Дз Халили Виктории Б01-302 по матану.
Задание 1, 3 семестр

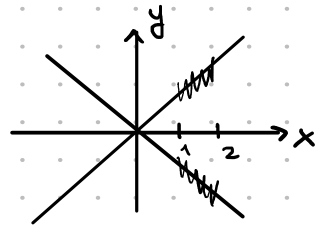
I.

Т1 $x^2 = y^2$

а) Сколько $y: \mathbb{R} \to \mathbb{R}$?

Ответ: беск. много

Пример: $f(x) = \begin{cases} x, & x \in X \\ -x, & x \notin X \end{cases}$ $X \in \mathbb{R}$ — таких функций беск. много



б) Сколько непр. $y: \mathbb{R} \to \mathbb{R}$?

Ответ: 4 : $y = x$; $y = -x$; $y = |x|$; $y = -|x|$

в) Сколько непр. $y: \mathbb{R} \to \mathbb{R}$ и $y(1) = 1$?

Ответ: 2 : $y = x$; $y = |x|$

г) Сколько непр. $y: [1,2] \to \mathbb{R}$ и $y(1) = 1$?

Ответ: 1 : $y = x$

§3

№61 2) Найти u'_x, u'_y в $(0,1,0)$ Проверим: $\cos 0 = 1$ — верно

$$x\cos y + y\cos u + u\cos x = 1$$

$$\cos y\,dx - \cancel{x\sin y\,dy} + \cos u\,dy - y\cancel{\sin u}\,du + \cos x\,du - \cancel{u\sin x\,dx} = 0$$

$x = 0$
$y = 1$
$u = 0$

$$\cos 1\,dx + dy + du = 0$$

$$du = -\cos 1\,dx - dy \Rightarrow \text{Ответ: } u'_x = -\cos 1;\ u'_y = -1$$

№64 2) Найти du в а) $(1,1,1)$ б) $(1,1,-2)$

$$x^3 + 2y^3 + u^3 - 3xyu + 2y - 3 = 0$$

$$3x^2dx + 6y^2dy + 3u^2du - 3xy\,du - 3xu\,dy - 3yu\,dx + 2dy = 0$$

$$(3x^2 - 3yu)\,dx + (6y^2 - 3xu + 2)\,dy + (3u^2 - 3xy)\,du = 0 \quad (1)$$

б) Проверим: $1 + 2 - 8 + 6 + 2 - 3 = 0$

$(1) \to (3+6)\,dx + (6+6+2)\,dy + (12-3)\,du = 0$

$9dx + 14dy + 9du = 0$

$du = -dx - \frac{14}{9}dy$

а) Проверим: $1 + 2 + 1 - 3 + 2 - 3 = 0$

$(1) \to 3dy = 0 \Rightarrow du$ не сущ.

Ответ: а) в т.$(1,1,1)$ du не сущ.
б) в т.$(1,1,-2)$ $du = -dx - \frac{14}{9}dy$

№69 $f(x-y;\, y-z;\, z-x) = 0$; $f(u,v,w)$ — дифф.; $z = z(x,y)$
Найти $dz(x,y)$

$$f'_u(dx - dy) + f'_v(dy - dz) + f'_w(dz - dx) = 0$$

$$dx(f'_u - f'_w) + dy(f'_v - f'_u) + dz(f'_w - f'_v) = 0$$

$$\Rightarrow \text{Ответ: } dz = \frac{(f'_u - f'_w)\,dx + (f'_v - f'_u)\,dy}{f'_v - f'_w}$$

№75 $u=u(x,y)$ $v=v(x,y)$ Найти: u'_x, u'_y, v'_x, v'_y в т. $(1,2)$

$$\begin{cases} xe^{u+v}+2uv=1 \\ ye^{u-v}-\frac{u}{1+v}=2x \end{cases} ; \quad x=1;\ y=2;\ u=v=0$$

$$\begin{cases} e^{u+v}+xe^{u+v}(u'_x+v'_x)+2uv'_x+2u'_xu=0 \\ xe^{u+v}(u'_y+v'_y)+2uv'_y+2u'_yv=0 \\ e^{u-v}+ye^{u-v}(u'_y-v'_y)-\frac{u'_y(1+v)-v'_yu}{(1+v)^2}=0 \\ ye^{u-v}(u'_x-v'_x)-\frac{u'_x(1+v)-v'_xu}{(1+v)^2}=2 \end{cases} \quad \begin{cases} 1+u'_x+v'_x=0 \\ u'_y+v'_y=0 \\ 1+2(u'_y-v'_y)-u'_y=0 \\ 2(u'_x-v'_x)-u'_x=2 \end{cases} \quad \begin{cases} u'_x=0 \\ v'_x=-1 \\ u'_y=1/3 \\ v'_y=-1/3 \end{cases}$$

Ответ: $u'_x=0$; $u'_y=1/3$; $v'_x=-1$; $v'_y=-1/3$

§4

№43 5) Найти: d^2u в т. $(1,-1,-1)$

$$u^3+2yu+xy=0$$
$$3u^2du+2ydu+2udy+xdy+ydx=0$$
$$3du-2du-2dy+dy-dx=0$$
$$-du+dy-dx=0 \quad \leftarrow \text{в т. } (1,-1,-1)$$
$$\underline{du=dy-dx}$$
$$6udu^2+3u^2d^2u+2dudy+2yd^2u+2dudy+2ud^2y+dxdy+xd^2y+dxdy+yd^2x=0$$
$$d^2u(3u^2+2y)+4dudy+d^2y(2u+x)+2dxdy+yd^2x+6udu^2=0$$

в т. $(\underset{x}{1},\underset{y}{-1},\underset{u}{-1})$:
$$d^2u+4dudy-d^2y+2dxdy-d^2x-6du^2=0$$
$$d^2u+4dy^2-4dydx-d^2y+2dxdy-d^2x-6(dy-dx)^2=0$$
$$d^2u+4dy^2-2dydx-d^2y-d^2x-6(dy^2-2dydx+dx^2)=0$$
$$d^2u-2dy^2-6dydx-\cancel{d^2y}-\cancel{d^2x}-6dx^2=0$$
$$d^2u=6dx^2+6dxdy+2dy^2 \Rightarrow$$

Ответ: $d^2u=6dx^2+6dxdy+2dy^2$

№46 1) $f(x+u,y+u)=0$ $\quad u=u(x,y)$ Найти: $d^2u(x,y)$
$$f'_1(dx+du)+f'_2(dy+du)=0$$
$$f'_1dx+f'_1du+f'_2dy+f'_2du=0 \Rightarrow du=-\frac{f'_1dx+f'_2dy}{f'_1+f'_2}$$
$$dx+du=\frac{f'_1dx+f'_2dx-f'_1dx-f'_2dy}{f'_1+f'_2}=\frac{f'_2(dx-dy)}{f'_1+f'_2}$$
$$dy+du=\frac{f'_1dy+f'_2dy-f'_1dx-f'_2dy}{f'_1+f'_2}=-\frac{f'_1(dx-dy)}{f'_1+f'_2}$$

$$df=f'_1\cdot d(x+u)+f'_2\cdot d(y+u)$$
$$d^2f=f''_{11}\cdot d((x+u))^2+2f''_{12}\,d(x+u)\cdot d(y+u)+f''_{22}\cdot d((y+u))^2+f'_1d^2u+f'_2d^2u=0$$
$$-d^2u(f'_1+f'_2)=f''_{11}\cdot\frac{f'^2_2(dx-dy)^2}{(f'_1+f'_2)^2}-2\cdot f''_{12}\cdot\frac{f'_1\cdot f'_2(dx-dy)^2}{(f'_1+f'_2)^2}+f''_{22}\cdot\frac{f'^2_1(dx-dy)^2}{(f'_1+f'_2)^2} \Rightarrow$$

$\Rightarrow$ Ответ: $d^2u=-\frac{(dx-dy)^2}{(f'_1+f'_2)^3}\left(f''_{11}\cdot f'^2_2-2\cdot f''_{12}f'_1f'_2+f''_{22}f'^2_1\right)$

II.

§3 N105 Найти: $\frac{\partial(x,y,z)}{\partial(r,\varphi,\psi)}$: $\begin{array}{l} x = r\cos^p\varphi\cos^q\psi \\ y = r\sin^p\varphi\cos^q\psi \\ z = r\sin^q\psi \end{array}$ $p, q \in \mathbb{N}$

$$\frac{\partial(x,y,z)}{\partial(r,\varphi,\psi)} = \begin{vmatrix} \cos^p\varphi\cdot\cos^q\psi & -rp\cdot\cos^{p-1}\varphi\cdot\sin\varphi\cdot\cos^q\psi & -rq\sin\psi\cos^p\varphi\cos^{q-1}\psi \\ \sin^p\varphi\cos^q\psi & rp\cos\varphi\sin^{p-1}\varphi\cdot\cos^q\varphi & -rq\cdot\sin\psi\cdot\cos^{q-1}\psi\sin^p\varphi \\ \sin^q\psi & 0 & rq\cos\psi\sin^{q-1}\psi \end{vmatrix} =$$

$$= \sin^q\psi\left[r^2pq\sin^{p+1}\varphi\cos^{p-1}\varphi\sin\psi\cos^{2q-1}\psi + r^2pq\sin^{p-1}\varphi\cos^{p+1}\varphi\sin\psi\cos^{2q-1}\psi\right] +$$
$$+ rq\cos\psi\sin^{q-1}\psi\left[rp\sin^{p-1}\varphi\cos^{p+1}\varphi\cos^{2q}\psi + rp\sin^{p+1}\varphi\cos^{p-1}\varphi\cos^{2q}\varphi\right] =$$
$$= r^2pq\sin^{p-1}\varphi\cos^{p-1}\varphi\sin^{q+1}\varphi\sin^{q+1}\psi\cos^{2q-1}\psi + r^2pq\sin^{p-1}\varphi\cos^{p-1}\varphi\sin^{q-1}\varphi\cos^{2q+1}\psi =$$
$$= r^2pq\sin^{p-1}\varphi\cos^{p-1}\varphi\sin^{q-1}\psi\cos^{2q-1}\psi$$

