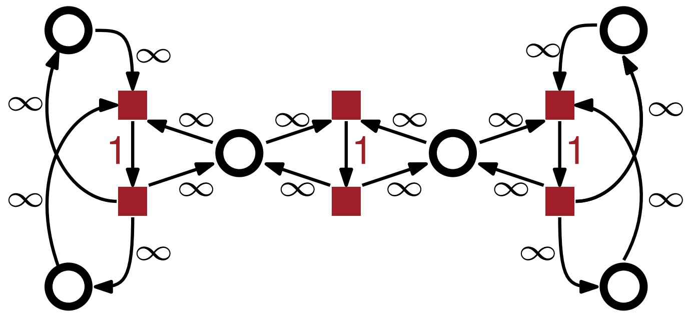

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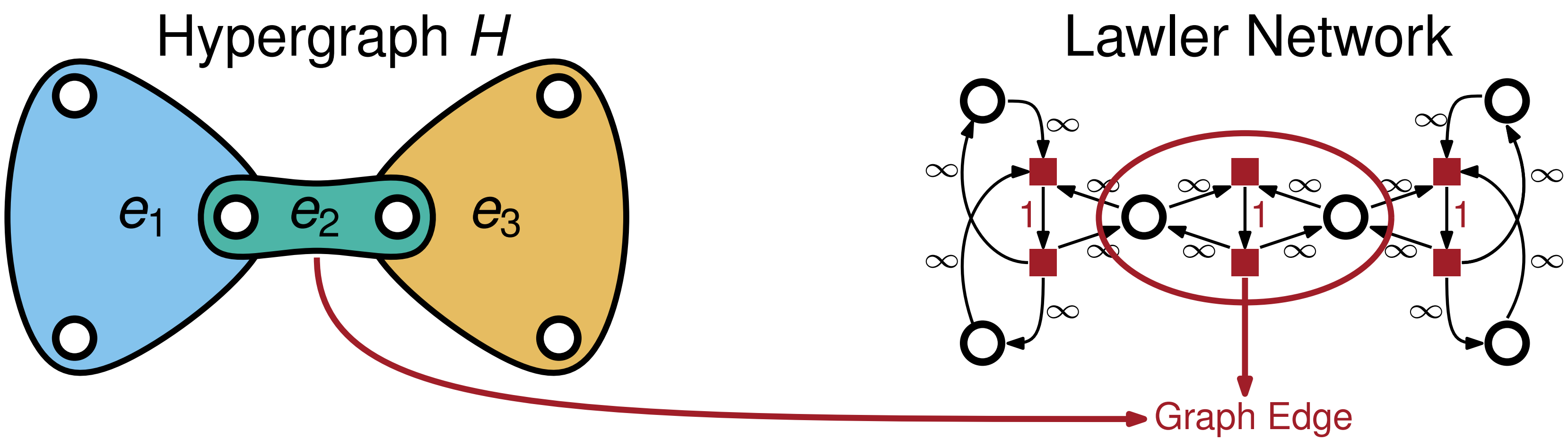

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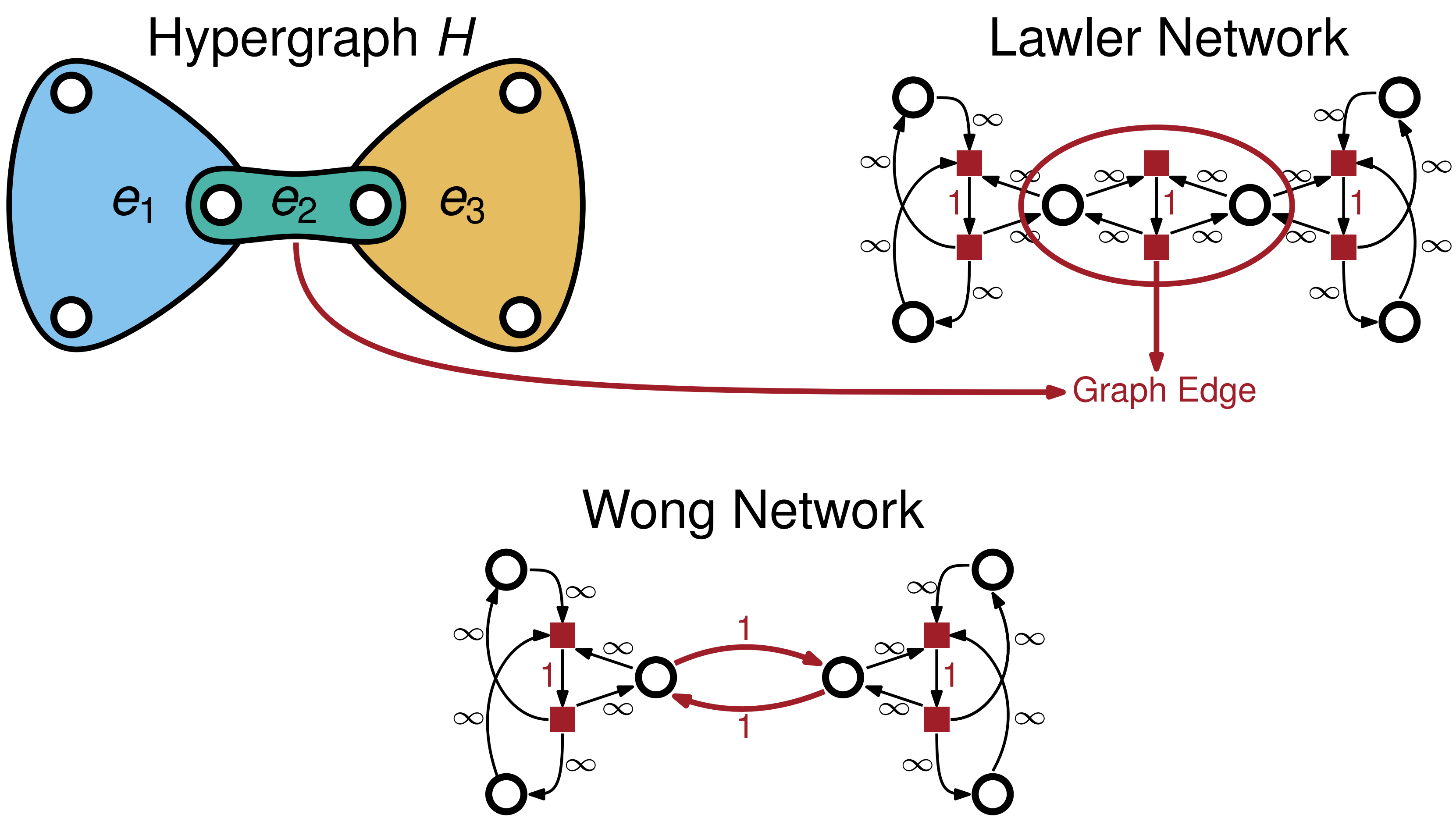
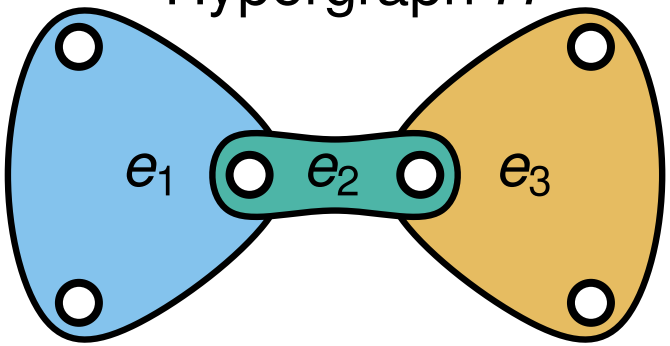
Hypergraph Flow Network - Graph Edges

