An Introduction to Newtonian Mechanics

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Preface

This book is intended as a supplement to classes that are usually given to normal Secondary 3/4 students in Singapore. Unlike other textbooks that are normally issued in the country, this one focuses on genuine explanations with rigour (at this level, where possible). Consequently, by reading this book you will not just know how to solve problems for the sake of obtaining a good grade on the exam. Rather, I hope to show you how beautiful physics is, even at a beginner level.

ChatGPT is credited for helping me make some of the diagrams in LaTeX (particularly the one used to visuaualise the derivation of the equation $p = \rho g h$. *No additional use of AI will be found in this book.*

Prerequisites

Reading the Material

I assume you have some acquaintance with basic mathematical manipulations in this book (your algebra should be good), along with some geometry (mainly vectors).

Reading the Remarks

I have added remarks throughout the book when I have felt that there is more to be elaborated upon or the derivation (at a secondary school level) is not sufficiently rigorous. These remarks may assume a more extensive mathematical background, such as vector calculus or some knowledge of differential equations. These remarks are intended to spark further interest in the subject, not to scare you (you do not have to read them as they will not be mentioned anywhere else in the book).

Problems

The lack of problems that are present in this book deserves an explanation. Do not get me wrong, problems are absolutely essential to learn not just physics, but science/mathematical subjects in general. You will not get anywhere by just reading a text, no matter who wrote it. Therefore, it may seem slightly odd that I have not included problems in this book. My reasoning is that your school would have probably given you numerous problem sheets (which you will not touch, until a day before the test), which have worked solutions. Consequently, there is no need for me to create new problems as

1. It would be a waste of paper

2. You already have access to an abundance of them.

In the improbable event that your institution has not provided you with any, I recommend supplementing your learning with Ten Year Series Problems (do all of them), and you should do just fine.

-Kieron Eisma

Contents

Chapter 1

Kinematics

1.1 Introduction

I would like to start this book with kinematics, because it is very important for what we are going to do next with mechanics and energy.

We start our discussion with the definition of displacement, velocity and acceleration, which I will simply just paste in, because there really is nothing much to gain(it should be 'common sense').

DEFINTIONS 1.1

Displacement: change in position of an object

Speed: rate of change of distance (with respect to time)

Velocity: rate of change of displacement (with respect to time)

Acceleration: rate of change of velocity (with respect to time)

Uniform acceleration: constant rate of change of velocity

From these definitions, we can make some formulas. For example, since we defined speed as the rate of change of distance, we can say

$$v = \frac{s}{t} \tag{1.1}$$

where s represents distance and t represents time.

Well how about velocity? It is essentially the same thing, except distance is replaced with displacement.

$$\vec{v} = \frac{\vec{s}}{t} \tag{1.2}$$

Note that the difference between Equation (1.2) and Equation (1.3) is that the former is a *scalar* quantity, while the latter is a *vector* quantity. In actual English, (1.2) tells you how fast something is going, while (1.3) tells you how fast and *where* it is going. Consider the following statements to help you gain some intuition.

I am on a plane that is travelling at 800km/h. This is a scalar because you did not specify where you are going.

I am on a plane travelling at 800 km/h from Paris to France (see the joke there?). This is a vector because you have specified your starting and ending points, in this rhetorical case, Paris and France, respectively.

Now that that is out of the way, our last job is to make an equation for acceleration. Acceleration is the rate of change of velocity with respect to time. Well the change in velocity is simply just the final velocity minus the initial velocity. This is analogous to the change in money of someone's bank account. You take how much he has currently, and compare it with how much he had earlier. We can thus write $\Delta\vec{V} = \vec{v_2} - \vec{v_1}$. Now all we need to do is just divide this quantity by the time taken to get from $\vec{v_1}$ to $\vec{v_2}$. From this information above, we deduce the equation:

$$\vec{a} = \frac{\vec{v_2} - \vec{v_1}}{t} \tag{1.3}$$

My notation is quite rigorous, some university level textbooks dont even indicate that these quantities are vectors, but I am doing this to **remind** you that you must account for their direction, before you can start doing that yourself. You have to crawl before you walk.

A useful thing to note is that the displacement of an object is given by the sum of the areas under a velocity time graph (regions under the graph are taken as negative), while the distance travelled by an object is given by the sum of the areas under a velocity (or speed-time) graph, where all areas are taken as positive.

REMARK:

Formally speaking, if we have a displacement function $s(t)$ which is twice-differentiable over some time interval $[0, T]$, then we can define the velocity and acceleration as follows:

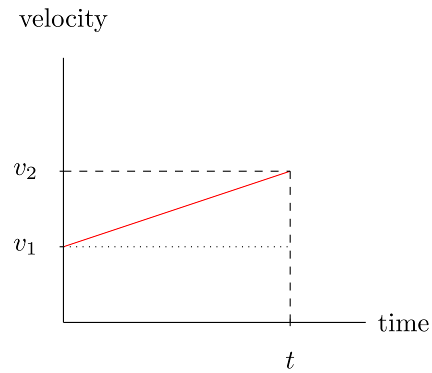
$\vec{v}(t) := \frac{d\vec{s}}{dt}$

$\vec{a}(t) := \frac{d\vec{v}}{dt} = \frac{d^2\vec{v}}{dt^2}$

Calculus by Michael Spivak, which is essentially a real analysis book, offers a great discussion on derivatives and how they can be related to speed and acceleration in Chapter 9.

1.2 Formulas

I will now derive some useful formulas that you can use in the future. These formulas are normally only introduced when students learn H2 physics, but the following formulas can be extremely useful in your 'O' Level examinations, and can save a lot of time. Note that these formulas only work when an object has a constant acceleration. In other words, they only work when the object is gaining velocity at a constant rate. Consider the situation below. Note that the acceleration of the object is in the direction of the motion (i.e it is accelerating, not decelerating). The necessary changes can be made to account decelerating motion by reversing the sign of a.