$\Rightarrow$ Ответ: $r^2pq(\sin\varphi\cos\varphi)^{p-1}\cdot(\sin\psi)^{q-1}\cdot(\cos\psi)^{2q-1}$

T3. $f: \mathbb{R}^2 \to \mathbb{R}^2$

а) Показать, что $J \neq 0$; отобр не явл. биективным

б) Найти: мн-во значений f

а) $\begin{array}{l} u = e^x\cos y \\ v = e^x\sin y \end{array}$

$$J = \begin{vmatrix} u'_x & u'_y \\ v'_x & v'_y \end{vmatrix} = \begin{vmatrix} e^x\cos y & -e^x\sin y \\ e^x\sin y & e^x\cos y \end{vmatrix} = e^{2x} > 0$$

при этом отобр. не биективно из-за периодичности:

$$u(r,\varphi) = u(r,\varphi+2\pi)$$
$$v(r,\varphi) = v(r,\varphi+2\pi)$$

ч.т.д

б) $\begin{array}{l} u = \operatorname{Re} e^{x+iy} \\ v = \operatorname{Im} e^{x+iy} \end{array}$ $\Rightarrow$ e^z принимает все знач, кроме 0

Ответ: $\mathbb{R}^2 \setminus \{0\}$

T4. Найти: $r'_x, r'_y, \varphi'_x, \varphi'_y$ через r, φ

$x = r\cos\varphi$; $y = r\sin\varphi$

$$\begin{array}{l} 1 = r'_x\cdot\cos\varphi - r\sin\varphi\cdot\varphi'_x \\ 0 = r'_y\cdot\cos\varphi - r\sin\varphi\cdot\varphi'_y \\ 0 = r'_x\cdot\sin\varphi + r\cos\varphi\cdot\varphi'_x \\ 1 = r'_y\cdot\sin\varphi + r\cos\varphi\cdot\varphi'_y \end{array} \Rightarrow \begin{array}{ll} \begin{cases} \cos\varphi\, r'_x - r\sin\varphi\cdot\varphi'_x = 1 \\ \sin\varphi\, r'_x + r\cos\varphi\,\varphi'_x = 0 \end{cases} & (1) \\ \begin{cases} \cos\varphi\, r'_y - r\sin\varphi\cdot\varphi'_y = 0 \\ \sin\varphi\cdot r'_y + r\cos\varphi\cdot\varphi'_y = 1 \end{cases} & (2) \end{array}$$

(1)

$$\Delta = \begin{vmatrix} \cos\varphi & -r\sin\varphi \\ \sin\varphi & r\cos\varphi \end{vmatrix} = r\cos^2\varphi + r\sin^2\varphi = r$$

$$\Delta r = \begin{vmatrix} 1 & -r\sin\varphi \\ 0 & r\cos\varphi \end{vmatrix} = r\cos\varphi \qquad r'_x = \frac{\Delta r}{\Delta} = \cos\varphi$$

$$\Delta\varphi = \begin{vmatrix} \cos\varphi & 1 \\ \sin\varphi & 0 \end{vmatrix} = -\sin\varphi \qquad \varphi'_x = \frac{\Delta y}{\Delta} = \frac{-\sin\varphi}{r}$$

(2)

$$\Delta = \begin{vmatrix} \cos\varphi & -r\sin\varphi \\ \sin\varphi & r\cos\varphi \end{vmatrix} = r$$

$$\Delta r = \begin{vmatrix} 0 & -r\sin\varphi \\ 1 & r\cos\varphi \end{vmatrix} = r\sin\varphi \qquad r'_y = \frac{\Delta r}{\Delta} = \sin\varphi$$

$$\Delta\varphi = \begin{vmatrix} \cos\varphi & 0 \\ \sin\varphi & 1 \end{vmatrix} = \cos\varphi \qquad \varphi'_y = \frac{\Delta\varphi}{\Delta} = \frac{\cos\varphi}{r}$$

Ответ: $r'_x = \cos\varphi$; $r'_y = \sin\varphi$; $\varphi'_x = -\frac{\sin\varphi}{r}$; $\varphi'_y = \frac{\cos\varphi}{r}$

§3

№86 Решить, преобр. к полярным коорд.

$$x\frac{\partial u}{\partial y}-y\frac{\partial u}{\partial x}=0 \qquad \begin{cases} x=r\cos\varphi \\ y=r\sin\varphi \end{cases}$$

(*) $x\frac{\partial u}{\partial y}=y\frac{\partial u}{\partial x}$

$$u(x,y)=\tilde{u}(r(x,y),\varphi(x,y))$$

$$r(x,y)=\sqrt{x^2+y^2}$$

$$\varphi(x,y)=\operatorname{arctg}\left(\frac{y}{x}\right)+\pi k,\ k\in\mathbb{Z}$$

$$\frac{\partial r}{\partial x}=\frac{2x}{2\sqrt{x^2+y^2}}=\frac{x}{r};\quad \frac{\partial r}{y}=\frac{y}{r}$$

$$\frac{\partial\varphi}{\partial x}=\frac{-y/x^2}{1+y^2/x^2}=-\frac{y}{x^2+y^2}=-\frac{y}{r^2};\quad \frac{\partial\varphi}{\partial y}=\frac{x}{r^2}$$

$$\Rightarrow \frac{\partial u}{\partial x}=\frac{\partial\tilde{u}}{\partial x}=\frac{\partial\tilde{u}}{\partial r}\cdot\frac{\partial r}{\partial x}+\frac{\partial\tilde{u}}{\partial\varphi}\cdot\frac{\partial\varphi}{\partial x}=\frac{x}{r}\cdot\frac{\partial\tilde{u}}{\partial r}-\frac{y}{r^2}\frac{\partial\tilde{u}}{\partial\varphi}$$

$$\frac{\partial u}{\partial y}=\frac{\partial\tilde{u}}{\partial y}=\frac{\partial\tilde{u}}{\partial r}\cdot\frac{\partial r}{\partial y}+\frac{\partial\tilde{u}}{\partial\varphi}\cdot\frac{\partial\varphi}{\partial y}=\frac{y}{r}\cdot\frac{\partial\tilde{u}}{\partial r}+\frac{x}{r^2}\cdot\frac{\partial\tilde{u}}{\partial\varphi}$$

$$\overset{(*)}{\Rightarrow}\quad \frac{xy}{r}\cdot\frac{\partial\tilde{u}}{\partial r}+\frac{x^2}{r^2}\cdot\frac{\partial\tilde{u}}{\partial\varphi}-\frac{xy}{r}\cdot\frac{\partial\tilde{u}}{\partial r}+\frac{y^2}{r^2}\cdot\frac{\partial\tilde{u}}{\partial\varphi}=0$$

произв. диф. ф-ия

$$\frac{x^2+y^2}{r^2}\cdot\frac{\partial\tilde{u}}{\partial\varphi}=0\ \Rightarrow\ \frac{\partial\tilde{u}}{\partial\varphi}=0\ \Rightarrow\ \tilde{u}(r,\varphi)=\tilde{f}(r)=\tilde{f}(x^2+y^2)=u(x,y)$$

Ответ: $u=\tilde{f}(x^2+y^2)$; $\tilde{f}$ – произв. диф. ф-ия

№88 1) Решить ур-ие, преобр. к u,v

(*) $\frac{\partial z}{\partial x}-\frac{\partial z}{\partial y}=0$, $u=x+y$ $v=x-y$

$$z(x,y)=\tilde{z}(u,v)=\tilde{z}(x+y,x-y)$$

$$\frac{\partial u}{\partial x}=1\quad \frac{\partial u}{\partial y}=1\quad \frac{\partial v}{\partial x}=1\quad \frac{\partial v}{\partial y}=-1$$

$$\frac{\partial z}{\partial x}=\frac{\partial\tilde{z}}{\partial x}=\frac{\partial\tilde{z}}{\partial u}\cdot\frac{\partial u}{\partial x}+\frac{\partial\tilde{z}}{\partial v}\cdot\frac{\partial v}{\partial x}=\frac{\partial\tilde{z}}{\partial u}+\frac{\partial\tilde{z}}{\partial v};\ \frac{\partial z}{\partial y}=\frac{\partial\tilde{z}}{\partial u}\cdot\frac{\partial u}{\partial y}+\frac{\partial\tilde{z}}{\partial v}\cdot\frac{\partial v}{\partial y}=\frac{\partial\tilde{z}}{\partial u}-\frac{\partial\tilde{z}}{\partial v}$$

$$\overset{(*)}{\Rightarrow}\ \frac{\partial\tilde{z}}{\partial u}+\frac{\partial\tilde{z}}{\partial v}-\frac{\partial\tilde{z}}{\partial u}+\frac{\partial\tilde{z}}{\partial v}=2\cdot\frac{\partial\tilde{z}}{\partial v}=0\ \Rightarrow\ \frac{\partial\tilde{z}}{\partial v}=0\ \Rightarrow\ \tilde{z}(u,v)=f(u)=f(x+y)=z(x,y)$$

$\Rightarrow$ Ответ: $z(x,y)=f(x+y)$; f – произв. диф. ф-ия.

