Statistics for Computing

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MSc in Computer Science

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RANDOM VARIABLES AND PROBABILITY DISTRIBUTIONS

CHAPTER 03

Contents

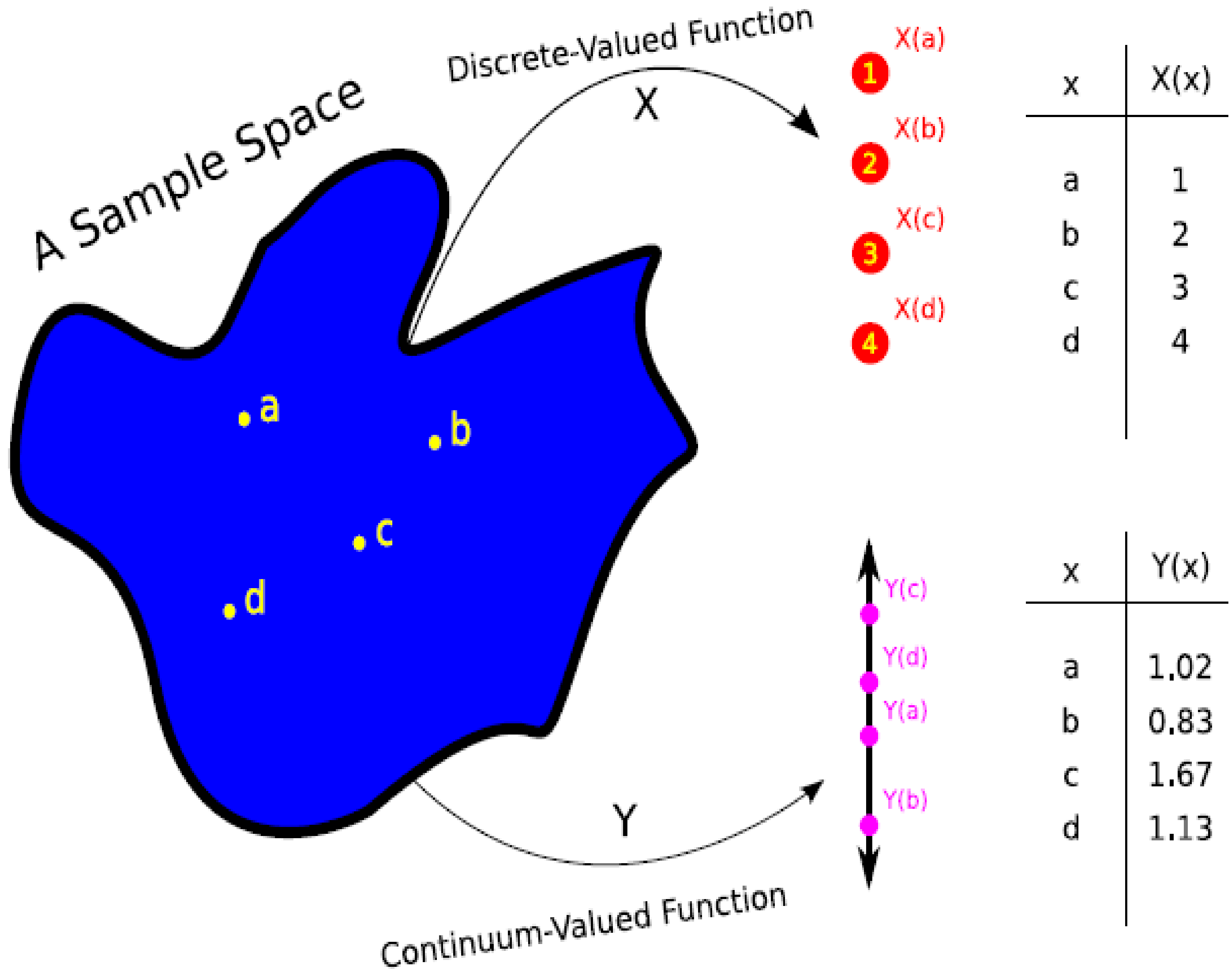
- Random Variables
- Probability Distributions
- Examples

Random Variable

- A *Random Variable* is a function, which assigns unique numerical values to all possible outcomes of a random experiment under fixed conditions.

- A ***random variable***, usually written *X*, is a variable whose possible values are numerical outcomes of a random phenomenon.

- A Random Variable is;
  - Discrete if it has either finite or countably infinite values.
  - Continuous if it takes values in a continuum.



A Sample Space
a
b
c
d
Discrete-Valued Function
X
X(a)
1
X(b)
2
X(c)
3
X(d)
4
x X(x)
a 1
b 2
c 3
d 4
Y
Continuum-Valued Function
Y(c)
Y(d)
Y(a)
Y(b)
x Y(x)
a 1.02
b 0.83
c 1.67
d 1.13

Random variable: Mathematical definition

For a given probability space $(\Omega, \varepsilon, P())$ a random variable, denoted by X or X( ), is a function with Ω domain and co-domain the real line.

Example :

Consider the experiment of tossing a single coin. Let the random variable X denoted the number of heads.

$$\boxed{\Omega = \{Head, Tail\} \Rightarrow \; Domain}$$

$$X = \begin{cases} 1 & X = head \\ 0 & X = tail \end{cases} \qquad (Co - domain)$$

Example :

Suppose that a coin is tossed three times and the sequence of heads and tails is noted.

The sample space for this experiment evaluates to:

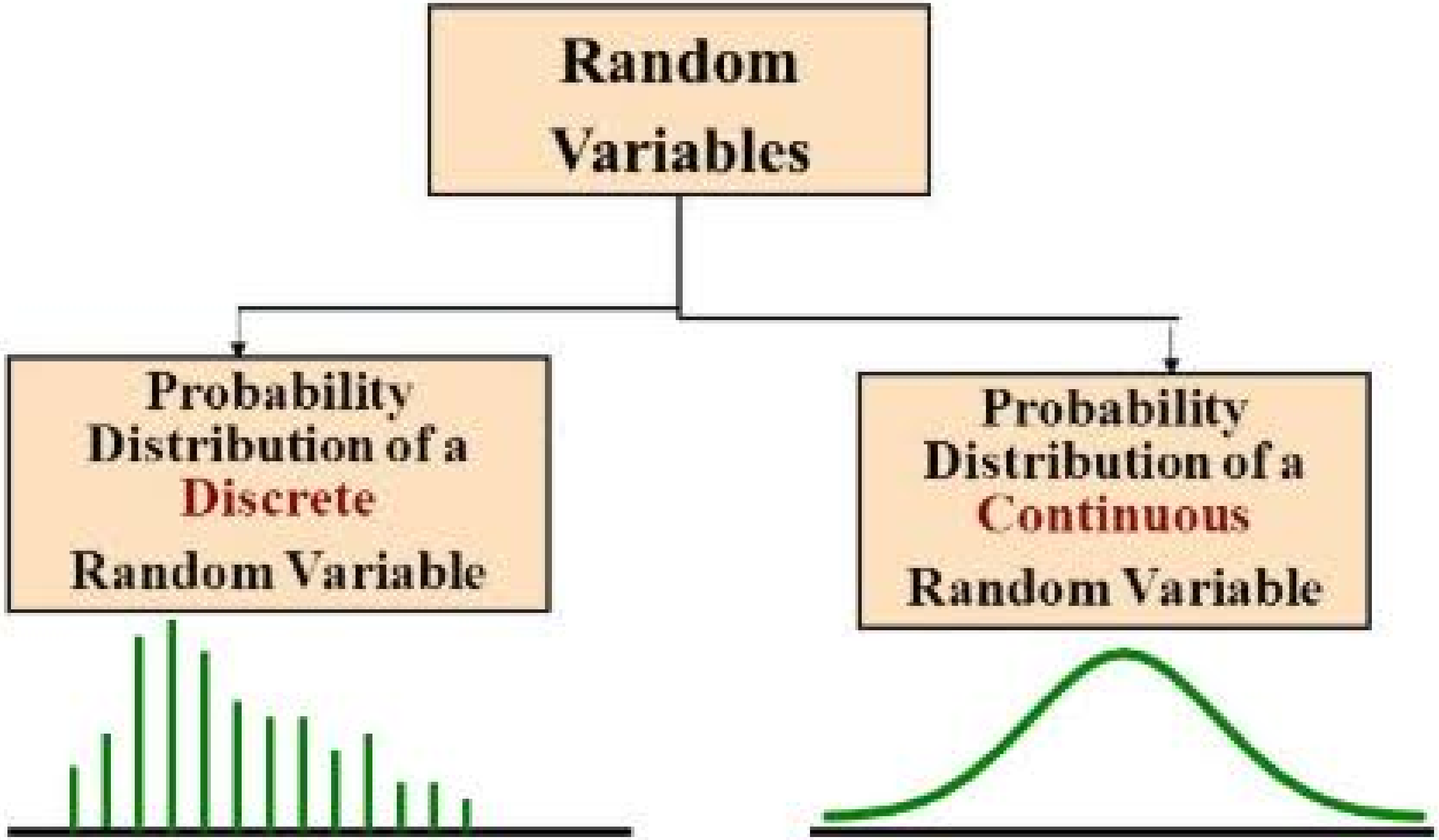
S={HHH, HHT, HTH, HTT, THH, THT, TTH, TTT}.

