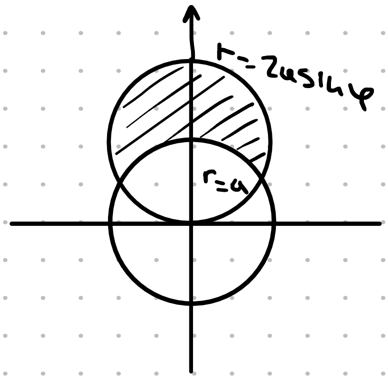
ИЗ 93 8.34 не делать!

Перейти к полярным координатам

$$\iint_G f(x,y)\,dx\,dy \quad G=\{a^2<x^2+y^2<2ay\}$$



Пр. $x=r\cos\varphi$ $\quad r\geq a$

$y=r\sin\varphi$ $\quad x^2+(y-a)^2\leq a^2$

$r^2\leq 2ar\sin\varphi$

$r\leq 2a\sin\varphi$

$a=2a\sin\varphi$

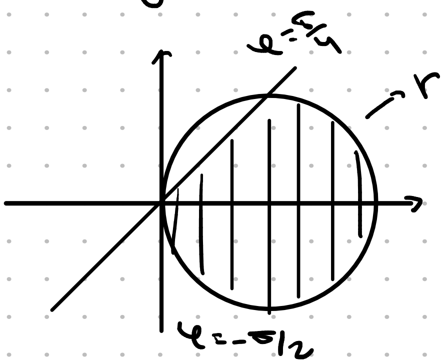
$\sin\varphi=1/2$

$\varphi_1=\pi/6 \quad \varphi_2=5\pi/6$ $\qquad \varphi=\arcsin\left(\frac{r}{2a}\right)$

$$\int_{\pi/6}^{5\pi/6}d\varphi\int_a^{2a\sin\varphi} r f(r\cos\varphi, r\sin\varphi)\,dr \qquad \int_a^{2a} r\,dr\int_{\arcsin(\frac{r}{2a})}^{\pi-\arcsin(\frac{r}{2a})} f(r\cos\varphi, r\sin\varphi)\,d\varphi$$

Пр. $\iint_G y\,dx\,dy \quad G=\{x^2+y^2<2x,\ x>y\}$

$(x-1)^2+y^2<1$



$$\int_{-\pi/2}^{\pi/4}d\varphi\int_0^{2\cos\varphi} r\,r\sin\varphi\,dr=\int_{-\pi/2}^{\pi/4}\sin\varphi\,d\varphi\left.\frac{r^3}{3}\right|_0^{2\cos\varphi}=$$

$$=\int_{-\pi/2}^{\pi/4}\frac{(2\cos\varphi)^3}{3}\sin\varphi\,d\varphi=\int_{-\pi/2}^{\pi/4}-\frac{8}{3}t^3dt=-\frac{8}{3}\left.\frac{t^4}{4}\right|_0^{1/\sqrt{2}}=-\frac{1}{6}$$

$\cos\varphi=t$

$dt=-\sin\varphi\,d\varphi$

$$\mu G=S_G=\iint_G 1\,dx\,dy$$

Найти S: $(x^2+y^2)^2=2a^2(x^2-y^2)$

$x^2+y^2=a^2 \quad (x^2+y^2>a^2) \quad a>0$

$r^4=2a^2\cdot r^2\cdot\cos2\varphi$

$r=a\sqrt{2\cos2\varphi}$

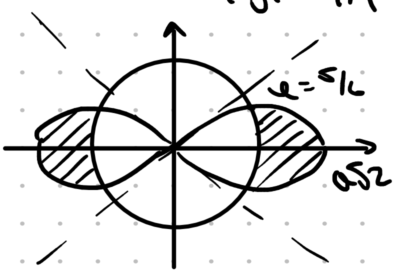
$\cos2\varphi\geq0$

$|y|\leq|x|$

$a=a\sqrt{2\cos2\varphi}$

$\cos2\varphi=1/2$

$\varphi=\pi/6$



$$S=\iint_G d\varphi\,dr=\int_0^{\pi/6}d\varphi\int_0^{a\sqrt{2\cos2\varphi}} r\,dr=\int_0^{\pi/6}d\varphi\left.\frac{r^2}{2}\right|_0^{a\sqrt{2\cos2\varphi}}=$$

$$=4\int_0^{\pi/6}\frac{a^2}{2}(2\cos2\varphi-1)\,d\varphi=2a^2(\sin2\varphi-\varphi)\Big|_0^{\pi/6}=$$

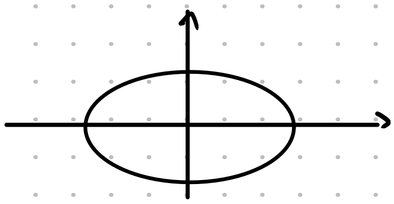
$$=2a^2(\sqrt{3}/2-\pi/6)=a^2\left(\sqrt{3}-\frac{\pi}{3}\right)$$

Полярные коорд. с растяжением.