From the previous section, we know that the displacement of the object is simply the area under the graph. Therefore, if we want to find the displacement of the object as it's velocity increases from v_1 to v_2, we simply need to compute the area under the graph.

The area under the graph is just the area for a trapezium, so we can say

$$s = \frac{v_1 + v_2}{2} t \tag{1.4}$$

Note that this can be expressed as

$$s = \frac{1}{2}(v_2 - v_1)t + v_1 t \tag{1.5}$$

.

Envoking Equation (1.4) and manipulating gives

$$v_2 - v_1 = at \tag{1.6}$$

.

Now substituting Equation (1.6) into Equation (1.5) yields

$$s = v_1 t + \frac{1}{2}at^2 \tag{1.7}$$

Let's now derive the second important formula to find v_2.

Equation (1.7) also implies that

$$t = \frac{v_2 - v_1}{a} \tag{1.8}$$

.

Substituting Equation (1.9) into Equation (1.5) gives

$$s = \frac{v_1 + v_2}{2} \cdot \frac{v_1 - v_2}{a} \tag{1.9}$$

$$s = \frac{(v_1 + v_2)(v_1 - v_2)}{2a} \tag{1.10}$$

.

Using the fact that $(a^2 - b^2) = (a + b)(a - b)$,

$$s = \frac{v_2^2 - v_1^2}{2a} \tag{1.11}$$

.

Rearranging, we find that

$$v_2^2 = v_1^2 + 2as \tag{1.12}$$

Again, these two equations are assuming that the acceleration is in the positive direction, and the necessary correction can be made by flipping the sign of a should the opposite be the case.

REMARK

For a more rigorous derivation of Equation (1.8), a differential equation can be set up such that $\frac{d^2x}{dt^2} = a$, with initial conditions $v(0) = v_1$ and $s(0) = 0$.

Chapter 2

Newton's Laws

In 1687, Newton published his three laws of motion in his Principia Mathematica. These laws are some of the most important things that have ever been deduced in the realm of physics. These laws will always keep coming back up, and have helped scientists of the past to deduce things like the Euler-Lagrange equation. The three laws of motion are written down below; don't worry, we will explore them in depth in this chapter. It is important to note that these definitions have been slightly modified to meet the 'O' Level specification. Readers who have studied physics at a higher level will thus likely find some discrepancies in the following definitions that I have stated.

The First Law A body moves with constant velocity (which may be zero) unless acted on by a net force.

The Second Law If a net force acts on an object, the object will move in the direction of the net-force, and it's acceleration is directly proportional to the force that acts on the object.

The Third Law For every force on one body, there is an equal and opposite force on another body.

2.1 Newton's First Law

Intuitively, Newton's First Law says that if you do not do anything to an object, then it will continue to do it's own thing. It is important to note that this law only holds in an inertial frame of reference, which is a frame of reference that is stationary with respect to everything else in the system. Readers who are exclusively using this book for their 'O' Level presentation may skip the following subsection.

2.1.1 Inertial Frames

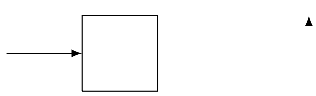
Let's do a simple thought experiment to illustrate the importance of an inertial frame of reference. Imagine a train car, perhaps a container one with no windows whatsoever. Suppose there is a rollerblader inside the train car, where the floor has no friction on the wheels of her rollerblades. Let's imagine what happens when the train starts to move. With respect to the train car, it appears that the skater is moving without a force acting on her; she slides to the back of the car. However, with reference to the ground, she is stationary, in accordance with Newton's First Law. If this is a little hard to visualise, get a small piece of cardboard and place a spherical object on top, perhaps a marble. If you move the piece of cardboard, the marble will appear to move with respect to the cardboard. However, with respect to the surface that you are moving the piece of cardboard on, the marble is stationary.

This example illustrates the importance of an inertial frame of reference, and how Newton's First Law will only hold in such a frame. This is why the First Law is sometimes referred to as the *Law of Inertia*.

In more advanced physics, you will learn more about frames of reference, but this is beyond the scope of this text.

2.1.2 The Resultant, or Net Force

Notice in the definition I did not write *a force*, but rather a *net force*. This is something that needs to be treated with *extreme caution*. Consider the following situation below.



Since both forces have the same *magnitude* and act in *directions*, the total net force F is given by

$F = \vec{F} + (-\vec{F})$,

$F = 0$.

There is thus no net force, so the object will not move. The purpose of this paragraph is not to confuse you with funny vector notation; it is to show you that applying a force on an object will not cause it to accelerate. The object will only accelerate in some direction if and only if there is a net force that acts on it. You experience this everyday; if you push on a wall, does your house move? Conversely, if the two forces in the figure above were in the same direction, they would not cancel each other out, and the object would start moving. We will study this situation in the following section.

2.2 Newton's Second Law

I showed you that in the previous section that an object will only move if there is a net force that is acting upon it. Most of the things we see currently, like buildings and bridges are designed to *prevent* such a force from acting. After all, this would be catastrophic. However, there are many systems in nature where a net force does act, such as in the solar system. The following equation that I am about to introduce is probably the most important one that you will know for the next few years of your physics education.

$$F = ma \tag{2.1}$$

Where a=acceleration, m=mass and F=net force.

It might not look like much, but this one equation is extremely important. Newton deduced this equation from experimental data, and it has allowed humanity to do things incomprehensible to our ancestors.

This equation is important because it allows us to easily determine the net force given an object's acceleration. Instead of having to add up all the little forces vectorially to determine the net force, we just need to take the product of the object's mass and acceleration. Conversely, if we manage to find the net force, we will be able to determine the resulting acceleration.

WORKED EXAMPLE 1.2.2

Consider a particle with mass 5kg. An applied force of 5N is impressed on the object. The frictional force acts in the opposite direction, and has a magnitude of 3N. Find the magnitude of the acceleration of the particle.

