CS 228 : Logic in Computer Science

Krishna. S

Some Real Life Stories

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- Design error in the control software (race condition)

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- Intel offered to replace all flawed processors

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  - uncaught exception: data conversion from 64-bit float to 16-bit signed int

Toyota Prius (2010)

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  - software “glitch” found in anti-lock braking system
  - Eventually fixed via software update in total 185,000 cars recalled, at huge cost

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  - software "glitch" led several homes to a frozen state, reported in NY times, Jan 13, 2016. May be, old fashioned mechanical thermostats better!

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    - conventional computers and networks
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  - for safety
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- Programming error direct cause of failure
- Software critical
  - for safety
  - for business
  - for performance
- High costs incurred: financial, loss of life
- Failures avoidable

Formal Methods

Intuitive Description

"Applied Mathematics for modelling and analysing ICT systems"

Formal methods offer a large potential for:

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Formal methods offer a large potential for:

- obtaining an early integration of verification in the design process
- providing more effective verification techniques (higher coverage)
- reducing the verification time

Simulation and Testing

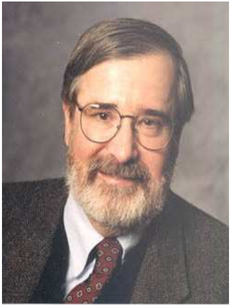
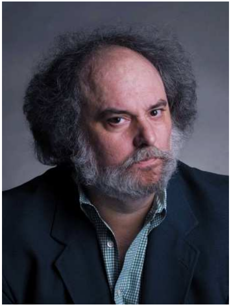
Basic procedure

- Take a model
- Simulate it with certain inputs
- Observe what happens, and if this is desired

Important Drawbacks

- possible behaviours very large/infinite
- unexplored behaviours may contain fatal bug
- can show presence of errors, not their absence

Model Checking

- Year 2008 : ACM confers the Turing Award to the pioneers of Model Checking: Ed Clarke, Allen Emerson, and Joseph Sifakis
- Why?

Model checking

- Model checking has evolved in last 25 years into a widely used verification and debugging technique for software and hardware.
- Cost of *not* doing formal verification is high!
  - The France Telecom example
  - Ariane rocket: kaboom due to integer overflow!
  - Toyota/Ford recalls

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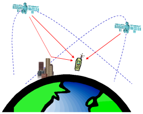
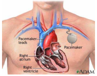
- Model checking used (and further developed) by companies/institutes such as IBM, Intel, NASA, Cadence, Microsoft, and Siemens, and has culminated in many freely downloadable software tools that allow automated verification.

What is Model Checking?





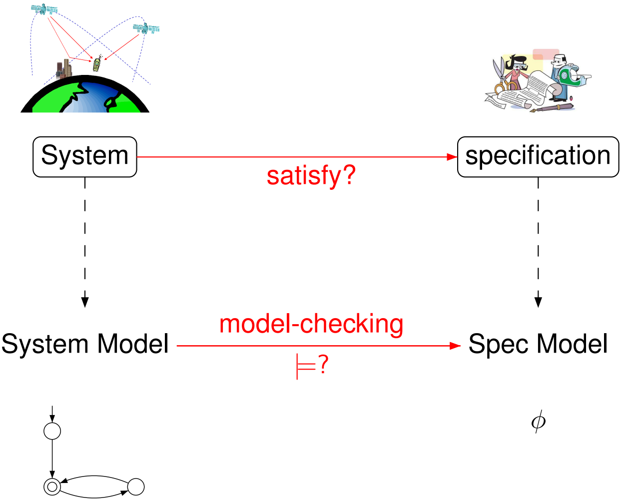
System

satisfy?

specification

good/bad properties

What is Model Checking?



Model Checker as a Black Box

- Inputs to Model checker : A finite state system M, and a property P to be checked.
- Question : Does M satisfy P?
- Possible Outputs
  - Yes, M satisfies P
  - No, here is a counter example!.

What are Models?

Transition Systems

- States labeled with propositions
- Transition relation between states
- Action-labeled transitions to facilitate composition

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Expressivity

- Programs are transition systems
- Multi-threading programs are transition systems
- Communicating processes are transition systems
- Hardware circuits are transition systems
- What else?

What are Properties?

Example properties

- Can the system reach a deadlock?
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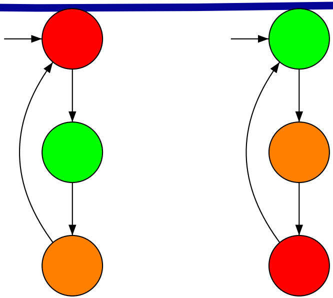
Logics of Relevance

- Classical Logics
  - First Order Logic
  - Monadic Second Order Logic
- Temporal Logics
  - Propositional Logic, enriched with modal operators such as $\Box$ (always) and $\Diamond$ (eventually)
  - Interpreted over state sequences (linear)
  - Or over infinite trees (branching)

Two Traffic Lights



1. The traffic lights are never green simultaneously
   $\forall x(\neg(green_1(x) \land green_2(x)))$ or $\Box(\neg(green_1 \land green_2))$
2. The first traffic light is infinitely often green
   $\forall x \exists y(x < y \land green_1(y))$ or $\Box\Diamond green_1$
3. Between every two occurrences of traffic light 1 becoming red, traffic light 2 becomes red once.

The Model Checking Process

- Modeling Phase
  - model the system under consideration
  - as a first sanity check, perform some simulations
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  - formalise property to be checked
- Running Phase
  - run the model checker to check the validity of the property in the model
- Analysis Phase
  - property satisfied? $\rightarrow$ check next property (if any)
  - property violated? $\rightarrow$
    - analyse generated counter example by simulation
    - refine the model, design, property, ... and repeat entire procedure
  - out of memory? $\rightarrow$ try to reduce the model and try again

The Pros of Model Checking

- widely applicable (hardware, software...)
- allows for partial verification (only relevant properties)
- potential “push-button” technology (tools)
- rapidly increasing industrial interest
- in case of property violation, a counter example is provided
- sound mathematical foundations
- not biased to the most possible scenarios (like testing)

The Cons of Model Checking

- model checking is only as “good” as the system model
- no guarantee about completeness of results (incomplete specifications)

Neverthless:
Model Checking is an effective technique to expose potential design errors

Striking Model-Checking Examples

- Security : Needham-Schroeder encryption protocol
  - error that remained undiscovered for 17 years revealed (model checker SAL)
- Transportation Systems
  - Train model containing 10^{47} states (model checker UPPAAL)
- Model Checkers for C, JAVA, C++
  - used (and developed) by Microsoft, Intel, NASA
  - successful application area: device drivers (model checker SLAM)
- Dutch storm surge barrier in Nieuwe Waterweg
- Software in current/next generation of space missiles
  - NASA's
    - Java Pathfinder, Deep Space Habitat, Lab for Reliable Software

Relevant Topics

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- What are properties?
  - Safety, Liveness, fairness
- How to check regular properties?
  - finite state automata and regular safety properties
  - Buchi automata and ω-regular properties

Relevant Topics

- How to express properties succintly?
  - First Order Logic (FO) : syntax, semantics
  - Monadic Second Order Logic (MSO) : syntax, semantics
  - Linear-Temporal-Logic (LTL) : syntax, semantics
  - What can be expressed in each logic?
  - Satisfiability and Model checking : algorithms, complexity

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  - Satisfiability and Model checking : algorithms, complexity
- How to make models succint?
  - Equivalences and partial-orders on transition systems
  - Which properties are preserved?
  - Minimization algorithms

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- Q6: How is logic L used in computer science?
- Q7: What are the techniques needed to go about these questions?

Some Members of the mini-zoo

- Propositional Logic
- First Order Logic
- Monadic Second Order Logic
- Propositional Dynamic Logic
- Linear Temporal Logic
- Computational Tree Logic

More if time permits!

References

- To start with, the text book of Huth and Ryan : Logic for CS.
- As we go ahead, lecture notes/monographs/other text books.
- Classes : Slot 4. Tutorial: To discuss.