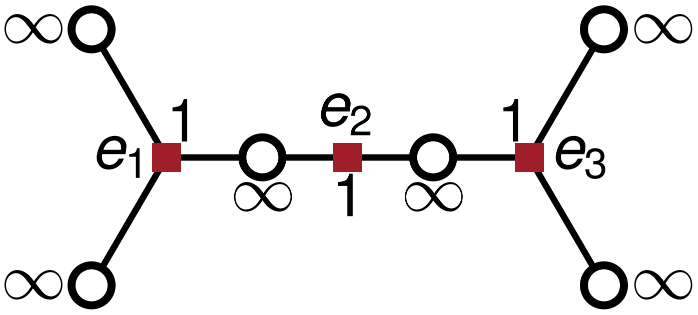


Hypergraph Flow Network - Low Degree Vertices

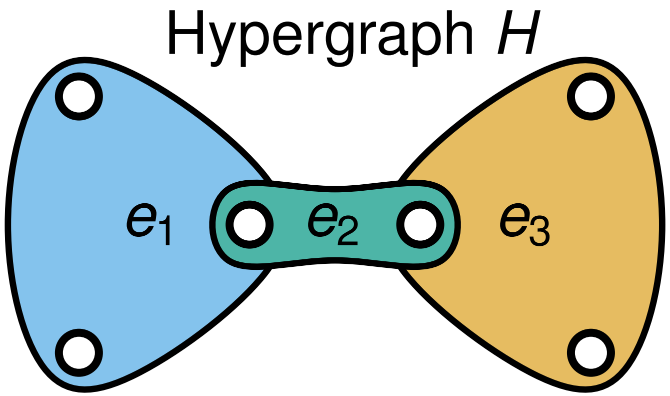
Hypergraph H

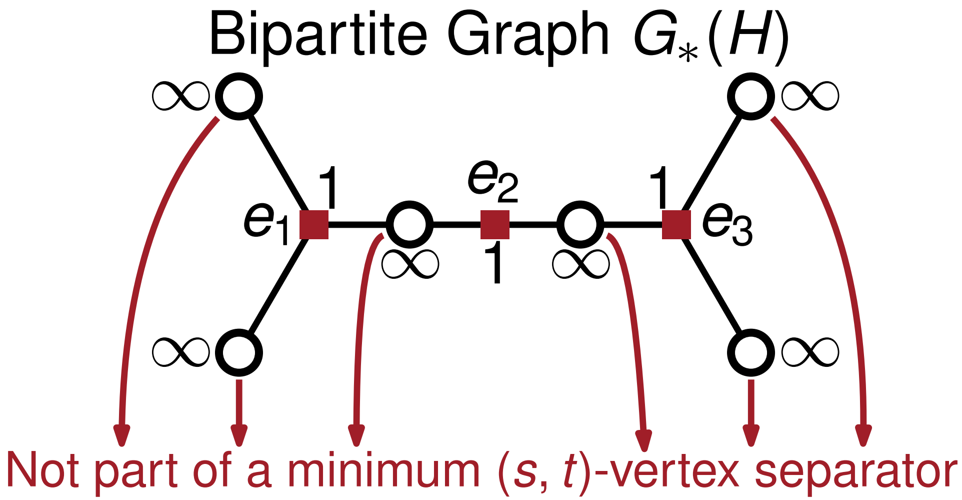


Bipartite Graph $G_*(H)$

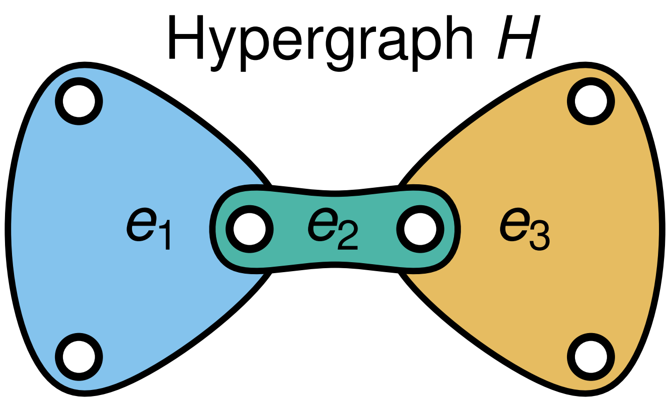


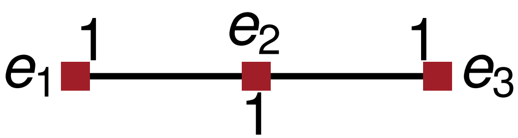
Hypergraph Flow Network - Low Degree Vertices





Hypergraph Flow Network - Low Degree Vertices





Remove all vertices by adding a clique

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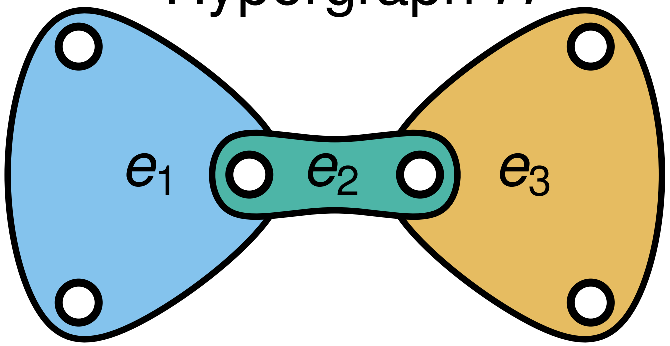
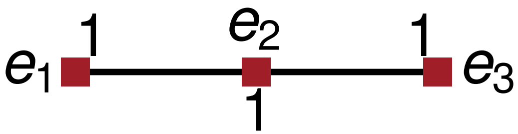
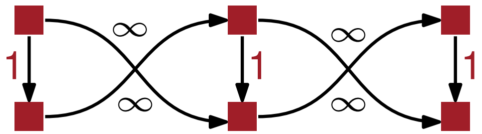
Hypergraph Flow Network - Low Degree Vertices





Remove all vertices by adding a clique

Our Network



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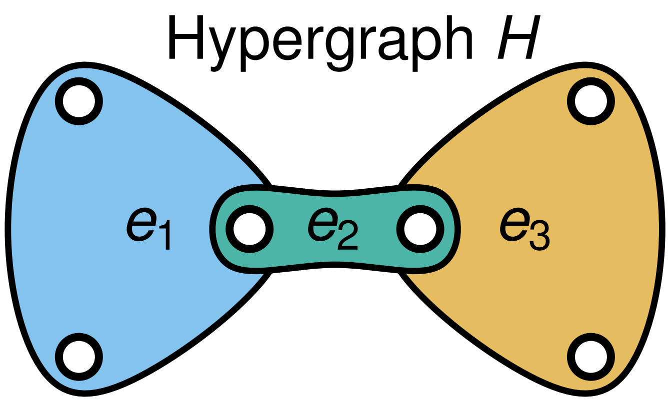
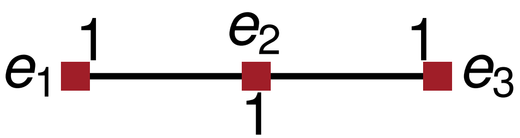
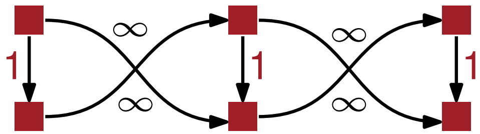
Hypergraph Flow Network - Low Degree Vertices





Remove all vertices by adding a clique

Our Network



A hypernode v induces ...

- ... $2d(v)$ edges in the Lawler Network
- ... $d(v)(d(v) - 1)$ edges in our network