- X is then a random variable taking on values in the set X = {0, 1, 2, 3}.

Example :

- X is then a random variable taking on values in the set X = {0, 1, 2, 3}.

OUT Come	HHH	HHT	HTH	THH	HTT	THT	TTH	TTT
X	3	2	2	2	1	1	1	0



Random Variables
Probability Distribution of a Discrete Random Variable
Probability Distribution of a Continuous Random Variable

Discrete Random Variable

- A random variable is called a discrete random variable if its set of possible outcomes is countable.

Example:

- Flip a coin and count the number of heads.
  - ✓ Number of heads is represented by an *integer* value - a number between 0 and plus infinity.
  - ✓ Therefore, the number of heads is a discrete random variable.
- Number of calls per a minute in a phone exchange.

$$X=\{0,1,2,3,4,5,\ldots..\}$$

Continuous Variable :

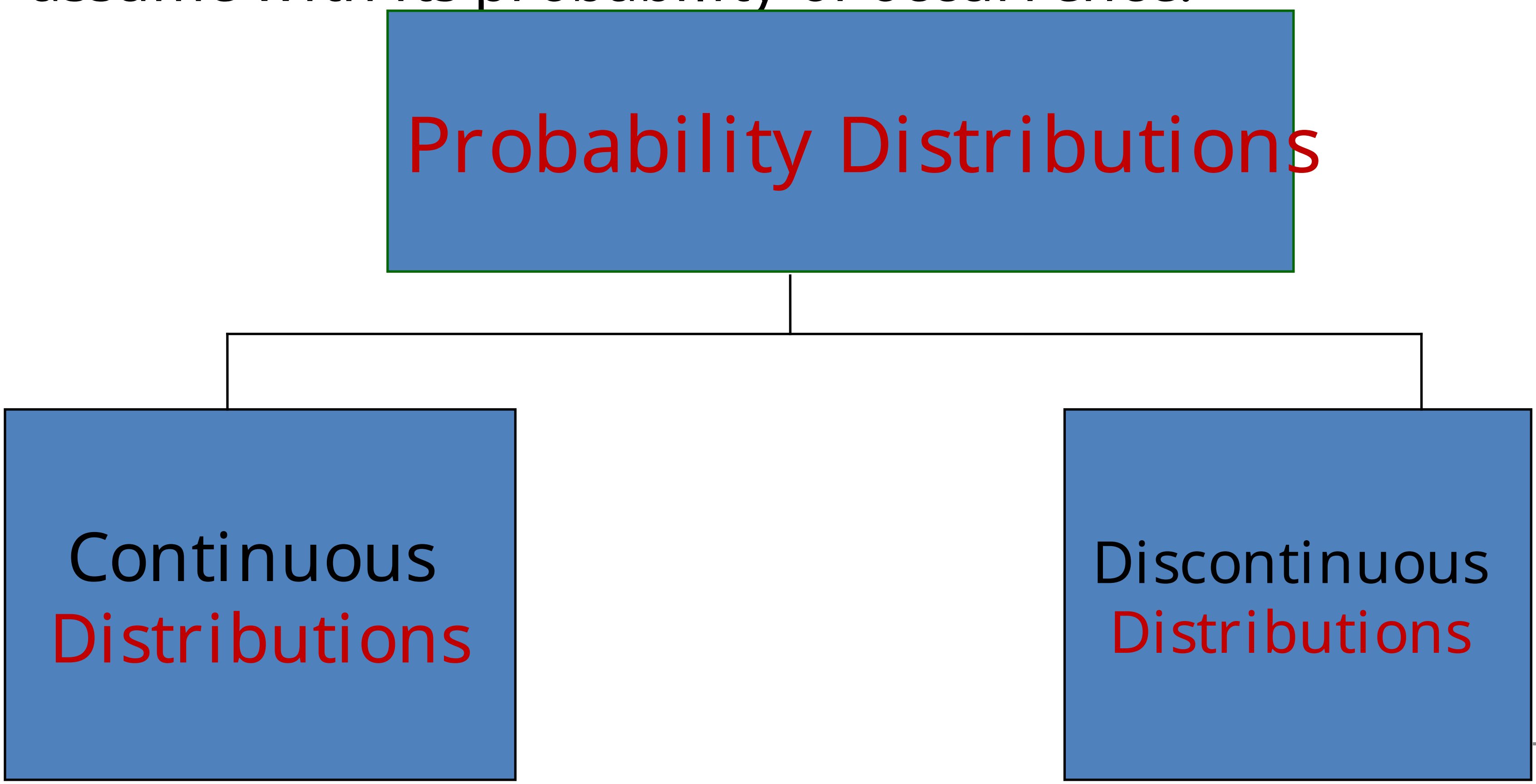
- Continuous random variables, in contrast, can take on any value within a range of values.

Example:

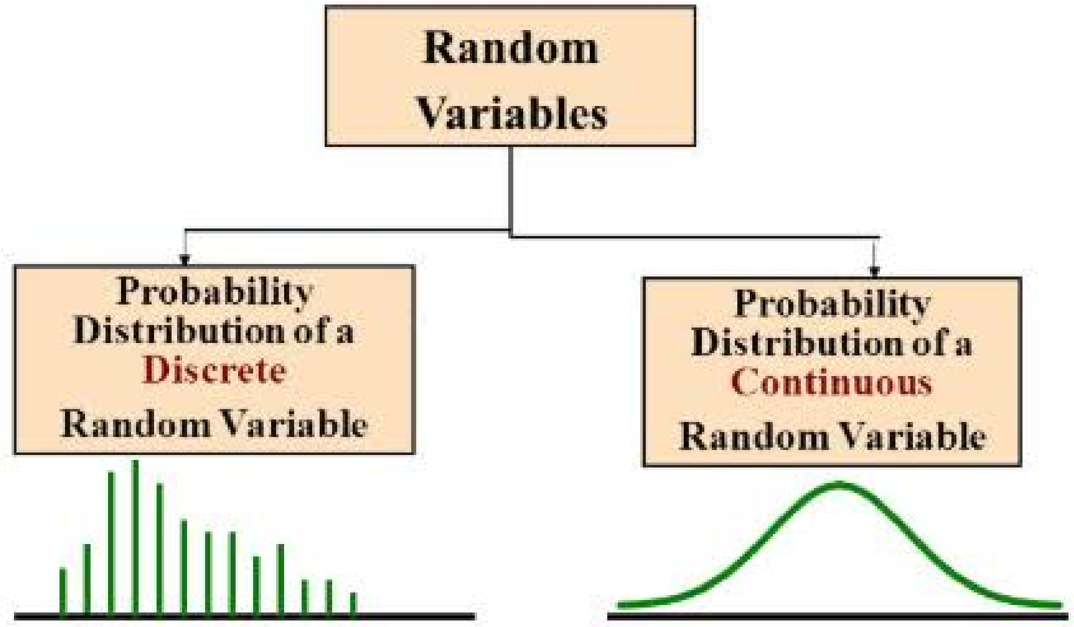
- ✓ height of students in class
- ✓ weight of students in class
- ✓ time it takes to get to school
- ✓ Distance traveled between classes

Probability Distributions

- A **probability distribution** is a table or an equation that links each possible value that a random variable can assume with its probability of occurrence.