Propositional Logic

Syntax

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- Parantheses as required
- Example : $[p \wedge (q \vee r)] \rightarrow [\neg r \wedge p]$
- $\neg$ binds tighter than $\vee, \wedge$, which bind tighter than $\rightarrow$. In the absence of parantheses, $p \rightarrow q \rightarrow r$ is read as $p \rightarrow (q \rightarrow r)$

Natural Deduction

- If it rains, Alice is outside and does not have any raingear with her, she will get wet. $\varphi = (R \wedge \mathit{AliceOut} \wedge \neg \mathit{RG}) \rightarrow \mathit{AliceWet}$

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- So, Alice has her rain gear with her. *RG*
- Thus, $\chi = \varphi \wedge \psi \rightarrow RG$. You can deduce *RG* from $\varphi \wedge \psi$.
- Is χ valid? Is χ satisfiable?

Two Examples of Natural Deduction

Solve Sudoku

Consider the following kid's version of Sudoku.

	2	4	
1			3
4			2
	1	3	

Rules:

- Each row must contain all numbers 1-4
- Each column must contain all numbers 1-4
- Each 2×2 block must contain all numbers 1-4
- No cell contains 2 or more numbers

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- Proposition $P(i, j, n)$ is true when cell (i, j) has number n

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  - Row 1: $[P(1,1,1) \vee P(1,2,1) \vee P(1,3,1) \vee P(1,4,1)] \wedge$
    $[P(1,1,2) \vee P(1,2,2) \vee P(1,3,2) \vee P(1,4,2)] \wedge$
    $[P(1,1,3) \vee P(1,2,3) \vee P(1,3,3) \vee P(1,4,3)] \wedge$
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    $[P(1,1,4) \vee P(1,2,4) \vee P(1,3,4) \vee P(1,4,4)]$
  - Row 2: $[P(2,1,1) \vee \ldots$
  - Row 3: $[P(3,1,1) \vee \ldots$
  - Row 4: $[P(4,1,1) \vee \ldots$

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- Lower left block contains all numbers 1-4:

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- For cell(1,1):

$$P(1,1,1) \rightarrow [\neg P(1,1,2) \wedge \neg P(1,1,3) \wedge \neg P(1,1,4)] \wedge$$

$$P(1,1,2) \rightarrow [\neg P(1,1,1) \wedge \neg P(1,1,3) \wedge \neg P(1,1,4)] \wedge$$

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$$P(1,1,4) \rightarrow [\neg P(1,1,1) \wedge \neg P(1,1,2) \wedge \neg P(1,1,3)] \wedge$$

- Similar for other cells

Encoding as Propositional Satisfiability

Encoding Initial Configuration:

$$P(1,2,2) \wedge P(1,3,4) \wedge P(2,1,1) \wedge P(2,4,3) \wedge$$

$$P(3,1,4) \wedge P(3,4,2) \wedge P(4,2,1) \wedge P(4,3,3)$$

Solving Sodoku

To solve the puzzle, just conjunct all the above formulae and find a satisfiable truth assignment!

Gold Rush

(**Box1**)	*The gold is not here*
(**Box2**)	*The gold is not here*
(**Box3**)	*The gold is in Box 2*

Only one message is true; the other two are false. Which box has the gold?

Solve Gold Rush

- Propositions $M1, M2, M3$ representing messages in boxes 1,2,3
- Propositions $G1, G2, G3$ representing gold in boxes 1,2,3
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  - $\neg(M1 \wedge M2 \wedge M3)$,

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  - $M1 \leftrightarrow \neg G1$, $M2 \leftrightarrow \neg G2$, $M3 \leftrightarrow G2$
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  - $(\neg M1 \land \neg M2) \lor (\neg M1 \land \neg M3) \lor (\neg M2 \land \neg M3)$

Solve Gold Rush

- Propositions $M1, M2, M3$ representing messages in boxes 1,2,3
- Propositions $G1, G2, G3$ representing gold in boxes 1,2,3
- Formalize what is given to you
  - $M1 \leftrightarrow \neg G1$, $M2 \leftrightarrow \neg G2$, $M3 \leftrightarrow G2$
  - $\neg(M1 \wedge M2 \wedge M3)$, $M1 \vee M2 \vee M3$,
  - $(\neg M1 \wedge \neg M2) \vee (\neg M1 \wedge \neg M3) \vee (\neg M2 \wedge \neg M3)$
  - Conjunct all these, and call the formula φ.

Solve Gold Rush

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  - Is there a unique satisfiable assignment for φ?

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  - $M1 \leftrightarrow \neg G1$, $M2 \leftrightarrow \neg G2$, $M3 \leftrightarrow G2$
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  - $(\neg M1 \wedge \neg M2) \vee (\neg M1 \wedge \neg M3) \vee (\neg M2 \wedge \neg M3)$
  - Conjunct all these, and call the formula φ.
  - Is there a unique satisfiable assignment for φ?
  - For example, is $M1 = \mathit{true}$ a part of the satisfiable assignment?

A Proof Engine for Natural Deduction

- If it rains, Alice is outside and does not have any raingear with her, she will get wet. $\varphi = (R \wedge \mathit{AliceOut} \wedge \neg \mathit{RG}) \rightarrow \mathit{AliceWet}$
- It is raining, and Alice is outside, and is not wet. $\psi = (R \wedge \mathit{AliceOut} \wedge \neg \mathit{AliceWet})$
- So, Alice has her rain gear with her. *RG*
- Thus, $\chi = \varphi \wedge \psi \rightarrow \mathit{RG}$.
- Given φ, ψ, can we "prove" *RG*?

A Proof Engine

- Given a formula φ in propositional logic, how to “prove” φ if φ is valid?
- What is a proof engine?
- Show that this proof engine is sound and complete
  - Completeness: Any fact that can be captured using propositional logic can be proved by the proof engine
  - Soundness: Any formula that is proved to be valid by the proof engine is indeed valid

Natural Deduction

- In natural deduction, we have a collection of proof rules

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- $\varphi_1, \ldots, \varphi_n \vdash \psi$: This is called a sequent. $\varphi_1, \ldots, \varphi_n$ are premises, and ψ, the conclusion.
- Given $\varphi_1, \ldots, \varphi_n$, we can deduce or prove ψ. What was the sequent in the Alice example?

Natural Deduction

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- For example, $\neg p \rightarrow q, q \rightarrow r, \neg r \vdash p$ is a sequent. How do you prove this?

Natural Deduction

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- Given $\varphi_1, \ldots, \varphi_n$, we can deduce or prove ψ. What was the sequent in the Alice example?
- For example, $\neg p \to q, q \to r, \neg r \vdash p$ is a sequent. How do you prove this?
- Proof rules to be carefully chosen, for instance you should not end up proving something like $p \wedge q \vdash \neg q$

The Rules of the Proof Engine

Rules for Natural Deduction

The and introduction rule denoted $\wedge i$

$$\frac{\varphi \qquad \psi}{\varphi \wedge \psi}$$

Rules for Natural Deduction

The and elimination rule denoted $\wedge e_1$

$$\frac{\varphi \wedge \psi}{\varphi}$$

The and elimination rule denoted $\wedge e_2$

$$\frac{\varphi \wedge \psi}{\psi}$$

A first proof using $\wedge i, \wedge e_1, \wedge e_2$

- Show that $p \wedge q, r \vdash q \wedge r$

1.	$p \wedge q$	premise
2.		

A first proof using $\wedge i, \wedge e_1, \wedge e_2$

- Show that $p \wedge q, r \vdash q \wedge r$

1.	$p \wedge q$	premise
2.	r	premise
3.		

A first proof using $\wedge i, \wedge e_1, \wedge e_2$

- Show that $p \wedge q, r \vdash q \wedge r$

1.	$p \wedge q$	premise
2.	r	premise
3.	q	$\wedge e_2$ 1
4.		