If $d(v) \leq 3$, then $d(v)(d(v) - 1) \leq 2d(v)$

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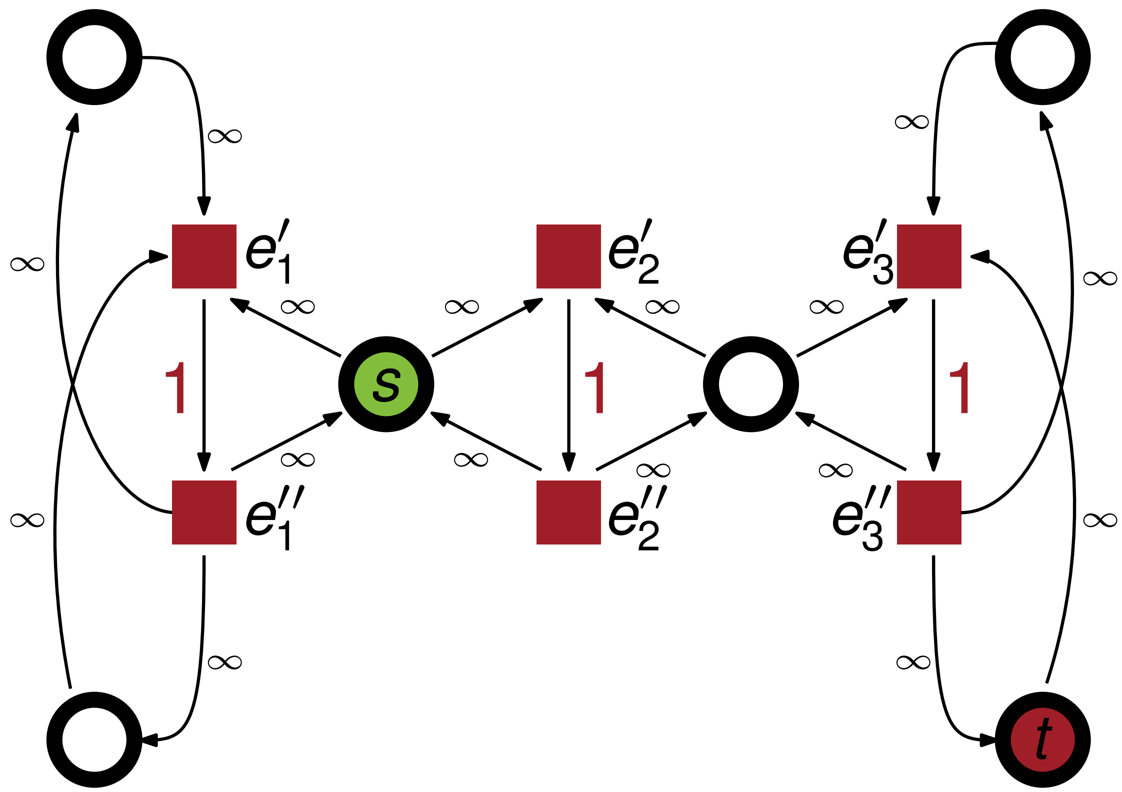
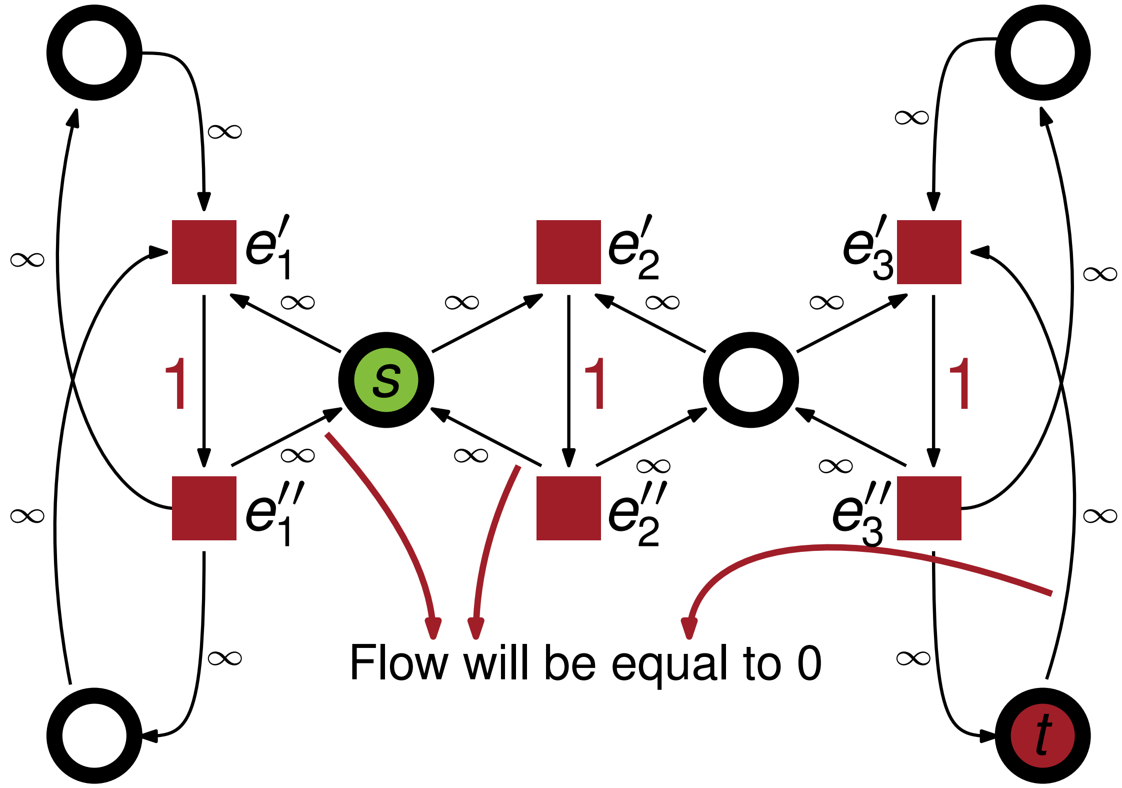
Removing Source and Sink Vertices



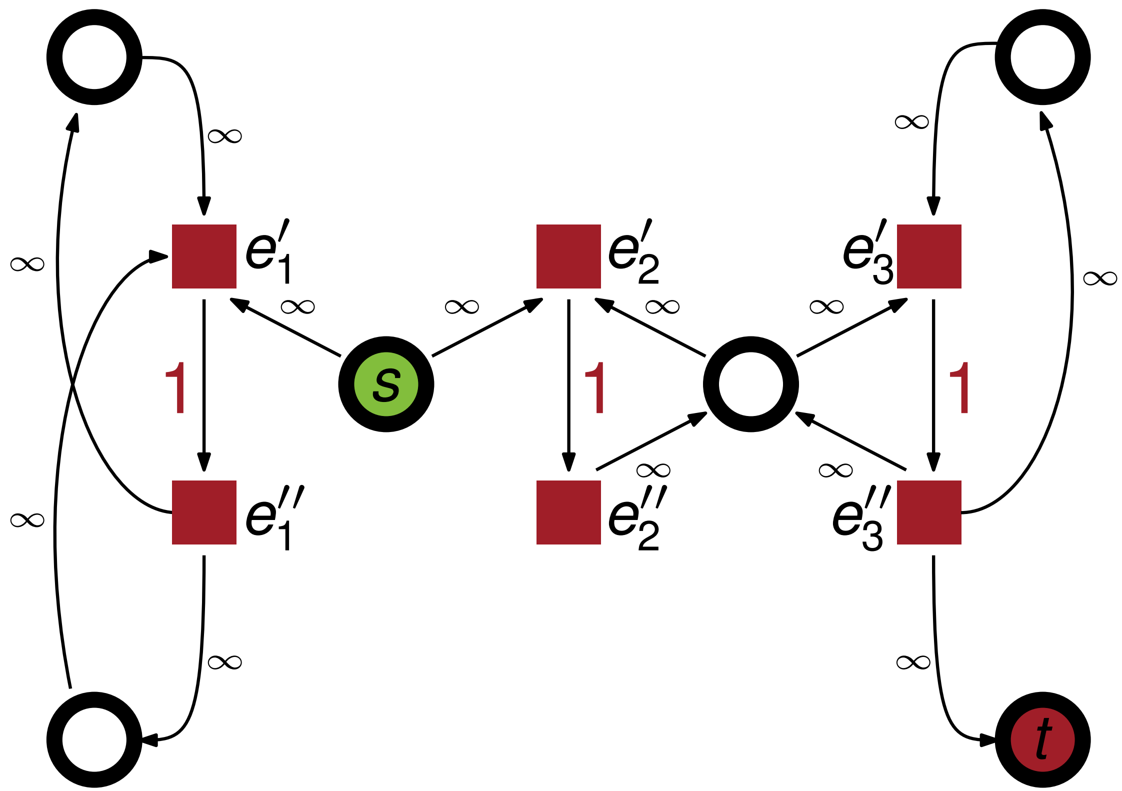
Lawler Network

Removing Source and Sink Vertices



Lawler Network

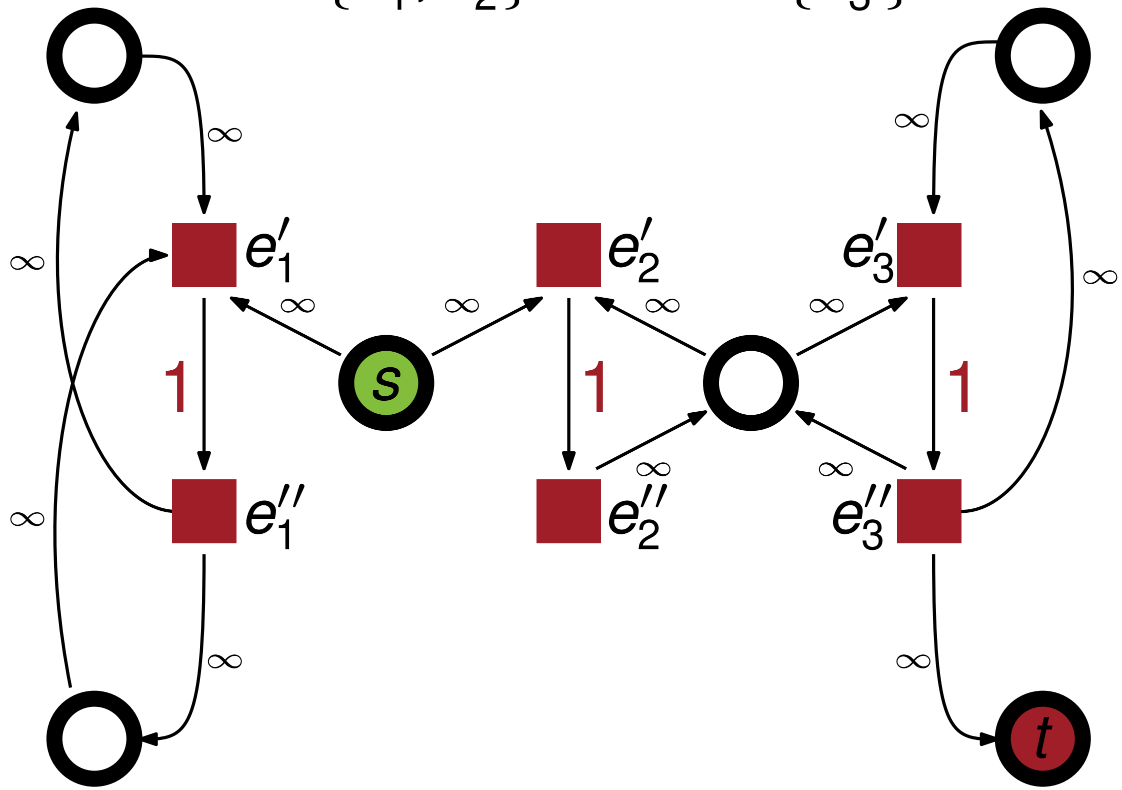
Removing Source and Sink Vertices



Lawler Network

Removing Source and Sink Vertices

Corresponds to *Multi-Source Multi-Sink* problem with
$S = \{e'_1, e'_2\}$ and $T = \{e''_3\}$