№91 Преобр. ур-ие, приняв: x – ф-ия, u,v – незав. перем-ые

$$(y-z)\frac{\partial z}{\partial x}+(y+z)\frac{\partial z}{\partial y}=0\ (*) \qquad u=y-z;\ v=y+z$$

$$x=x(u,v)=x(y-z,y+z)$$

$$dx=\frac{\partial x}{\partial u}du+\frac{\partial x}{\partial v}dv=\frac{\partial x}{\partial u}(dy-dz)+\frac{\partial x}{\partial v}(dy+dz)=dy\left(\frac{\partial x}{\partial u}+\frac{\partial x}{\partial v}\right)+dz\left(\frac{\partial x}{\partial v}-\frac{\partial x}{\partial u}\right)$$

$$\Rightarrow dz=\frac{dx}{\frac{\partial x}{\partial v}-\frac{\partial x}{\partial u}}-\frac{dy\left(\frac{\partial x}{\partial u}+\frac{\partial x}{\partial v}\right)}{\frac{\partial x}{\partial v}-\frac{\partial x}{\partial u}};\quad \frac{\partial z}{\partial x}=\frac{1}{\frac{\partial x}{\partial v}-\frac{\partial x}{\partial u}};\quad \frac{\partial z}{\partial y}=\frac{\frac{\partial x}{\partial u}+\frac{\partial x}{\partial v}}{\frac{\partial x}{\partial u}-\frac{\partial x}{\partial v}}$$

$$\overset{(*)}{\Rightarrow}\ u\frac{\partial z}{\partial x}+v\frac{\partial z}{\partial y}=0\quad \frac{1}{\frac{\partial x}{\partial v}-\frac{\partial x}{\partial u}}\left(u-v\left(\frac{\partial x}{\partial u}+\frac{\partial x}{\partial v}\right)\right)=0\quad \frac{u}{v}=\frac{\partial x}{\partial u}+\frac{\partial x}{\partial v}$$

$\Rightarrow$ Ответ: $\frac{u}{v}=\frac{\partial x}{\partial u}+\frac{\partial x}{\partial v}$

§4

№51 2) преобр. к полярным коорд.

$$x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=0$$

$x=r\cos\varphi$, $y=r\sin\varphi$; $r=\sqrt{x^2+y^2}$, $\varphi=\operatorname{arctg}\left(\frac{y}{x}\right)+\pi k,\ k\in\mathbb{Z}$

$$u(x,y)=v(r,\varphi)$$

из Т4: $r'_x=\cos\varphi$; $r'_y=\sin\varphi$; $\varphi'_x=-\frac{\sin\varphi}{r}$; $\varphi'_y=\frac{\cos\varphi}{r}$ $\Rightarrow$ $u'_x=u'_r\cos\varphi-u'_\varphi\cdot\frac{\sin\varphi}{r}$; $u'_y=u'_r\sin\varphi+u'_\varphi\cdot\frac{\cos\varphi}{r}$

$$u''_{xx}=(u'_x)'_r\cdot r'_x+(u'_x)'_\varphi\cdot\varphi'_x=\left(u''_{rr}\cos\varphi-\frac{u''_{\varphi r}\sin\varphi}{r}+\frac{u'_\varphi\sin\varphi}{r^2}\right)\cos\varphi+$$
$$+\left(u''_{r\varphi}\cos\varphi-u'_r\sin\varphi-\frac{u''_{\varphi\varphi}\sin\varphi}{r}+\frac{u'_\varphi\cos\varphi}{r}\right)\cdot\left(-\frac{\sin\varphi}{r}\right)=$$
$$=u''_{rr}\cos^2\varphi-u''_{\varphi r}\frac{\sin\varphi\cos\varphi}{r}+u'_r\cdot\frac{\sin^2\varphi}{r}+u''_{\varphi\varphi}\frac{\sin^2\varphi}{r^2}-u''_{r\varphi}\frac{\sin\varphi\cos\varphi}{r}$$

$$u''_{yy}=(u'_y)'_r\cdot r'_y+(u'_y)'_\varphi\cdot\varphi'_y=\left(u''_{rr}\sin\varphi+\frac{u''_{\varphi r}\cos\varphi}{r}-\frac{u'_\varphi\cos\varphi}{r^2}\right)\sin\varphi+$$
$$+\left(u''_{r\varphi}\sin\varphi+u'_r\cos\varphi+u''_{\varphi\varphi}\frac{\cos\varphi}{r}-u'_\varphi\frac{\sin\varphi}{r}\right)\cdot\frac{\cos\varphi}{r}=$$
$$=u''_{rr}\sin^2\varphi+u''_{\varphi r}\cdot\frac{\sin\varphi\cos\varphi}{r}+u''_{r\varphi}\frac{\sin\varphi\cos\varphi}{r}-u'_\varphi\frac{\sin 2\varphi}{r^2}+u''_{\varphi\varphi}\frac{\cos^2\varphi}{r^2}+u'_r\frac{\cos^2\varphi}{r}$$

$$u''_{yx}=(u'_y)'_r\cdot r'_x+(u'_y)'_\varphi\cdot\varphi'_x=\left(u''_{rr}\sin\varphi+\frac{u''_{\varphi r}\cos\varphi}{r}-\frac{u''_{\varphi r}\cos\varphi}{r^2}\right)\cos\varphi+$$
$$+\left(u''_{r\varphi}\sin\varphi+u'_r\cos\varphi+u''_{\varphi\varphi}\frac{\cos\varphi}{r}-u'_\varphi\frac{\sin\varphi}{r}\right)\left(-\frac{\sin\varphi}{r}\right)=$$
$$=u''_{rr}\frac{\sin 2\varphi}{2}+u''_{\varphi r}\frac{\cos 2\varphi}{r}-u''_{r\varphi}\frac{\sin^2\varphi}{r}+u'_\varphi\cdot\frac{\sin^2\varphi-\cos^2\varphi}{r^2}-u'_r\cdot\frac{\sin 2\varphi}{2r}-u''_{\varphi\varphi}\frac{\sin 2\varphi}{2r^2}$$

$$x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=0\Rightarrow u''_{xx}r^2\cos^2\varphi+u''_{yx}r^2\sin 2\varphi+u''_{yy}r^2\sin^2\varphi=0\Rightarrow$$

$$\Rightarrow r^2\cos^2\varphi\left(u''_{rr}\cos^2\varphi-u''_{\varphi r}\frac{\sin 2\varphi}{r}+u'_r\frac{\sin^2\varphi}{r}+u''_{\varphi\varphi}\frac{\sin^2\varphi}{r^2}\right)+$$
$$+r^2\sin 2\varphi\left(u''_{rr}\frac{\sin 2\varphi}{r}+u''_{\varphi r}\frac{\cos 2\varphi}{r}+u'_\varphi\frac{\sin^2\varphi-\cos^2\varphi}{r^2}-u'_r\frac{\sin 2\varphi}{2r}-u''_{\varphi\varphi}\frac{\sin^2\varphi}{r^2}\right)+$$
$$+r^2\sin^2\varphi\left(u''_{rr}\sin^2\varphi+u''_{\varphi r}\frac{\sin 2\varphi}{r}-u'_\varphi\frac{\sin 2\varphi}{r^2}+u'_r\frac{\cos^2\varphi}{r}+u''_{\varphi\varphi}\frac{\cos^2\varphi}{r^2}\right)=0$$

$\Rightarrow$ Ответ: $r^2u''_{rr}=0$

№52 1) преобр. u,v – новые незав. перем.

$$\frac{\partial^2 z}{\partial t^2}=a^2\frac{\partial^2 z}{\partial x^2};\quad u=x-at,\ v=x+at\quad z=z(u,v)$$

$\Rightarrow$ $z''_{tt}=a^2z''_{xx}$

$z'_t=z'_u\cdot u'_t+z'_v\cdot v'_t=-a\cdot z'_u+a\cdot z'_v$

$z''_{tt}=(z'_t)'_t=\frac{\partial}{\partial t}\left(\frac{\partial z}{\partial t}\right)=\frac{\partial}{\partial t}\cdot a\left(\frac{\partial z}{\partial v}-\frac{\partial z}{\partial u}\right)=$

$z'_x=z'_u\cdot u'_x+z'_v\cdot v'_x=z'_u+z'_v$

$z''_{xx}=z''_{uu}+2z''_{uv}+z''_{vv}$

$=a\cdot\left(\frac{\partial^2 z}{\partial t\partial v}-\frac{\partial^2 z}{\partial t\partial u}\right)=a\left((z'_t)'_v-(z'_t)'_u\right)=$

$=a\left(a\cdot z''_{vv}-a\cdot z''_{uv}+a\cdot z''_{uu}-a\cdot z''_{uv}\right)=$

$=a^2\left(z''_{vv}-2z''_{uv}+z''_{uu}\right);$

$z''_{tt}=a^2z''_{xx}\Rightarrow a^2\left(z''_{vv}-2z''_{uv}+z''_{uu}\right)=a^2\left(z''_{uu}+2z''_{uv}+z''_{vv}\right)$

$\Rightarrow z''_{uv}=0$

Ответ: $z''_{uv}=0$

III.

T5. В стац. т. кв. форма 2 диф - полож. полуопр.

а) Может ли она быть лок. мин?

Ответ: да, например 5.9.

б) —//— лок. макс?

Ответ: нет

в) —//— не быть т. лок. экстр?

Ответ: да, например для опр. упр. 5.9.

§5

№2 2) Иссл. на экстр.

$u = 3x^2y + y^3 - 12x - 15y + 3$

$u'_x = 6yx - 12 = 0 \Rightarrow yx = 2 \quad y = \frac{2}{x}$

$u'_y = 3x^2 + 3y^2 - 15 = 0 \Rightarrow x^2 + y^2 = 5 \quad x^2 + \frac{4}{x^2} = 5 \quad x^4 - 5x^2 + 4 = 0 \quad x = \pm 1; x = \pm 2$

$y = \pm 2; y = \pm 1$

$u''_{xx} = 6y$

$u''_{yy} = 6y \Rightarrow d^2u = 6y\,dx^2 + 6y\,dy^2 + 12x\,dx\,dy$

$u''_{xy} = 6x \qquad \frac{d^2u}{6} = y\,dx^2 + y\,dy^2 + 2x\,dx\,dy$

в т. $(1,2)$ $\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$ – полож. опр., лок. мин.

в т. $(2,1)$ $\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$ – неопр; не лок. экстр.

в т. $(-1,-2)$ $\begin{pmatrix} -2 & -1 \\ -1 & -2 \end{pmatrix}$ – отр. опр; лок. макс.

в т. $(-2,-1)$ $\begin{pmatrix} -1 & -2 \\ -2 & -1 \end{pmatrix}$ – неопр.; не лок. экстр.

Ответ: $(-1,-2)$ – лок. макс

$(1,2)$ – лок. мин

№9 Найти стац. точки и иссл. на экстр.