Solution From this information, the net force acting on the particle is

$$F = 5\text{N} - 3\text{N} \tag{2.2}$$

$$F = 2\text{N} \tag{2.3}$$

Note that the question never asked for us to determine the direction of the acceleration, but rather the magnitude. So all we need to do is apply our equation $F = ma$.

By Newton's Second Law,

$$2\text{N} = (5\text{kg})a, \tag{2.4}$$

$$a = \frac{2N}{5\text{kg}} \tag{2.5}$$

$$a = 0.400\text{m}/s^2 \text{ (3s.f.)} \tag{2.6}$$

and this is our final answer. □

WORKED EXAMPLE 2.2.2

A crate of mass 50kg is attached to a cable and dropped from rest from a crane. The tension in the cable is 200N.

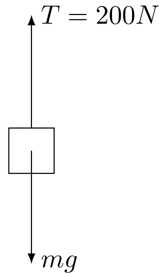
a) Calculate the acceleration of the crate

b) If the box falls through a height of 20m, calculate the final speed of the box towards the ground.

Solution:

a)

We first draw a free body diagram of the system below. Note that the tension in the cable acts in the opposite direction to the weight of the block.



Taking downward as positive, we formulate the equation for the net force:

$$F = mg - T \tag{2.7}$$

.

By Newton's Second Law, it follows that

$$ma - mg - T \tag{2.8}$$

$$a = \frac{mg - T}{m} \tag{2.9}$$

Substituting the values of m, g and T yields

$$a = \frac{(50\text{kg})(10\text{N/kg}) - 200\text{N}}{50\text{kg}} \tag{2.10}$$

and this gives us the final answer

$$a = 6.00\text{m}/s^2 \text{ (3s.f.)} \tag{2.11}$$

b) Recall Equation (1.13), $v_2^2 = v_1^2 + 2as$.

We know that the crate was dropped from rest, so $v_1 = 0$. Since the crate falls through a height of 20m, it follows that $s = 20\text{m}$. Therefore,

$$v_2^2 = 0 + 2(6.00\text{m}/s^2)(20\text{m}) \tag{2.12}$$

$$v_2 = 15.5\text{m/s (3s.f.)} \tag{2.13}$$

□

2.3 Newton's Third Law

Intuitively, what Newton's Third Law says is that if you apply a force on an object, the object will apply an equal and opposite force on you. This is very easy to illustrate. For instance, if you punch a table with your fist, the table will probably move (if you do it sideways), but your fist will definitely hurt. More rigorously, you exerted a force $\vec{F_1}$ on the table, and the table exerted a force $\vec{F_2} = -\vec{F_1}$, opposite to the force that you applied on the table. These kinds of forces are known as *action-reaction pairs*, and this is a concept that is heavily tested in the 'O' Level physics examinations.

WORKED EXAMPLE 1.2.3

Calculate the acceleration of the Earth (mass 6.0×10^{24}kg due to a falling 5kg mass.

Solution: This may seem like a very daunting problem, but if you paid attention to the paragraph above, you would have noticed that this is just a direct application of Newton's Third and Second Laws. The force that the Earth exerts on the mass is just it's weight;

$$W = (5.00\text{kg})(10\text{N/kg}) \tag{2.14}$$

$$W = 50.0\text{N} \tag{2.15}$$

Now by Newton's Second Law, we have

$$m_{\text{Earth}} a_{\text{Earth}} = 50.0\text{N} \tag{2.16}$$

Using the value for m_{Earth} that has been provided,

$$a_E = \frac{50.0\text{N}}{6.0 \times 10^{24}\text{kg}} \tag{2.17}$$

,

and this has a value of

$$a_E = 8.33 \times 10^{-24}\text{kg (3s.f.)} \tag{2.18}$$

towards the mass. □

The worked example above illustrates the power of Newton's Third Law; a small mass can move the entire planet, though very very slowly.

Chapter 3

Work and Energy

3.1 Introduction

There is no doubt that Newtonian Mechanics is useful and is one of the most profound inventions in the history of physics. However, this theory of nature has some limitations. While Newtonian Mechanics can be used to predict the subsequent motion of an object by simply taking the sum of all forces, the calculations can rapidly become complicated. To see this, consider a ball that rolls down a hill of some height. If we wanted to calculate the final velocity of this ball by just using Newton's Second Law, we would have to consider each individual interaction between the ground and the ball on each point on the slope. This is very impractical, so it seems that such a problem is unsolvable. However, much of what we see around us in the world today is based on similar analogues of this problem statement. If only there could be some quantity, that is conserved, and because of this, we can apply it to all physical systems and skip all the tedious calculations needed to solve the problem using Newton's Second Law. This brings us to our discussion of work.

DEFINITION 3.1

The **work** done by a constant force on an object is defined as the product of the force and the distance moved in the direction of the force.

Note that work is measured in the *Joule (J)* after the English physicist James Joule.

It follows from Definition 3.1 that

$$W = Fd \tag{3.1}$$

Where the symbol W represents the work done.

You should think about this definition very carefully. If you were to push on the walls of a building, they would not move (at least, hopefully). Therefore, you have done no work on the building, but you would probably feel tired after pushing the wall for a long amount of time. Consequently, work does not equate to being tired. The reason why you would feel tired is that the tiny fibres inside your muscles contract and relax, and they thus do work. However, the work done by you on the wall of the building remains zero.

With this measly equation, it seems that we cannot do so much. After all, we have simply defined some quantity as $W = Fd$, which could have been done by anybody. This will soon change.

WORKED EXAMPLE 3.1

Calculate the work done by a force of magnitude 5N on an object through a distance of 5m

Solution: Direct application of our definition of work done yields

$$W = (5\text{N})(5\text{m}) \tag{3.2}$$

$$W = 25.0\text{J (3s.f.)} \tag{3.3}$$

Which is our final answer. $\square$

3.2 The Work-Energy Theorem

I will now present one of the most important theorems in classical mechanics in the context of one dimension. While not directly taught in schools around the country, it can be used to solve problems in the blink of an eye, and also goes to show how important the concept of work is.