Lawler Network

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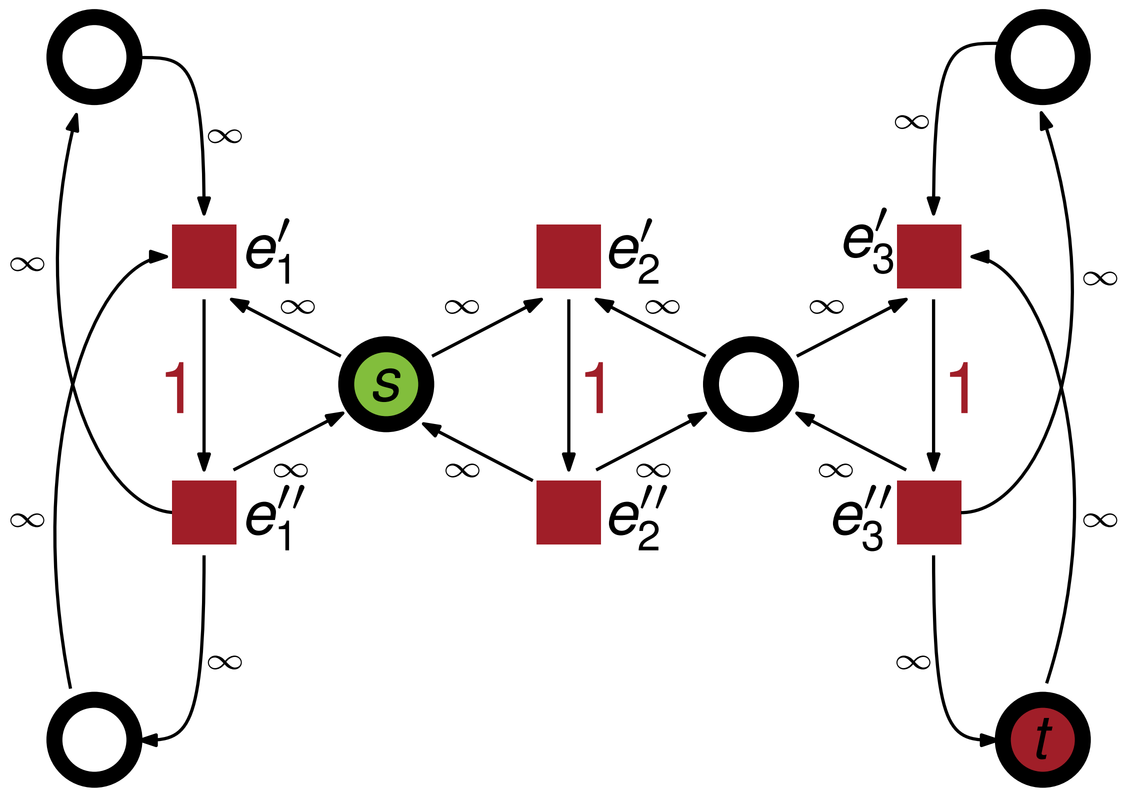
Reconstruction of Minimum (s, t)-Bipartition



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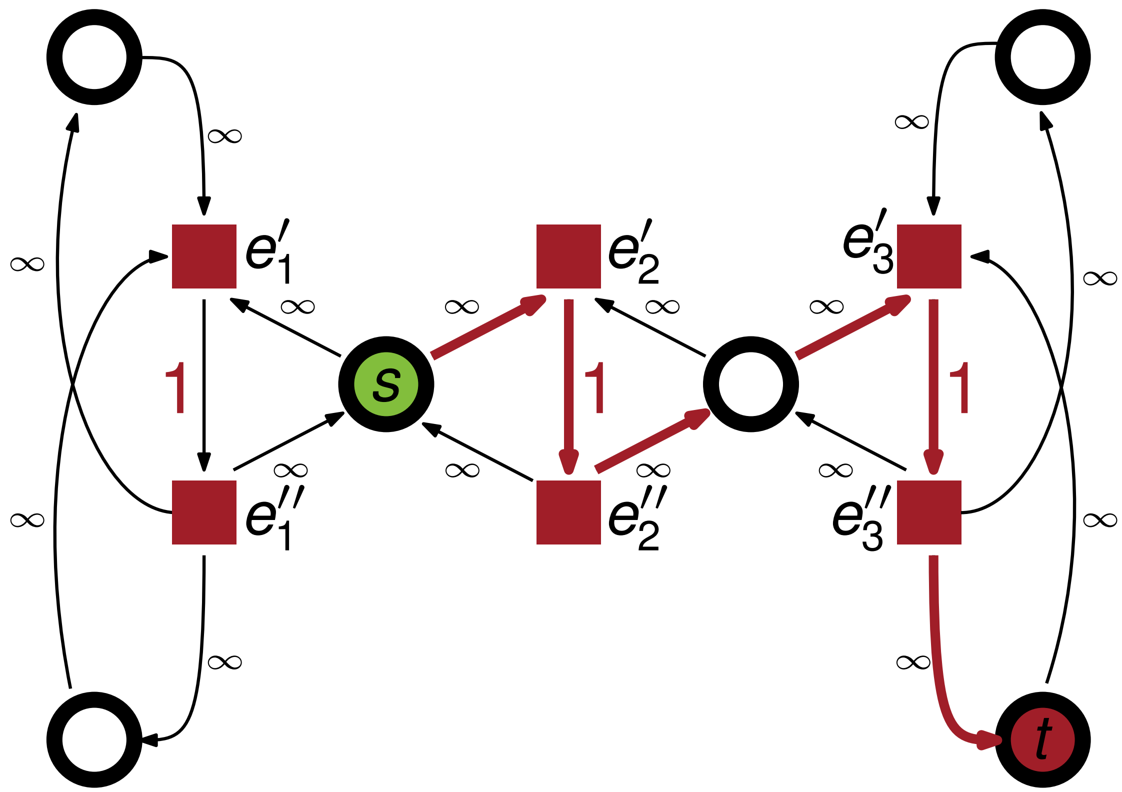
Reconstruction of Minimum (s, t)-Bipartition



