# Convex Polygon Midpoint Iteration

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## Polygon Iteration:

Polygons are among the first mathematical objects of interest. We call a polygon with n-vertices an n-gon. We can also specify it to be convex, meaning the curvature is only in one direction. In this paper we examine an iterative process that involves convex n-gons. Given an initial convex n-gon, $\mathcal{P}_n^0$, with vertices given by coordinate pairs, $(x_i, y_i)$, we create a new polygon by forming a new vertex at the midpoint of each side of the previous polygon. Each vertex is formed at the midpoint of a side, and then connected to the other new vetices to form an inscribed polygon. If we let this process of midpoint polygon iteration continue forever, where would it converge to?

## Hypothesis:

To examine this question, we wrote a polygon plotting program with an iterator using Maple. Using the program, we have observed that the iteration of polygons seems to converge to the center of mass of the original polygon. The center of mass of an n-gon can be calculated by treating each vertex as a vector, and then finding the vector average.

We hypothesize that the infinite iteration of midpoint polygons will converge to the center of mass of the original polygon. Specifically, for initial convex n-gon;

$$\mathcal{P}^0 = \{[x_i, y_i]\}_{i=1}^n$$

and the iteration process generally defined from $[x_i, y_i] \in \mathcal{P}^k$ as;

$$\mathcal{P}^{k+1} = \left\{ \left[ \frac{x_i + x_{i+1}}{2}, \frac{y_i + y_{i+1}}{2} \right] \right\}_{i=1}^{n-1} \bigcup \left\{ \left[ \frac{x_1 + x_n}{2}, \frac{y_1 + y_n}{2} \right] \right\}$$

Then we hypothesize;

$$\lim_{k \to \infty} \mathcal{P}^k = \left[ \frac{1}{n} \sum_{i=1}^n x_i, \frac{1}{n} \sum_{i=1}^n y_i \right]$$

where the $[x_i, y_i]$ in the limit are from $\mathcal{P}^0$.

## Analysis:

To examine the iteration of midpoint polygons, we use a matrix representation of the transformation. Here we will only explicitly prove our hypothesis for the case of a convex 3-gon; however, while doing so we will also set up some tools and propose conjectures needed for proving the more general case.

### The Matrix:

For a convex 3-gon, the matrix transformation to the next midpoint polygon is;

$$A = \frac{1}{2} \cdot \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1/2 & 1/2 & 0 \\ 0 & 1/2 & 1/2 \\ 1/2 & 0 & 1/2 \end{pmatrix}$$

We can apply this transformation to each vector of coordinates for the polygon. In other words, we have a vector of all the $x$-coordinates, and $A \cdot x$ gives us the vector of x-coordinates for the next polygon. We do this to both the $x$ and $y$ vectors and then reunite them to form the full polygon. We can see that this matrix creates the same operation as defined above, for example;

$$A \cdot x = \begin{pmatrix} 1/2 & 1/2 & 0 \\ 0 & 1/2 & 1/2 \\ 1/2 & 0 & 1/2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} \frac{x_1}{2} + \frac{x_2}{2} \\ \frac{x_2}{2} + \frac{x_3}{2} \\ \frac{x_1}{2} + \frac{x_3}{2} \end{pmatrix}$$

While we will only prove the 3-gon case, we must note here that the higher vertex cases have a matrix with the same structure. For example, the transition matrix for a 5-gon is;

$$\begin{pmatrix} 1/2 & 1/2 & 0 & 0 & 0 \\ 0 & 1/2 & 1/2 & 0 & 0 \\ 0 & 0 & 1/2 & 1/2 & 0 \\ 0 & 0 & 0 & 1/2 & 1/2 \\ 1/2 & 0 & 0 & 0 & 1/2 \end{pmatrix}$$

One important feature to note, which will be used to make simplifications and arguments about the higher cases, is that each row adds to 1. This can be seen in the case of the 3-gon transition matrix, as well as with the 5-gon matrix. It is easy to see that increasing the dimension of the matrix by one, results in an additional row which also sums to 1. Thus this property holds for all midpoint polygon transition matrices. This is an important property to note because it guarantees that $\lambda = 1$ is an eigenvalue with the associated eigenvector $\vec{v} = \begin{pmatrix} 1 & 1 & 1 & ... & 1 \end{pmatrix}$. One can confirm this by multiplying and using the definition. We will discuss the importance of eigenvalue one later.