$u = x^4 + y^4 - 2x^2$

$u'_x = 4x^3 - 4x = 0 \Rightarrow x = 0; \pm 1$

$u'_y = 4y^3 = 0 \qquad y = 0 \Rightarrow (0,0); (1,0); (-1;0)$ – стац. точки

$u''_{xx} = 12x^2 - 4$

$u''_{yy} = 12y^2 \Rightarrow d^2u = (12x^2 - 4)dx^2 + 12y^2 dy^2 \qquad \begin{pmatrix} 12x^2-4 & 0 \\ 0 & 12y^2 \end{pmatrix} \quad \Delta_1 = 12x^2 - 4$

$u''_{xy} = 0 \qquad \Delta_2 = 12y^2(12x^2 - 4)$

в т. $(0,0)$ $\Delta_1 < 0 \quad \Delta_2 = 0$ – отр. полуопр.

$u(\Delta x, \Delta y) - u(0,0) = \Delta x^4 + \Delta y^4 - 2\Delta x^2 = \Delta x^2(\Delta x^2 - 2) + \Delta y^4$

при $\Delta x = 0 \quad \Delta y \neq 0 \quad > 0$

при $0 < \Delta x < \sqrt{2} \quad \Delta y = 0 \quad < 0$

$\Rightarrow$ экстр. нет

в т. $(\pm 1, 0)$ $\Delta_1 > 0 \quad \Delta_2 = 0$ – полож. полуопр.

$u(\pm 1 + \Delta x, \Delta y) - u(\pm 1, 0) = (\pm 1 + \Delta x)^4 + \Delta y^4 - 2(\pm 1 + \Delta x)^2 + 1 = (\pm 1 + \Delta x)^2((\pm 1 + \Delta x)^2 - 2) + \Delta y^4 + 1 =$

$= (\Delta x^2 \pm 2\Delta x + 1)(\Delta x^2 \pm 2\Delta x - 1) + \Delta y^4 + 1 = \Delta x^4 \pm 2\Delta x^3 - \Delta x^2 \pm 2\Delta x^3 + 4\Delta x^2 \cancel{\mp 2\Delta x} + \Delta x^2 \cancel{\pm 2\Delta x} - 1 + \Delta y^4 + 1 =$

$= \Delta x^4 \pm 4\Delta x^3 + 4\Delta x^2 + \Delta y^4 = \Delta x^2(\Delta x^2 - 2) + \Delta y^4 > 0 \Rightarrow \min$

Ответ: лок. мин. в $(\pm 1, 0)$

№13 1) $u = x^2 + y^2 + (z+1)^2 - xy + x$

$u'_x = 2x - y + 1 = 0$
$u'_y = 2y - x = 0 \Rightarrow$ $x = -2/3$, $y = -1/3$, $z = -1$ $\quad (-2/3; -1/3; -1)$
$u'_z = 2z + 2 = 0$

$u''_{xx} = 2 \quad u''_{yy} = 2$
$u''_{xy} = -1 \quad u''_{yz} = 0$
$u''_{xz} = 0 \quad u''_{zz} = 2$

$d^2u = 2dx^2 + 2dy^2 + 2dz^2 - 2dxdy$

в т. $(-2/3; -1/3; -1)$: $\begin{pmatrix} 2 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$ $\Delta_1 > 0$, $\Delta_2 > 0$, $\Delta_3 > 0$ $\Rightarrow$ лок. мин.

Ответ: лок. мин. в $(-2/3; -1/3; -1)$

№18 1) Исс. неявную ф-ию, зад. ур-ием

$x^2 + y^2 + u^2 + 2x - 2y + 4u - 3 = 0$

$\cancel{2}x\,dx + \cancel{2}y\,dy + \cancel{2}u\,du + \cancel{2}dx - \cancel{2}dy + \overset{2}{\cancel{4}}du = 0$

$du(u+2) + dx(x+1) + dy(y-1) = 0$

$du = -\dfrac{dx(x+1) + dy(y-1)}{u+2} = 0 \Rightarrow x = -1;\ y = 1$

$1 \cancel{+x} + u^2 \cancel{-2} - 2 + 4u - 3 = 0$

$u^2 + 4u - 5 = 0 \Rightarrow u = 1, -5$

$d^2u(u+2) + \overset{0}{\cancel{du^2}} + dx^2 + dy^2 = 0$

$d^2u = -\dfrac{dx^2 + dy^2}{u+2}$

в т. $(-1, 1, 1)$ $d^2u = -\dfrac{dx^2}{3} - \dfrac{dy^2}{3}$ – отр. опр. $\Rightarrow$ max

в т. $(-1, 1, -5)$ $d^2u = \dfrac{dx^2}{3} + \dfrac{dy^2}{3}$ – полож. опр $\Rightarrow$ min

Ответ: $(-1, 1, 1)$ – лок. макс.
$(-1, 1, -5)$ – лок. мин.

№19 1) Найти усл. экстр. ф-ии $u = f(x,y)$

$u = xy \ ; \ x + y - 1 = 0$

$y = 1 - x \Rightarrow f(x,y) = x(1-x) = x - x^2 = f_0(x)$

$f_0(x) = -x^2 + x = -\left(x^2 - x + \frac{1}{4} - \frac{1}{4}\right) = -\left(x - \frac{1}{2}\right)^2 + \frac{1}{4}$

$f_0(x)$ принимает наиб. знач. в $x = 1/2$, других экстр. нет.

$\Rightarrow$ лок. макс. $f(x,y)$ в $x = 1/2$; $y = 1 - 1/2 = 1/2$

Ответ: $(1/2, 1/2)$ – лок. макс

№21 2) $u = 1 - 4x - 8y \ ; \ x^2 - 8y^2 - 8 = 0$

$\mathcal{L} = 1 - 4x - 8y + \lambda(x^2 - 8y^2 - 8)$

$\begin{cases} \mathcal{L}'_x = -4 + 2\lambda x = 0 \\ \mathcal{L}'_y = -8 - 16\lambda y = 0 \\ x^2 - 8y^2 - 8 = 0 \end{cases}$ $\quad \lambda x = 2$, $\lambda y = -\frac{1}{2}$, $\left(\frac{2}{\lambda}\right)^2 - 8\left(\frac{1}{2\lambda}\right)^2 - 8 = 0$

$\frac{4}{\lambda^2} - \frac{2}{\lambda^2} - 8 = 0 \quad \frac{2}{\lambda^2} = 8 \quad \lambda = \pm 1/2$

$\Rightarrow x = \pm 4;\ y = \mp 1$

$\mathcal{L}''_{xx} = 2\lambda \ ; \ \mathcal{L}''_{yy} = -16\lambda \ ; \ \mathcal{L}''_{xy} = 0 \ ; \ d^2\mathcal{L} = \pm dx^2 \mp 8dy^2$ $\begin{pmatrix} \pm 1 & 0 \\ 0 & \mp 8 \end{pmatrix}$ $\Delta_1 > 0$, $\Delta_2 < 0$ или $\Delta_1 < 0$, $\Delta_2 < 0$ неопр. форма

$2x\,dx - 16y\,dy = 0 \ ; \ dx = 8\frac{y}{x}dy$

при $\lambda = 1/2$; $y/x = -1/4$
$d^2\mathcal{L} = 4dy^2 - 8dy^2 = -4dy^2$ – отр.
$(4, -1)$ – лок. макс.

при $\lambda = -1/2$; $y/x = -1/4$
$d^2\mathcal{L} = -4dy^2 + 8dy^2 = 4dy^2$ полож.
$(-4, 1)$ – лок. мин

Ответ: $(4, -1)$ – лок. макс.
$(-4, 1)$ – лок. мин.

№25 б) $u = x - y + 2z$; $x^2 + y^2 + 2z^2 = 16$

$\mathcal{L} = x - y + 2z + \lambda(x^2 + y^2 + 2z^2 - 16)$

$$\begin{cases} \mathcal{L}'_x = 1 + 2\lambda x = 0 & x = -\frac{1}{2\lambda} \\ \mathcal{L}'_y = -1 + 2\lambda y = 0 & y = \frac{1}{2\lambda} \\ \mathcal{L}'_z = 2 + 4\lambda z = 0 & z = -\frac{1}{2\lambda} \\ x^2 + y^2 + 2z^2 = 16 \end{cases}$$

$\frac{1}{4\lambda^2} + \frac{1}{4\lambda^2} + \frac{2}{4\lambda^2} = 16$ $\lambda^2 = \frac{1}{16}$

$\lambda = \pm\frac{1}{4} \Rightarrow x = \mp 2$; $y = \pm 2$; $z = \mp 2$

$\mathcal{L}''_{xx} = 2\lambda$ $\mathcal{L}''_{yy} = 2\lambda$ $\mathcal{L}''_{zz} = 4\lambda$

$\mathcal{L}''_{xy} = 0$ $\mathcal{L}''_{yz} = 0$ $\mathcal{L}''_{xz} = 0$

$\lambda = 1/4$

$d^2\mathcal{L} = \frac{1}{2}(dx^2 + dy^2 + dz^2)$ – полож. опр; лок. мин.

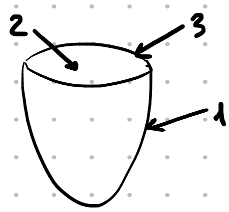
$\lambda = -1/4$

$d^2\mathcal{L} = -1/2\,(dx^2 + dy^2 + dz^2)$ – отр. опр; лок. макс.

Ответ: $(-2, 2, -2)$ – лок. мин.
$(2, -2, 2)$ – лок. макс.