A first proof using $\wedge i, \wedge e_1, \wedge e_2$

- Show that $p \wedge q, r \vdash q \wedge r$

1.	$p \wedge q$	premise
2.	r	premise
3.	q	$\wedge e_2$ 1
4.	$q \wedge r$	$\wedge i$ 3,2

Rules for Natural Deduction

The rule of double negation elimination $\neg\neg e$

$$\frac{\neg\neg\varphi}{\varphi}$$

The rule of double negation introduction $\neg\neg i$

$$\frac{\varphi}{\neg\neg\varphi}$$

Rules for Natural Deduction

The implies elimination rule or Modus Ponens MP

$$\frac{\varphi \qquad \varphi \rightarrow \psi}{\psi}$$

Another Proof

- Show that $p, p \to q, p \to (q \to \neg\neg r) \vdash r$

1.	$p \to (q \to \neg\neg r)$	premise
2.		

Another Proof

- Show that $p, p \to q, p \to (q \to \neg\neg r) \vdash r$

1.	$p \to (q \to \neg\neg r)$	premise
2.	$p \to q$	premise
3.		

Another Proof

- Show that $p, p \to q, p \to (q \to \neg\neg r) \vdash r$

1.	$p \to (q \to \neg\neg r)$	premise
2.	$p \to q$	premise
3.	p	premise
4.		

Another Proof

- Show that $p, p \to q, p \to (q \to \neg\neg r) \vdash r$

1.	$p \to (q \to \neg\neg r)$	premise
2.	$p \to q$	premise
3.	p	premise
4.	$q \to \neg\neg r$	MP 1,3
5.		

Another Proof

- Show that $p, p \to q, p \to (q \to \neg\neg r) \vdash r$

1.	$p \to (q \to \neg\neg r)$	premise
2.	$p \to q$	premise
3.	p	premise
4.	$q \to \neg\neg r$	MP 1,3
5.	q	MP 2,3
6.		

Another Proof

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1.	$p \to (q \to \neg\neg r)$	premise
2.	$p \to q$	premise
3.	p	premise
4.	$q \to \neg\neg r$	MP 1,3
5.	q	MP 2,3
6.	$\neg\neg r$	MP 4,5
7.		

Another Proof

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1.	$p \to (q \to \neg\neg r)$	premise
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3.	p	premise
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5.	q	MP 2,3
6.	$\neg\neg r$	MP 4,5
7.	r	$\neg\neg e$ 6

CS 228 : Logic in Computer Science

Krishna. S

Rules for Natural Deduction

Another implies elimination rule or Modus Tollens MT

$$\frac{\varphi \rightarrow \psi \qquad \neg\psi}{\neg\varphi}$$

A Proof

- Show that $p \rightarrow \neg q, q \vdash \neg p$

1.	$p \rightarrow \neg q$	premise
2.		

A Proof

- Show that $p \rightarrow \neg q, q \vdash \neg p$

1.	$p \rightarrow \neg q$	premise
2.	q	premise
3.		

A Proof

- Show that $p \to \neg q, q \vdash \neg p$

1.	$p \to \neg q$	premise
2.	q	premise
3.	$\neg\neg q$	$\neg\neg i$ 2
4.		

A Proof

- Show that $p \to \neg q, q \vdash \neg p$

1.	$p \to \neg q$	premise
2.	q	premise
3.	$\neg\neg q$	$\neg\neg i$ 2
4.	$\neg p$	MT 1,3

More Rules

- Thanks to MT, we have $p \to q, \neg q \vdash \neg p$.

More Rules

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- So far, no proof rule that can do this.
- Given $p \to q$, let us assume $\neg q$. Can we then prove $\neg p$?

More Rules

- Thanks to MT, we have $p \to q, \neg q \vdash \neg p$.
- Can we prove $p \to q \vdash \neg q \to \neg p$?
- So far, no proof rule that can do this.
- Given $p \to q$, let us assume $\neg q$. Can we then prove $\neg p$?
- Yes, using MT.

The implies introduction rule $\rightarrow i$

- $p \rightarrow q \vdash \neg q \rightarrow \neg p$

1.	$p \rightarrow q$	premise
2.	$\neg q$	assumption
3.	$\neg p$	MT 1,2
4.	$\neg q \rightarrow \neg p$	$\rightarrow i$ 2-3

More on $\rightarrow i$

- $\vdash (q \rightarrow r) \rightarrow [(\neg q \rightarrow \neg p) \rightarrow (p \rightarrow r)]$

1.	*true*	premise
2.		

More on $\rightarrow i$

- $\vdash (q \rightarrow r) \rightarrow [(\neg q \rightarrow \neg p) \rightarrow (p \rightarrow r)]$

1.	*true*	premise
2.	$q \rightarrow r$	assumption
3.		

More on $\rightarrow i$

- $\vdash (q \rightarrow r) \rightarrow [(\neg q \rightarrow \neg p) \rightarrow (p \rightarrow r)]$

1.	*true*	premise
2.	$q \rightarrow r$	assumption
3.	$\neg q \rightarrow \neg p$	assumption
4.		

More on $\rightarrow i$

- $\vdash (q \rightarrow r) \rightarrow [(\neg q \rightarrow \neg p) \rightarrow (p \rightarrow r)]$

1.	$true$	premise
2.	$q \rightarrow r$	assumption
3.	$\neg q \rightarrow \neg p$	assumption
4.	p	assumption
5.		

More on $\rightarrow i$

- $\vdash (q \rightarrow r) \rightarrow [(\neg q \rightarrow \neg p) \rightarrow (p \rightarrow r)]$

1.	$true$	premise
2.	$q \rightarrow r$	assumption
3.	$\neg q \rightarrow \neg p$	assumption
4.	p	assumption
5.	$\neg\neg p$	$\neg\neg i$ 4
6.		

More on $\rightarrow i$

- $\vdash (q \rightarrow r) \rightarrow [(\neg q \rightarrow \neg p) \rightarrow (p \rightarrow r)]$

1.	$true$	premise
2.	$q \rightarrow r$	assumption
3.	$\neg q \rightarrow \neg p$	assumption
4.	p	assumption
5.	$\neg\neg p$	$\neg\neg i$ 4
6.	$\neg\neg q$	MT 3,5
7.		

More on $\rightarrow i$

- $\vdash (q \rightarrow r) \rightarrow [(\neg q \rightarrow \neg p) \rightarrow (p \rightarrow r)]$

1.	$true$	premise
2.	$q \rightarrow r$	assumption
3.	$\neg q \rightarrow \neg p$	assumption
4.	p	assumption
5.	$\neg\neg p$	$\neg\neg i$ 4
6.	$\neg\neg q$	MT 3,5
7.	q	$\neg\neg e$ 6
8.		

More on $\rightarrow i$

- $\vdash (q \rightarrow r) \rightarrow [(\neg q \rightarrow \neg p) \rightarrow (p \rightarrow r)]$

1.	*true*	premise
2.	$q \rightarrow r$	assumption
3.	$\neg q \rightarrow \neg p$	assumption
4.	p	assumption
5.	$\neg\neg p$	$\neg\neg i$ 4
6.	$\neg\neg q$	MT 3,5
7.	q	$\neg\neg e$ 6
8.	r	MP 2,7

More on $\rightarrow i$

- $\vdash (q \rightarrow r) \rightarrow [(\neg q \rightarrow \neg p) \rightarrow (p \rightarrow r)]$

1.	$true$	premise
2.	$q \rightarrow r$	assumption
3.	$\neg q \rightarrow \neg p$	assumption
4.	p	assumption
5.	$\neg\neg p$	$\neg\neg i$ 4
6.	$\neg\neg q$	MT 3,5
7.	q	$\neg\neg e$ 6
8.	r	MP 2,7
9.	$p \rightarrow r$	$\rightarrow i$ 4-8

More on $\rightarrow i$

- $\vdash (q \rightarrow r) \rightarrow [(\neg q \rightarrow \neg p) \rightarrow (p \rightarrow r)]$

1.	$true$	premise
2.	$q \rightarrow r$	assumption
3.	$\neg q \rightarrow \neg p$	assumption
4.	p	assumption
5.	$\neg\neg p$	$\neg\neg i$ 4
6.	$\neg\neg q$	MT 3,5
7.	q	$\neg\neg e$ 6
8.	r	MP 2,7
9.	$p \rightarrow r$	$\rightarrow i$ 4-8
10.	$(\neg q \rightarrow \neg p) \rightarrow (p \rightarrow r)$	$\rightarrow i$ 3-9
11.		