Properties of probability density function



i. $$0 \leq P(X = x_i) = p_i \leq 1$$

ii. $$\sum_{i=1}^{n} P(X = x_i) = \sum_{i=1}^{n} p_i = 1$$

Discrete Probability Distributions

- The probability distribution of a discrete random variable can always be represented by a table.

Example:
suppose you flip a coin two times. This simple exercise can have four possible outcomes:

HH, HT, TH, and TT.

- variable X - number of heads that result from the coin flips.

Discrete Probability Distributions

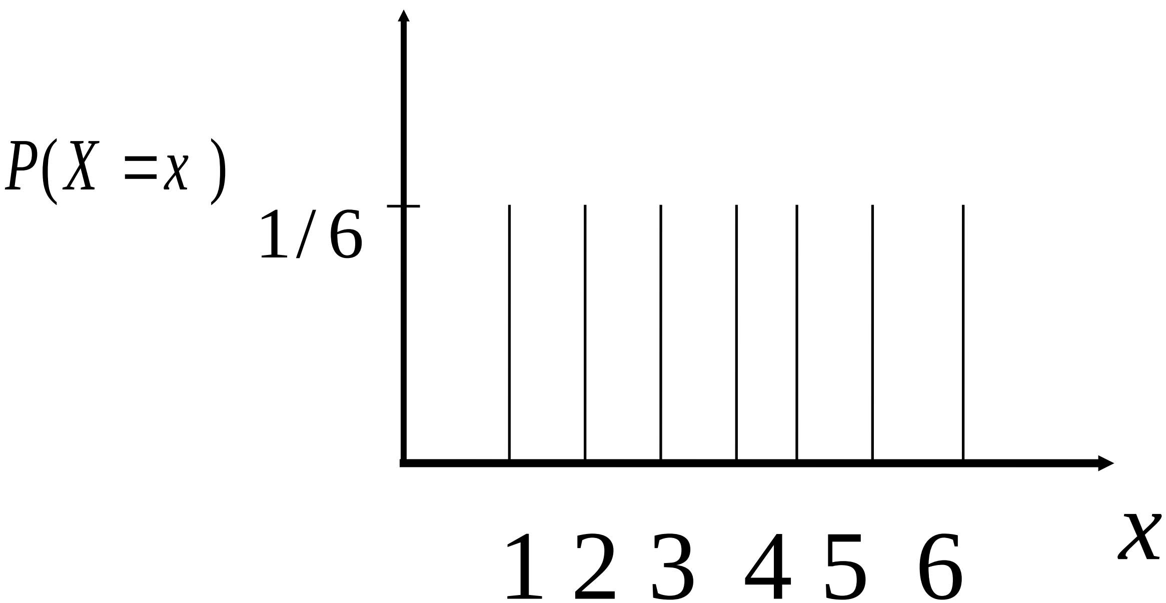
- The variable X can take on the values 0, 1, or 2;

Number of heads	Probability, P(X=x)
0 1 2	0.25 0.5 0.25

Sketch the probability distribution

Consider the experiment of tossing a die. Let X denote the value appears on the upper face.

X	1 2 3 4 5 6
P(X=x)	1/6 1/6 1/6 1/6 1/6 1/6



Consider the experiment of tossing a die. Let X denote the value appears on the upper face.

X	1	2	3	4	5	6
P(X=x)	1/6	1/6	1/6	1/6	1/6	1/6

$$P(X = x) \begin{cases} 1/6 & if\ i = 1,2,\ldots,6 \\ 0 & otherwise \end{cases}$$

Example (01) :

Using the above equation,

$$P(X=x)=\begin{cases} k & \text{if } x=1,2,3,4 \\ 0 & \text{others} \end{cases}$$

Find the value of k.

Hint: $0 \le P(X=x_i)=p_i \le 1$ *and* $\sum_{i=1}^{n} P(X=x_i)=\sum_{i=1}^{n} p_i=1$

Example (02) :

For the following probability distribution function,

$$P(X = x) = ax \quad for\ x = 1,\ 2,\ 3,\ 4$$

Find the value of a.

Hint: $0 \leq P(X = x_i) = p_i \leq 1$ *and* $\sum_{i=1}^{n} P(X = x_i) = \sum_{i=1}^{n} p_i = 1$

Example (03) :

A discrete random variable X has the following probability distribution.

X	1	2	3
P(X=x)	m	4m	2m

Where m is a constant.

1. Find the value of m.
2. Find $P(X \geq 2)$
3. Find $P(X < 2)$

Hint: $0 \leq P(X = x_i) = p_i \leq 1$ *and* $\sum_{i=1}^{n} P(X = x_i) = \sum_{i=1}^{n} p_i = 1$

Continuous Probability Distribution

- If a random variable is a continuous variable , its probability distribution is called a continuous probability distribution .

- A continuous probability distribution differs from a discrete probability distribution in several ways.
  - The continuous probability distribution cannot be expressed in tabular form.
  - An equation or formula is used to describe a continuous probability distribution.

- In the following figure probability that X assumes a value between a and b is equal to the shaded area under the density function between the ordinates at x=a and x=b.

Probability Densitv Function is aiven bv

$$F(x) = P(a \le x \le b) = \int_a^b f(x)dx \ge 0$$

Definition

The function $f(x)$ is a probability density function for the continuous random variable X, defined over the set of real numbers R. The pdf $f(x)$ has two important properties

$$\boxed{f(x) \geq 0,\ \ for\ all\ \ x \in R.}$$

$$\boxed{\int_{-\infty}^{\infty} f(x)dx = 1.}$$

$$\boxed{P(a < X < b) = \int_a^b f(x)dx}$$

Example

Suppose that the error in the reaction temperature, in , for a controlled laboratory experiment is a continuous random variable X having the probability density function

$$f(x) = \begin{cases} \dfrac{x^2}{3}, & -1 < x < 2, \\ 0, & otherwise. \end{cases}$$

1. Verify condition 2 of above definition.

2. Find $P(0 < X \leq 1)$.

$$f(x) = \begin{cases} \dfrac{x^2}{3}, & -1 < x < 2, \\ 0, & otherwise. \end{cases}$$

$$P(0 < X \leq 1).$$

Consider the function

$$f(x) = \begin{cases} 6x(1-x) & \text{if } 0 \le x \le 1, \\ 0 & \text{otherwise.} \end{cases}$$

a Check that $f(x)$ has the two required properties for a pdf, and sketch its graph.

b Suppose that the continuous random variable X has the pdf $f(x)$. Obtain the following probabilities *without calculation*:

i $\Pr(X \le -3)$

ii $\Pr(0 \le X \le 1)$

iii $\Pr(0.5 \le X \le 1)$.

MATHEMATICAL EXPECTATION

Mean of a random variable

- Let X be a random variable with probability distribution $f(x)$. The mean or expected value of X is

$$\mu = E(X) = \sum_x xP(x) \quad \text{if X is discrete,}$$

$$\mu = E(X) = \int_{-\infty}^{\infty} xf(x)dx \quad \text{if X is continuous.}$$

Example:

- Assuming that the two fair coins were tossed, we find that the sample space for our experiment is

$$S = \{HH, HT, TH, TT\}.$$

- Let X represent the number of heads in the sample. So we can write x=0,1,2

x	0	1	2
$P(X = x)$	1/4	1/2	1/4

x	0	1	2
$P(X = x)$	1/4	1/2	1/4

0	1	2
1/4	1/2	1/4

Then $\mu = E(X) = \sum_x xP(x)$

$$E(X) = (0)\left(\frac{1}{4}\right) + (1)\left(\frac{1}{2}\right) + (2)\left(\frac{1}{4}\right) = 1$$

- This result means that a person who tosses 2 coins, on the average get 1 head.

Example:

- Let X be the random variable that denotes the life in hours of a certain electronic device. The probability density function is

$$f(x) = \begin{cases} \dfrac{20{,}000}{x^3}, & x > 100, \\ 0, & elsewhere. \end{cases}$$

- Find the expected life of this type of device.

$$f(x) = \begin{cases} \dfrac{20{,}000}{x^3}, & x > 100, \\ 0, & elsewhere. \end{cases}$$

Example (04) :

A discrete random variable X has the following probability distribution.

