Proof of Note 7

1. $f(u,v) = -f(v,u) \Rightarrow f(u,u) = 0$
2. $\sum_{u \in V} f(u,v) = \sum_{u \in V} f(v,u) = 0$
3. $(u,v),(v,u) \notin E \Rightarrow c(u,v) = 0 = c(v,u)$
   $\Rightarrow f(u,v) \leq c(u,v) = 0, \quad f(v,u) \leq 0 \Rightarrow d)$

Proof of Lemma 6

a)

$$\begin{aligned} 2 \cdot f(X,X) &= \sum_{x \in X} \sum_{y \in X} f(x,y) + \sum_{y \in X} \sum_{x \in X} f(y,x) \\ &= \sum_{x \in X} \sum_{y \in X} f(x,y) + f(y,x) = 0 \end{aligned}$$

b)

$$f(X,Y) = \sum_{x \in X} \sum_{y \in Y} f(x,y)$$

$$= \sum_{x \in X} \sum_{y \in Y} -f(y,x) = -\sum_{y \in Y} \sum_{x \in X} f(y,x) = -f(Y,X)$$

c)

$$f(X \cup Y, Z) = \sum_{x \in X} \sum_{z \in Z} f(x,z) + \sum_{y \in Y} \sum_{z \in Z} f(y,z) = f(X,Z) + f(Y,Z)$$

Proof of Lemma 7

1. Capacity constraints:

$$\begin{aligned} (f + f')(u,v) = f(u,v) + f'(u,v) &\leq f(u,v) + c_f(u,v) \\ &\leq f(u,v) + c(u,v) - f(u,v) = c(u,v) \end{aligned}$$

2. Skew symmetrie:

$$(f + f')(u,v) = f(u,v) + f'(u,v) = -f(v,u) - f'(v,u) = -(f + f')(v,u)$$

3. Flow conservation:

$$\forall u \in V - \{s,t\} \text{ we have}$$

$$\sum_{v\in V}(f+f')(u,v) = \sum_{v\in V} f(u,v) + \sum_{v\in V} f'(u,v) = 0$$

4.

$$|f+f'| = \sum_{v\in V}(f+f')(v,t)$$

$$= \sum_{v\in V} f(v,t) + \sum_{v\in V} f'(v,t) = |f| + |f'|$$

Proof of Lemma 8

1. Capacity constraint:

$$(u,v) \in P : f_P(u,v) = c_f(P) = \min\{c_f(u,v) | (u,v) \in P\} \leq (u,v)$$

$$(u,v) \notin P : f_P(u,v) \leq 0 \leq c(u,v) - f(u,v) = c_f(u,v)$$

2. Skew symmetrie: clear

3. Flow cons:

$$u \notin P : \sum_{v\in V} f_P(u,v) = 0 \qquad\qquad u \in P : s \rightsquigarrow x \to u \to y \rightsquigarrow t$$

$$\sum_{v\in V} f_P(u,v) = f_P(u,x) + f_P(u,y) \underset{\text{skew}}{=} f_P(u,y) - f_P(x,u) = c_f(P) - c_f(P) = 0$$