DEFINITION 3.2.1:

Energy is the capacity to do work, and is a scalar quantity.

DEFINITION 3.2.2 The **kinetic energy** of a particle is the energy possessed by the particle due to it's speed. It is a scalar quantity and is measured in Joules (J). We define the kinetic energy E_k for particles at non-relativistic speeds ($v << c$) by $E_k = \frac{1}{2}mv^2$

REMARK

More advanced textbooks sometimes denote the kinetic energy (at non-relativistic speeds) as K or T, but they all mean the same thing.

The Work Energy Theorem. The total work done by a constant force on a particle of mass m is given by the change in the object's kinetic energy.

Proof: By Newton's Second Law, the force will produce an acceleration given by

$$a = \frac{F}{m} \tag{3.4}$$

From Equation (1.13), we have

$$v_2^2 = v_1^2 + 2\left(\frac{F}{m}\right)s \tag{3.5}$$

This yields

$$mv_2^2 = mv_1^2 + 2Fs \tag{3.6}$$

Applying Definition 3.1,

$$2W_{\text{tot.}} = mv_2^2 - mv_1^2 \tag{3.7}$$

and we obtain

$$W_{\text{tot.}} = \frac{1}{2}mv_2^2 - \frac{1}{2}mv_1^2 \tag{3.8}$$

By Definition 3.2.2, the kinetic energy is $E_k = \frac{1}{2}mv^2$ gives

$$W_{\text{tot.}} = E_{k2} - E_{k1} \tag{3.9}$$

$$W_{\text{tot.}} = \Delta E_k \tag{3.10}$$

and this is precisely the Work-Energy Theorem. ■

In other words, if a force F acts over a distance s on an object of mass m with some initial kinetic energy, you can use the Work-Energy Theorem to determine it's final kinetic energy, and as a result, it's final speed. In accordance with the 'O' Level syllabus, I must introduce a definition of kinetic energy:

It is important for readers to note that both work and kinetic energy are scalar quantities. That is, quantities that do not depend on direction. Intuitively, if you get hit by a ball moving at some speed from your friend, it does not matter which way it moves and hits you, the impact will be equally painful.

WORKED EXAMPLE 3.2

A particle of mass 10kg has an initial speed of 5m/s and is acted upon by a force. The total work done by this force on the particle is 200J. Calculate the final speed of the particle.

Solution: Let the mass of the particle be m, initial and final speeds be v_1 and v_2 respectively, and let the work done be W.

From the Work-Energy Theorem, we have

$$W = E_{k2} - E_{k1} \tag{3.11}$$

$$W = \frac{1}{2}mv_2^2 - \frac{1}{2}mv_1^2 \tag{3.12}$$

$$\frac{1}{2}mv_2^2 = W + \frac{1}{2}mv_1^2 \tag{3.13}$$

$$v_2^2 = v_1^2 + \frac{2W}{m} \tag{3.14}$$

$$v_2 = \sqrt{v_1^2 + \frac{2W}{m}} \tag{3.15}$$

Plugging the values into Equation (3.13), we find that

$$v_2 = 8.06\text{m/s (3s.f.)} \tag{3.16}$$

which is our final answer. □

3.3 Potential Energy

DEFINITION 3.3

The **potential energy** of a body refers to the stored energy that depends upon the relative position of various parts of a system. It is a scalar quantity, and is measured in Joules (J)

Potential energy refers to, at least intuitively, 'how much money is stored in the bank'. The potential energy of an object can therefore be increased or decreased, depending on the situation. One of the best examples of potential energy is a battery. Chemical potential energy is stored within it, which is then used to drive electrons around a complete circuit when the battery is in use. Another form is the energy stored within a spring then it is stretched and compressed. When you let go of a spring in one of the two states mentioned above, it will vibrate and undergo oscillatory motion for a short period of time until friction with the surrounding air molecules or internal resistance dampens the oscillations until they are negligible.

3.3.1 Gravitational Potential Energy

Consider an object of mass m which moves through a vertical height $h << R_{Earth}$ with no change in speed. Since the height through which this object rises is significantly less than Earth's radius, the gravitational field strength can be very well approximated to be constant.

The downward force acting on the weight is mg.

Using our formula $W = Fd$, and replacing F with $-mg$ (we can do this as the gravitational force due to the object's weight acts in the exact opposite direction to the object's increase in vertical height), we find that

$$W_{grav.} = -mgh \tag{3.17}$$

Since the total work on the system must be zero by the Work-Energy Theorem because the object undergoes no change in speed, it follows that

$$W_{grav.} + W_{obj.=0} \tag{3.18}$$

and thus

$$W_{obj.} = mgh \tag{3.19}$$

The work done on the object is the energy that is stored within it, giving us the formula for gravitational potential energy,

$$E_p = mgh \tag{3.20}$$

REMARK:

You may be asking the very good question 'but what is h?'. After all, the object could be 2m above a microwave which is on the third floor, but at the same time, 10m above a pedestrian on the street! So which one do we choose? The answer is that it is entirely your decision! The height is relative, but it is preferred among students who do higher level physics to use a suitable reference point at the particle's initial position instead of, say, 10,000 m above it, because that way you can drop any annoying terms in the following equations that you must perform to answer the problem. That being said, in 'O' Level Physics, the situation will usually be with respect to the ground or some platform, and the setter will give you the relevant heights of that object above the surface. Unless, of course, that is what they want you to determine in the question.

3.3.2 Elastic Potential Energy

This subsection does not have to be read by readers who are solely focused on obtaining a good grade on their 'O' Level examinations. Therefore, this subsection may be skipped entirely.