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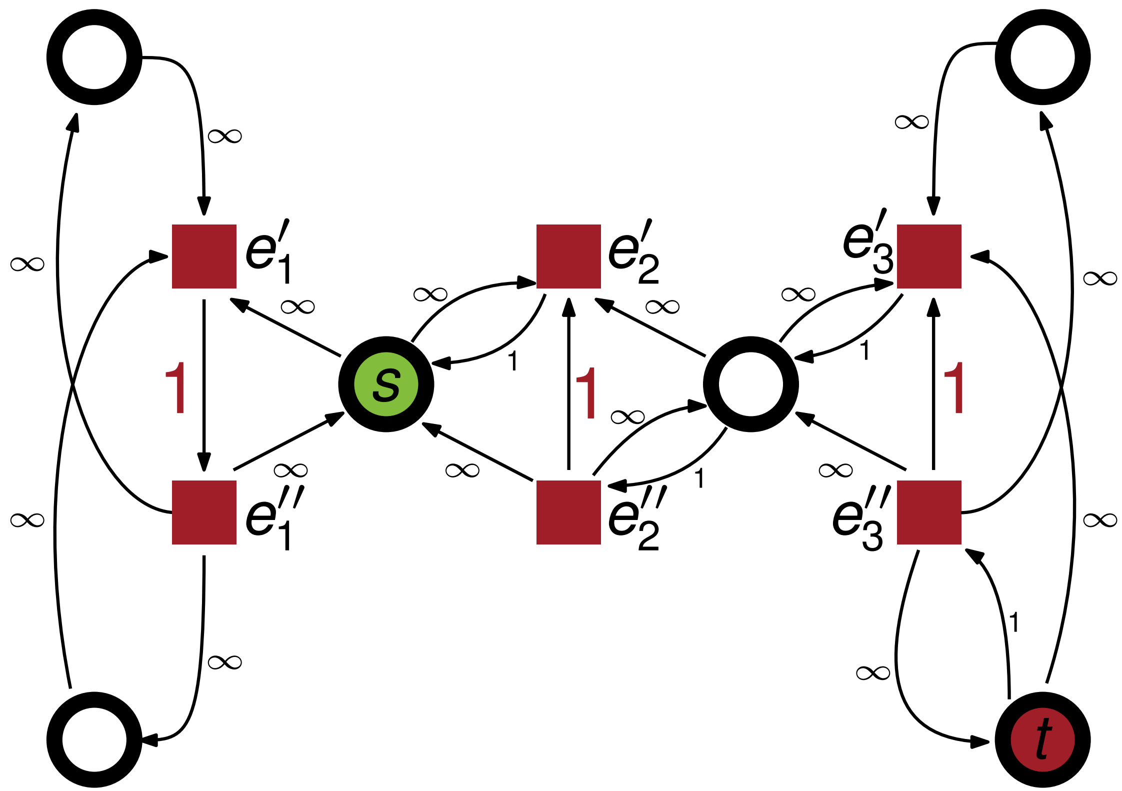
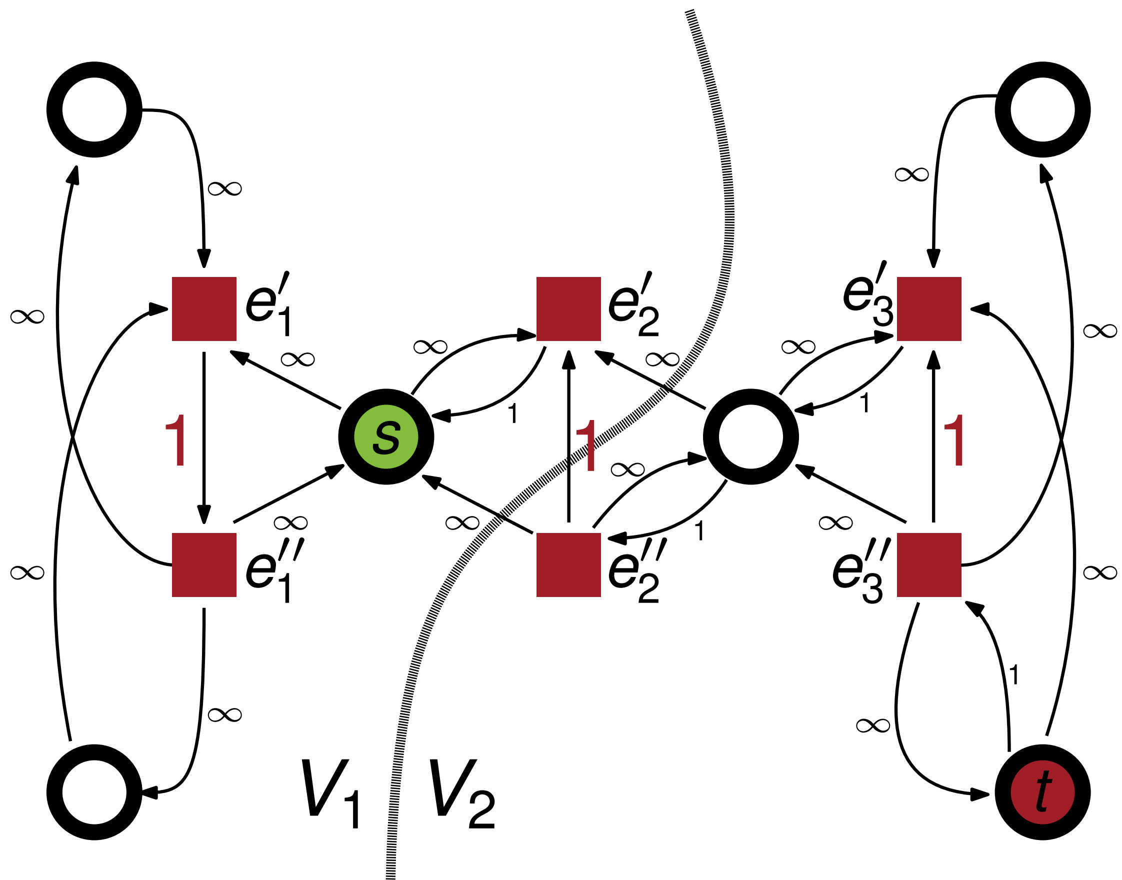
Reconstruction of Minimum (s, t)-Bipartition



Reconstruction of Minimum (s, t)-Bipartition



All nodes *reachable* from s are part of V_1 and $V_2 = V \setminus V_1$

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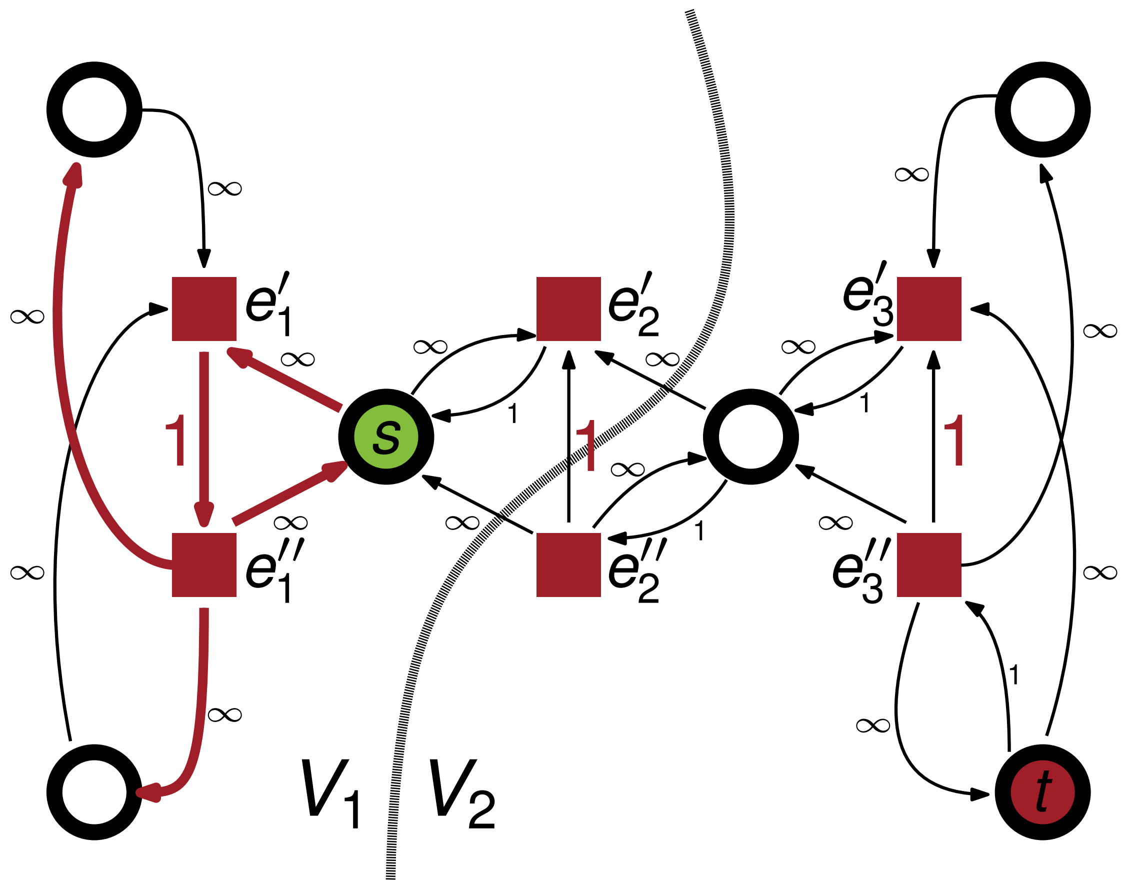
Reconstruction of Minimum (s, t)-Bipartition

For each hypernode $v \in V_1$, there exists at least one $e \in I(v)$ with $e'' \in V_1$