№31 3) Найти наиб. и наим. знач. u на мн-ве

$u = x + y + z$ $x^2 + y^2 \leq z \leq 1$



1) $u = x + y + z$ $x^2 + y^2 = z$ $x^2 + y^2 < 1$

$u = x + y + x^2 + y^2$

$u'_x = 1 + 2x = 0$
$u'_y = 1 + 2y = 0$ $\Rightarrow$ стац.т. $(-1/2; -1/2)$ $u = -1/2$

2) $u = x + y + z$ $x^2 + y^2 < z$ $z = 1$

$u = x + y + 1$

стац. точки: $\varnothing$

3) $u = x + y + z$ $x^2 + y^2 = z$ $z = 1$

$u = x + y + 1$; $x^2 + y^2 = 1$

$\mathcal{L} = x + y + 1 + \lambda(x^2 + y^2 - 1)$

$$\begin{cases} \mathcal{L}'_x = 1 + 2\lambda x = 0 \\ \mathcal{L}'_y = 1 + 2\lambda y = 0 \\ x^2 + y^2 = 1 \end{cases}$$

$x = -\frac{1}{2\lambda} = y$

$\frac{1}{2\lambda^2} = 1$ $\lambda^2 = \frac{1}{2}$ $\lambda = \pm\frac{1}{\sqrt{2}}$ $x = y = \mp\frac{1}{\sqrt{2}}$

$u\left(\frac{1}{\sqrt{2}}; \frac{1}{\sqrt{2}}\right) = 1 + \sqrt{2}$ $u\left(-\frac{1}{\sqrt{2}}; -\frac{1}{\sqrt{2}}\right) = 1 - \sqrt{2}$

$\Rightarrow$ Ответ: $\max u = 1 + \sqrt{2}$
$\min u = -1/2$

IV §8

№23 Пример f непр. на изм. мн-ве G, но не инт. на нем

$$f(x,y)=\begin{cases} 1/y^2, & 0<x<y<1 \\ -1/x^2, & 0<y<x<1 \\ 0 & \text{иначе} \end{cases} \qquad G=[0,1]\times[0,1]$$

непр. на G

$$\int_0^1 dy\int_0^1 f\,dx=\int_0^1 dy\left(\int_0^y f\,dx+\int_y^1 f\,dx\right)=\int_0^1 dy\left(\int_0^y \frac{dx}{y^2}-\int_y^1\frac{dx}{x^2}\right)=\int_0^1 dy\left(\frac{x}{y^2}\Big|_0^y+\frac{1}{x}\Big|_y^1\right)=\int_0^1 dy=1$$

$$\int_0^1 dx\int_0^1 f\,dy=\int_0^1 dx\left(\int_0^x f\,dy+\int_x^1 f\,dy\right)=\int_0^1 dx\left(-\int_0^x\frac{dy}{x^2}+\int_x^1\frac{dx}{y^2}\right)=\int_0^1 dx\left(-\frac{y}{x^2}\Big|_0^x-\frac{1}{y}\Big|_x^1\right)=\int_0^1 dx\left(-\frac{1}{x}-1+\frac{1}{x}\right)=-\int_0^1 dx=-1$$

$1\neq -1 \Rightarrow$ f не инт. на G — пример подходит

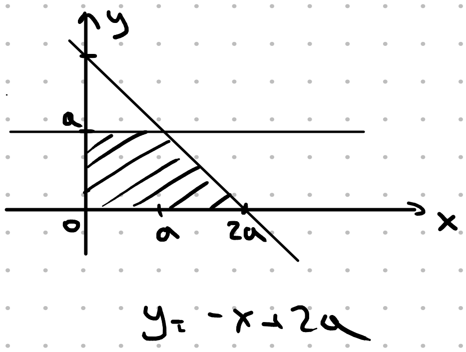
№76 $f(x,y)$ опр на $P=[a,b]\times[c,d]$

а) Ответ: нет (8.23)

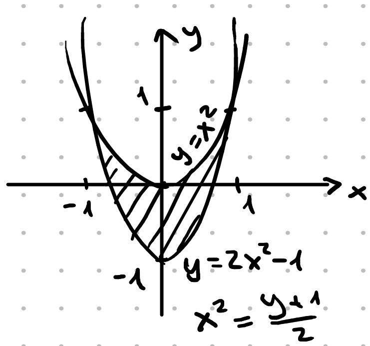
б) Ответ: нет, контрпример: $f(0,y)=\begin{cases}1, & y\in\mathbb{Q}\\ 0, & y\in \mathbb{R}/\mathbb{Q}\end{cases}$ — не инт. при $x=0$

№79 1) G: $x=0,\ y=0,\ y=a,\ x+y=2a$

$$\int_0^a dx\int_0^a f(x,y)\,dy+\int_a^{2a}dx\int_0^{-x+2a}f(x,y)\,dy=$$
$$=\int_0^a dy\int_0^{-y+2a}f(x,y)\,dx$$

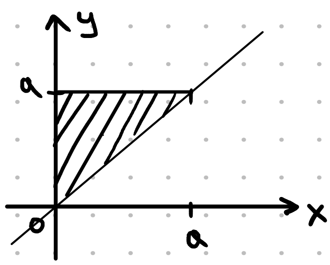


№83 15) $$\int_{-1}^{1}dx\int_{x^2}^{2x^2-1}f(x,y)\,dy=$$
$$\int_{-1}^{0}dy\int_{-\sqrt{\frac{y+1}{2}}}^{\sqrt{\frac{y+1}{2}}}f(x,y)\,dx+\int_0^1 dy\int_{-\sqrt{\frac{y+1}{2}}}^{-\sqrt{y}}f(x,y)\,dy+\int_0^1 dy\int_{\sqrt{\frac{y+1}{2}}}^{\sqrt{y}}f(x,y)\,dy$$



№85 3) $$\int_0^a dx\int_x^a (a^2-y^2)^\alpha dy= \qquad a>0;\ \alpha>0$$

$$=\int_0^a dy\int_0^y (a^2-y^2)^\alpha dx=\int_0^a y(a^2-y^2)^\alpha dy=-\frac{1}{2}\int_0^a (a^2-y^2)^\alpha d(a^2-y^2)=$$
$$=-\frac{1}{2}\cdot\frac{(a^2-y^2)^{\alpha+1}}{\alpha+1}\Big|_0^a=\frac{a^{2\alpha+2}}{2(\alpha+1)}$$



№90

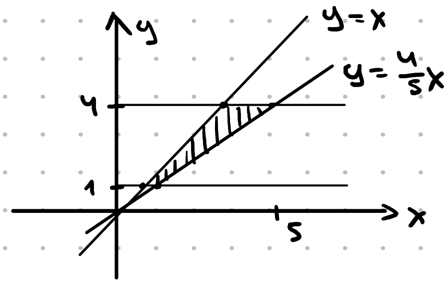
1) $\iint\limits_G (x\sin y + y\cos x)\,dx\,dy \qquad G=[0,\frac{\pi}{2}]\times[0,\frac{\pi}{2}]$

$$\int_0^{\pi/2}dx\int_0^{\pi/2}(x\sin y+y\cos x)\,dy = \int_0^{\pi/2}dy\int_0^{\pi/2}x\sin y\,dx + \int_0^{\pi/2}dx\int_0^{\pi/2}y\cos x\,dy = \frac{\pi^2}{8}\int_0^{\pi/2}\sin y\,dy+\frac{\pi^2}{8}\int_0^{\pi/2}\cos x\,dx =$$

$$=\frac{\pi^2}{8}\left(-\cos y\Big|_0^{\pi/2}+\sin x\Big|_0^{\pi/2}\right) = \frac{\pi^2}{4}$$

$\Rightarrow$ Ответ: $\frac{\pi^2}{4}$

8) $\iint\limits_G\sqrt{x-y}\,dx\,dy \qquad G=\{\frac{4}{5}x\le y\le x,\ 1\le y\le 4\}$



$$\int_1^4 dy\int_y^{5/4y}\sqrt{x-y}\,dx = \int_1^4 dy\cdot\frac{2}{3}\cdot(x-y)^{3/2}\Big|_y^{5/4y} =$$

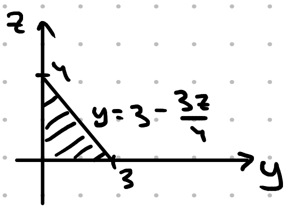
$$=\int_1^4 dy\cdot\frac{2}{3}\cdot\frac{1}{8}\cdot y^{3/2} = \frac{1}{12}\int_1^4 y^{3/2}\,dy = \frac{1}{12}\cdot\frac{2}{5}\cdot y^{5/2}\Big|_1^4 =$$

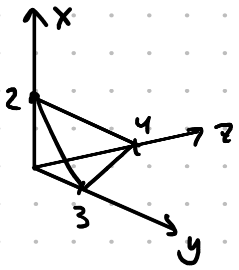
$$=\frac{1}{30}\cdot(32-1)=\frac{31}{30}$$

Ответ: $\frac{31}{30}$

№133 2) $zyx \longrightarrow xyz$

$$\int_0^4 dz\int_0^{3-3z/4}dy\int_0^{2-2y/3-z/2}f(x,y,z)\,dx = \int_0^2 dx\int_0^{3-\frac{3}{2}x}dy\int_0^{-2x+4-4y/3}f(x,y,z)\,dz \Rightarrow$$





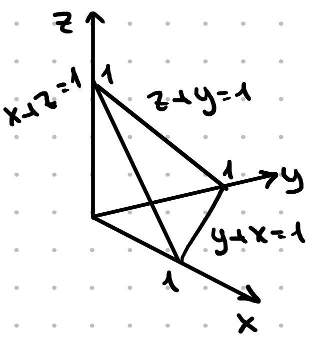
$0\le x\le 2-\frac{2y}{3}-\frac{z}{2}$

$z=-2x+4-4y/3$

$y=\frac{3}{2}(2-x-z/2)$

$\Rightarrow$ Ответ: $\int_0^2 dx\int_0^{3-3/2x} dy\int_0^{-2x+4-\frac{4y}{3}} f(x,y,z)\,dz$

№139 2) $f(x,y,z)=(1+x+y+z)^{-3} \qquad G:\ x+y+z=1,\ x=0,\ y=0,\ z=0$



$$\int_0^1 dx\int_0^{1-x}dy\int_0^{1-x-y}(1+x+y+z)^{-3}dz = \int_0^1 dx\int_0^{1-x}dy\cdot\left(-\frac{1}{2}\right)\cdot(1+x+y+z)^{-2}\Big|_0^{1-x-y} =$$

$$=-\frac{1}{2}\int_0^1 dx\int_0^{1-x}dy\cdot\left(2^{-2}-(1+x+y)^{-2}\right) = \frac{1}{2}\int_0^1 dx\left(\frac{1}{(1+x+y)^2}\Big|_0^{1-x}+\frac{1}{4}\Big|_0^{1-x}\right) =$$

$$=-\frac{1}{2}\int_0^1 dx\left(\frac{1}{4}-\frac{1}{(1+x)^2}+\frac{1-x}{4}\right) = -\frac{1}{2}\left(\frac{1}{4}-\ln 2+\frac{1}{4}+\frac{1}{8}\right) = \frac{\ln 2}{2}-\frac{5}{16}$$