Hooke's Law

Hooke's Law is an empirical law that states that the force needed to stretch or compress a string by some distance is directly proportional to that distance. In other words, Hooke's Law states that

$$F_S = kx \tag{3.21}$$

where the constant k is referred to as the *spring force constant.* Intuitively, the value k tells you how hard it is to stretch (or squish) the particular spring. You should note that F_S can be negative or positive; if you attach a spring to the wall and compress it, the force will act outward, while if you stretch it, the force will act inward. We really should therefore write

$$F_S = -kx \tag{3.22}$$

where x represents the displacement from equilibrium. This modified form demonstrates how the force produced by the spring acts in the opposite direction to it's displacement, and is an example of Simple Harmonic Motion (SHM), which is a very important topic in physics.

REMARK

We can actually solve for what Simple Harmonic Motion looks like. Consider a mass m attached to the end of a spring which obeys Hooke's Law, and suppose that no other net forces act on the system (no friction).

In differential form, Newton's Second Law states that

$$F = m\ddot{x} \tag{3.23}$$

where I have adopted Newton's convention that $\dot{x} = \frac{dx}{dt}$, $\ddot{x} = \frac{d^2x}{dt^2}$.

The force from the spring is given by Hooke's Law by our assumption,

$$F_S = -kx \tag{3.24}$$

Since no other (net) forces act on the mass, we may equate Equation 3.22 with Equation 3.21, giving

$$-kx = m\ddot{x} \tag{3.25}$$

and we obtain

$$m\ddot{x} + kx = 0 \tag{3.26}$$

, or simply

$$\ddot{x} + \frac{k}{m}x = 0 \tag{3.27}$$

This is a linear homogeneous second-order differential equation, and we have just reduced a physics problem into a math problem. A common procedure to solve such an equation is to guess a solution, for example suppose $x(t) = Ae^{i\omega t}$ is a solution to this equation, for some unknown constant ω where i denotes $\sqrt{-1}$ (if we can solve for ω, then we have found a valid solution to the differential equation. Note that the constant A is determined by the inital conditions). We then take successive derivatives, giving $\dot{x} = i\omega e^{i\omega t}$, and $\ddot{x} = -\omega^2 e^{i\omega t}$. Substituting this into Equation 3.25, we obtain

$$-\omega^2 Ae^{i\omega t} + \frac{k}{m}Ae^{i\omega t} = 0 \tag{3.28}$$

Dividing through by $Ae^{i\omega t}$ and rearranging terms, we find that

$$\omega^2 = \frac{k}{m} \tag{3.29}$$

and thus

$$\omega = \pm\sqrt{\frac{k}{m}} \tag{3.30}$$

We first take the positive root, then the negative root. Using Euler's Formula

$$e^{i\omega t} = \cos(\omega t) + i\sin(\omega t) \tag{3.31}$$

Thus the positive and negative roots give

$$x_1(t) = A\cos\left(\sqrt{\frac{k}{m}}t\right) + Ai\sin\left(\sqrt{\frac{k}{m}}t\right) \tag{3.32}$$

$$x_2(t) = B\cos\left(\sqrt{\frac{k}{m}}t\right) - Bi\sin\left(\sqrt{\frac{k}{m}}t\right) \tag{3.33}$$

by virtue of the fact that for all $x \in \mathbb{R}$, $\cos x = \cos(-x)$ and $\sin x = -\sin(-x)$.

These are both valid solutions to Equation 3.25 (the constants A and B cancel out), as you should verify. Using the Principle of Superposition, the function

$$x(t) := x_1(t) + x_2(t) \tag{3.34}$$

is also a valid solution to the differential equation.

$$x(t) = C\cos\left(\sqrt{\frac{k}{m}}t\right) + D\sin\left(\sqrt{\frac{k}{m}}t\right) \tag{3.35}$$

where $C := A + B$ and $D := (A - B)i$ are again determined by the initial conditions. Using R-formula, we can rewrite Equation 3.33 into a more familiar form

$$x(t) = K\cos\left(\sqrt{\frac{k}{m}}t + \phi\right) \tag{3.36}$$

Equation 3.34 describes the position of the mass with time. You may recall from trigonometry that the period of $x(t)$ is given by

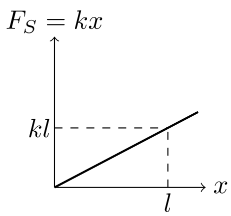
$$T = \frac{2\pi}{\sqrt{\frac{k}{m}}} = 2\pi\sqrt{\frac{m}{k}} \tag{3.37}$$

thus the frequency of oscillation (defined by $f := \frac{1}{T}$) is given by

$$f = \frac{1}{2\pi}\sqrt{\frac{k}{m}} \tag{3.38}$$

A more elaborate (but essentially the same) derivation of the SHM Equation can be found in the graduate-level textbook *Classical Mechanics by John R. Taylor*.

Consider the following graph below, which shows how the value of F_S varies with distance x away from the spring's equilibrium position.



Since the force characteristic is linear by Hooke's Law, it follows that the average force exerted by the spring is given by the average of the initial and final forces;

$$F_{S_{\text{avg}}} = \frac{kl + 0}{2} = \frac{kl}{2} \tag{3.39}$$

Hence, the total work done on the spring is given by

$$W_{\text{tot}} = \frac{1}{2}kl(l) = \frac{1}{2}kl^2 \tag{3.40}$$

because the situation is equivalent to that of a force of magnitude $kl/2$ acting through a distance l (we evaluated the average force). Therefore the energy stored in the spring, the *elastic potential energy* is given by

$$E_S = \frac{1}{2}kl^2 \tag{3.41}$$

where l is the extension.

REMARK 1 This derivation is not rigorous. This remark is meant for readers who are not satisfied with my derivation above, and have taken some vector calculus.

Define work as the line integral

$$W = \int_C \vec{F} \cdot d\vec{s} \tag{3.42}$$

In this case $\vec{F} = -kx\hat{i}$ and $d\vec{s} = dx\hat{i}$, so the dot product yields $\vec{F} \cdot d\vec{s} = -kxdx$ and the line integral becomes

$$W = \int_0^l -kxdx = -\frac{1}{2}kl^2 \tag{3.43}$$

So the work done *by* the spring is negative, and the potential energy stored inside is

$$E_S = \frac{1}{2}kl^2 \tag{3.44}$$

Which is the desired result. This is an example of a quadratic energy potential function.