X	2	3
P(X=x)	a	1-a

Where a is a constant and E(X)=2.6

1. Find the value of a.

Theorem

- Let X be a random variable with probability distribution $f(x)$. The expected value of the random variable $g(X)$ is

$$\boldsymbol{\mu_{g(X)} = E[g(X)] = \sum_x g(x) f(x),} \quad \textbf{if X is discrete.}$$

$$\boldsymbol{\mu_{g(X)} = E[g(X)] = \int_{-\infty}^{\infty} g(x) f(x),} \quad \textbf{if X is continuous.}$$

Theorem

Let X be a continuous random variable with mean μ_X . Then

$$\boldsymbol{E(aX+b)=aE(X)+b=a\mu_X+b}$$

for any real numbers a,b.

Proof

For a continuous random variable X, the mean of a function of X, say $g(X)$, given by

$$E[g(X)] = \int_{-\infty}^{\infty} g(x)\, f_X(x)\, dx.$$

So, for $g(X) = aX + b$, we find that

$$\begin{aligned} E(aX+b) &= \int_{-\infty}^{\infty} (ax+b)\, f_X(x)\, dx \\ &= \int_{-\infty}^{\infty} ax\, f_X(x)\, dx + \int_{-\infty}^{\infty} b\, f_X(x)\, dx \\ &= a\int_{-\infty}^{\infty} x\, f_X(x)\, dx + b\int_{-\infty}^{\infty} f_X(x)\, dx \\ &= a\mu_X + b. \end{aligned}$$

- Let X be a random variable with density function

$$f(x) = \begin{cases} \dfrac{x^2}{3}, & -1 < x < 2, \\ 0, & elsewhere. \end{cases}$$

- Find the expected value of g(X)=4X+3.

Variance and Covariance of Random Variables

- Let X be a random variable with probability distribution $f(x)$ and mean μ.
- The variance of X is

$$\sigma^2 = E[(X-\mu)^2] = \sum_x (x-\mu)^2 f(x), \qquad \textit{if } X \textit{ is discrete},$$

$$\sigma^2 = E[(X-\mu)^2] = \int_{-\infty}^{\infty} (x-\mu)^2 f(x), \qquad \textit{if } X \textit{ is continuous}.$$

- The positive square root of the variance, σ, is called the standard deviation of X.

Theorem

The variance of a random variable X is

$$\sigma^2 = E(X^2) - (E(X))^2 = E(X^2) - \mu^2$$

Theorem

Let X be a random variable with variance σ^2. Then

$$Var\,(aX + b) = a^2 Var\,(X)$$

for any real numbers a and b.

Proof

Define $Y = aX + b$. Then $\mathrm{var}(Y) = \mathrm{E}[(Y - \mu_Y)^2]$. We know that $\mu_Y = a\mu_X + b$. Hence,

$$\begin{aligned}
\mathrm{var}(aX + b) &= \mathrm{E}\left[\left(aX + b - (a\mu_X + b)\right)^2\right] \\
&= \mathrm{E}\left[a^2 (X - \mu_X)^2\right] \\
&= a^2\, \mathrm{E}[(X - \mu_X)^2] \\
&= a^2\, \mathrm{var}(X) \\
&= a^2 \sigma_X^2.
\end{aligned}$$

□

Example :

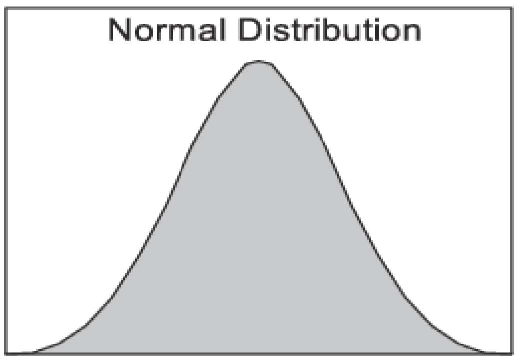
- The weekly demand for Pepsi, in thousands of liters, from a local chain of efficiency stores, is a continuous random variable X having the probability density

$$f(x) = \begin{cases} 2(x-1), & 1 < x < 2, \\ 0, & elsewhere. \end{cases}$$

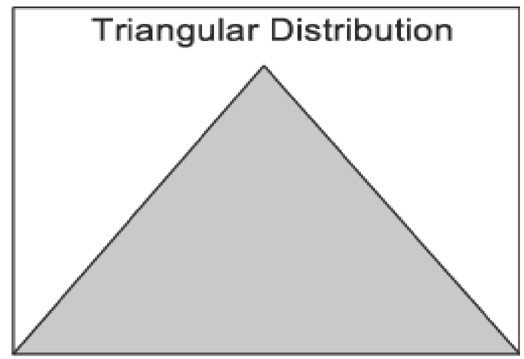
Find the mean and variance of X.

$$f(x) = \begin{cases} 2(x-1), & 1 < x < 2, \\ 0, & elsewhere\,. \end{cases}$$

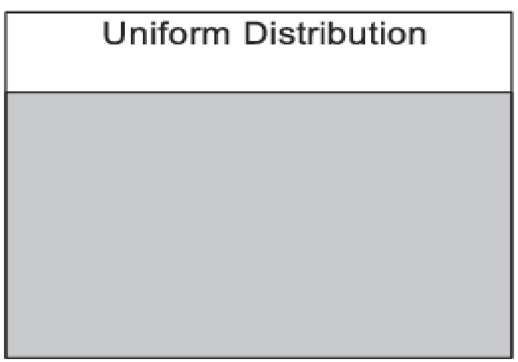
PROBABILITY DISTRIBUTIONS



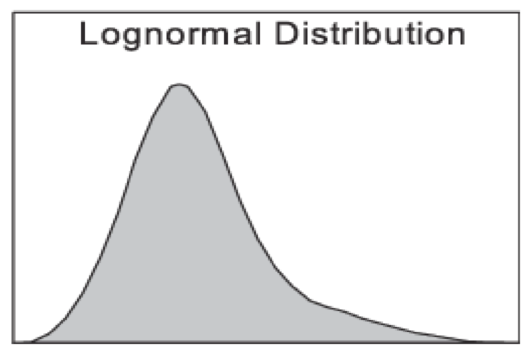
Normal distribution



Triangular distribution



Uniform distribution



Log-normal distribution

Statistical Experiment

All statistical experiments have three things in common:

- The experiment can have more than one possible outcome.
- Each possible outcome can be specified in advance.
- The outcome of the experiment depends on chance.

Example

- A coin toss has all the attributes of a statistical experiment.
- There is more than one possible outcome.
- We can specify each possible outcome in advance - heads or tails. And there is an element of chance.
- We cannot know the outcome until we actually flip the coin.

Probability Distribution

A probability distribution is a table or an equation that *links each outcome* of a statistical experiment with its probability of occurrence.

Example :

- Consider a simple experiment in which we flip a coin two times.

- Suppose the random variable X is defined as the number of heads that result from two coin flips.

- Then, the above table represents the probability distribution of the r

Number of heads	Probability
0	0.25
1	0.50
2	0.25

Probability Distribution

There are two probability distributions.

- *Discrete probability Distribution*

- Continuous probability Distribution

If a variable can take on any value between two specified values, it is called a **continuous variable**; otherwise, it is called a **discrete variable**.

Some examples will clarify the <u>difference between discrete and continuous variables</u>.

1. Suppose we flip a coin and count the number of heads. The number of heads could be any integer value between 0 and plus infinity.

- However, it could not be any number between 0 and plus infinity. We could not, for example, get 2.5 heads. Therefore, the number of heads must be a discrete variable.

2. Suppose the fire department that all fire fighters must weigh between 75 and 85 kg.

- The weight of a fire fighter would be an example of a continuous variable; since a fire fighter's weight could take on any value between 75 and 80 kg.

Discrete probability Distribution

1. Discrete Uniform Distribution
2. Bernoulli Probability Distribution
3. Binomial Probability Distribution
4. Poisson Probability Distribution

Discrete Uniform Distribution

- The simplest of all discrete probability distributions is one where the random variable assumes each of its values with an equal probability.

- Such a probability distribution is called a discrete uniform distribution.

- If the random variable X assumes the values $x_1, x_2, \ldots, x_k$, with equal probabilities, then the discrete uniform distribution is given by

$$\boxed{P(X = x) = \frac{1}{k}, \quad x = x_1, x_2, \ldots, x_k}$$

Example 01:

When a fair die is tossed, each element of the sample space $S = \{1,2,3,4,5,6\}$ occurs with probability 1/6.