$\Rightarrow$ Ответ. $\frac{\ln 2}{2}-\frac{5}{16}$

№175 2) Вычисл. инт. по кубу $Q_n=[0,a]^n\in\mathbb{R}^n$ $\quad n\geq 2$

$$\int\limits_{Q_n}\sum_{k=1}^n x_k\,dx=\int\limits_{Q_n}\sum_{k=1}^n x_k\,dx_1dx_2dx_3\dots dx_n=\int_0^a dx_n\int_0^a dx_{n-1}\int_0^a dx_{n-2}\dots\int_0^a\sum_{k=1}^n x_k\,dx_1=$$

$$=\int_0^a dx_n\int_0^a dx_{n-1}\dots\int_0^a dx_2\left(\frac{a^2}{2}+x_2a+x_3a+\dots x_na\right)=\int_0^a dx_n\int_0^a dx_{n-1}\dots\int_0^a dx_3\left(\frac{a^3}{2}+\frac{a^2}{2}a+x_3a^2+\dots x_na^2\right)=$$

$$=\dots=\int_0^a dx_n\left(\frac{a^n}{2}+\frac{a^n}{2}+\frac{a^n}{2}+\dots+a^{n-1}x_n\right)=\frac{a^{n+1}}{2}\cdot n\ \Rightarrow\ \text{Ответ: } \frac{a^{n+1}}{2}n$$

№176 2) Вычисл. инт. по пирамиде $\Pi_n=\{0\leq x_n\leq x_{n-1}\leq\dots\leq x_2\leq x_1\leq a\}$

$$\int\limits_{\Pi_n}x_1x_2\dots x_n\,dx=\int\limits_{\Pi_n}x_1x_2\dots x_n\,dx_1dx_2\dots dx_n=\int_0^a x_1dx_1\int_0^{x_1}x_2dx_2\dots\int_0^{x_{n-1}}x_ndx_n$$

$$I_1=\int_0^a x_1dx_1=a^2/2\ ;\quad I_2=\int_0^a x_1dx_1\int_0^{x_1}x_2dx_2=\int_0^a x_1dx_1\cdot\frac{x_1^2}{2}=\int_0^a\frac{x_1^3}{2}=\frac{a^4}{2\cdot 4}$$

Докажем по индукции:

$$I_n(a)=\frac{a^{2n}}{(2n)!!}$$

$n=1,2$ – проверили

$$I_{n+1}(a)=\int_0^a x_1dx_1\underbrace{\int_0^{x_1}x_2dx_2\dots\int_0^{x_n}x_{n+1}dx_{n+1}}_{I_n(x_1)}$$

$\Rightarrow$ по предполож. индукции

$$I_{n+1}(a)=\int_0^a x_1\cdot\frac{x_1^{2n}}{(2n)!!}=\int_0^a\frac{x_1^{2n+1}}{(2n)!!}=\frac{a^{2n+2}}{(2n+2)!!}\ -\ \text{ч.т.д.}$$

Ответ: $\dfrac{a^{2n}}{(2n)!!}$

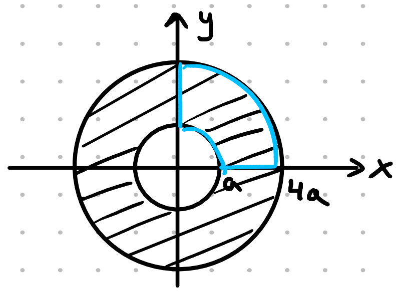
№106 3) $\displaystyle\iint\limits_G|xy|\,dxdy\qquad G=\{a^2\leq x^2+y^2\leq 4a^2\}\qquad x=r\cos\varphi$

$a^2\leq r^2\leq 4a^2\qquad y=r\sin\varphi$

$$\iint\limits_G|r^2\cos\varphi\sin\varphi|\,r\,drd\varphi=4\int_0^{\pi/2}d\varphi\int_a^{2a}r^3\cdot\cos\varphi\sin\varphi\,dr$$

$$=4\int_0^{\pi/2}\sin\varphi\cos\varphi\,d\varphi\int_a^{2a}r^3dr=15a^4\int_0^{\pi/2}\sin\varphi\,d(\sin\varphi)=$$

$$=15a^4\cdot\frac{1}{2}\cdot\sin^2\varphi\Big|_0^{\pi/2}=\frac{15a^4}{2}$$



Ответ: $\dfrac{15a^4}{2}$

№107 2) $\displaystyle\iint\limits_G y\,dxdy\qquad G=\{x^2+y^2\leq 2x\ ,\ x>y\}\qquad x=r\cos\varphi$

$y=r\sin\varphi$

$(x-1)^2+y^2\leq 1\ ,\ y<x$

$-2r\cos\varphi+r^2\leq 0\ ,\ \sin\varphi<\cos\varphi$

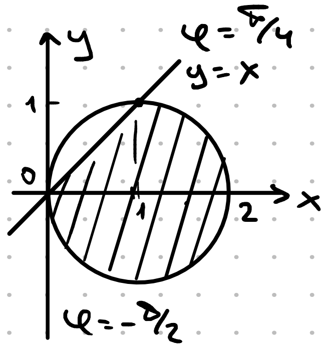
$2r\cos\varphi\geq r^2\ ;\ \cos\varphi>\sin\varphi$

$r\leq 2\cos\varphi\ ;\ \varphi\in\left(-\frac{3\pi}{4},\frac{\pi}{4}\right)$

$$\int_{-\pi/2}^{\pi/4}d\varphi\int_0^{2\cos\varphi}r^2\sin\varphi\,dr=\int_{-\pi/2}^{\pi/4}\sin\varphi\,d\varphi\int_0^{2\cos\varphi}r^2dr=$$

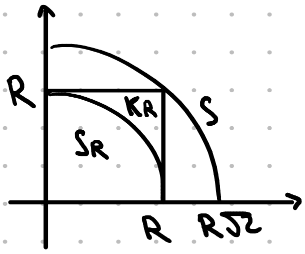
$$=\int_{-\pi/2}^{\pi/4}\sin\varphi\,d\varphi\,\frac{1}{3}(8\cos^3\varphi)=$$

$$=-\frac{8}{3}\int_{-\pi/2}^{\pi/4}\cos^3\varphi\,d(\cos\varphi)=-\frac{8}{12}\cdot\cos^4\varphi\Big|_{-\pi/2}^{\pi/4}=-\frac{1}{6}$$



Ответ: $-\dfrac{1}{6}$

№110 3) $\int\limits_0^{+\infty} e^{-x^2}dx = I - ?$



$$\iint\limits_{S_R} e^{-x^2-y^2}dx\,dy \le \iint\limits_{K_R} e^{-x^2-y^2}dx\,dy \le \iint\limits_{S_{R\sqrt{2}}} e^{-x^2-y^2}dx\,dy$$

$$\iint\limits_{K_R} e^{-x^2-y^2}dx\,dy = \int\limits_0^{\pi/2}d\varphi\int\limits_0^R re^{-r^2}dr = \frac{\pi}{2\cdot 2}\int\limits_0^{R^2}e^{-t}dt = \frac{\pi}{4}(1-e^{-R^2})$$

$$\Rightarrow \frac{\pi}{4}(1-e^{-R^2}) \le I_R^2 \le \frac{\pi}{4}(1-e^{-2R^2}) \qquad R\to+\infty$$

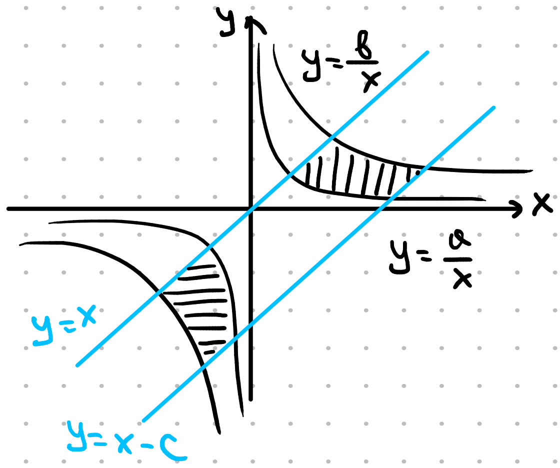
$$\frac{\pi}{4} \le I^2 \le \frac{\pi}{4}\ ; \quad I^2 = \frac{\pi}{4} \Rightarrow I = \frac{\sqrt{\pi}}{2}$$

Ответ: $\frac{\sqrt{\pi}}{2}$

№124 1) $f(x,y) = x+y$ $\quad G:\ xy=a,\ xy=b,\ y=x,\ y=x-c$

$0<a<b\ ;\ 0<c$

$\iint\limits_G f(x,y)\,dx\,dy\ - ?$



Пусть $u=xy$, $x-y=v$

$\Rightarrow G:\ u=a,\ u=b,\ v=0,\ v=c$

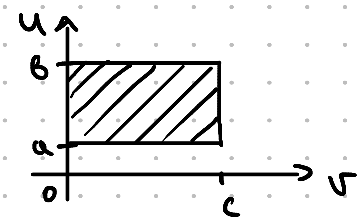
$u = x(x-v) = x^2-vx$

$1 = 2x\frac{\partial x}{\partial u} - v\frac{\partial x}{\partial u} \Rightarrow \frac{\partial x}{\partial u} = \frac{1}{2x-v}$

$0 = 2x\frac{\partial x}{\partial v} - v\frac{\partial x}{\partial u} - x \Rightarrow \frac{\partial x}{\partial v} = \frac{x}{2x-v}$

$u = y(y+v) = y^2+vy \Rightarrow \frac{\partial y}{\partial u} = \frac{1}{2y+u}\ ;\ \frac{\partial y}{\partial v} = -\frac{y}{2y+v}$

$$\frac{D(x,y)}{D(u,v)} = \begin{vmatrix} \frac{1}{2x-v} & \frac{x}{2x-v} \\ \frac{1}{2y+u} & \frac{-y}{2y+v} \end{vmatrix} = \left|-\frac{y+x}{(2x-v)(2y+v)}\right| = \frac{1}{x+y}$$



$$\iint\limits_G f(x,y)\,dx\,dy = \iint\limits_G du\,dv = \int\limits_a^b du\int\limits_0^c dv = c(b-a)$$