More on $\rightarrow i$

- $\vdash (q \rightarrow r) \rightarrow [(\neg q \rightarrow \neg p) \rightarrow (p \rightarrow r)]$

1.	*true*	premise
2.	$q \rightarrow r$	assumption
3.	$\neg q \rightarrow \neg p$	assumption
4.	p	assumption
5.	$\neg\neg p$	$\neg\neg i$ 4
6.	$\neg\neg q$	MT 3,5
7.	q	$\neg\neg e$ 6
8.	r	MP 2,7
9.	$p \rightarrow r$	$\rightarrow i$ 4-8
10.	$(\neg q \rightarrow \neg p) \rightarrow (p \rightarrow r)$	$\rightarrow i$ 3-9
11.	$(q \rightarrow r) \rightarrow [(\neg q \rightarrow \neg p) \rightarrow (p \rightarrow r)]$	$\rightarrow i$ 2-10

Transforming Proofs

- $(q \to r), (\neg q \to \neg p), p \vdash r$
- Transform any proof $\varphi_1, \ldots, \varphi_n \vdash \psi$ to $\vdash \varphi_1 \to (\varphi_2 \to \ldots (\varphi_n \to \psi) \ldots))$ by adding n lines of the rule $\to i$

More Examples

- $p \to (q \to r) \vdash (p \land q) \to r$

1.	$p \to (q \to r)$	premise
2.		

More Examples

- $p \to (q \to r) \vdash (p \wedge q) \to r$

1.	$p \to (q \to r)$	premise
2.	$p \wedge q$	assumption
3.		

More Examples

- $p \to (q \to r) \vdash (p \land q) \to r$

1.	$p \to (q \to r)$	premise
2.	$p \land q$	assumption
3.	p	$\land e_1$ 2
4.		

More Examples

- $p \to (q \to r) \vdash (p \land q) \to r$

1.	$p \to (q \to r)$	premise
2.	$p \land q$	assumption
3.	p	$\land e_1$ 2
4.	q	$\land e_2$ 2
5.		

More Examples

- $p \to (q \to r) \vdash (p \wedge q) \to r$

1.	$p \to (q \to r)$	premise
2.	$p \wedge q$	assumption
3.	p	$\wedge e_1$ 2
4.	q	$\wedge e_2$ 2
5.	$q \to r$	MP 1,3
6.		

More Examples

- $p \to (q \to r) \vdash (p \wedge q) \to r$

1.	$p \to (q \to r)$	premise
2.	$p \wedge q$	assumption
3.	p	$\wedge e_1$ 2
4.	q	$\wedge e_2$ 2
5.	$q \to r$	MP 1,3
6.	r	MP 4,5
7.		

More Examples

- $p \to (q \to r) \vdash (p \wedge q) \to r$

1.	$p \to (q \to r)$	premise
2.	$p \wedge q$	assumption
3.	p	$\wedge e_1$ 2
4.	q	$\wedge e_2$ 2
5.	$q \to r$	MP 1,3
6.	r	MP 4,5
7.	$p \wedge q \to r$	$\to i$ 2-6

More Rules

The or introduction rule $\vee i_1$

$$\frac{\varphi}{\varphi \vee \psi}$$

The or introduction rule $\vee i_2$

$$\frac{\psi}{\varphi \vee \psi}$$

More Rules

The or elimination rule $\vee e$

$$\frac{\varphi \vee \psi \qquad \varphi \vdash \chi \qquad \psi \vdash \chi}{\chi}$$

Or Elimination Example

- $q \to r \vdash (p \vee q) \to (p \vee r)$

1.	$q \to r$	premise
2.		

Or Elimination Example

- $q \to r \vdash (p \vee q) \to (p \vee r)$

1.	$q \to r$	premise
2.	$p \vee q$	assumption
3.		

Or Elimination Example

- $q \rightarrow r \vdash (p \vee q) \rightarrow (p \vee r)$

1.	$q \rightarrow r$	premise
2.	$p \vee q$	assumption
3.	p	$\vee e$ (1)
4.		

Or Elimination Example

- $q \to r \vdash (p \vee q) \to (p \vee r)$

1.	$q \to r$	premise
2.	$p \vee q$	assumption
3.	p	$\vee e$ (1)
4.	$p \vee r$	$\vee i_1$ 3
5.		

Or Elimination Example

- $q \to r \vdash (p \vee q) \to (p \vee r)$

1.	$q \to r$	premise
2.	$p \vee q$	assumption
3.	p	$\vee e$ (1)
4.	$p \vee r$	$\vee i_1$ 3
5.	q	$\vee e$ (2)
6.		

Or Elimination Example

- $q \to r \vdash (p \vee q) \to (p \vee r)$

1.	$q \to r$	premise
2.	$p \vee q$	assumption
3.	p	$\vee\ e\ (1)$
4.	$p \vee r$	$\vee i_1$ 3
5.	q	$\vee\ e\ (2)$
6.	r	MP 1,5
7.		

Or Elimination Example

- $q \rightarrow r \vdash (p \vee q) \rightarrow (p \vee r)$

1.	$q \rightarrow r$	premise
2.	$p \vee q$	assumption
3.	p	$\vee\ e\ (1)$
4.	$p \vee r$	$\vee i_1$ 3
5.	q	$\vee\ e\ (2)$
6.	r	MP 1,5
7.	$p \vee r$	$\vee i_2$ 6

Or Elimination Example

- $q \to r \vdash (p \vee q) \to (p \vee r)$

1.	$q \to r$	premise
2.	$p \vee q$	assumption
3.	p	$\vee\ e\ (1)$
4.	$p \vee r$	$\vee i_1$ 3
5.	q	$\vee\ e\ (2)$
6.	r	MP 1,5
7.	$p \vee r$	$\vee i_2$ 6
8.	$p \vee r$	$\vee e$ 2, 3-4, 5-7

Or Elimination Example

- $q \rightarrow r \vdash (p \vee q) \rightarrow (p \vee r)$

1.	$q \rightarrow r$	premise
2.	$p \vee q$	assumption
3.	p	$\vee\ e$ (1)
4.	$p \vee r$	$\vee i_1$ 3
5.	q	$\vee\ e$ (2)
6.	r	MP 1,5
7.	$p \vee r$	$\vee i_2$ 6
8.	$p \vee r$	$\vee e$ 2, 3-4, 5-7
9.		

Or Elimination Example

- $q \rightarrow r \vdash (p \vee q) \rightarrow (p \vee r)$

1.	$q \rightarrow r$	premise
2.	$p \vee q$	assumption
3.	p	$\vee\ e\ (1)$
4.	$p \vee r$	$\vee i_1$ 3
5.	q	$\vee\ e\ (2)$
6.	r	MP 1,5
7.	$p \vee r$	$\vee i_2$ 6
8.	$p \vee r$	$\vee e$ 2, 3-4, 5-7
9.	$(p \vee q) \rightarrow (p \vee r)$	$\rightarrow i$ 2-8

Associativity Using Or Elimination

- $(p \vee q) \vee r \vdash p \vee (q \vee r)$

1.	$(p \vee q) \vee r$	premise
2.		

Associativity Using Or Elimination

- $(p \vee q) \vee r \vdash p \vee (q \vee r)$

1.	$(p \vee q) \vee r$	premise
2.	$p \vee q$	$\vee e$ (1)
3.		