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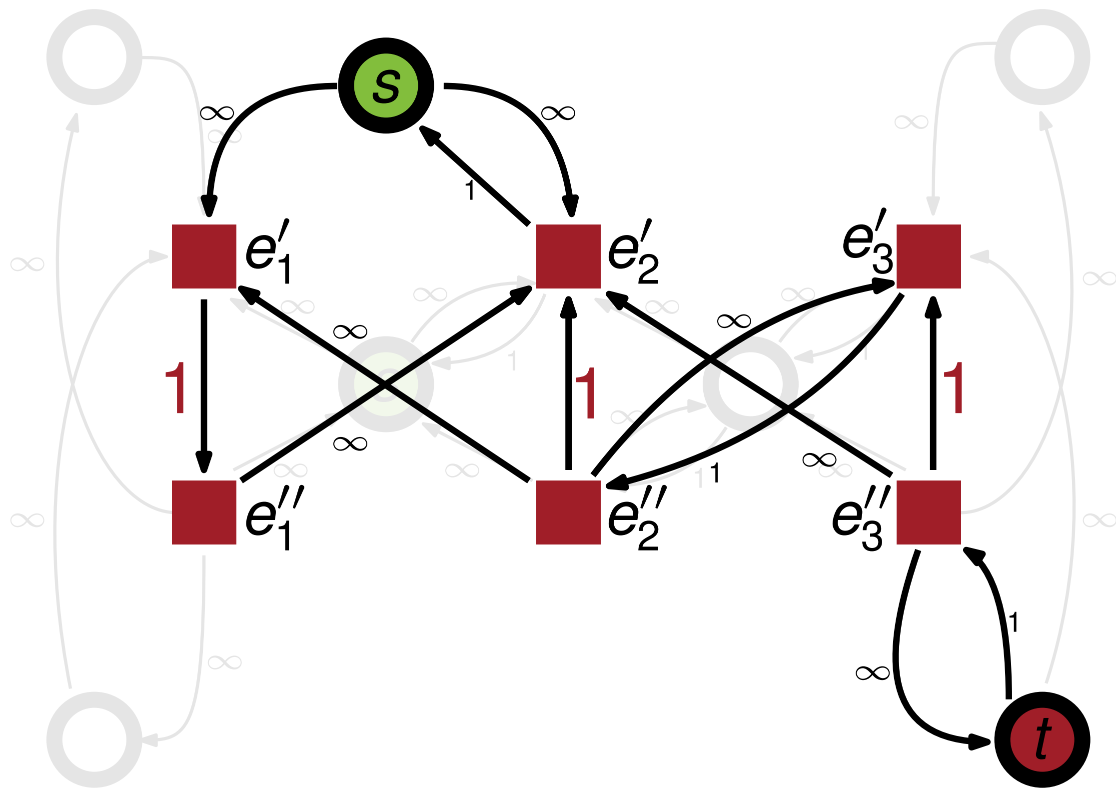
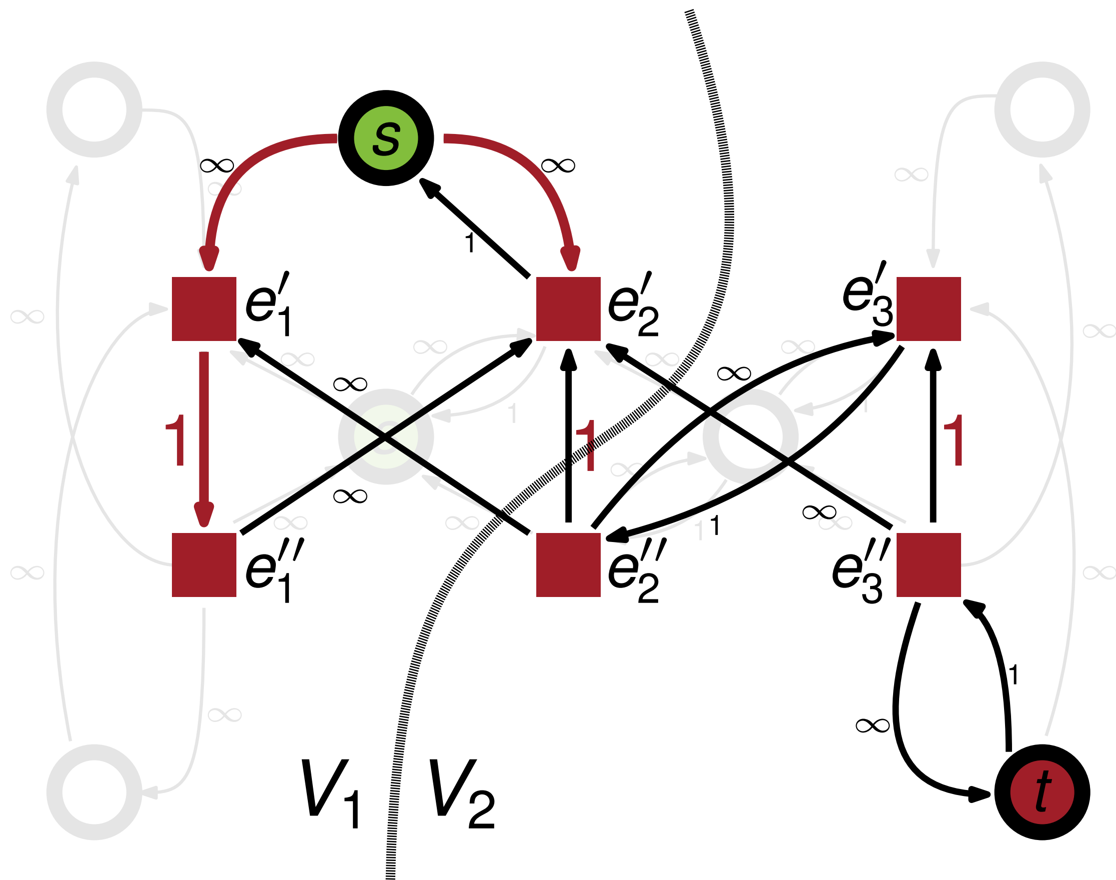
Reconstruction of Minimum (s, t)-Bipartition



Reconstruction of Minimum (s, t)-Bipartition



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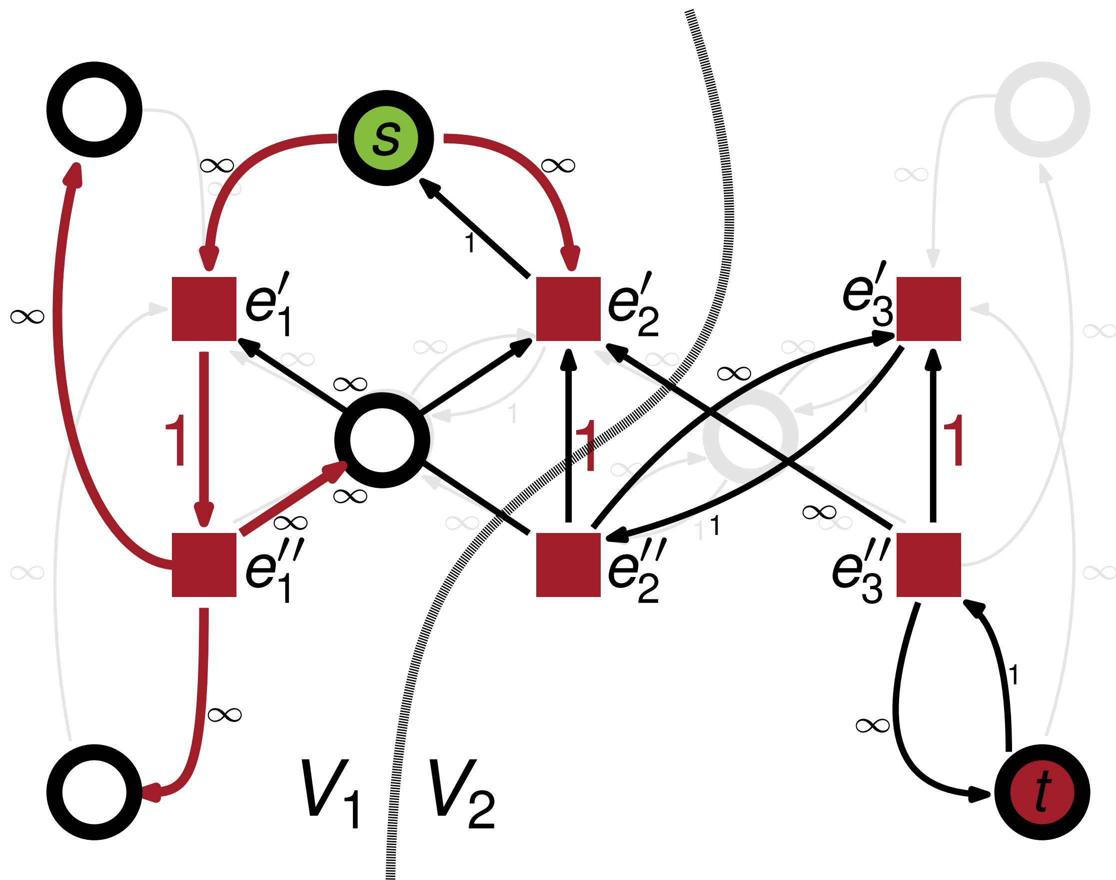
Reconstruction of Minimum (s, t)-Bipartition