- Therefore, we have a uniform distribution, with

$$P(X = x) = \frac{1}{6}, \qquad x = 1,2,3,4,5,6.$$

- The graphic representation of the uniform distribution by means of a histogram always turns out to be a set of rectangles with equal heights.

- The mean and variance of the discrete uniform distribution are

$$\boxed{\mu = \frac{1}{k}\sum_{i=1}^{k} x_i, \quad and \quad \sigma^2 = \frac{1}{k}\sum_{i=1}^{k}(x_i - \mu)^2}$$

Binomial Distribution

A **binomial experiment** (also known as a **Bernoulli trial**) is a *statistical experiment* that has the following properties:

- The experiment consists of *n* repeated trials.
- Each trial can result in just two possible outcomes. We call one of these outcomes a success and the other, a failure.
- The probability of success, denoted by *P*, is the same on every trial.

- The trials are independent; that is, the outcome on one

Consider the following statistical experiment.

You flip a coin 2 times and count the number of times the coin lands on heads. This is a binomial experiment because:

- The experiment consists of repeated trials. We flip a coin 2 times.
- Each trial can result in just two possible outcomes - heads or tails.
- The probability of success is constant 0.5 on every trial.
- The trials are independent; that is, getting heads on one trial does not affect whether we get heads on other trials.

Notation

- The following notation is helpful, when we talk about binomial probability.

- *x*: The number of successes that result from the binomial experiment.

- *n*: The number of trials in the binomial experiment.

- *p*: The probability of success on an individual trial.

- *q*: The probability of failure on an individual trial. (This is equal to 1 - *P*.)

Probability density function

$$X \sim bin(n, p)$$

$$P(X = k) = \overset{n}{\underset{k}{C}} p^k (1 - p)^{n-k}$$

- *n*: The total number of trials in the binomial experiment.
- k : number of successes trails k=0,1,2,...n
- *p*: The probability of success on an individual trial.

Example 01:

A fair coin is tossed 6 times. Find the probability of getting

a). exactly 4 heads. b). more than 3 heads

c). more than or equal 4heads

d). more than or equal 1head

Example 02:

The probability that the student graduate is 0.4.

Determine the probability that outcome of 5 students,

a. None b. one c. At least one d. All are graduate.

The binomial distribution has the following properties:

$$X \sim bin(n, p)$$

The mean of the distribution

$$(\mu_x) = n * P$$

The variance of the distribution

$$(\sigma^2_x) = n * P * (1 - P)$$

Example

1. The probability that Kamal hits a target is ¼. He hits 100 times. Find the expected number of times he will hit the target and find the standard deviation.

The binomial distribution has the following properties:

$$X \sim bin(n, p)$$

The mean of the distribution

$$(\mu_x) = n * P$$

The variance of the distribution

$$(\sigma^2_x) = n * P * (1 - P)$$

Example

A student takes an exam of 18 multiple choice questions with 4 choices per question. Find the expected number of correct answers and its standard deviation.

Poisson distribution

A **Poisson experiment** is a statistical experiment that has the following properties:

- A discrete random variable X is said to follow a Poisson distribution if it assumes only nonnegative integer values.

- The random variable x denotes the number of occurrences over a given span.

- There is only one parameter, **λ** , which is the average rate of occurrence.

- The occurrence of the event is not dependent on another occurrence of that event.

Following are some situations where the application of Poisson distribution is suitable.

- Number of telephone calls arriving at a switch board in a given interval of time.
- Number of printing errors in each page of book.
- Number of customers arriving at a Bank in some unit of time.

The probability distribution of the Poisson random variable X, representing the number of outcomes occurring in a given time interval or specified region denoted by t is

$$\boxed{p(x;\lambda t) = \frac{e^{-\lambda t}(\lambda t)^x}{x!}, \quad x = 0,1,2,\ldots\ldots}$$

- When t=1;

$$X \sim Poiss(\lambda)$$

$$P(X = x) = \frac{e^{-\lambda}\lambda^x}{x!} \quad x = 0,\ 1,\ 2,.$$

- where **λ** is the average rate of occurrence

The Poisson distribution has the following properties:

Theorem

Both the mean and variance of the Poisson distribution $p(x;\lambda t)$ are λt.

$$X \sim Poiss(\lambda t)$$

$$Mean = E(X) = \lambda t$$

$$Variance = Var(X) = \lambda t$$

Example 01:

The number of telephone calls made to a switch board during an afternoon can be distributed by Poisson distribution with a mean of eight calls per five minutes period.

Find the probability that in the next five minutes,

a. No calls b. Five c. at least three d. at most four calls are made.

Example 01:

The number of telephone calls made to a switch board during an afternoon can be distributed by Poisson distribution with a mean of eight calls per five minutes period.

Find the probability that in the next five minutes,

a. No calls b. Five c. at least three d. at most four calls are made.

Example 02:

Sri Lanka Insurance Company receives on average four claims a week in a Bandarawela town. Assuming that the number of claims follow a Poisson distribution, find the probability that the insurance company receives,

a). No b). More than 4 c). At least three d). Exactly 4 ;

claim in a given week.

Example 03:

Sri Lanka Insurance Company receives on average four claims a week in a Bandarawela town. Assuming that the number of claims follow a Poisson distribution, find the probability that the insurance company receives,

a). No b). More than 4 c). At least three d). Exactly 4

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Sri Lanka Insurance Company receives on average four claims a week in a Bandarawela town. Assuming that the number of claims follow a Poisson distribution, find the probability that the insurance company receives,

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claim in a given week.

Example 03:

Sri Lanka Insurance Company receives on average four claims a week in a Bandarawela town. Assuming that the number of claims follow a Poisson distribution, find the probability that the insurance company receives,

a). No b). More than 4 c). At least three d). Exactly 4

claim in a given <u>two week</u>.

The Poisson Distribution as an Approximation for the Binomial Distribution

On a particular production line, the probability that an item is depictive is 0.01.

Using suitable approximation, find the probability that, in a batch of 200 items,

a). There are no defective items

b). There are more than five defective items.

c). There are exactly 175 defective items.

Theorem

Let be a binomial random variable with probability distribution $b(x; n, p)$.

When $n \to \infty, p \to 0, and \ np \xrightarrow{n \to \infty} \mu$ remains constant,

$$X \sim bin\,(n, p)$$

$$when \ \ n \uparrow\uparrow \ \ and \ \ p \downarrow\downarrow$$

$$\lambda = np$$

$$X \sim Poiss(\lambda)$$

On a particular production line, the probability that an item is depictive is 0.01.

Using suitable approximation, find the probability that, in a batch of 200 items,

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Continuous Probability Distributions

- Uniform Distribution
- Normal Distribution

A continuous probability distribution differs from a discrete probability distribution in several ways.

- As a result, a continuous probability distribution cannot be expressed in tabular form.

- Instead, an equation or formula is used to describe a continuous probability distribution.

Uniform Distribution

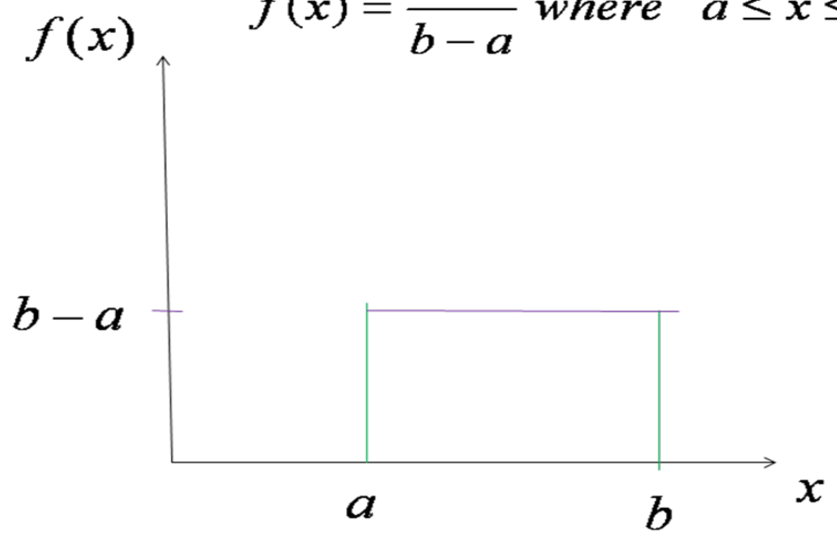
- One of the simplest continuous distributions in all of statistics is the continuous uniform distribution.

