Laufzeit gPP-Algorithmus

Proof of Lemma 16. u overflowing node. $U := \left\{ v \in V | u \underset{G_f}{\rightsquigarrow} v \right\}$, $\bar{U} := V \setminus U$.

Claim 1: $\forall (v, w) \in U \times \bar{U}$ we have $f(w, v) \leq 0$.

$$f(w,v) > 0 \Rightarrow f(v,w) < 0 \Rightarrow c_f(v,w) = \underbrace{c(v,w)}_{\geq 0} - f(v,w) > 0$$

$$\Rightarrow (v,w) \in E_f \Rightarrow w \in U \lightning$$
$$f(\bar{U}, U) \leq 0 \Rightarrow e(U) := \sum_{u \in U} e(u) = f(V, U) \underset{\text{L6e}}{=} f(\bar{U}, U) + \underbrace{f(U,U)}_{\underset{\text{L6a}}{=} 0} = f(\bar{U}, U) \leq 0$$

f preflow: $e(u) \geq 0 \quad \forall u \in V \setminus \{s\}$

Then $s \notin U \Rightarrow e(v) = 0 \quad \forall v \in U \lightning (u \text{ overflowing})$ □

Proof of Lemma 17. $n = |V| : h(s) = n, h(t) = 0$ unchanged ✓

u overflowing $\underset{\text{Lemma 16}}{\Rightarrow} \exists P = (u = v_0, v_1, v_2, \ldots, v_k = s)$ in $G_f, k \leq n-1$ with

$$h(v_i) = h(v_{i+1}) + 1 \quad \forall i \in \{0, \ldots, k-1\}$$
$$h(u) = h(v_0) \leq h(v_k) + k \leq \underbrace{h(s)}_{=n} + k = n + k \leq 2n - 1.$$

□

Proof of Corollary 6. $\textsc{Lift}(u)$ increases $h(u)$ by at least 1 (Lemma 14) and $h(u) \leq 2n-1$ (Lemma 17). s, t are never lifted. □

Proof of Lemma 18. Let $(u, v) \in E$. As in the proof of Lemma 11 we get: Between any two successive saturating pushes from u to v, $h(u)$ and $h(v)$ increase by at least 2.

$$(u,v) \in E : A = (a_k)_{k=1,2,\ldots} \quad a_k := h(u) + h(v)$$

when the k-th saturating push between u and v occurs.

$$a_1 \geq 1, a_k \underset{\text{L17}}{\leq} 2n - 1 + 2n - 2 = 4n - 3$$

$$a_k \text{ is odd}, a_i \neq a_j \ i \neq j \Rightarrow k \leq \left\lfloor \frac{4n-3}{2} + 1 \right\rfloor = 2n - 1$$

□

Proof of Lemma 19. $n = |V|, m = |E|, X = \{u \in V | e(u) > 0\}$ overflowing nodes.

Potential function $\phi = \sum_{u \in X} f(u)$

$\textsc{InitPreflow}()$: $\phi = 0$

$\textsc{Lift}(\mathrm{u})$: X is unchanged, $h(u) \underset{\text{L14}}{\leq} 2n - 1 \Rightarrow \phi$ increases by at most $2n - 1$.

$\underset{\text{saturating}}{\textsc{Push}(u, v)}$: h is unchanged, v may become overflowing, u may stay overflowing. $h(v) \leq 2n - 1 \Rightarrow \phi$ increases by at most $2n - 1$.

$\underset{\text{unsaturating}}{\textsc{Push}(u, v)}$: h is unchanged, u is removed from X, v may be added to X. $\Rightarrow \phi$ decreases by 1.

Cor. 6, L18 $\Rightarrow$ total increase of ϕ is at most

$$(2n-1) \cdot \underbrace{2n^2}_{\#\textsc{Lift}} + (2n-1) \underbrace{2n \cdot m}_{\#\text{ sat }\textsc{Push}} \leq 4n^2(n+m) \Rightarrow \#\text{ of non. sat. pushes is } \leq 4n^2(n+m)$$

$\square$