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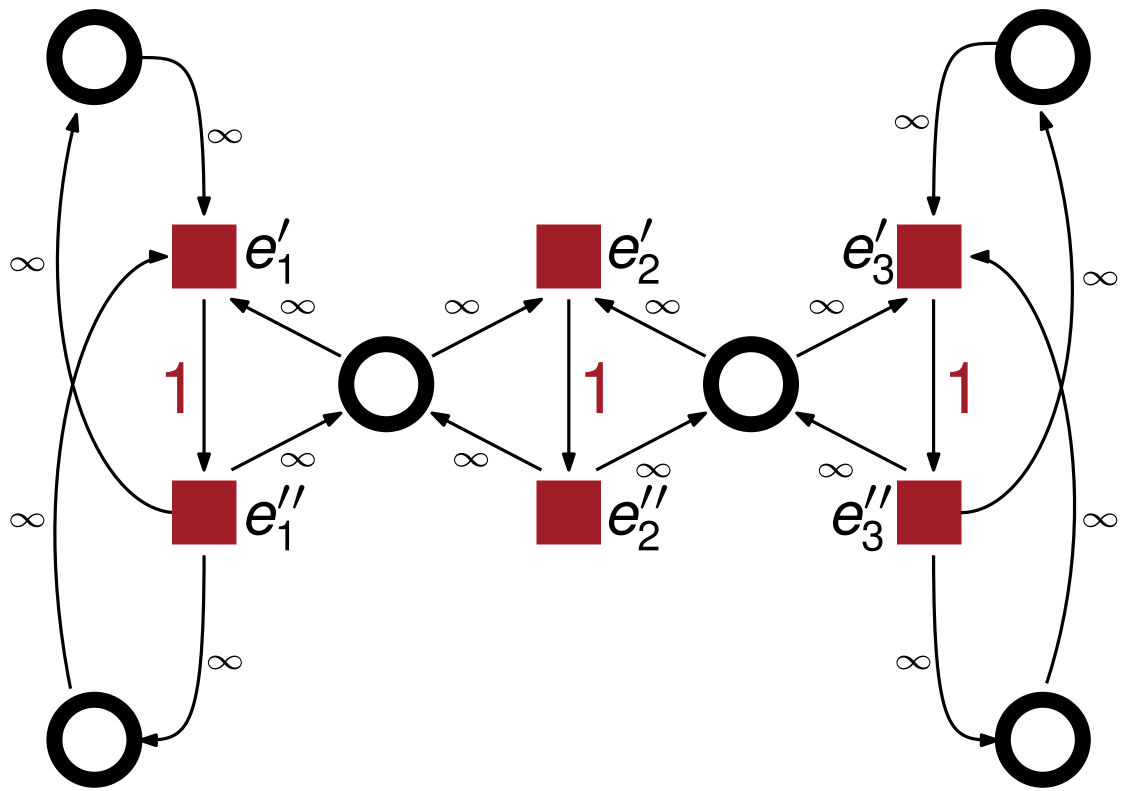
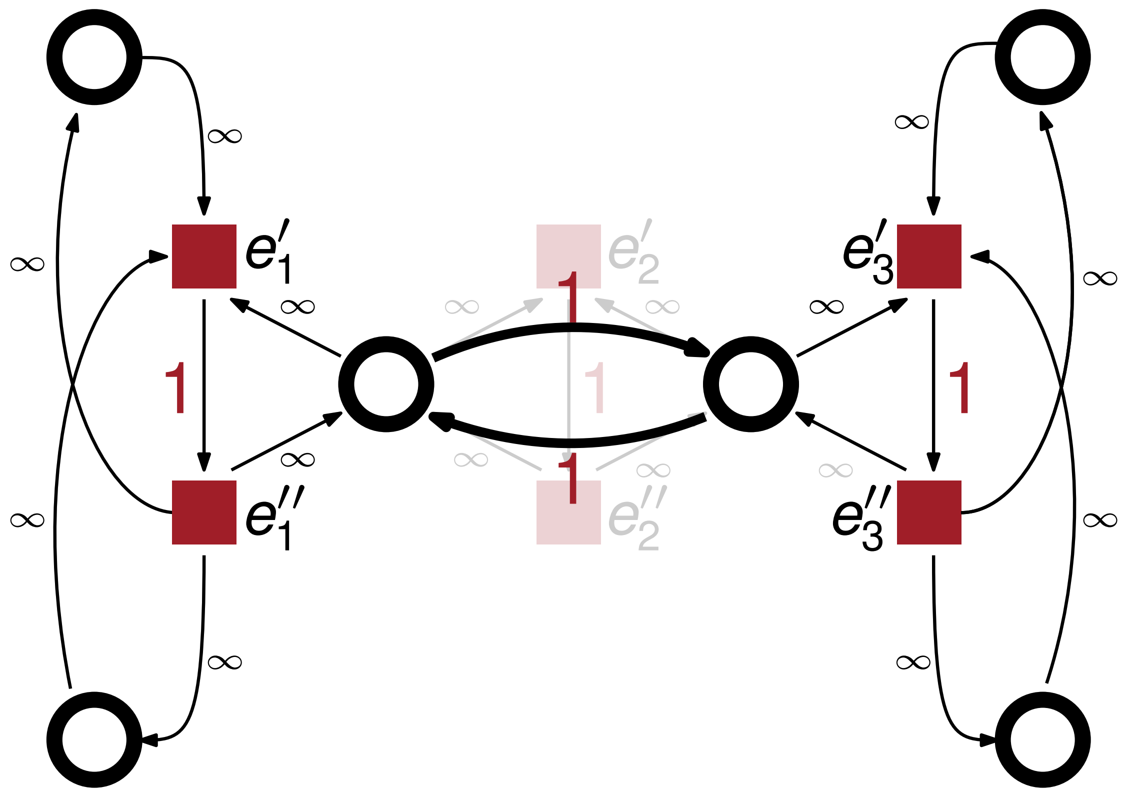
Hypergraph Flow Network - Summary