3.4 The Conservation of Energy

We are now ready to tackle one of the most profound and beautiful conclusions in physics: **The Conservation of Energy**

DEFINITION 3.4

The Law of Conservation of Energy states that, energy cannot be created or destroyed, energy can be converted from one store to another, and that the total energy of *an isolated system* remains constant.

An isolated system is exactly what you would think it is, a system of particles that are completely isolated from the outside world. There is thus no exchange of energy with an isolated system.

Quantitatively, the Conservation of (mechanical) Energy states that

$$E_{k1} + E_{p1} = E_{k2} + E_{p2} \tag{3.45}$$

So the sum of the initial kinetic and potential energies of the system is the same of the sum of the final kinetic and potential energies of the system, which implies that the total energy of the system must be constant. This is in accordance with the definition.

EXAMPLE 3.4.1

A ball of mass 10.0kg falls from rest through a vertical height of 10.0m in vacuum. The gravitational field strength can be taken to be a constant, at 10N/kg. Calculate the final speed of the ball using the equations developed in Section 1.4 (a kinematic approach), and by using the Conservation of Mechanical Energy.

Solution:

USING KINEMATICS

Recall Equation 1.12 from Section 1.4:

$$v_2^2 = v_1^2 + 2as \tag{3.46}$$

In this case, the acceleration a of the ball (I will take downwards as positive) is given by the gravitational field strength g, because by Newton's Second Law (cf. Section 2.2), the downward resultant force is given by $F = ma = mg$ (the ball is in vacuum, so no other force acts on it apart from it's weight), so it follows that $a = g$ by dividing through by m. Furthermore, the ball is dropped from rest, so $v_1 = 0$. Applying all these initial conditions to our equation above yields the relation

$$v_2^2 = 2gh \tag{3.47}$$

where h is defined to be the height the ball is dropped through. Taking the square root, we obtain

$$v_2 = \sqrt{2gh} \tag{3.48}$$

Substituting our values,

$$v_2 = \sqrt{(10\text{kg})(10.0\text{N/kg})} \tag{3.49}$$

$$v_2 = 10.0\text{m/s (3s.f.)} \tag{3.50}$$

which is our final answer.

USING ENERGY CONSIDERATIONS

The Conservation of Mechanical Energy states that

$$E_{k1} + E_{p1} = E_{k2} + E_{p2} \tag{3.51}$$

For the sake of convenience, let's measure the height of the ball from it's final point; i.e. it's final position after falling 10.0m. Now $E_{k1} = 0$, because $E_{k1} = \frac{1}{2}mv_1^2$ and $v_1 = 0$ as the ball was dropped from rest. Since we are measuring the height of the ball from it's final point, $E_{p2} = 0$, since $h_2 = 0$ and $E_{p2} = mgh_2$. Our energy relation therefore simplifies to become

$$E_{p1} = E_{k2} \tag{3.52}$$

Remember, we were asked to find the ball's final speed, or in other words, to determine the value of v_2.

Writing down the definitions of E_{p1} and E_{k2} yields

$$mgh_1 = \frac{1}{2}mv_2^2 \tag{3.53}$$

dividing both sides by m and solving for v_2 gives the relation

$$v_2^2 = 2gh_1 \tag{3.54}$$

or in other words

$$v_2 = \sqrt{2gh_1} \tag{3.55}$$

which is the exact same equation that we obtained using the kinematic approach, but this time, much easier to derive! Since the equations are the same, it follows that we must have

$$v_2 = 10.0\text{m/s (3s.f.)} \tag{3.56}$$

so our answers match, as they should! ■

3.5 Power

The work done only tells you how much energy you have transferred to the object. It does not tell you how long it has taken. For instance, if you raise a box of mass 5kg through a vertical height of 10m, then it follows that the work you have done is given by

$$W = (5kg)(10\text{N/kg})(10\text{m}) = 500\text{J}$$

This does not tell you how long you have taken to do so. The amount of work required to lift this $5kg$ box in 1 minute is the same as the amount of work required to lift the box in 1000 years. Therefore, we must have some way to measure how *quickly* work is being done, leading to our following discussion on the concept of power.

DEFINITION 3.5

Power is the **rate** of work done or **energy transferred** per unit time. It is a scalar quantity, and is measured in Watts (W)

From Definition 3.5, we deduce the formula for the average power performed:

$$P = \frac{W}{t} \tag{3.57}$$

This equation tells you how much energy is being transferred per second. If you buy a 60W lightbulb from the hardware store, it will convert 60 Joules of electrical energy per second into heat and light energy (in the form of emitted photons, which are tiny particles of light).

EXAMPLE 3.5

A box of mass 10kg is raised upward through a vertical height of 20m in 20s by a pulley. It's speed remains constant throughout the motion. Calculate the average power of the pulley. Take $g = 10\text{N/kg}$.

Solution By the Work Energy Theorem, the total work done is zero because there is no change in the kinetic energy of the box. Hence we must have

$$W_{\text{pullley}} = -W_{\text{grav.}} \tag{3.58}$$

otherwise the total work done would be nonzero, which contradicts the statement of the Work Energy Theorem.

From Section 3.3.1, we have $W_{\text{grav.}} = -mgh$, so it follows that

$$W_{\text{pullley}} = -(-mgh) = mgh \tag{3.59}$$

By Definition,

$$P_{\text{pulley}} = \frac{mgh}{t} \tag{3.60}$$

Hence,

$$P_{\text{pulley}} = \frac{(10\text{kg})(20\text{m})(10\text{N/kg})}{20\text{s}} \tag{3.61}$$

Giving

$$P_{\text{pulley}} = 100\text{W (3s.f.)} \tag{3.62}$$

which is our final answer. □

REMARK

Formally speaking, we define power as the rate of work done at a given time, i.e.

$$P = \frac{dW}{dt} \tag{3.63}$$

so the definition that I gave above in Equation 3.39 is simply the formula for the average power.

