Preflow Algorithms

(Sketch) Proof of Note 14. capacity constraints: $d_f(u,v) \le c_f(u,v) \Rightarrow f(u,v) + d_f(u,v) \le c(u,v)$

skew symm.: clear

flow conservation relaxation: $d_f(u,v) \le e(u) \Rightarrow e(u) - d_f(u,v) \ge 0$ ∎

Proof of Note 15. Let $e(u) = f(V,u) > 0 \Rightarrow \exists v \in V$ so that $f(u,v) > 0$.

$$\Rightarrow c_f(u,v) = c(u,v) - f(u,v) \underset{\text{skew symm}}{=} \underbrace{c(u,v)}_{\ge 0} + \underbrace{f(v,u)}_{>0} > 0 \Rightarrow (u,v) \in E_f$$

∎

Proof of Note 17. f preflow ✓, e is excess flow ✓

$$h(u) = \begin{Bmatrix} |V|, & \text{if } u = s \\ 0, & \text{otherwise} \end{Bmatrix} \Rightarrow h(u) > h(v) + 1 \text{ only if } u = s$$

But $c_f(s,v) = c(s,v) - \underbrace{f(s,v)}_{=c(s,v)} = 0 \Rightarrow (s,v) \notin E_f$. ∎

Proof of Lemma 13. By def of a height function we have $h(u) \le h(v) + 1 \forall (u,v) \in E_f$.

Suppose we can not apply push to node u. Then, for all $(u,v) \in E_f$ (there is always such an edge by Note 15) we must have $h(u) < h(v) + 1$.

$$\Rightarrow h(u) \le h(v) \Rightarrow \text{ Lift can be applied to } u$$

∎

Proof of Lemma 14. (Proof: see proof of lemma 10) ∎

Proof of Lemma 15. Suppose to the contrary that there is a path from s to t in G_f : $(s = v_0, v_1, \ldots, v_k = t) = P$. Wlog P is simple and thus $k < |V|$.

Because h is a height function $h(v_i) \le h(u_{i-1}) + 1 \ \forall i \in \{0, \ldots, k-1\}$.

But then $|V| = h(s) \le \underbrace{h(t)}_{=0} + k = k < |V|$ ↯ ∎

Proof of Theorem 15. If gPP terminates, then neither LIFT nor PUSH is applicable.

By L13 $e(u) = 0\ \forall u \in V \setminus \{s, t\}$. By L14 h is a height function, by Notes 17 and 14 f is a preflow. Therefore f is a flow.

Because h is a height function , by L15 there is no path from s to t in G_f. By Theorem 12, f is a maximum flow. ■