Sheet 11

Exercise 29

Proof (Lower Bound): Suppose C_n has a matching of size k. $\Rightarrow n-2k$ nodes are unmatched. These $n-2k$ nodes fall into k „buckets" between matched edges around C_n.

If a bucket contains more than one node, the matching is not maximal.
$\Rightarrow$ If $n-2k \geq k+1 \Rightarrow$ matching is not maximal
$\Rightarrow$ For a maximal matching it must hold: $n-2k \leq k \Rightarrow k \geq \frac{n}{3}$.
$\Rightarrow$ lower bound for the size of a maximal matching is $\left\lceil \frac{n}{3} \right\rceil$ $\square$

Exercise 30

Let $G=(V,E)$ be a bipartite graph.
Define $\big(G'=(V',E'),s,t,c\big)$ as follows:

$$V' = V_1 \cup V_2 \cup \{s,t\}$$

$$E' = \big\{(s,i) | i \in V_1\big\} \cup \big\{(j,t) | v \in V_2\big\} \cup E$$

$$c(u,v) = \begin{cases} \infty, & \text{if } u \in V_1, v \in V_2 \\ 1, & \text{otherwise} \end{cases}$$

Proof: Let H be a vertex cover of G.
Define

$$Q = \big\{(s,i) | i \in H \cap V_1\big\} \cup \big\{(i,t) | i \in H \cap V_2\big\}$$

Since H is a vertex cover, each directed path from s to t in G' contains one edge in Q.
$\Rightarrow Q$ is a valid cut of size $|H|$.

Let (S,T) be a cut of capacity k in G'.
Since edges $e \in E \setminus E'$ have a capacity of ∞, the cut (S,T) only contains edges from $E' \setminus E$. Define

$$H = \big\{i | (s,i) \in S \times T \cap E'\big\} \cup \big\{i | (t,i) \in S \times T \cap E'\big\}$$

Observe that each edge $(i,j) \in E \setminus E'$ defines a directed path $s \to i \to j \to t$ in G'.
$\Rightarrow (s,i) \in S \times T$ or $(j,t) \in S \times T$.

$\Rightarrow$ either $i \in H$ or $j \in H$.
$\Rightarrow$ H is a vertex cover of size k in G.

By the max-flow-min-cut Theorem it now holds that the maximum number of independent edges in G equals the minimum number of nodes in a vertex cover of G. $\square$