You have most definitely seen the power labels on lightbulbs or any household electrical appliance, but this quantity need not solely be for electrical purposes (as you will see in later chapters of the syllabus document). Power can also be used for cars, in fact, the term 1 horsepower (the strength of an engine) means 736kW.

It is important to note that from this definition, we can derive an alternative expression for power:

From the definition of work done, we have

$$P = \frac{Fd}{t} \tag{3.64}$$

But since the speed (or velocity if the motion is 1D and in the positive direction) is given by

$$v = \frac{d}{t} \tag{3.65}$$

It follows that

$$P = Fv$$

Chapter 4

Pressure and density

4.1 Introduction

Pressure and density are some of the most important things that determine whether a planet or a region is liveable, and even the *shape* of the creatures that inhabit the region. Simply look up the photos of an anglerfish and a goldfish, and you will see what I mean. We start this chapter with a definition.

DEFINITION 4.1

The **density** of an object is the amount of mass per unit volume of the object.

This definition of density is perfectly reasonable for the 'O' Levels, but a very fussy reader may want something like

DEFINITION 4.1*

The **average density** of an object is the average mass per unit volume of the object.

From Definition 4.1, we deduce the formula for average density:

$$\rho = \frac{m}{V} \tag{4.1}$$

Where the Greek letter rho (ρ) represents the average density of the object, and V represents the total volume of the object.

Before we progress to the real crux of this chapter, we need one more definition.

Definition 4.2

The **pressure** due to a force is the total force per unit area. This quantity has units of (Nm^{-2}), and this is denoted by the pascal (Pa)

From this definition, we deduce the second formula,

$$P = \frac{F}{A} \tag{4.2}$$

Again, when I say pressure in this book, since this is supposed to be an introductory text to the subject, I mean the average pressure, not the *exact pressure* at a particular point.

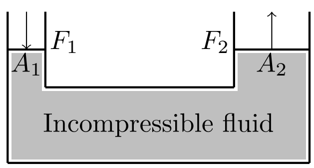
4.2 Pascal's Principle

In 1663, the French mathematician Blaise Pascal published his findings on what we now know as **Pascal's Principle:**

Pascal's principle: if pressure is applied to an enclosed fluid, the pressure is transmitted equally throughout to all other parts of the fluid.

REMARK: It should be noted that the fluid is assumed to be incompressible, as is the case with water.

4.2.1 Derivation of the Hydraulic Press Equation



Pascal's Principle has an important use in hydraulic presses. Consider a system which encloses an incompressible fluid in a pipe, with pistons on either end (see above). Let the left-hand piston (P1) have area A_1 and the right-hand piston (P2) have area A_2. Suppose we apply a force F_1 on the left-hand piston. The pressure on this piston is therefore (by (4.2)) given by $\frac{F_1}{A_1}$. But of course, the pressure at P1 must be equal to the pressure at P2. This follows by invoking Pascal's Principle, as the liquid, perhaps oil, is assumed to be incompressible. Consequently, the (upward) force generated at P2 due to the pressure exerted on P2 by the fluid is given by

$$F_2 = A_2 \frac{F_1}{A_1} \tag{4.3}$$

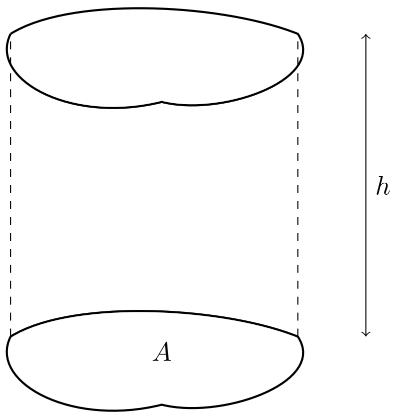
. (From 4.2, $F = AP$). Dividing both sides of Equation 4.3 by the quantity A_2, we obtain

$$\frac{F_1}{A_1} = \frac{F_2}{A_2} \tag{4.4}$$

You *can* memorise Equation 4.4, but I would rather you just note that the pressure at both ends is equal, and 4.4 immediately follows by setting the two force-area quotients equal to one another (though, not rigorously). I leave you to make a decision.

4.3 Pressure at a Depth

You should refer to the figure below for the following discussion.



Consider a column of liquid of height h and density ρ. For the sake of argument, suppose that the gravitational field strength remains constant, and call it g. Let the base area of this column be A (A can be anything, except zero). The volume of water V is given by

$$V = Ah \tag{4.5}$$

So since $m = \rho V$ by Equation 4.1, we obtain an equation for the mass of the water column:

$$m = \rho Ah \tag{4.6}$$

Consequently, the weight W of the water column is given by

$$W = \rho Ahg \tag{4.7}$$

Now the pressure at the bottom of the column is given by

$$p = \frac{W}{A} = \frac{\rho Ahg}{A} = \rho gh \tag{4.8}$$

by Equation 4.2. The part of Equation 4.8 that you should memorise is $p = \rho gh$.

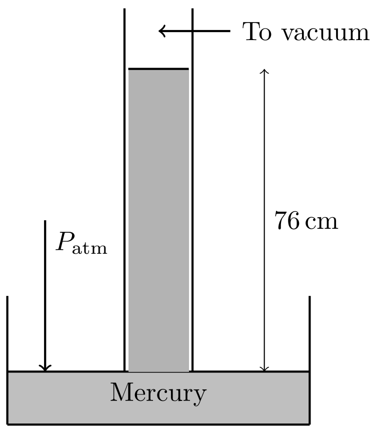
REMARK: This derivation is *not* rigorous. For a more complete treatment, one should confer to the book Young & Freedman University Physics 13th Edition, page 376.

4.3.1 Applications to a Manometer

Manometers are devices that are used to measure pressure. In the case study below I will demonstrate how to calculate atmospheric pressure using a mercury manometer. Note that they can also be in the shape of a tube.