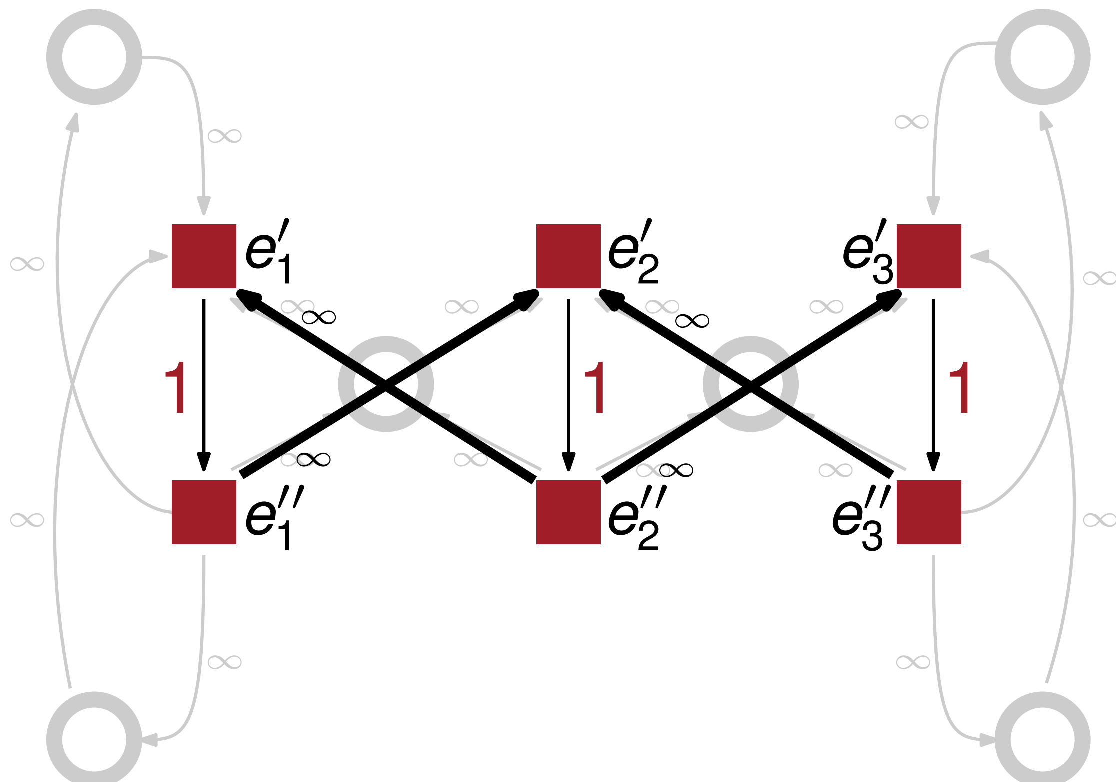
Lawler Network

Hypergraph Flow Network - Summary



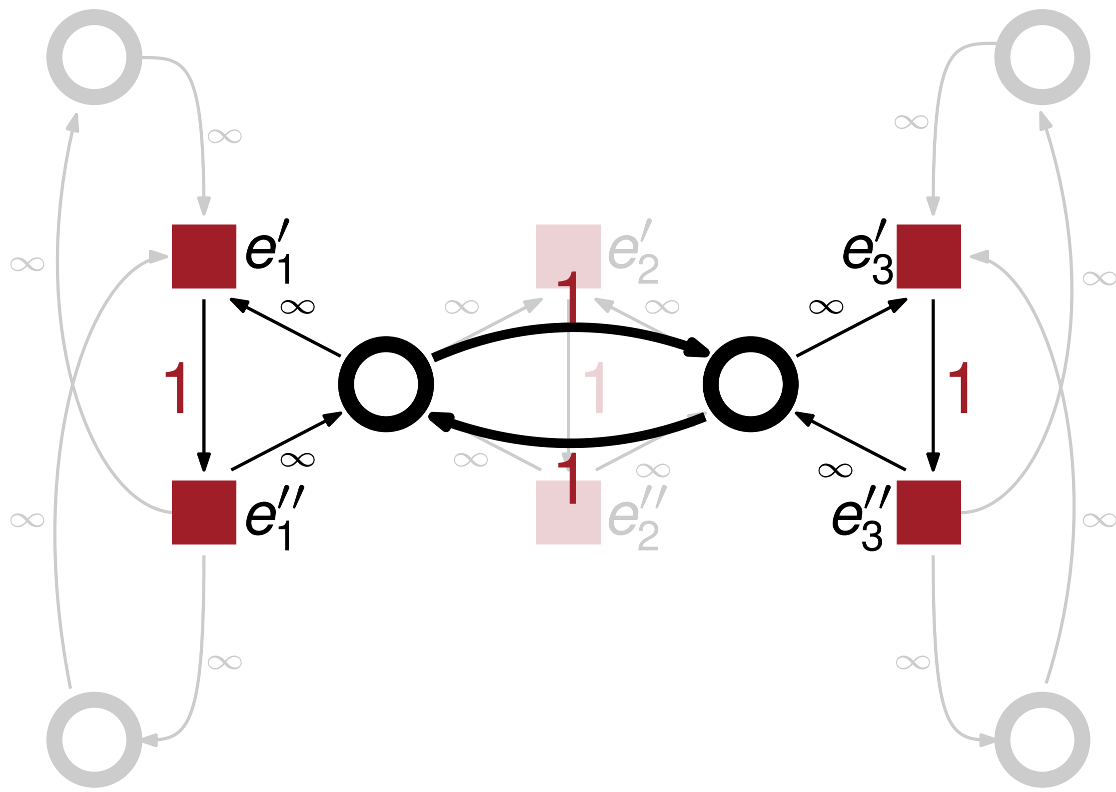
Wong Network

Hypergraph Flow Network - Summary



Our Network

Hypergraph Flow Network - Summary



Hybrid Network

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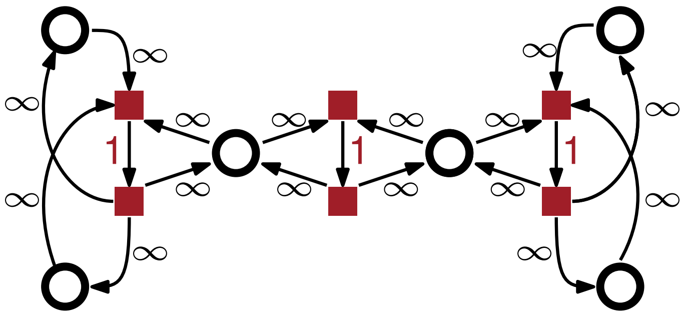
Hypergraph Flow Network - Summary

Lawler Network



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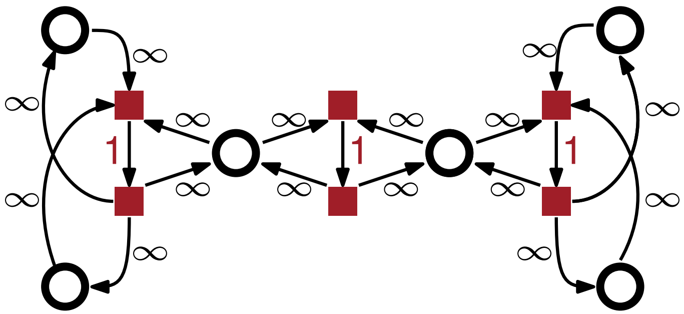
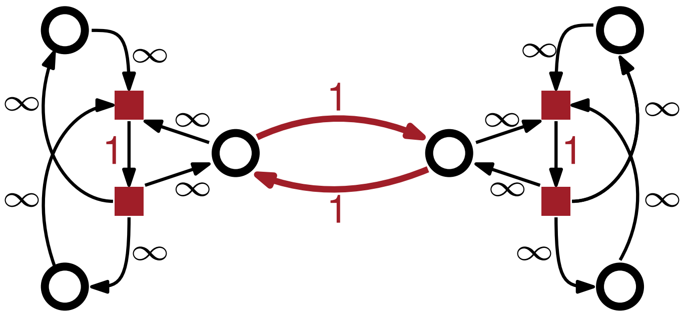
Hypergraph Flow Network - Summary

Lawler Network



Wong Network



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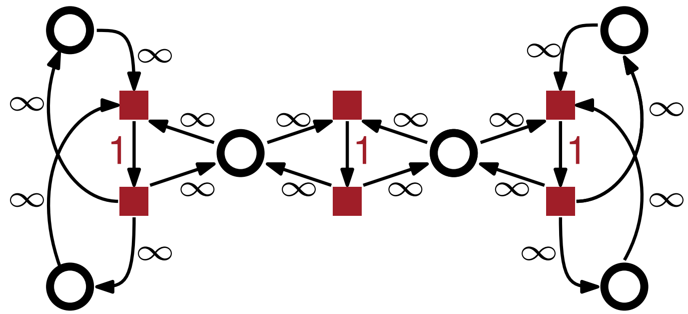
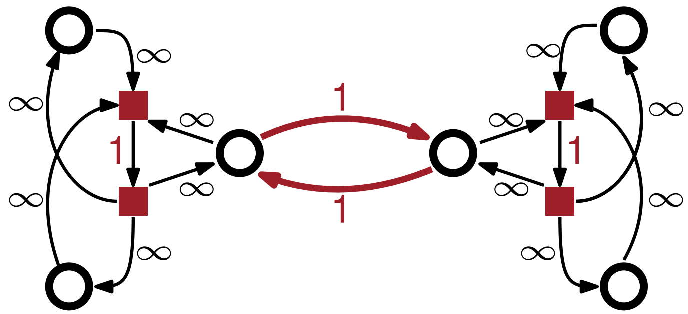
Hypergraph Flow Network - Summary

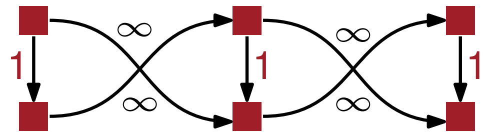
Lawler Network



Wong Network



Our Network



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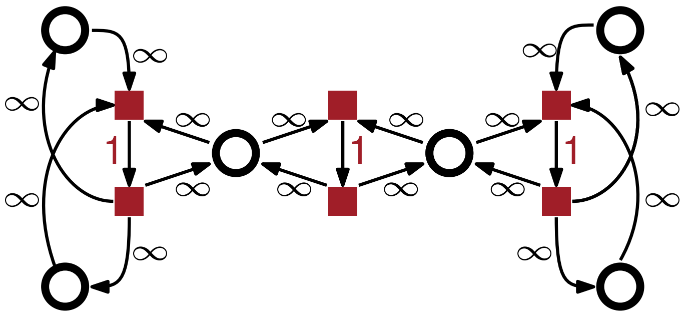
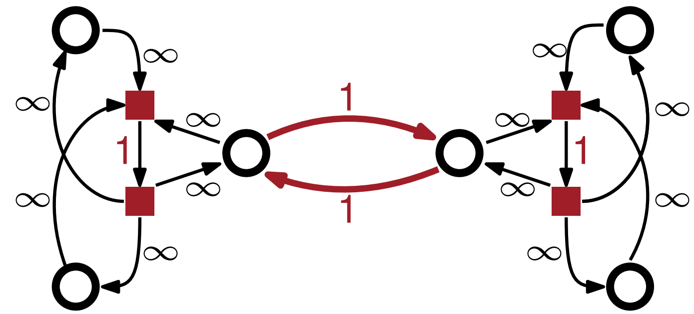
Hypergraph Flow Network - Summary

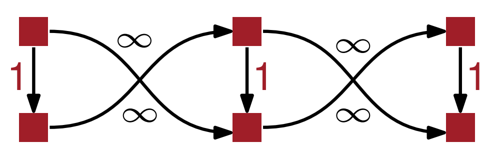
Lawler Network



Wong Network



Our Network



Hybrid Network