- This family of distributions is used to describe situations where the possible outcomes are all equally likely to occur.

- If X is a continues random variable with a uniform distribution defined by,

$$f(x;A,B) = \begin{cases} \frac{1}{B-A}, & A \leq x \leq B, \\ 0, & elsewhere. \end{cases}$$

$$f(x) = \frac{1}{b-a} \quad where \quad a \leq x \leq b$$



Theorem 1

The mean and variance of the uniform distribution are

$$\mu = \frac{A+B}{2}, \quad and \quad \sigma^2 = \frac{(B-A)^2}{12}.$$

$$X \sim uni(A, B)$$

$$Mean = E(X) = \frac{1}{2}(B + A)$$

$$Variance = Var(X) = \frac{1}{12}(B - A)^2$$

Example 01:

If $x \sim uni(2,5)$, write down the probability density function of X. Also find

a). E(X) b). Var (X)

Example 02:

The continuous random variable X has a distribution such as $x \sim uni(-2,3)$, Find

a). E(X) b). Var (X) c). P(-1<X<1.5)

Example 01:

If $x \sim uni(2,5)$, write down the probability density function of X. Also find

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Normal Distribution

- The normal probability density function usually called the Normal distribution is one of the widely used probability models.

- Most phenomena such as
  - ✓ Average marks of student
  - ✓ Diameters of machine parts
  - ✓ Life time of television bulb
  - ✓ Weights of packages are normally distributed.

- The normal distribution is also useful for approximating other distribution such as the binomial.

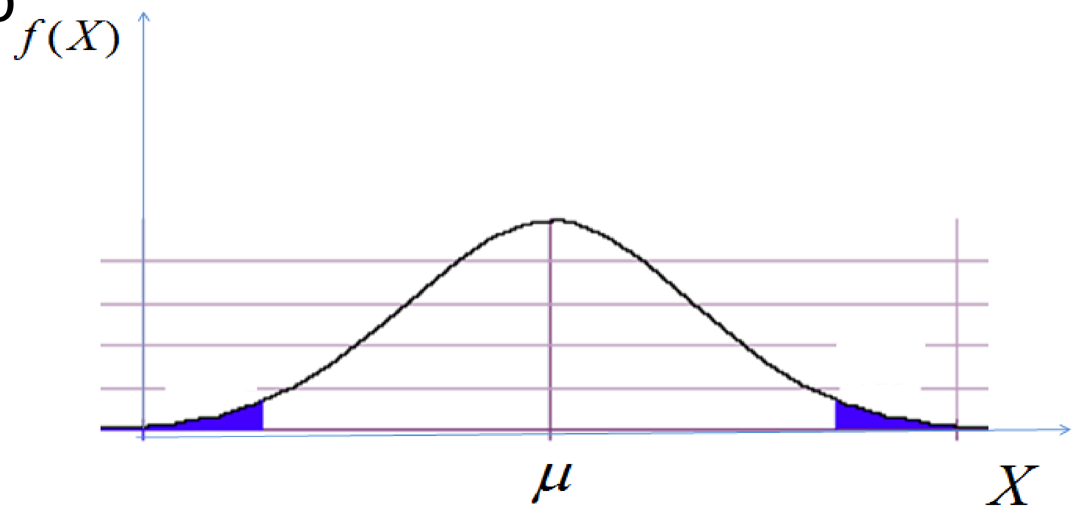
- The normal probability density function for a continuous random variable X is,

$$f(X)=\frac{1}{\sqrt{2\pi\sigma^2}}e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} \quad for\ -\infty < x < \infty$$

$$\mu - mean\, of\ x\ (E(X)=\mu)$$

$$\sigma - Standard\ deviation\, of\ X = (\mathrm{var}(X)=\sigma^2)$$

- The curve defined by the above function is a bell – shaped symmetrical distribution



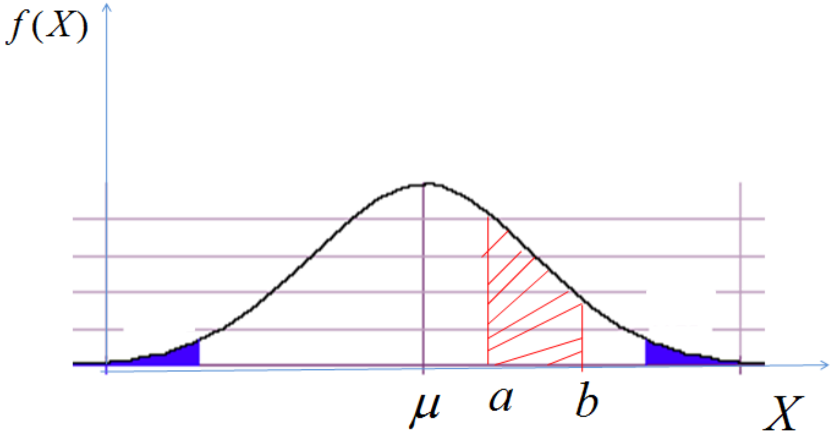
The NORMAL distribution has the following properties:

- It is symmetric about the mean value.
- It is bell- shaped and has one mode.
- The total area under the normal distribution curve equal one.
- The two tails of the distribution approach the horizontal axis but not touch the axis.

- The probability that a normally distributed random variable X assumes values between a and b is $P(a \le X \le B)$ which is equal to the proportion of total area under the curve between the limits a and b and this can be determined by integral calculus.

$$P(a \le X \le B) = \int_a^b f(X)\,dx$$

$$= \int_a^b \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx$$



Standard Normal Distribution

- In practical situations it is not available to use the density function directly for finding area under the normal curve.

- Tables that allow us to compute areas under the normal probability density function are based on the standard normal distribution.

- The standard normal distribution has the same features as any normal distribution.

- The mean of the standard normal distribution is 0 and variance is one.

A random variable X any mean and standard deviation can be transformed to a standardized random variable Z by using the relation,

$$X \sim N(\mu, \sigma^2)$$

$$\mu - mean\, of\;\; x$$

$$\sigma - \;S\tan dard\; deviation\, of\;\; X$$

$$X \sim N(\mu, \sigma^2)$$

$$\frac{X - \mu}{\sigma} \sim N(0,1)$$

$$Z = \frac{X - \mu}{\sigma} \Rightarrow Z \sim N(0,1)$$

- If X is normally distributed, Z is also normally distributed random variable with mean 0 and standard deviation 1.

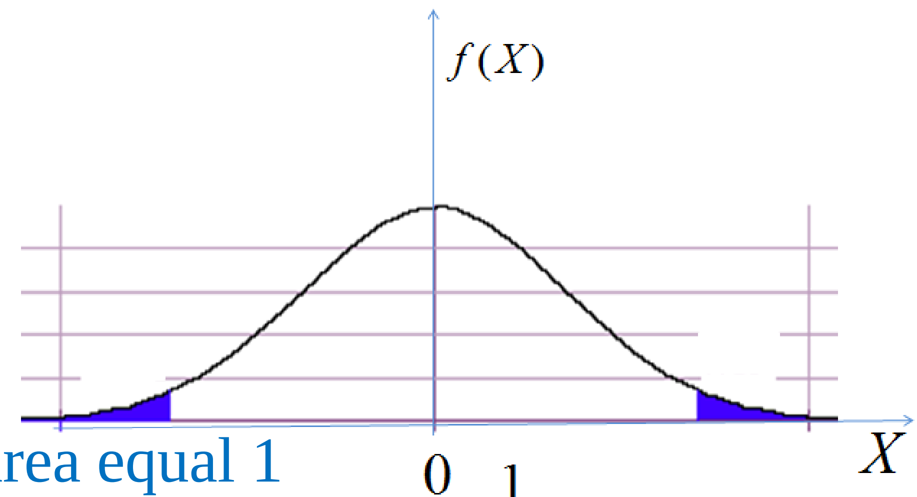
- If Z is the standardized normal distributed random variable , the Z has the probability density function,

$$f(Z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(Z)^2} \qquad for\ -\infty < Z < \infty$$

$$E(Z) = 0$$

$$\text{var}(Z) = 1$$

$$Z \sim N(0,1)$$



Probability of surface area equal 1

Example 09:

Using normal distribution table find ;