Associativity Using Or Elimination

- $(p \vee q) \vee r \vdash p \vee (q \vee r)$

1.	$(p \vee q) \vee r$	premise
2.	$p \vee q$	$\vee e$ (1)
3.	p	$\vee e$ (1.1)
4.		

Associativity Using Or Elimination

- $(p \vee q) \vee r \vdash p \vee (q \vee r)$

1.	$(p \vee q) \vee r$	premise
2.	$p \vee q$	$\vee e\,(1)$
3.	p	$\vee e\,(1.1)$
4.	$p \vee (q \vee r)$	$\vee i_1$ 3
5.		

Associativity Using Or Elimination

- $(p \lor q) \lor r \vdash p \lor (q \lor r)$

1.	$(p \lor q) \lor r$	premise
2.	$p \lor q$	$\lor e$ (1)
3.	p	$\lor e$ (1.1)
4.	$p \lor (q \lor r)$	$\lor i_1$ 3
5.	q	$\lor e$ (1.2)
6.		

Associativity Using Or Elimination

- $(p \vee q) \vee r \vdash p \vee (q \vee r)$

1.	$(p \vee q) \vee r$	premise
2.	$p \vee q$	$\vee e\,(1)$
3.	p	$\vee e\,(1.1)$
4.	$p \vee (q \vee r)$	$\vee i_1\ 3$
5.	q	$\vee e\,(1.2)$
6.	$q \vee r$	$\vee i_1\ 5$
7.		

Associativity Using Or Elimination

- $(p \vee q) \vee r \vdash p \vee (q \vee r)$

1.	$(p \vee q) \vee r$	premise
2.	$p \vee q$	$\vee\, e\,(1)$
3.	p	$\vee\, e\,(1.1)$
4.	$p \vee (q \vee r)$	$\vee i_1\; 3$
5.	q	$\vee\, e\,(1.2)$
6.	$q \vee r$	$\vee i_1\; 5$
7.	$p \vee (q \vee r)$	$\vee i_2\; 6$
8.		

Associativity Using Or Elimination

- $(p \vee q) \vee r \vdash p \vee (q \vee r)$

1.	$(p \vee q) \vee r$	premise
2.	$p \vee q$	$\vee e$ (1)
3.	p	$\vee e$ (1.1)
4.	$p \vee (q \vee r)$	$\vee i_1$ 3
5.	q	$\vee e$ (1.2)
6.	$q \vee r$	$\vee i_1$ 5
7.	$p \vee (q \vee r)$	$\vee i_2$ 6
8.	$p \vee (q \vee r)$	$\vee e$ 2, 3-4, 5-7
9.		

Associativity Using Or Elimination

- $(p \vee q) \vee r \vdash p \vee (q \vee r)$

1.	$(p \vee q) \vee r$	premise
2.	$p \vee q$	$\vee\ e\ (1)$
3.	p	$\vee\ e\ (1.1)$
4.	$p \vee (q \vee r)$	$\vee i_1\ 3$
5.	q	$\vee\ e\ (1.2)$
6.	$q \vee r$	$\vee i_1\ 5$
7.	$p \vee (q \vee r)$	$\vee i_2\ 6$
8.	$p \vee (q \vee r)$	$\vee e$ 2, 3-4, 5-7
9.	r	$\vee\ e\ (2)$
10.		

Associativity Using Or Elimination

- $(p \vee q) \vee r \vdash p \vee (q \vee r)$

1.	$(p \vee q) \vee r$	premise
2.	$p \vee q$	$\vee e$ (1)
3.	p	$\vee e$ (1.1)
4.	$p \vee (q \vee r)$	$\vee i_1$ 3
5.	q	$\vee e$ (1.2)
6.	$q \vee r$	$\vee i_1$ 5
7.	$p \vee (q \vee r)$	$\vee i_2$ 6
8.	$p \vee (q \vee r)$	$\vee e$ 2, 3-4, 5-7
9.	r	$\vee e$ (2)
10.	$q \vee r$	$\vee i_2$ 9
11.		

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1.	$(p \vee q) \vee r$	premise
2.	$p \vee q$	$\vee e$ (1)
3.	p	$\vee e$ (1.1)
4.	$p \vee (q \vee r)$	$\vee i_1$ 3
5.	q	$\vee e$ (1.2)
6.	$q \vee r$	$\vee i_1$ 5
7.	$p \vee (q \vee r)$	$\vee i_2$ 6
8.	$p \vee (q \vee r)$	$\vee e$ 2, 3-4, 5-7
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Associativity Using Or Elimination

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4.	$p \vee (q \vee r)$	$\vee i_1$ 3
5.	q	$\vee\ e\ (1.2)$
6.	$q \vee r$	$\vee i_1$ 5
7.	$p \vee (q \vee r)$	$\vee i_2$ 6
8.	$p \vee (q \vee r)$	$\vee e$ 2, 3-4, 5-7
9.	r	$\vee\ e\ (2)$
10.	$q \vee r$	$\vee i_2$ 9
11.	$p \vee (q \vee r)$	$\vee i_2$ 10
12.	$p \vee (q \vee r)$	$\vee e$ 1, 2-8, 9-11

Basic Rules So Far

- $\wedge$i, $\wedge e_1$, $\wedge e_2$ (and introduction and elimination)
- $\neg\neg e$, $\neg\neg i$ (double negation elimination and introduction)
- MP (Modus Ponens)
- $\rightarrow i$ (Implies Introduction : remember opening boxes)
- $\vee i_1, \vee i_2, \vee e$ (Or introduction and elimination)

The Copy Rule

- $\vdash p \to (q \to p)$

1.	*true*	premise
2.		

The Copy Rule

- $\vdash p \rightarrow (q \rightarrow p)$

1.	*true*	premise
2.	p	assumption
3.		

The Copy Rule

- $\vdash p \to (q \to p)$

1.	*true*	premise
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The Copy Rule

- $\vdash p \to (q \to p)$

1.	*true*	premise
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3.	q	assumption
4.	p	copy 2
5.		

The Copy Rule

- $\vdash p \rightarrow (q \rightarrow p)$

1.	*true*	premise
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3.	q	assumption
4.	p	copy 2
5.	$q \rightarrow p$	$\rightarrow i$ 3-4
6.		

The Copy Rule

- $\vdash p \rightarrow (q \rightarrow p)$

1.	*true*	premise
2.	p	assumption
3.	q	assumption
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6.	$p \rightarrow (q \rightarrow p)$	$\rightarrow i$ 2-5

The Rules of Single Negation

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- Any two contradictions are equivalent : $p \wedge \neg p$ is equivalent to $\neg r \wedge r$. Contradictions denoted by $\perp$.
- $\perp \rightarrow \varphi$ for any formula φ.

Rules with $\perp$

The $\perp$ elimination rule $\perp e$

$$\frac{\perp}{\psi}$$

The $\perp$ introduction rule $\perp i$

$$\frac{\varphi \qquad \neg\varphi}{\perp}$$

An Example

- $\neg p \vee q \vdash p \rightarrow q$

1.	$\neg p \vee q$	premise
2.		

An Example

- $\neg p \lor q \vdash p \to q$

1.	$\neg p \lor q$	premise
2.	$\neg p$	$\lor e$ (1)
3.		

An Example

- $\neg p \vee q \vdash p \rightarrow q$

1.	$\neg p \vee q$	premise
2.	$\neg p$	$\vee e$ (1)
3.	p	assumption
4.		

An Example

- $\neg p \vee q \vdash p \to q$

1.	$\neg p \vee q$	premise
2.	$\neg p$	$\vee e$ (1)
3.	p	assumption
4.	$\perp$	$\perp i$ 2,3
5.		

An Example

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1.	$\neg p \vee q$	premise
2.	$\neg p$	$\vee e$ (1)
3.	p	assumption
4.	$\perp$	$\perp i$ 2,3
5.	q	$\perp e$ 4
6.		