**3-gon Eigenvalues and Vectors:**

We will now compute the other eigenvalues and vectors for our 3-gon transition matrix $A$. We already know $\lambda_1 = 1$ is an eigenvalue with the one eigenvector. We have two other eigenvectors to find. We note that;

$$\text{Trace}(A) = \frac{1}{2} + \frac{1}{2} + \frac{1}{2} = \frac{3}{2} = \lambda_1 + \lambda_2 + \lambda_3 \Rightarrow \lambda_2 + \lambda_3 = \frac{1}{2}$$

We can also use the fact that the determinant equals the product sum of eigenvalues;

$$\text{Det}(A) = \frac{1}{4} = \lambda_2 \cdot \lambda_3 \Rightarrow \lambda_2 = \frac{1}{4\lambda_3}$$

Substituting the determinant equation into the trace equation, (and simplifying) we have;

$$\frac{1}{2} = \frac{1}{4\lambda_3} + \lambda_3$$

$$\Rightarrow 4\lambda_3 - 2\lambda_3 + 1 = 0 \Rightarrow \lambda_3 = \frac{1}{4} \pm \frac{i\sqrt{3}}{4}$$

The eigenvalue we have found is complex, so we have both of the remaining eigenvalues by the complex conjugate eigenvalue theorem. The computations for the eigenvectors are lengthy, and are omitted here. They may be found be solving $(A - \lambda_2 I)v = 0$, during which we choose the free variable in the third column to be 1. Using Maple we end up with;

$$v_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \qquad v_2 = \begin{pmatrix} \frac{4}{(i\sqrt{3}-1)^2} \\ \frac{2}{i\sqrt{3}-1} \\ 1 \end{pmatrix} \qquad v_3 = \begin{pmatrix} \frac{4}{(i\sqrt{3}+1)^2} \\ -\frac{2}{i\sqrt{3}+1} \\ 1 \end{pmatrix}$$

We can now write the transition matrix in a diagonalized form.

Our transition matrix $A$ is not in a good form to prove limiting behavior. To do this, we form a matrix, $Q$, of eigenvectors, and $\Lambda$ of eigenvalues, so that $A = Q\Lambda Q^{-1}$. We have;

$$A = \begin{pmatrix} 1 & \frac{4}{(i\sqrt{3}-1)^2} & \frac{4}{(i\sqrt{3}+1)^2} \\ 1 & \frac{2}{i\sqrt{3}-1} & -\frac{2}{i\sqrt{3}+1} \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{4} + \frac{i\sqrt{3}}{4} & 0 \\ 0 & 0 & \frac{1}{4} - \frac{i\sqrt{3}}{4} \end{pmatrix} Q^{-1}$$

Now note by design, each application of $A = Q\Lambda Q^{-1}$ leads to the inner matrices neutralizing to the identity matrix. So each application of the transition matrix is equivalent to adding another power to the diagonal matrix $\Lambda$. For example,

$$A^3 = (Q\Lambda Q^{-1})^3 = Q\Lambda Q^{-1} Q\Lambda Q^{-1} Q\Lambda Q^{-1} = Q\Lambda^3 Q^{-1}$$

We can now use this to examine the limiting behavior of polygon iteration process. We have;

$$\lim_{k\to\infty} \mathcal{P}^k \cong \lim_{k\to\infty} A^k = \lim_{k\to\infty} Q\Lambda^k Q^{-1} = \lim_{k\to\infty} Q \begin{pmatrix} 1^k & 0 & 0 \\ 0 & (\frac{1}{4} + \frac{i\sqrt{3}}{4})^k & 0 \\ 0 & 0 & (\frac{1}{4} - \frac{i\sqrt{3}}{4})^k \end{pmatrix} Q^{-1}$$