$P(z<0.75)$ $P(z>1.25)$ $P(z<-1.4)$ $P(z<2)$ $P(Z \geq -2)$

$P(0 \leq Z \leq 1.86)$ $P(-0.52 \leq Z \leq 0)$ $P(-0.30 \leq Z \leq 1.76)$

$P(-0.52 \leq Z \leq 0.52)$

P(z<0.75)

P(z>1.25)

P(z<-1.4)

P(z<2)

(1)

CUMULATIVE NORMAL DISTRIBUTION

$$\Phi(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-t^2/2} dt$$

x	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
.0	.5000	.5040	.5080	.5120	.5160	.5199	.5239	.5279	.5319	.5359
.1	.5398	.5438	.5478	.5517	.5557	.5596	.5636	.5675	.5714	.5753
.2	.5793	.5832	.5871	.5910	.5948	.5987	.6026	.6064	.6103	.6141
.3	.6179	.6217	.6255	.6293	.6331	.6368	.6406	.6443	.6480	.6517
.4	.6554	.6591	.6628	.6664	.6700	.6736	.6772	.6808	.6844	.6879
.5	.6915	.6950	.6985	.7019	.7054	.7088	.7123	.7157	.7190	.7224
.6	.7257	.7291	.7324	.7357	.7389	.7422	.7454	.7486	.7517	.7549
.7	.7580	.7611	.7642	.7673	.7704	.7734	.7764	.7794	.7823	.7852
.8	.7881	.7910	.7939	.7967	.7995	.8023	.8051	.8078	.8106	.8133
.9	.8159	.8186	.8212	.8238	.8264	.8289	.8315	.8340	.8365	.8389
1.0	.8413	.8438	.8461	.8485	.8508	.8531	.8554	.8577	.8599	.8621
1.1	.8643	.8665	.8686	.8708	.8729	.8749	.8770	.8790	.8810	.8830
1.2	.8849	.8869	.8888	.8907	.8925	.8944	.8962	.8980	.8997	.9015
1.3	.9032	.9049	.9066	.9082	.9099	.9115	.9131	.9147	.9162	.9177
1.4	.9192	.9207	.9222	.9236	.9251	.9265	.9279	.9292	.9306	.9319
1.5	.9332	.9345	.9357	.9370	.9382	.9394	.9406	.9418	.9429	.9441
1.6	.9452	.9463	.9474	.9484	.9495	.9505	.9515	.9525	.9535	.9545
1.7	.9554	.9564	.9573	.9582	.9591	.9599	.9608	.9616	.9625	.9633
1.8	.9641	.9649	.9656	.9664	.9671	.9678	.9686	.9693	.9699	.9706
1.9	.9713	.9719	.9726	.9732	.9738	.9744	.9750	.9756	.9761	.9767
2.0	.9772	.9778	.9783	.9788	.9793	.9798	.9803	.9808	.9812	.9817
2.1	.9821	.9826	.9830	.9834	.9838	.9842	.9846	.9850	.9854	.9857
2.2	.9861	.9864	.9868	.9871	.9875	.9878	.9881	.9884	.9887	.9890
2.3	.9893	.9896	.9898	.9901	.9904	.9906	.9909	.9911	.9913	.9916
2.4	.9918	.9920	.9922	.9925	.9927	.9929	.9931	.9932	.9934	.9936
2.5	.9938	.9940	.9941	.9943	.9945	.9946	.9948	.9949	.9951	.9952
2.6	.9953	.9955	.9956	.9957	.9959	.9960	.9961	.9962	.9963	.9964
2.7	.9965	.9966	.9967	.9968	.9969	.9970	.9971	.9972	.9973	.9974
2.8	.9974	.9975	.9976	.9977	.9977	.9978	.9979	.9979	.9980	.9981
2.9	.9981	.9982	.9982	.9983	.9984	.9984	.9985	.9985	.9986	.9986
3.0	.9987	.9987	.9987	.9988	.9988	.9989	.9989	.9989	.9990	.9990
3.1	.9990	.9991	.9991	.9991	.9992	.9992	.9992	.9992	.9993	.9993
3.2	.9993	.9993	.9994	.9994	.9994	.9994	.9994	.9995	.9995	.9995
3.3	.9995	.9995	.9995	.9996	.9996	.9996	.9996	.9996	.9996	.9997
3.4	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9998

$P(Z \geq -2)$

$P(0 \leq Z \leq 1.86)$

$P(-0.52 \leq Z \leq 0)$

(1)

CUMULATIVE NORMAL DISTRIBUTION

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.4	.6554	.6591	.6628	.6664	.6700	.6736	.6772	.6808	.6844	.6879
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2.9	.9981	.9982	.9982	.9983	.9984	.9984	.9985	.9985	.9986	.9986
3.0	.9987	.9987	.9987	.9988	.9988	.9989	.9989	.9989	.9990	.9990
3.1	.9990	.9991	.9991	.9991	.9992	.9992	.9992	.9992	.9993	.9993
3.2	.9993	.9993	.9994	.9994	.9994	.9994	.9994	.9995	.9995	.9995
3.3	.9995	.9995	.9995	.9996	.9996	.9996	.9996	.9996	.9996	.9997
3.4	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9998

$$P(-0.30 \le Z \le 1.76)$$

(1)

CUMULATIVE NORMAL DISTRIBUTION

$$\Phi(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-t^2/2} dt$$

x	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
.0	.5000	.5040	.5080	.5120	.5160	.5199	.5239	.5279	.5319	.5359
.1	.5398	.5438	.5478	.5517	.5557	.5596	.5636	.5675	.5714	.5753
.2	.5793	.5832	.5871	.5910	.5948	.5987	.6026	.6064	.6103	.6141
.3	.6179	.6217	.6255	.6293	.6331	.6368	.6406	.6443	.6480	.6517
.4	.6554	.6591	.6628	.6664	.6700	.6736	.6772	.6808	.6844	.6879
.5	.6915	.6950	.6985	.7019	.7054	.7088	.7123	.7157	.7190	.7224
.6	.7257	.7291	.7324	.7357	.7389	.7422	.7454	.7486	.7517	.7549
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2.6	.9953	.9955	.9956	.9957	.9959	.9960	.9961	.9962	.9963	.9964
2.7	.9965	.9966	.9967	.9968	.9969	.9970	.9971	.9972	.9973	.9974
2.8	.9974	.9975	.9976	.9977	.9977	.9978	.9979	.9979	.9980	.9981
2.9	.9981	.9982	.9982	.9983	.9984	.9984	.9985	.9985	.9986	.9986
3.0	.9987	.9987	.9987	.9988	.9988	.9989	.9989	.9989	.9990	.9990
3.1	.9990	.9991	.9991	.9991	.9992	.9992	.9992	.9992	.9993	.9993
3.2	.9993	.9993	.9994	.9994	.9994	.9994	.9994	.9995	.9995	.9995
3.3	.9995	.9995	.9995	.9996	.9996	.9996	.9996	.9996	.9996	.9997
3.4	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9998

$$P(-0.52 \le Z \le 0.52)$$

The random variable

$$X \sim N(12, 2^2)$$

Find,

a) $P(X<15)$

b) $P(X>10)$

c) $P(9<X<13)$

d) $P(9 \le X \le 12)$

CUMULATIVE NORMAL DISTRIBUTION

$$\Phi(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-t^2/2} dt$$

x	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
.0	.5000	.5040	.5080	.5120	.5160	.5199	.5239	.5279	.5319	.5359
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2.6	.9953	.9955	.9956	.9957	.9959	.9960	.9961	.9962	.9963	.9964
2.7	.9965	.9966	.9967	.9968	.9969	.9970	.9971	.9972	.9973	.9974
2.8	.9974	.9975	.9976	.9977	.9977	.9978	.9979	.9979	.9980	.9981
2.9	.9981	.9982	.9982	.9983	.9984	.9984	.9985	.9985	.9986	.9986
3.0	.9987	.9987	.9987	.9988	.9988	.9989	.9989	.9989	.9990	.9990
3.1	.9990	.9991	.9991	.9991	.9992	.9992	.9992	.9992	.9993	.9993
3.2	.9993	.9993	.9994	.9994	.9994	.9994	.9994	.9995	.9995	.9995
3.3	.9995	.9995	.9995	.9996	.9996	.9996	.9996	.9996	.9996	.9997
3.4	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9998

x	1.282	1.645	1.960	2.326	2.576	3.090	3.291	3.891	4.417
$\Phi(x)$	.90	.95	.975	.99	.995	.999	.9995	.99995	.999995
$2[1-\Phi(x)]$	.20	.10	.05	.02	.01	.002	.001	.0001	.00001

Computing Probabilities for a Normal Distribution

Example 1: lengths of a particular species of worm are normally distributed with mean 140cm and standard deviation 10cm. Find the probability that a worm is selected at random is,

a. less than 120cm long,
b. more than 148cm long,
c. between 148cm and 154cm long.