Ответ: $c(b-a)$

5) $f(x,y) = x^4-y^4$ $\quad G = \{x>0,\ 1\le xy\le 2,\ 1\le x^2-y^2\le 2\}$

$u=xy\ ;\ v=x^2-y^2$

$$\frac{D(u,v)}{D(x,y)} = \begin{vmatrix} y & x \\ 2x & -2y \end{vmatrix} = 2(x^2+y^2) \qquad \frac{f}{2(x^2+y^2)} = \frac{x^2-y^2}{2} = \frac{v}{2}$$

$$\iint\limits_G f(x,y)\,dx\,dy = \frac{1}{2}\int\limits_1^2 du\int\limits_1^2 v\,dv = \frac{1}{2}\cdot\frac{3}{2}\cdot\int\limits_1^2 du = \frac{3}{4}$$

Ответ: $3/4$

№144 б) $f(x,y,z)=\sqrt{x^2+y^2+z^2}$ $\quad G=\{x^2+y^2+z^2\le z\}$

$x=r\cos\psi\cos\varphi$
$y=r\cos\psi\sin\varphi$ $\Rightarrow$ $f=r$; G: $r^2\le r\sin\psi$
$z=r\sin\psi$

$0\le r\le\sin\psi$
$0\le\varphi\le 2\pi$
$0\le\psi\le\pi/2$

$$\int_0^{2\pi}d\varphi\int_0^{\pi/2}d\psi\int_0^{\sin\psi}r^2\cos\psi\, r\,dr=\int_0^{2\pi}d\varphi\int_0^{\pi/2}\cos\psi\,d\psi\cdot\frac{1}{4}\cdot r^4\Big|_0^{\sin\psi}=\frac{1}{4}\int_0^{2\pi}d\varphi\int_0^{\pi/2}\sin^4\psi\,d(\sin\psi)=$$

$$=\frac{1}{4}\int_0^{2\pi}d\varphi\cdot\frac{1}{5}\cdot\sin^5\psi\Big|_0^{\pi/2}=\frac{1}{20}\cdot 2\pi=\frac{\pi}{10}$$

Ответ: $\frac{\pi}{10}$

№145 1) $\iiint f\left(\sqrt{x^2+y^2+z^2}\right)dx\,dy\,dz$ $\quad G=\{\sqrt{x^2+y^2}\le z\le\sqrt{2-x^2-y^2}\}$

$J=r^2\cos\psi$ $\quad G$ $\quad -\frac{\pi}{2}\le\psi\le\frac{\pi}{2}$; $-\pi\le\varphi\le\pi$

$x=r\cos\psi\cos\varphi$
$y=r\cos\psi\sin\varphi$
$z=r\sin\psi$

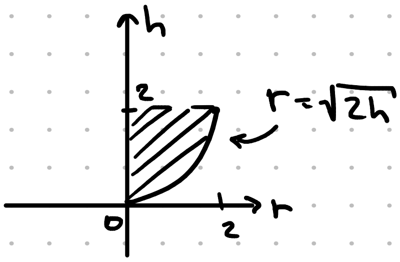
$f(r)$; $r\cos\psi\le r\sin\psi$ $\qquad r^2\sin^2\psi\le 2-r^2\cos^2\psi$
$\cos\psi\le\sin\psi$ $\qquad r^2\le 2$
$\pi/4\le\psi\le\pi/2$ $\qquad 0\le r\le\sqrt{2}$

$$\int_{-\pi}^{\pi}d\varphi\int_{\pi/4}^{\pi/2}d\psi\int_0^{\sqrt2}f(r)\cdot r^2\cos\psi\,dr=2\pi\int_{\pi/4}^{\pi/2}\cos\psi\,d\psi\int_0^{\sqrt2}r^2f(r)\,dr=2\pi\left(1-\frac{\sqrt2}{2}\right)\int_0^{\sqrt2}r^2f(r)\,dr$$

Ответ: $2\pi\left(1-\frac{\sqrt2}{2}\right)\int_0^{\sqrt2}r^2f(r)\,dr$

№146 3) $f(x,y,z)=x^2+y^2$ $\quad G=\{\frac{x^2+y^2}{2}\le z\le 2\}$

$J=r$
$x=r\cos\varphi$
$y=r\sin\varphi$ $\quad f=r^2$; G: $\frac{r^2}{2}\le h\le 2$;
$z=h$ $\quad 0\le\varphi\le 2\pi$



$$\int_0^{2\pi}d\varphi\int_0^2dh\int_0^{\sqrt{2h}}r^3dr=2\pi\int_0^2dh\cdot\frac{1}{4}\cdot r^4\Big|_0^{\sqrt{2h}}=\frac{\pi}{2}\int_0^2 4h^2dh=$$

$$=2\pi\cdot\frac{1}{3}\cdot h^3\Big|_0^2=\frac{2\pi}{3}\cdot 8=\frac{16\pi}{3}$$

Ответ: $\frac{16\pi}{3}$

№147 Показать, что при переходе к обобщ. сф. коорд. $\mathcal{J} = abcr^2\cos\psi$

$x = ar\cos\psi\cos\varphi$, $y = br\cos\psi\sin\varphi$, $z = cr\sin\psi$

$\mathcal{J} = \dfrac{D(x,y,z)}{D(r,\psi,\varphi)}$; $x'_r = a\cos\psi\cos\varphi$, $x'_\psi = -ar\sin\psi\cos\varphi$, $x'_\varphi = -ar\cos\psi\sin\varphi$
$y'_r = b\cos\psi\sin\varphi$, $y'_\psi = -br\sin\psi\sin\varphi$, $y'_\varphi = br\cos\psi\cos\varphi$
$z'_r = c\sin\psi$, $z'_\psi = cr\cos\psi$, $z'_\varphi = 0$

$$\frac{D(x,y,z)}{D(r,\psi,\varphi)} = \begin{vmatrix} a\cos\psi\cos\varphi & -ar\sin\psi\cos\varphi & -ar\cos\psi\sin\varphi \\ b\cos\psi\sin\varphi & -br\sin\psi\sin\varphi & br\cos\psi\cos\varphi \\ c\sin\psi & cr\cos\psi & 0 \end{vmatrix} =$$

$$= c\sin\psi\left(abr^2\sin\psi\cos\psi\cos^2\varphi + abr^2\sin\psi\cos\psi\sin^2\varphi\right) +$$
$$+ cr\cos\psi\left(abr\cos^2\psi\cos^2\varphi + abr\cos^2\psi\sin^2\varphi\right) =$$
$$= c\sin\psi\cdot abr^2\sin\psi\cos\psi + cr\cos\psi\cdot abr\cos^2\psi =$$
$$= cabr\cos\psi\, r(\cos^2\psi + \sin^2\psi) = abcr^2\cos\psi = \mathcal{J} \quad \text{— ч.т.д}$$

№148 2) $\iiint\limits_G (x^2+y^2)\,dx\,dy\,dz$ $\quad G = \left\{\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \le 1\right\}$

$x = ar\cos\psi\cos\varphi$, $y = br\cos\psi\sin\varphi$, $z = cr\sin\psi$; $\mathcal{J} = abcr^2\cos\psi$

$\Rightarrow$ $0 \le r \le 1$; $0 \le \varphi \le 2\pi$; $-\pi/2 \le \psi \le \pi/2$; $x^2+y^2 - r^2\cos^2\psi(a^2\cos^2\varphi + b^2\sin^2\varphi)$

$\dfrac{D(x,y,z)}{D(r,\varphi,\psi)} = abcr^2\cos\psi$

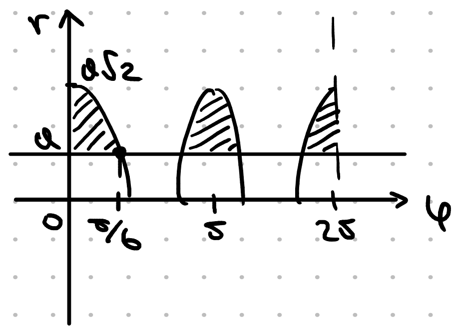
$$\int_0^{2\pi} d\varphi \int_{-\pi/2}^{\pi/2} d\psi \int_0^1 abcr\cos^3\psi\,(b^2 + \cos^2\varphi(a^2-b^2))\,dr = \int_0^{2\pi}(a^2\cos^2\varphi + b^2\sin^2\varphi)\,d\varphi \int_{-\pi/2}^{\pi/2}\cos^3\psi\,d\psi \int_0^1 abcr^4\,dr =$$
$$= \pi(a^2+b^2)\frac{4}{3}\cdot\frac{abc}{5} = \frac{4\pi abc(a^2+b^2)}{15}$$

Ответ: $\frac{4}{15}\pi(a^2+b^2)abc$

§9

№6 2) S?