An Example

- $\neg p \vee q \vdash p \rightarrow q$

1.	$\neg p \vee q$	premise
2.	$\neg p$	$\vee e$ (1)
3.	p	assumption
4.	$\bot$	$\bot i$ 2,3
5.	q	$\bot e$ 4
6.	$p \rightarrow q$	$\rightarrow i$ 3-5
7.	q	$\vee e$ (2)
8.		

An Example

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2.	$\neg p$	$\vee e$ (1)
3.	p	assumption
4.	$\perp$	$\perp i$ 2,3
5.	q	$\perp e$ 4
6.	$p \rightarrow q$	$\rightarrow i$ 3-5
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8.	p	assumption
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An Example

- $\neg p \vee q \vdash p \rightarrow q$

1.	$\neg p \vee q$	premise
2.	$\neg p$	$\vee e$ (1)
3.	p	assumption
4.	$\perp$	$\perp i$ 2,3
5.	q	$\perp e$ 4
6.	$p \rightarrow q$	$\rightarrow i$ 3-5
7.	q	$\vee e$ (2)
8.	p	assumption
9.	q	copy 7
10.	$p \rightarrow q$	$\rightarrow i$ 8-9
11.	$p \rightarrow q$	$\vee e$ 1, 2-6, 7-10

Introducing Negations (PBC)

- In the course of a proof, if you assume φ (by opening a box) and obtain $\perp$ in the box, then we conclude $\neg\varphi$
- This rule is denoted $\neg i$ and is read as $\neg$ introduction.
- Also known as Proof By Contradiction

An Example

- $p \to \neg p \vdash \neg p$

1.	$p \to \neg p$	premise
2.		

An Example

- $p \to \neg p \vdash \neg p$

1.	$p \to \neg p$	premise
2.	p	assumption
3.		

An Example

- $p \to \neg p \vdash \neg p$

1.	$p \to \neg p$	premise
2.	p	assumption
3.	$\neg p$	MP 1,2
4.		

An Example

- $p \to \neg p \vdash \neg p$

1.	$p \to \neg p$	premise
2.	p	assumption
3.	$\neg p$	MP 1,2
4.	$\bot$	$\bot i$ 2,3
5.	$\neg p$	$\neg i$ 2-4

The Last One

Law of the Excluded Middle (LEM)

$$\frac{}{\varphi \vee \neg\varphi}$$

Summary of Basic Rules

- $\wedge i$, $\wedge e_1$, $\wedge e_2$,
- $\neg\neg e$
- MP
- $\rightarrow i$
- $\vee i_1, \vee i_2, \vee e$
- Copy, $\neg i$ or PBC
- $\bot e, \bot i$

Derived Rules

- MT (derive using MP, $\bot i$ and $\neg i$)
- $\neg\neg i$ (derive using $\bot i$ and $\neg i$)
- LEM (derive using $\vee i_1, \bot i, \neg i, \vee i_2, \neg\neg e$)

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- Now we show that whatever can be proved makes sense semantically too.

Semantics

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CS 228 : Logic in Computer Science

Krishna. S

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Soundness of Propositional Logic

$\varphi_1, \ldots, \varphi_n \vdash \psi \Rightarrow \varphi_1, \ldots, \varphi_n \models \psi$

Whenever ψ can be proved from $\varphi_1, \ldots, \varphi_n$, then ψ evaluates to true whenever $\varphi_1, \ldots, \varphi_n$ evaluate to true

Soundness

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- Consider now a proof with k lines.

Soundness : Case $\wedge i$

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- Assume ψ was obtained using $\wedge i$. Then ψ is of the form $\psi_1 \wedge \psi_2$.
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- We have the shorter proofs $\varphi_1, \ldots, \varphi_n \vdash \psi_1$ and $\varphi_1, \ldots, \varphi_n \vdash \psi_2$

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- We have the shorter proofs $\varphi_1, \ldots, \varphi_n \vdash \psi_1$ and $\varphi_1, \ldots, \varphi_n \vdash \psi_2$
- By inductive hypothesis, we have $\varphi_1, \ldots, \varphi_n \models \psi_1$ and $\varphi_1, \ldots, \varphi_n \models \psi_2$. By semantics, we have $\varphi_1, \ldots, \varphi_n \models \psi_1 \wedge \psi_2$.

Soundness : Case $\rightarrow i$

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Soundness : Case $\rightarrow i$

- Assume ψ was obtained using $\rightarrow i$. Then ψ is of the form $\psi_1 \rightarrow \psi_2$.
- A box starting with ψ_1 was opened at some line $k_1 < k$.
- The last line in the box was ψ_2.

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- The line just after the box was ψ.
- Consider adding ψ_1 in the premises along with $\varphi_1, \ldots, \varphi_n$. Then we will get a proof $\varphi_1, \ldots, \varphi_n, \psi_1 \vdash \psi_2$, of length $k - 1$. By inductive hypothesis, $\varphi_1, \ldots, \varphi_n, \psi_1 \models \psi_2$. By semantics, this is same as $\varphi_1, \ldots, \varphi_n \models \psi_1 \rightarrow \psi_2$

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- The equivalence of $\varphi_1, \ldots, \varphi_n \vdash \psi_1 \rightarrow \psi_2$ and $\varphi_1, \ldots, \varphi_n, \psi_1 \vdash \psi_2$ gives the proof.

Soundness : Other cases

Completeness

$$\varphi_1, \ldots, \varphi_n \models \psi \Rightarrow \varphi_1, \ldots, \varphi_n \vdash \psi$$

Whenever $\varphi_1, \ldots, \varphi_n$ semantically entail ψ, then ψ can be proved from $\varphi_1, \ldots, \varphi_n$. The proof rules are complete

Completeness : 3 steps

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- If $\not\models \varphi_1 \to (\varphi_2 \to (\ldots (\varphi_n \to \psi) \ldots))$, then ψ evaluates to false when all of $\varphi_1, \ldots, \varphi_n$ evaluate to true, a contradiction.

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- Hence, $\models \varphi_1 \to (\varphi_2 \to (\ldots(\varphi_n \to \psi)\ldots))$.

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- Using this insight, we have to give a proof of ψ.

Completeness : Step 2

Truth Table to Proof

Let φ be a formula with variables $p_1, \ldots, p_n$. Let $\mathcal{T}$ be the truth table of φ, and let l be a line number in $\mathcal{T}$. Let $\hat{p}_i$ represent p_i if p_i is assigned true in line l, and let it denote $\neg p_i$ if p_i is assigned false in line l. Then

1. $\hat{p}_1, \ldots, \hat{p}_n \vdash \varphi$ if φ evaluates to true in line l
2. $\hat{p}_1, \ldots, \hat{p}_n \vdash \neg\varphi$ if φ evaluates to false in line l

CS 228 : Logic in Computer Science

Krishna. S

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- Step 2: Show that $\vdash \varphi_1 \to (\varphi_2 \to (\ldots(\varphi_n \to \psi)\ldots))$
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Let φ be a formula with variables $p_1, \ldots, p_n$. Let $\mathcal{T}$ be the truth table of φ, and let l be a line number in $\mathcal{T}$. Let $\hat{p}_i$ represent p_i if p_i is assigned true in line l, and let it denote $\neg p_i$ if p_i is assigned false in line l. Then

1. $\hat{p}_1, \ldots, \hat{p}_n \vdash \varphi$ if φ evaluates to true in line l
2. $\hat{p}_1, \ldots, \hat{p}_n \vdash \neg\varphi$ if φ evaluates to false in line l

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  - Assume φ evaluates to true in line l of $\mathcal{T}$. Then φ_1 evaluates to false in line l. By inductive hypothesis, $\hat{p}_1, \dots, \hat{p}_n \vdash \neg\varphi_1$.
  - Assume φ evaluates to false in line l of $\mathcal{T}$. Then φ_1 evaluates to true in line l. By inductive hypothesis, $\hat{p}_1, \dots, \hat{p}_n \vdash \varphi_1$. Use the $\neg\neg i$ rule to obtain a proof of $\hat{p}_1, \dots, \hat{p}_n \vdash \neg\neg\varphi_1$.