Now we note the first entry on the diagonal is $1^k$, which immediately converges to 1. For the complex eigenvalues, we consider their polar form. The imaginary part of the values represents rotation, which we can ignore. Each power of $k$ will apply a small rotation, but more importantly will apply an addition power to the modulus, $r_i$. In this case we have;

$$r_1 = \sqrt{\left(\frac{1}{4}\right)^2 + \left(\frac{\sqrt{3}}{4}\right)^2} = \sqrt{\frac{1}{4}} = \frac{1}{2} < 1$$

This is also the modulus for $r_2$. We note that the modulus is bounded by zero and one, so each application of $k$ reduces the modulus, and we have $\lambda_2^k \to 0$ and $\lambda_3^k \to 0$. This means that;

$$\lim_{k\to\infty} Q \begin{pmatrix} 1^k & 0 & 0 \\ 0 & (\frac{1}{4} + \frac{i\sqrt{3}}{4})^k & 0 \\ 0 & 0 & (\frac{1}{4} - \frac{i\sqrt{3}}{4})^k \end{pmatrix} Q^{-1} = Q \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} Q^{-1}$$

There is an important generalization here; our limiting diagonalized matrix has one a single one. This one originates from the eigenvalue $\lambda_1 = 1$, and when we multiply out the first two matrices, the eigenvalue preserves its eigenvector;

$$Q \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

We have noted previously that every transition matrix for our midpoint polygon, regardless of the number of vertices, will have eigenvalue one with the eigenvector of all ones as well. So the column of ones is something that can be used in a more general proof; However we require the remaining eigenvalues to converge to zero. We discuss this in the concluding paragraph.

Our simplified matrix above reduces the number of calculations we must do. The matrix consisting of a column of ones in the first column, has the effect of taking the top row of a matrix and replicating it across a new matrix, each row identical to the top row of the input matrix. For us, this means that we are only concerned with the top row of $Q^{-1}$; and in general, we would only need to compute the top row of $Q^{-1}$ for any n-gon we apply this to.

Using maple we see that the top row of $Q^{-1}$ is $\begin{pmatrix} 1/3 & 1/3 & 1/3 \end{pmatrix}$; this makes our limiting matrix;

$$Q\begin{pmatrix}1 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}Q^{-1} = \begin{pmatrix}1 & 0 & 0\\ 1 & 0 & 0\\ 1 & 0 & 0\end{pmatrix}Q^{-1} = \begin{pmatrix}1/3 & 1/3 & 1/3\\ 1/3 & 1/3 & 1/3\\ 1/3 & 1/3 & 1/3\end{pmatrix}$$

For our initial convex 3-gon, the limiting polygon would be given by applying this matrix to the vector of $x$ coordinates and $y$ coordinates separately. We would receive a vector with all entries being the same, so we reduce it down;

$$\lim_{k\to\infty} \mathcal{P}^k = \left[\frac{x_1 + x_2 + x_3}{3}, \frac{y_1 + y_2 + y_3}{3}\right]$$

So our hypothesis holds for the case of a 3-gon.

**Following Remarks:**

We note that there are two key points of interest in this proof which could potentially be used to generalize it.

The first point is that the transition matrix for any n-gon will always have eigenvalue one; however to take advantage of this we also require that the remaining eigenvalues be zero-convergent. Using Maple, we have inspected several cases, and there is an apparent pattern in both the eigenvalues and eigenvectors. Notably, the remaining eigenvalues are all zero-convergent. Proving that all the remaining eigenvalues are zero-convergent would be a key part of the generalized proof for the convergence of any convex n-gon. We did not inspect this issue further.

The second point of interest is that the first row of $Q^{-1}$ miraculously consisted of only the element $1/3$; which also happened to be the reciprocal of the number of vertices we had. Inspecting with Maple, we found this to be consistent across all cases examined. This also lines up with visual inspections of convergence (and the rest of the proof). So we can hypothesize that the top row of $Q^{-1}$ will always consist of $1/n$, repeating for each element.

Our paper shows that a convex 3-gon will converge to the center of mass of the original polygon. Our iterations and simulations on Maple also suggest that this is the case for all convex n-gons. The proof structure from the 3-gon suggests a way to prove a more general case of any n-gon.