$x = ar\cos\varphi \qquad \dfrac{D(x,y)}{D(r,\varphi)} = abr$

$y = br\sin\varphi$

Пр. площадь эллипса $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$



$r \le 1$

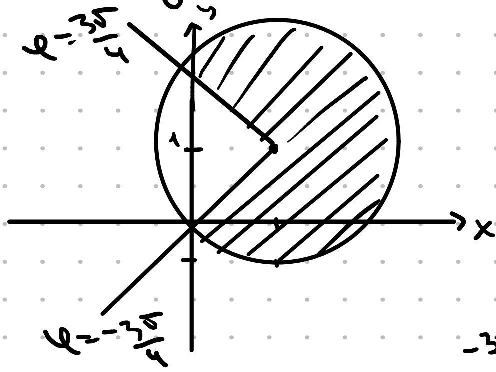
$$S = \iint\limits_{G} dx\,dy = \int_0^{2\pi} d\varphi \int_0^1 abr\,dr = ab \cdot 2\pi \int_0^1 r\,dr = \underline{\pi ab}$$

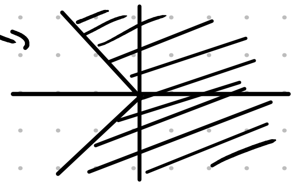
Полярные коорд. со сдвигом.

$x = x_0 + r\cos\varphi \qquad \dfrac{D(x,y)}{D(r,\varphi)} = r$

$y = y_0 + r\sin\varphi$

Пр. $\iint\limits_{G} (x^2+y^2)\,dx\,dy \qquad G: \begin{cases} x^2+y^2 < 2(x+y) \\ x-1+|y-1| > 0 \end{cases} \qquad (x-1)^2+(y-1)^2 < 2$





$x = 1 + r\cos\varphi$

$y = 1 + r\sin\varphi$

$0 < r < \sqrt{2}$

$-\frac{3\pi}{4} < \varphi < \frac{3\pi}{4}$

$$\int_{-3\pi/4}^{3\pi/4} d\varphi \int_0^{\sqrt{2}} r\left(1 + 2r\cos\varphi + \underline{r^2\cos^2\varphi} + 1 + 2r\sin\varphi + \underline{r^2\sin^2\varphi}\right) =$$

$$= \int_{-3\pi/4}^{3\pi/4} d\varphi \int_0^{\sqrt{2}} (r^3 + 2r)\,dr + \int_{-3\pi/4}^{3\pi/4} d\varphi \int_0^{\sqrt{2}} 2r^2(\sin\varphi + \cos\varphi)\,dr = \frac{3\pi}{2}\left(r^2 + \frac{r^4}{4}\right)\Big|_0^{\sqrt{2}} + \int_0^{\sqrt{2}} 2r^2 dr \cdot \underbrace{\int_{-3\pi/4}^{3\pi/4} (\sin\varphi + \cos\varphi)\,d\varphi}_{s=0 \text{ (нечет)}} =$$

$$= \frac{9\pi}{2} + \frac{4\sqrt{2}}{3} \cdot 2\sin\frac{3\pi}{4} - \frac{\sqrt{2}}{2} \cdot 2 \cdot \frac{4\sqrt{2}}{3} = \frac{8}{3}$$

Ответ: $\frac{8}{3}$

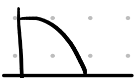
Обобщенные полярн. коорд.

