Proof of Lemma 10. d valid: $\quad d(u) \leq d(v) + 1 \quad \forall (u,v) \in E_f$.
(by induction on the number of augment + retreat operations.)

(IB) $d(u) = \delta_G(u,t) \underset{f \equiv 0}{=} \delta_{G_f}(u,t)$ implies $d(u) \leq d(v) + 1 \quad \forall (u,v) \in E_f$.

(IS)

a) augmentation along $(u,v) \in E_f$.

- remove (u,v) from E_f
- create an additional edge $(v,u) \in E_f$ and thus an additional inequality $d(v) \leq d(u) + 1$
  We know: $d(u) = d(v) + 1$ because (u,v) is admissible. Thus $d(v) = d(u) - 1 < d(u) + 1$.

b) retreat from u to $\pi(u)$ modifies $d(u)$. Let $d'(u) = \min\{d(v) + 1 | (u,v) \in E_f\}$
Thus, after retreat from $u \quad d'(u) \leq d(v) + 1 \quad \forall (u,v) \in E_f$.
The alg retreats from u, when u has no admissible edges $(u,v) \in E_f$.

By (IH) we have $d(u) < d(v) + 1 \quad \forall (u,v) \in E_f$ and $d'(u) > d(u)$.
From this $d(w) \leq d'(u) + 1 \quad \forall (w,u) \in E_f$.

■

Proof of Theorem 13. The algorithm terminates with $d(s) \geq |V|$. By Note 10b there is no path from s to t in G_f. By Theorem 12 f is a max flow. ■

Proof of NM. The algorithm performs a retreat at node u only after it passed through the complete list $N(u)$. The relabelling passes through $N(u)$ a second time, resulting in a total time of $\mathcal{O}(|N(u)|)$ for each retreat. ■

Proof of Lemma 11. We show:
(*) Between two consecutive saturations of an edge (u,v) both $d(u)$ and $d(v)$ must increase by at least 2 units.

$$\overset{d(u)}{u} \longrightarrow \overset{d(v)+1}{v} \rightsquigarrow u \leftarrow v$$

now v must push flow to u:

$$\overset{d(u)+1}{u} \longleftarrow \overset{d'(v)}{v}$$

i.e. $d(v)$ increased by 2

$$\overset{d'(u)=d(u)+2}{u} \longleftarrow \overset{d'(v)=d(u)+1}{v}$$ ■

If the algorithm retreats from any node u at most k times, we can conclude that any edge can be saturated at most $\frac{k}{\cancel{2}} \rightsquigarrow k$ times. $\rightsquigarrow$ at most $k|E|$ saturations.

Sketch of Proof of Lemma 12. Each update (retreat from u) of $d(u)$ increases $d(u)$ by at least 1. Nodes u with $d(u) \geq |V|$ are never considered again. $\rightsquigarrow$ a)

b) L11 and L12a) $\Rightarrow$ algorithm saturates at most $\frac{|V|\cdot|E|}{\cancel{2}}$ edges. Since each augmentation saturates at least 1 edge:
\# augmentations $\leq |V| \cdot |E|$ ■

Proof of Theorem 14. $n = |V|, m = |E|$ L12, NM: the total effort to find admissable edges and to update the distance labels is $\mathcal{O}(nm)$. By L12:
Number of augmentations $\leq nm$, each of which takes $\mathcal{O}(n)$ time. So the total effort for augmentation is $\mathcal{O}(n^2 m)$. The total number of retreats is $\mathcal{O}(n^2)$. Each advance adds an edge to P, each retreat deletes an edge from P. Since P has at most n edges, the alg. requires at most $\mathcal{O}(n^2 + n^2 \cdot m)$ advance operations. ■