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  - If φ evaluates to false in line l, then φ_1 evaluates to true and φ_2 to false. Let $\{q_1, \ldots, q_k\}$ be the variables of φ_1 and let $\{r_1, \ldots, r_j\}$ be the variables in φ_2. $\{q_1, \ldots, q_k\} \cup \{r_1, \ldots, r_j\} = \{p_1, \ldots, p_n\}$.

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  - By inductive hypothesis, $\hat{q}_1, \ldots, \hat{q}_k \models \varphi_1$ and $\hat{r}_1, \ldots, \hat{r}_j \models \neg\varphi_2$. Then, $\hat{p}_1, \ldots, \hat{p}_n \models \varphi_1 \wedge \neg\varphi_2$.

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  - By inductive hypothesis, $\hat{q}_1, \ldots, \hat{q}_k \models \varphi_1$ and $\hat{r}_1, \ldots, \hat{r}_j \models \neg\varphi_2$. Then, $\hat{p}_1, \ldots, \hat{p}_n \models \varphi_1 \wedge \neg\varphi_2$.
  - Prove that $\varphi_1 \wedge \neg\varphi_2 \vdash \neg(\varphi_1 \to \varphi_2)$.

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  - If both φ_1, φ_2 evaluate to false, then we have $\hat{p}_1, \ldots, \hat{p}_n \models \neg\varphi_1 \wedge \neg\varphi_2$. Proving $\neg\varphi_1 \wedge \neg\varphi_2 \vdash \varphi_1 \rightarrow \varphi_2$, we are done.

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  - If both φ_1, φ_2 evaluate to false, then we have $\hat{p}_1, \ldots, \hat{p}_n \models \neg\varphi_1 \wedge \neg\varphi_2$. Proving $\neg\varphi_1 \wedge \neg\varphi_2 \vdash \varphi_1 \rightarrow \varphi_2$, we are done.
  - Last, if φ_1 evaluates to false and φ_2 evaluates to true, then we have $\hat{p}_1, \ldots, \hat{p}_n \models \neg\varphi_1 \wedge \varphi_2$. Proving $\neg\varphi_1 \wedge \varphi_2 \vdash \varphi_1 \rightarrow \varphi_2$, we are done.

Truth Table to Proof

- Prove the cases $\wedge, \vee$.

On An Example

We know $\models (p \wedge q) \rightarrow p$. Using this fact, show that $\vdash (p \wedge q) \rightarrow p$.

- $p, q \vdash (p \wedge q) \rightarrow p$
- $\neg p, q \vdash (p \wedge q) \rightarrow p$
- $p, \neg q \vdash (p \wedge q) \rightarrow p$
- $\neg p, \neg q \vdash (p \wedge q) \rightarrow p$

Now, combine the 4 proofs above to give a single proof for
$\vdash (p \wedge q) \rightarrow p$.

Completeness : Steps 2, 3

- Step 2: From $\models \varphi_1 \to (\varphi_2 \to (\ldots(\varphi_n \to \psi)\ldots))$, use LEM on all the propositional variables of $\varphi_1, \ldots, \varphi_n, \psi$ to obtain $\vdash \varphi_1 \to (\varphi_2 \to (\ldots(\varphi_n \to \psi)\ldots))$.

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- Step 3: Take the proof $\vdash \varphi_1 \to (\varphi_2 \to (\dots(\varphi_n \to \psi)\dots))$. This proof has *n* nested boxes, the *i*th box opening with the assumption φ_i. The last box closes with the last line ψ. Hence, the line immediately after the last box is $\varphi_n \to \psi$.

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- In a similar way, the $(n-1)$th box has as its last line $\varphi_n \to \psi$, and hence, the line immediately after this box is $\varphi_{n-1} \to (\varphi_n \to \psi)$ and so on.

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- In a similar way, the $(n-1)$th box has as its last line $\varphi_n \to \psi$, and hence, the line immediately after this box is $\varphi_{n-1} \to (\varphi_n \to \psi)$ and so on.
- Add premises $\varphi_1, \ldots, \varphi_n$ on the top. Use MP on the premises, and the lines after boxes 1 to n in order to obtain ψ.

Summary

Propositional Logic is sound and complete.

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Every formula F is equivalent to some formula F_1 in CNF and some formula F_2 in DNF.

CNF Algorithm

Given a formula F, $(x \to [\neg(y \vee z) \wedge \neg(y \to x)])$

- Replace all subformulae of the form $F \to G$ with $\neg F \vee G$, and all subformulae of the form $F \leftrightarrow G$ with $(\neg F \vee G) \wedge (\neg G \vee F)$. When there are no more occurrences of $\to, \leftrightarrow$, proceed to the next step.

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- Get rid of all double negations : Replace all subformulae
  - $\neg\neg G$ with G,
  - $\neg(G \wedge H)$ with $\neg G \vee \neg H$
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  - $\neg(G \lor H)$ with $\neg G \land \neg H$

  When there are no more such subformulae, proceed to the next step.
- Distribute $\lor$ wherever possible.

The resultant formula F_1 is in CNF and is provably equivalent to F.
$[(\neg x \lor \neg y) \land (\neg x \lor \neg z)] \land [(\neg x \lor y) \land (\neg x \lor \neg x)]$

The Hardness of SAT

- Given a formula φ how to check if φ is satisfiable?
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- SAT is NP-complete

Polynomial Time Formula Classes

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CS 228 : Logic in Computer Science

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- Consider subformulae of the form $(p_1 \wedge \cdots \wedge p_m) \to \bot$. If there is one such subformula with all p_i marked, then say Unsat, otherwise say Sat.

An Example

$$(\top \to A) \land (C \to D) \land ((A \land B) \to C) \land ((C \land D) \to \bot) \land (\top \to B).$$

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The Horn Algorithm

The Horn algorithm concludes Sat iff H is satisfiable.

Complexity of Horn

- Given a Horn formula ψ with *n* propositions, how many times do you have to read ψ?
- Step 1: Read once
- Step 2: Read atmost *n* times
- Step 3: Read once

2-CNF

- 2-CNF : CNF where each clause has at most 2 literals.

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- Let C_1, C_2 be two clauses. Assume $p \in C_1$ and $\neg p \in C_2$ for some literal p. Then the clause $R = (C_1 - \{p\}) \cup (C_2 - \{\neg p\})$ is a resolvent of C_1 and C_2.

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- Let $C_1 = \{p_1, \neg p_2, p_3\}$ and $C_2 = \{p_2, \neg p_3, p_4\}$. As $p_3 \in C_1$ and $\neg p_3 \in C_2$, we can find the resolvent. The resolvent is $\{p_1, p_2, \neg p_2, p_4\}$.

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- Resolvent not unique : $\{p_1, p_3, \neg p_3, p_4\}$ is also a resolvent.

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- Let F be a formula in CNF. Let R be a resolvent of two clauses of F. Then $F \vdash R$ (Prove!)

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Show that resolution can be used to determine whether any given formula is unsatisfiable.

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- $Res^0(F) = F$, there are finitely many clauses that can be derived from F.
- There is some $m \geqslant 0$ such that $Res^m(F) = Res^{m+1}(F)$. Denote it by $Res^*(F)$.

Example

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- $Res^1(F) = F \cup \{p_1, p_2, \neg p_2\} \cup \{p_1, \neg p_3, p_3\}$.

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CS 228 : Logic in Computer Science

Krishna. S

Resolution

- Given a propositional logic formula φ, is it unsatisfiable?

Resolution

- Given a propositional logic formula φ, is it unsatisfiable?
- How does a solver do it?
- Assume it is in CNF

Resolution

- Let C_1, C_2 be two clauses. Assume $p \in C_1$ and $\neg p \in C_2$ for some literal p.

