FIFO Preflow Push Algorithm

Proof of Theorem 17. Correctness: Theorem 15 ✓

Partition the examination of nodes into phases:

Phase 1: examination of nodes active after InitPreflow

Phase $i, i > 1$: examination of all nodes active („in Q“) when phase $i - 1$ is done.

Note: During each phase a node is examined at most once.

$n = |V|$:

Claim 1: the number of phases is $\leq 4n^2 + n$

Proof. potential function $\phi := \max\{h(u) \mid e(u) > 0\}$
Change of ϕ over an entire phase:

1st case: FIFO-PPA performs at least one Lift in the phase

$$h(u) \leq 2n - 1 \quad \forall u \in V$$

at most $2n - 1$ Lift operations per node. $\rightsquigarrow$ total increase of ϕ over all such phases is at most $(2n - 1) \cdot n < 2n^2$

2nd case: FIFO-PPA performs no Lift:
$\Rightarrow$ after examination of u : $e(u) = 0$
Push(u, v) : $h(u) = h(v) + 1$
$\Rightarrow$ ϕ decreases by at least 1.

Initial value of ϕ: $\leq n$

$\Rightarrow$ total number of phases is at most $\underbrace{2n^2}_{\text{with Lift}} + \underbrace{2n^2 + n}_{\text{without Lift}} = 4n^2 + n$

In each phase:

- at most one non-saturating push per node
- each node is considered at most once in a phase

$\Rightarrow$ # non-saturating pushes $\leq n(4n^2 + n) = \mathcal{O}(n^3)$
saturating pushes $\underset{\text{L18}}{=} \mathcal{O}(n \cdot m) = \mathcal{O}(n^3)$
Lift $\underset{\text{C6}}{=} \mathcal{O}(n^3)$

$\Rightarrow$ total runtime $\mathcal{O}(n^3)$ $\square$

$\square$