G: $(x^2+y^2)^2 = 2a^2(x^2-y^2)$, $x^2+y^2 = a^2$; $\sqrt{x^2+y^2} \geqslant a > 0$
$x = r\cos\varphi$, $y = r\sin\varphi$, $\varphi \in (0, 2\pi)$
$\Rightarrow$ G: $r^2 = 2a^2\cos 2\varphi$, $r^2 = a^2$, $r \geqslant a > 0$
$r = a\sqrt{2\cos 2\varphi}$ $\quad r = a$



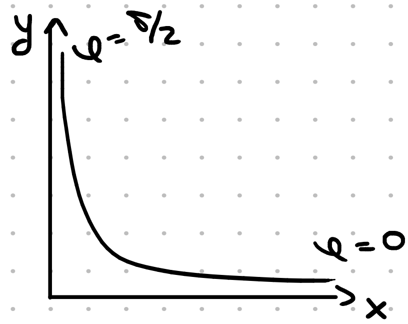
$$S = 4\int_0^{\pi/6} d\varphi \int_a^{a\sqrt{2\cos 2\varphi}} r\,dr = 2\int_0^{\pi/6} d\varphi \cdot a^2(2\cos 2\varphi - 1) =$$
$$= 2a^2\int_0^{\pi/6}(2\cos 2\varphi - 1)\,d\varphi = 2a^2\left(\sin 2\varphi\Big|_0^{\pi/6} - \varphi\Big|_0^{\pi/6}\right) =$$
$$= 2a^2\left(\frac{\sqrt{3}}{2} - \frac{\pi}{6}\right) = \frac{3\sqrt{3}-\pi}{3}a^2$$

Ответ: $\frac{3\sqrt{3}-\pi}{3}a^2$

№8 б) $\sqrt[4]{\frac{x}{a}}+\sqrt[4]{\frac{y}{b}}=1$, $x=0$, $y=0$, $a>0$, $b>0$, $S?$

$x=ar\cos^8\varphi$, $y=br\sin^8\varphi$

$\Rightarrow G:\ \sqrt[4]{r}\le 1,\ \varphi\in(0,\pi/2)$



$$\frac{D(x,y)}{D(r,\varphi)}=\begin{vmatrix} a\cos^8\varphi & -8ar\cos^7\varphi\sin\varphi \\ a\sin^8\varphi & 8br\sin^7\varphi\cos\varphi\end{vmatrix}=8abr\cos^7\varphi\sin^7\varphi$$

$$S=\int_0^{\pi/2}d\varphi\int_0^1 8abr\cos^7\varphi\sin^7\varphi=4ab\int_0^{\pi/2}\cos^7\varphi\sin^7\varphi\,d\varphi=$$
$$=4ab\cdot I_0\Big|_0^{\pi/2}\ \ominus$$

$$I_0=\int\cos\varphi\sin^7\varphi(1-\sin^2\varphi)^3d\varphi=\begin{bmatrix}u=\sin\varphi\\ du=\cos\varphi d\varphi\end{bmatrix}=\int u^7(1-u^2)^3du=$$

$$=\int(-u^{13}+3u^{11}-3u^9+u^7)du=-\frac{\sin^{14}\varphi}{14}+\frac{\sin^{12}\varphi}{4}-\frac{3\sin^{10}\varphi}{10}+\frac{\sin^8\varphi}{8}$$

$$\ominus\ 4ab\cdot\left(-\frac{1}{14}+\frac{1}{4}-\frac{3}{10}+\frac{1}{8}\right)=ab\left(-\frac{2}{7}+1-\frac{6}{5}+\frac{1}{2}\right)=ab\left(\frac{1}{2}-\frac{1}{5}-\frac{2}{7}\right)=\frac{ab}{70}$$

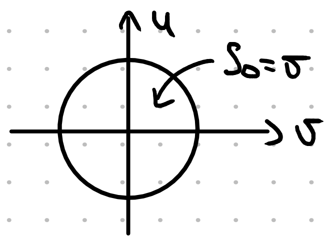
Ответ: $\frac{ab}{70}$

№10 $S?$ $(a_1x+b_1y+c_1)^2+(a_2x+b_2y+c_2)^2=1$

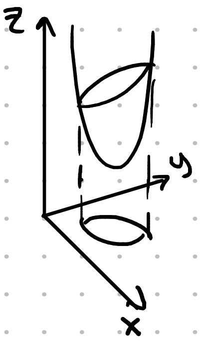
$u=a_1x+b_1y+c_1$; $v=a_2x+b_2x+c_2$ $\Delta=a_1b_2-b_2a_1\neq 0$

$$\frac{D(x,y)}{D(u,v)}=\left|\frac{1}{a_1b_2-b_2a_1}\right|=\frac{1}{|\Delta|}\ ;\quad u^2+v^2=1$$

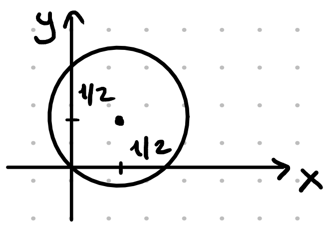
$S=S_0\cdot\frac{1}{|\Delta|}=\frac{\pi}{|\Delta|}$ Ответ: $\frac{\pi}{|\Delta|}$



№13 5) $V?$ $G:\ z=x^2+y^2,\ z=x+y$



$$\iint_G(x+y-x^2-y^2)dxdy\qquad G:\ x^2+y^2<x+y;\ (x-1/2)^2+(y-1/2)^2<\frac{1}{2}$$



$$\begin{cases}x=1/2+r\cos\varphi\\ y=1/2+r\sin\varphi\end{cases}\quad r=\frac{1}{\sqrt2}$$

$$\Rightarrow x+y-x^2-y^2=\frac{1}{2}-(x-1/2)^2-(y-1/2)^2=\frac{1}{2}-r^2$$

$$\int_0^{2\pi}d\varphi\int_0^{1/\sqrt2}r\left(\frac{1}{2}-r^2\right)dr=2\pi\left(\frac{1}{4}r^2\Big|_0^{1/\sqrt2}-\frac{1}{4}r^4\Big|_0^{1/\sqrt2}\right)=2\pi\left(\frac{1}{8}-\frac{1}{16}\right)=\frac{\pi}{8}$$

Ответ: $\frac{\pi}{8}$

№16 3) V? $(x^2+y^2+z^2)^2 = a^3x$ $a>0$

$x = r\cos\psi\cos\varphi$
$y = r\cos\psi\sin\varphi$ $\quad J = r^2\cos\psi \Rightarrow r^3 = a^3\cos\psi\cos\varphi$ $\quad \psi\in(-\pi/2, \pi/2)$
$z = r\sin\psi$ $\quad \varphi\in(-\pi/2,\pi/2)$ т.к. $\cos\varphi > 0$

$$\int_{-\pi/2}^{\pi/2} d\varphi \int_{-\pi/2}^{\pi/2} d\psi \int_0^{(a^3\cos\psi\cos\varphi)^{1/3}} r^2\cos\psi\, dr = \frac{1}{3}\int_{-\pi/2}^{\pi/2} d\varphi \int_{-\pi/2}^{\pi/2} \cos\psi\, d\psi \cdot a^3\cdot\cos\psi\cdot\cos\varphi =$$

$$= \frac{a^3}{3}\int_{-\pi/2}^{\pi/2}\cos\varphi\, d\varphi \int_{-\pi/2}^{\pi/2}\cos^2\psi\, d\psi = \frac{2a^3}{3}\cdot\frac{1}{2}\left(\cancel{\int_{-\pi/2}^{\pi/2}\cos 2x\, dx}^{\,0} + \int_{-\pi/2}^{\pi/2} dx\right) = \frac{\pi a^3}{3}$$

Ответ: $\frac{\pi a^3}{3}$

№21 V? $a_1x+b_1y+c_1z = \pm d_1,$
$a_2x+b_2y+c_2z = \pm d_2,$
$a_3x+b_3y+c_3z = \pm d_3$, если $\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \neq 0$

Пусть $p = a_1x+b_1y+c_1z$
$q = a_2x+b_2y+c_2z$ $\quad , \quad \frac{D(x,y,z)}{D(p,q,u)} = \frac{1}{|\Delta|}$
$u = a_3x+b_3y+c_3z$

$\Rightarrow$ $p = \pm d_1,\ q = \pm d_2,\ u = \pm d_3$

$$V = \int_{-d_1}^{d_1} dp \int_{-d_2}^{d_2} dq \int_{-d_3}^{d_3} \frac{du}{|\Delta|} = \frac{8d_1d_2d_3}{|\Delta|}$$

Ответ: $\frac{8d_1d_2d_3}{|\Delta|}$

№63 4) Найти коорд. ц.масс с плотн. ρ

$\sqrt{x^2+y^2} \le z \le h$, $\rho = \rho_0 z^2$

$x = r\cos\varphi$
$y = r\sin\varphi$ $\quad \varphi\in(0,2\pi)$; $\quad 0\le r\le h$; $\quad \rho = \rho_0\cdot z^2$
$z = z$ $\quad J = r$ $\quad r \le z \le h$

$$M = \int_0^{2\pi} d\varphi \int_0^h dr \int_r^h r\rho_0 z^2\, dz = \rho_0\cdot 2\pi\int_0^h r\, dr \frac{1}{3}z^3\Big|_r^h = \frac{2\pi\rho_0}{3}\int_0^h (h^3-r^3)r\, dr =$$

$$= \frac{2\pi\rho_0}{3}\left(\frac{h^3}{2}\cdot r^2\Big|_0^h - \frac{1}{5}r^5\Big|_0^h\right) = \frac{2\pi\rho_0}{3}\left(\frac{h^5}{2}-\frac{h^5}{5}\right) = \frac{h^5\pi\rho_0}{5}$$

$$x_c = \frac{1}{M}\int_0^{2\pi} d\varphi\int_0^h dr\int_r^h r^2\cos\varphi\,\rho_0 z^2\, dz = \frac{\rho_0}{M}\int_0^{2\pi}\cancel{\cos\varphi\, d\varphi}^{\,0}\int_0^h r^2\, dr\,\frac{1}{3}(h^3-r^3) = 0$$

$$y_c = \frac{1}{M}\int_0^{2\pi} d\varphi\int_0^h dr\int_r^h r^2\cancel{\sin\varphi}^{\,0}\,\rho_0 z^2\, dz = 0$$

$$z_c = \frac{1}{M}\int_0^{2\pi} d\varphi\int_0^h dr\int_r^h r\rho_0 z^3\, dz = \frac{\rho_0}{M}2\pi\int_0^h dr\cdot\frac{1}{4}(rh^4 - r^5) = \frac{\rho_0\pi}{2M}\left(\frac{h^4}{2}r^2\Big|_0^h - \frac{r^6}{6}\Big|_0^h\right) =$$

$$= \frac{\rho_0\pi}{2M}\left(\frac{h^6}{2}-\frac{h^6}{6}\right) = \frac{\rho_0\pi h^6}{6M} = \frac{\cancel{\rho_0\pi} h^6}{6}\cdot\frac{5}{h^5\cancel{\pi\rho_0}} = \frac{5}{6}h$$

Ответ: $x_c = y_c = 0$, $z_c = \frac{5}{6}h$