Proof of Lemma 9

Let (S,T) be a cut of G. Then

$$f(S,T) \underset{\text{L6c}}{=} f(S,V) - f(S,S) \underset{\text{L6c)}}{=} f(s,V) + \underbrace{f(S-s,V)}_{=0,\text{ Def 16a) flow cons}} = f(s,V) \underset{\text{Def 16b}}{=} |f|.$$

Proof of Corollary 4

Let f be any flow, let (S,T) be any cut. Then

$$|f| \underset{\text{L9}}{=} f(S,T) = \sum_{u\in S}\sum_{v\in T} \underbrace{f(u,v)}_{\leq c(u,v)} \underset{\text{cap constr}}{\leq} \sum_{u\in S}\sum_{v\in T} c(u,v) = c(S,T)$$

Proof of Theorem 12

We show a) $\Rightarrow$ b) $\Rightarrow$ c) $\Rightarrow$ a)
„a) $\Rightarrow$ b)“
Suppose t.t.c. that f is maximum flow, but G_f has augmenting path P. Then, by corollary 3, $f + f_P$, with

$$f_P(u,v) = \begin{cases} c_f(P), & \text{if } (u,v) \in P \\ -c_f(P), & \text{if } (v,u) \in P \\ 0, \text{otherwise} \end{cases}$$

is a flow in G with $|f + f_P| > |f|$. Contradiction!
„b) $\Rightarrow$ c)“
Let f be such that G_f does not contain any augmenting path. Define

$$S := \{v \in V | s \underset{G_f}{\rightsquigarrow} v\}, \qquad\qquad T := V - S$$

Then $s \in S, t \in T$ and $S \cup T = V, S \cap T = \emptyset,$ so (S,T) is a cut.
For each $(u,v) \in S \times T$ we have $f(u,v) = c(u,v)$ because $c_f(u,v) = c(u,v) - f(u,v) = 0$

$$\overbrace{s \rightsquigarrow u}^{S} \not\rightarrow \overbrace{v \quad t}^{T}$$

By Lemma 9 $|f| = f(S,T) = c(S,T)$.
„c) $\Rightarrow$ a)“
By corollary 4 $|f| \leq c(S,T)$ for all cuts (S,T) in G.
So $|f| = c(S,T)$ implies that f is maximum.

Proof of corollary 5

FordFulkerson starts with an admissible flow $f \equiv 0$.

As long as an augmenting path P exists in G_f we have that the function adds a flow f_P to f, resulting in an admissible flow $f + f_P$, with $|f + f_P| > |f|$. (Cor 3)

After the while-loop G_f does not contain an augmenting path and therefore f is maximum (Theorem 12). In each iteration of the while-loop the value of $|f|$ is increased by at least 1 (c is integral!). The computation of an augmenting path can be done in $\mathcal{O}(|E|)$.
$\rightsquigarrow$ Total $\mathcal{O}(|E| \cdot |f|)$.

Proof of Note 10

a) Let $u \in V$, $P = (u = u_0, u_1, u_2, \ldots, u_k = t)$ be a shortest path from u to t in G_f. Then $\delta_{G_f}(u, t) = k$ and

$$\begin{aligned} d(u_k) &= 0 = \delta_{G_f}(u_k, t) \\ d(u_{k-1}) &\leq d(u_k) + 1 = 1 \\ d(u_{k-2}) &\leq d(u_{k-1}) + 1 \leq 2 \\ &\vdots \\ d(u) = d(u_0) &\leq d(u_1) + 1 \leq k = \delta_{G_f}(u, t) \end{aligned}$$

b) By a) $\delta_{G_f}(s, t) \geq d(s) \geq |V|$, and thus no simple path from s to t in G_f exists.

c) Let P be n admissible path in G_f from s to t. Then P is an augmenting path ($c_f(u, v) > 0!$)). Since $d(u) = d(v) + 1 \ \forall (u, v) \in P$ we have that $d(s) =$ number of edges of P. Since $d(s) \leq \delta_{G_f}(s, t)$, P must be a shortest path.