Resolution

- Let C_1, C_2 be two clauses. Assume $p \in C_1$ and $\neg p \in C_2$ for some literal p. Then the clause $R = (C_1 - \{p\}) \cup (C_2 - \{\neg p\})$ is a resolvent of C_1 and C_2.
- Let $C_1 = \{p_1, \neg p_2, p_3\}$ and $C_2 = \{p_2, \neg p_3, p_4\}$. As $p_3 \in C_1$ and $\neg p_3 \in C_2$, we can find the resolvent. The resolvent is $\{p_1, p_2, \neg p_2, p_4\}$.
- Resolvent not unique : $\{p_1, p_3, \neg p_3, p_4\}$ is also a resolvent.

3 rules in Resolution

- Let G be any formula. Let F be the CNF formula resulting from the CNF algorithm applied to G. Then $G \vdash F$ (Prove!)

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- Let F be a formula in CNF, and let C be a clause in F. Then $F \vdash C$ (Prove!)
- Let F be a formula in CNF. Let R be a resolvent of two clauses of F. Then $F \vdash R$ (Prove!)

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Show that resolution can be used to determine whether any given formula is unsatisfiable.

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- $Res^0(F) = F$, there are finitely many clauses that can be derived from F.
- There is some $m \geqslant 0$ such that $Res^m(F) = Res^{m+1}(F)$. Denote it by $Res^*(F)$.

Example

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Resolution

Let F be a formula in CNF. If $\emptyset \in \mathit{Res}^*(F)$, then F is unsatisfiable.

- If $\emptyset \in \mathit{Res}^*(F)$. Then $\emptyset \in \mathit{Res}^n(F)$ for some n.

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Let F be a formula in CNF. If $\emptyset \in Res^*(F)$, then F is unsatisfiable.

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- Since $\emptyset \notin Res^0(F)$ ($\emptyset$ is not a clause), there is an $m > 0$ such that $\emptyset \notin Res^m(F)$ and $\emptyset \in Res^{m+1}(F)$.

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- Since $\emptyset \notin Res^0(F)$ ($\emptyset$ is not a clause), there is an $m > 0$ such that $\emptyset \notin Res^m(F)$ and $\emptyset \in Res^{m+1}(F)$.
- Then $\{p\}, \{\neg p\} \in Res^m(F)$. By the rules of resolution, we have $F \vdash p, \neg p$, and thus $F \vdash \bot$. Hence, F is unsatisfiable.

Resolution

Prove the converse: F is unsatisfiable implies $\emptyset \in \mathit{Res}^*(F)$.

(Discuss substitution before the proof)

Resolution

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- Let F have k clauses $C_1, \ldots, C_k$.

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- Hence, $F = \{\{p\}, \{\neg p\}\}$. Clearly, $\emptyset \in Res(F)$.

Resolution

- Inductive hypothesis : If F has $\leqslant n$ variables and is unsat, then $\emptyset \in Res^*(F)$.

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  - Let G_0 be the conjunction of all C_i in F such that $\neg p_{n+1} \notin C_i$.
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- Let $F_0 = \{C_i - \{p_{n+1}\} \mid C_i \in G_0\}$
- Let $F_1 = \{C_i - \{\neg p_{n+1}\} \mid C_i \in G_1\}$

Resolution

Let $F = \{\{p_1, p_3\}, \{p_2\}, \{\neg p_1, \neg p_2, p_3\}, \{\neg p_2, \neg p_3\}\}$ and $n = 2$.

- $G_0 = \{\{p_1, p_3\}, \{p_2\}, \{\neg p_1, \neg p_2, p_3\}\}$, $G_1 = \{\{p_2\}, \{\neg p_2, \neg p_3\}\}$.
- $F_0 = \{\{p_1\}, \{p_2\}, \{\neg p_1, \neg p_2\}\}$ and $F_1 = \{\{p_2\}, \{\neg p_2\}\}$
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- If $p_{n+1} = \mathit{true}$ in F, then F is equisatisfiable with F_1
- Hence F is satisfiable iff $F_0 \vee F_1$ is.
- As F is unsatisfiable, F_0 and F_1 are both unsatisfiable.

Resolution

- By induction hypothesis, $\emptyset \in \mathit{Res}^*(F_0)$ and $\emptyset \in \mathit{Res}^*(F_1)$.

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- By induction hypothesis, $\emptyset \in Res^*(F_0)$ and $\emptyset \in Res^*(F_1)$.
- Hence, $\emptyset \in Res^*(G_0)$ or $\{p_{n+1}\} \in Res^*(G_0)$, and $\emptyset \in Res^*(G_1)$ or $\{\neg p_{n+1}\} \in Res^*(G_1)$.

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- Else, $\{p_{n+1}\} \in \mathit{Res}^*(G_0)$ and $\{\neg p_{n+1}\} \in \mathit{Res}^*(G_1)$.
- Hence $\emptyset \in \mathit{Res}^*(F)$.

Resolution Summary

Given a formula ψ, convert it into CNF, say ζ. ψ is satisfiable iff $\emptyset \notin \mathit{Res}^*(\zeta)$.

- If ψ is unsat, we might get $\emptyset$ before reaching $\mathit{Res}^*(\zeta)$.
- If ψ is sat, then truth tables are faster : stop when some row evaluates to 1.

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- A disjunction of literals $L_1 \vee L_2 \vee \ldots L_n$ is valid iff ...
- A conjunction of literals $L_1 \wedge L_2 \wedge \ldots L_n$ is satisfiable iff ...

Normal Forms : CNF Validity

Let $\varphi = C_1 \wedge C_2 \wedge \cdots \wedge C_n$ be in CNF.

- Checking if φ is satisfiable is NP-complete.
- Checking if φ is valid is polynomial time. Why?
- Question raised in class : If validity is polytime, so should be satisfiability. Is this true?

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- Checking if φ is satisfiable is NP-complete.
- Checking if φ is valid is polynomial time. Why?
- Question raised in class : If validity check is polynomial time, so should be satisfiability. Is this true?
- If φ is valid, it is indeed satisfiable
- If φ is not valid, then...?

Normal Forms : DNF Satisfiability

Let $\varphi = D_1 \vee D_2 \vee \cdots \vee D_n$ be in DNF.

- Checking if φ is valid is NP-complete. Why?
- Checking if φ is satisfiable is polynomial time. Why?

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Assume you are given the truth table of a formula φ. Then it is very easy to obtain the equivalent CNF/DNF of φ.

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- Consider for example $\varphi = p \leftrightarrow q$.
- Truth table of φ : φ is false when $p = T, q = F$ and $p = F, q = T$.
- CNF equivalent is $(\neg p \vee q) \wedge (p \vee \neg q)$.

CNF to DNF Sizes

- $\varphi = (p_1 \vee q_1) \wedge (p_2 \vee q_2) \wedge \ldots (p_n \vee q_n)$
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- Prove that any equivalent DNF formula has 2^n clauses

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- What is the equivalent DNF formula?
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- Call an assignment *minimal* if it maps exactly one of p_i, q_i to 1

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- Prove that any equivalent DNF formula has 2^n clauses
- Call an assignment *minimal* if it maps exactly one of p_i, q_i to 1
- There are 2^n *minimal* assignments, satisfying clauses in φ'
- Show that no two *minimal* assignments satisfy the same clause of φ' (hence there must be 2^n clauses in φ')

CNF to DNF Sizes

- Let α and β be two minimal assignments such that $\alpha(p_i) \neq \beta(p_i)$ for $i \in \{1, \ldots, n\}$

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- $\mathsf{min}(\alpha, \beta)(p)$=$\mathsf{min}(\alpha(p), \beta(p))$ for each variable p with the assumption that $0 < 1$, 0 represents false and 1 represents true
- $\mathsf{min}(\alpha, \beta) \not\models p_i \vee q_i$, $\mathsf{min}(\alpha, \beta) \not\models \varphi'$
- However, if $\alpha \models D_j$ and $\beta \models D_j$ for some clause D_j of φ', then $\mathsf{min}(\alpha, \beta) \models D_j$ and hence $\mathsf{min}(\alpha, \beta) \models \varphi'$, a contradiction.

Think of an example where DNF to CNF explodes.