Example 2: A biologist has studied a particular tropical insect and she has discovered that its lifespan is normally distributed. The mean life span of this insect is 72 days and stranded deviation of it's life span is eight days. Find the next insect studied lived,

a. Fewer than 70 days
b. More than 76 days
c. Between 68 and 78 days

CUMULATIVE NORMAL DISTRIBUTION

$$\Phi(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-t^2/2} dt$$

x	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
.0	.5000	.5040	.5080	.5120	.5160	.5199	.5239	.5279	.5319	.5359
.1	.5398	.5438	.5478	.5517	.5557	.5596	.5636	.5675	.5714	.5753
.2	.5793	.5832	.5871	.5910	.5948	.5987	.6026	.6064	.6103	.6141
.3	.6179	.6217	.6255	.6293	.6331	.6368	.6406	.6443	.6480	.6517
.4	.6554	.6591	.6628	.6664	.6700	.6736	.6772	.6808	.6844	.6879
.5	.6915	.6950	.6985	.7019	.7054	.7088	.7123	.7157	.7190	.7224
.6	.7257	.7291	.7324	.7357	.7389	.7422	.7454	.7486	.7517	.7549
.7	.7580	.7611	.7642	.7673	.7704	.7734	.7764	.7794	.7823	.7852
.8	.7881	.7910	.7939	.7967	.7995	.8023	.8051	.8078	.8106	.8133
.9	.8159	.8186	.8212	.8238	.8264	.8289	.8315	.8340	.8365	.8389
1.0	.8413	.8438	.8461	.8485	.8508	.8531	.8554	.8577	.8599	.8621
1.1	.8643	.8665	.8686	.8708	.8729	.8749	.8770	.8790	.8810	.8830
1.2	.8849	.8869	.8888	.8907	.8925	.8944	.8962	.8980	.8997	.9015
1.3	.9032	.9049	.9066	.9082	.9099	.9115	.9131	.9147	.9162	.9177
1.4	.9192	.9207	.9222	.9236	.9251	.9265	.9279	.9292	.9306	.9319
1.5	.9332	.9345	.9357	.9370	.9382	.9394	.9406	.9418	.9429	.9441
1.6	.9452	.9463	.9474	.9484	.9495	.9505	.9515	.9525	.9535	.9545
1.7	.9554	.9564	.9573	.9582	.9591	.9599	.9608	.9616	.9625	.9633
1.8	.9641	.9649	.9656	.9664	.9671	.9678	.9686	.9693	.9699	.9706
1.9	.9713	.9719	.9726	.9732	.9738	.9744	.9750	.9756	.9761	.9767
2.0	.9772	.9778	.9783	.9788	.9793	.9798	.9803	.9808	.9812	.9817
2.1	.9821	.9826	.9830	.9834	.9838	.9842	.9846	.9850	.9854	.9857
2.2	.9861	.9864	.9868	.9871	.9875	.9878	.9881	.9884	.9887	.9890
2.3	.9893	.9896	.9898	.9901	.9904	.9906	.9909	.9911	.9913	.9916
2.4	.9918	.9920	.9922	.9925	.9927	.9929	.9931	.9932	.9934	.9936
2.5	.9938	.9940	.9941	.9943	.9945	.9946	.9948	.9949	.9951	.9952
2.6	.9953	.9955	.9956	.9957	.9959	.9960	.9961	.9962	.9963	.9964
2.7	.9965	.9966	.9967	.9968	.9969	.9970	.9971	.9972	.9973	.9974
2.8	.9974	.9975	.9976	.9977	.9977	.9978	.9979	.9979	.9980	.9981
2.9	.9981	.9982	.9982	.9983	.9984	.9984	.9985	.9985	.9986	.9986
3.0	.9987	.9987	.9987	.9988	.9988	.9989	.9989	.9989	.9990	.9990
3.1	.9990	.9991	.9991	.9991	.9992	.9992	.9992	.9992	.9993	.9993
3.2	.9993	.9993	.9994	.9994	.9994	.9994	.9994	.9995	.9995	.9995
3.3	.9995	.9995	.9995	.9996	.9996	.9996	.9996	.9996	.9996	.9997
3.4	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9998

Computing Probabilities for a Normal Distribution

Example 1: lengths of a particular species of worm are normally distributed with mean 140cm and standard deviation 10cm. Find the probability that a worm is selected at random is,

a. less than 120cm long,
b. more than 148cm long,
c. between 148cm and 154cm long.

Computing Probabilities for a Normal Distribution

Example 2: A biologist has studied a particular tropical insect and she has discovered that its lifespan is normally distributed. The mean life span of this insect is 72 days and stranded deviation of it's life span is eight days. Find the next insect studied lived,

a. Fewer than 70 days
b. More than 76 days
c. Between 68 and 78 days

Example 3:

The weights of 1000 packages in a brand of serial are normally distributed with mean of 32 grams and standard deviation of 1.3 grams.

a. What is the probability that the weight of a package is between 32 grams and 34 grams.

b. Find the expected number of packages of weight between 32 grams and 34 grams.

c. Find the expected number of packages of weight more than 36 grams.

Example 3:

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CUMULATIVE NORMAL DISTRIBUTION

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x	.00	.01	.02	.03	.04	.05	.06	.07	.08
.0	.5000	.5040	.5080	.5120	.5160	.5199	.5239	.5279	.5319
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.5	.6915	.6950	.6985	.7019	.7054	.7088	.7123	.7157	.7190
.6	.7257	.7291	.7324	.7357	.7389	.7422	.7454	.7486	.7517
.7	.7580	.7611	.7642	.7673	.7704	.7734	.7764	.7794	.7823
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1.3	.9032	.9049	.9066	.9082	.9099	.9115	.9131	.9147	.9162
1.4	.9192	.9207	.9222	.9236	.9251	.9265	.9279	.9292	.9306
1.5	.9332	.9345	.9357	.9370	.9382	.9394	.9406	.9418	.9429
1.6	.9452	.9463	.9474	.9484	.9495	.9505	.9515	.9525	.9535
1.7	.9554	.9564	.9573	.9582	.9591	.9599	.9608	.9616	.9625
1.8	.9641	.9649	.9656	.9664	.9671	.9678	.9686	.9693	.9699
1.9	.9713	.9719	.9726	.9732	.9738	.9744	.9750	.9756	.9761
2.0	.9772	.9778	.9783	.9788	.9793	.9798	.9803	.9808	.9812
2.1	.9821	.9826	.9830	.9834	.9838	.9842	.9846	.9850	.9854
2.2	.9861	.9864	.9868	.9871	.9875	.9878	.9881	.9884	.9887
2.3	.9893	.9896	.9898	.9901	.9904	.9906	.9909	.9911	.9913
2.4	.9918	.9920	.9922	.9925	.9927	.9929	.9931	.9932	.9934
2.5	.9938	.9940	.9941	.9943	.9945	.9946	.9948	.9949	.9951
2.6	.9953	.9955	.9956	.9957	.9959	.9960	.9961	.9962	.9963
2.7	.9965	.9966	.9967	.9968	.9969	.9970	.9971	.9972	.9973
2.8	.9974	.9975	.9976	.9977	.9977	.9978	.9979	.9979	.9980
2.9	.9981	.9982	.9982	.9983	.9984	.9984	.9985	.9985	.9986
3.0	.9987	.9987	.9987	.9988	.9988	.9989	.9989	.9989	.9990
3.1	.9990	.9991	.9991	.9991	.9992	.9992	.9992	.9992	.9993
3.2	.9993	.9993	.9994	.9994	.9994	.9994	.9994	.9995	.9995
3.3	.9995	.9995	.9995	.9996	.9996	.9996	.9996	.9996	.9996
3.4	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997

x	1.282	1.645	1.960	2.326	2.576	3.090	3.291	3.891	4.4