4.3.2 Calculating Atmospheric Pressure using a Manometer

A mercury manometer is placed at sea-level, as shown below. The top of the column is subjected to a vacuum, so there is no internal pressure. The height of the mercury column reads 76mm at equilibrium. The density of mercury is 13600kg/m^3. Calculate Earth's atmospheric pressure, P_{atm} at sea level.



Solution: Since the mercury in the manometer is in equilibrium, there is no net force acting on it. This amounts to saying that the pressure of the mercury column is equivalent to atmospheric pressure. Otherwise, one of the forces produced (either due to the atmosphere or the mercury column) would be greater than the other, which would create a net force, contradicting the initial condition that the manometer is in equilibrium. Using Equation 4.8, we obtain

$$P_{\text{atm}} = \rho g h \tag{4.9}$$

Substituting values:

$$P_{\text{atm}} = (13600\text{kg/m}^3)(10\text{N/kg})(76 \times 10^{-2}\text{m}) \tag{4.10}$$

we obtain

$$P_{\text{atm}} = 103 \times 10^3\text{Pa}(3\text{s.f.}) \tag{4.11}$$

which is our final answer. ■

Chapter 5

Moments

Most of classical physics is concerned with the motion of objects. Problems involving motion are very fun to work on, however we require many things in the world to be static. For instance, you would not want a skyscraper to rock back and forth. This brings us to the topic of moments.

5.1 The Moment

DEFINITION 5.1

The **moment M** of a force is defined as the product of the force F and the perpendicular distance d from a pivot to the line of action of the force, ($M = Fd$). It is measured in newton-meters, (Nm).

The definition above deserves two remarks. Intuitively, the moment of a force on an object about some pivot represents the 'rotating power' of the force on that object. This is also referred to as torque. (τ).

The second remark is that you should not get confused with the unit of measurement for the moment of a force and of energy. Always use the Joule (J) for energy, and Nm for moments to prevent confusion.

REMARK

The formal definition of torque is

$$\tau = \vec{F} \times \vec{r}$$

where $\times$ denotes the cross product between two vectors. For reasons that I will not elaborate on in this book (search the magnitude of the cross product), the magnitude of the torque is given by $Fr \sin\theta$, where θ denotes the angle between the two vectors $\vec{F}$ and $\vec{r}$. Consider the case where $\theta = 90°$. Then $\sin(\theta) = 1$, and we obtain $\tau = Fr$. This special case is the one that will be considered in this book, and the one that will (mostly) be given in the examination.

WORKED EXAMPLE 5.1

Calculate the moment of a perpendicular force of 5N on a acting on a 1m long rod from a pivot at the end of the rod.

Solution: By Definition 5.1, $M = Fd$. Here, we are given that $F = 5N$ and $d = 1$m. Hence $M = (5N)(1m) = 5$Nm. □

5.2 The Principle of Moments

DEFINITION 5.2

Principle of moments: when a body is in (rotational) equilibrium, the sum of clockwise moments about a pivot is equal to the sum of anticlockwise moments about the same pivot.

The only way to learn how to use this definition is to do many problems involving the principle of moments. When solving such problems, calculate the sum of all the anticlockwise (ACW) moments and the sum of all clockwise (CW) moments, and set $\sum M_{\text{CW}} = \sum M_{\text{ACW}}$

5.3 Centre of Gravity

DEFINITION 5.3

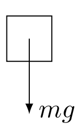
Centre of gravity: the imaginary point through which the entire mass of an object appears to act for any orientation.

Intuitively, the centre of gravity is the point with which you can balance the entire object on the tip of your finger. One method one can use to determine thee position of an object's centre of gravity is by using the *plumb line* method. I highly recommend watching videos on how this is done to get a good visualisation.

Another method which is more precise (and fun) is by using surface integrals, which you will learn in a Calculus 3 or 4 course in university. See Lang, Calculus of Several Variables pg 313.

5.3.1 Case Study: Rotating a Square

For a simple example of the utility of Definition 5.3, we consider a square as shown below. This square has weight mg, acting from the geometric centre (assuming uniform density)



Now consider rotating this square about the bottom-right corner through an angle less than 45°. The mg vector lies inside the base area of the square, and there is a non-zero distance between it and the pivot (taken as the corner of the square). Hence, with respect to the page, an anticlockwise moment acts about the right hand corner of the square, so it falls back to it's original position.

Now consider rotating this square about the bottom-right corner through an angle of 45°. Then the geometric centre (in this case, the centre of gravity) lies directly above the corner (which we take as the pivot). No net moment acts, as the distance between the force (due to the weight) and the pivot is zero (they lie directly atop one another). Consequently, the square balances on it's edge.

Now suppose the square is rotated through an angle greater than 45°. In this case, the weight-vector lies outside the base area of the square, and has a perpendicular component to the line between the pivot and itself. Therefore, a clockwise moment is produced, and the square topples over.

5.4 Applications to the Stability of an Object

As can be deduced from the case study above, the *wider* the base area of an object, the harder it is for it to topple over (you will have to rotate it more in order for it's centre of gravity to lie outside it's base area). Furthermore, if the centre of gravity of an object lies lower, it will also be harder to topple the object by the same reasoning. This gives us a definition of stability:

DEFINITION 5.4

Stability: the ability of an object to return to its original position after it has been slightly displaced.

This explains why it is much easier to say, balance a filled water bottle on its base rather than it's cap. The base has a greater area than the cap, so it is more difficult for the bottle's centre of gravity to be displaced outside the bottle's base rather than the cap.

Sports cars, as well as Formula 1 Cars stability to their advantage by bringing the car's centre of gravity as low as possible. This makes them more stable, enabling them to corner faster, which improves lap-time. In contrast, Sports-Utility Vehicles (SUVs) tend to fail the Moose Test, which involves cornering around a series of cones at $60 km/h$ as their centre of gravity is relatively high, causing them to roll over.