$x = ar\cos^k\varphi$

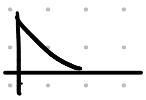
$y = ar\sin^k\varphi$

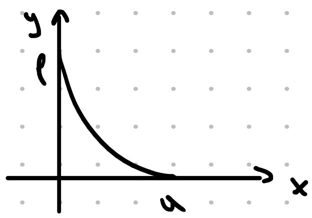
Пр. Найти площадь $\sqrt[4]{\dfrac{x}{a}} + \sqrt[4]{\dfrac{y}{b}} = 1 \quad x = 0, y = 0 \quad a, b > 0$

$x^2 + y^2 = 1$



$x + y = 1$





$x = ar\cos^8\varphi$

$y = br\sin^8\varphi$

$$S = \int_0^{\pi/2} d\varphi \int_0^1 \left|\frac{D(x,y)}{D(r,\varphi)}\right| dr$$

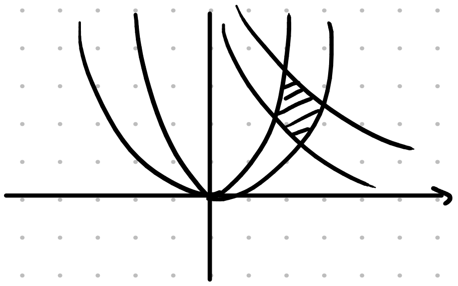
$$\left|\frac{D(x,y)}{D(r,\varphi)}\right| = \begin{vmatrix} a\cos^8\varphi & -ar8\cos^7\varphi\sin\varphi \\ b\sin^8\varphi & 8br\sin^7\varphi\cos\varphi \end{vmatrix} = 8abr(\cos^9\varphi\sin^7\varphi + \sin^9\varphi\cos^7\varphi) =$$

$$= 8abr\cos^7\varphi\sin^7\varphi = \frac{1}{2^7} \cdot 8abr\sin^7 2\varphi = \frac{1}{32}ab\int_0^{\pi} \sin^7 t\,dt = \frac{1}{64}\int (1-u^2)^3 du = \begin{bmatrix} u = \cos t \\ du = -\sin t\,dt \end{bmatrix}$$

$$= \ldots \frac{1}{32} \cdot \frac{16}{35} \cdot ab = \underline{\frac{1}{70}ab}$$

Пр. S?

$y = ax^2 \quad y = bx^2 \quad xy = p \quad xy = q \;;\; b > a > 0 \;;\; q > p > 0$



$u = \frac{y}{x^2} \quad v = xy$

$a < u < b$

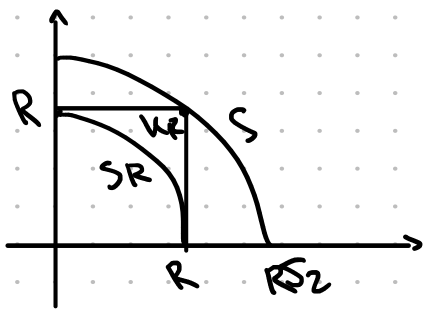
$p < v < q$

$$S = \int_a^b du \int_p^q \left| \frac{D(x,y)}{D(u,v)} \right| dv = \int_a^b du \int_p^q \frac{1}{3} u^{-1} dv =$$

$$= \frac{1}{3}(q-p)\ln\frac{b}{a}$$

$$\frac{D(x,y)}{D(u,v)} = \begin{vmatrix} -1/3\, u^{-4/3} v^{1/3} & 1/3\, u^{-1/3} v^{-2/3} \\ 1/3\, u^{-2/3} v^{2/3} & \frac{2}{3} u^{1/3} v^{-1/3} \end{vmatrix} = -\frac{2}{9}u^{-1} - \frac{1}{9}u^{-1} = -\frac{1}{3}u^{-1}$$

Интеграл Пуассона $\quad \int_0^{+\infty} e^{-x^2} dx = \frac{\sqrt{\pi}}{2}$



$$\iint_{S_R} e^{-x^2-y^2} dx\,dy \le \iint_{K_R} e^{-x^2-y^2} dx\,dy \le \iint_{S_{R\sqrt{2}}} e^{-x^2-y^2} dx\,dy$$

$$\iint_{K_R} e^{-x^2-y^2} dx\,dy = \int_0^{\pi/2} d\varphi \int_0^R r\, e^{-r^2} dr = \frac{\pi}{2 \cdot 2} \int_0^{R^2} e^{-t} dt = \frac{\pi}{4}\left(1 - e^{-R^2}\right)$$

$$\frac{\pi}{4}\left(1 - e^{-R^2}\right) \le I_R^2 \le \frac{\pi}{4}\left(1 - e^{-2R^2}\right) \qquad R \to \infty$$

$$\frac{\pi}{4} \le I^2 \le \frac{\pi}{4} \;;\; I^2 = \pi/4 \quad I = \frac{\sqrt{\pi}}{2}$$