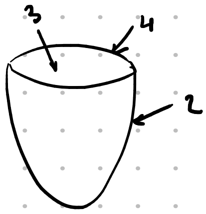
Найти min и max знач. $u=x+y+z$
при мн-ве $x^2+y^2 \le z \le 1$



1) стац. точки внутри $x^2+y^2<z$, $x^2+y^2<1$

2) $u=x+y+z$, $x^2+y^2=z$, $x^2+y^2<1$

$u=x+y+x^2+y^2$, $x^2+y^2<1$

Найти стац. точки

$u'_x=1+2x=0$ $\quad (-1/2, -1/2)$

$u'_y=1+2y=0$ $\quad u=-\frac{1}{2}$

3) $u=x+y+z$, $x^2+y^2<z$, $z=1$

$u=x+y+1$

Найти стац. точки: $\varnothing$

4) $u=x+y+z$, $x^2+y^2=z$, $z=1$

$u=x+y+1$, $x^2+y^2=1$

$\mathcal{L}=x+y+1+\lambda(x^2+y^2-1)$ Найти стац. т. ф-ии Лагранжа

$\mathcal{L}'_x=1+2\lambda x=0$

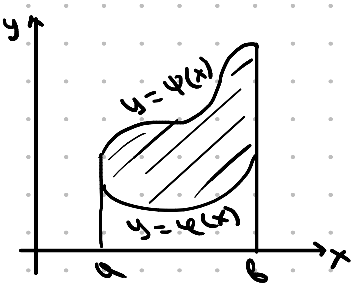
$\mathcal{L}'_y=1+2\lambda y=0$ $\quad 2\lambda=-\frac{1}{x}=-\frac{1}{y}$ $\quad x=y=\pm\frac{1}{\sqrt{2}}$

$x^2+y^2=1$ $\quad u\left(\frac{1}{\sqrt{2}}; \frac{1}{\sqrt{2}}\right)=1+\sqrt{2}$ $\quad u\left(-1/\sqrt{2}; -\frac{1}{\sqrt{2}}\right)=1-\sqrt{2}$

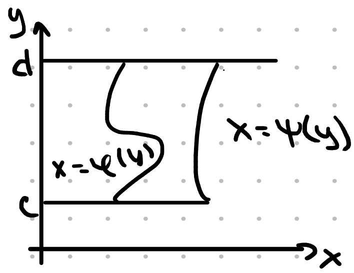
$\Rightarrow$ Ответ: $\max u = 1+\sqrt{2}$
$\min u = -1/2$

Двойные интегралы

Двойной интеграл вычисл. как повторный.



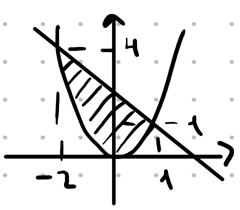
$$\iint\limits_G f(x,y)\,dxdy = \int_a^b \left\{\int_{\varphi(x)}^{\psi(x)} f(x,y)\,dy\right\} dx = \int_a^b dx \int_{\varphi(x)}^{\psi(x)} f(x,y)\,dy$$



$$\int_c^d dy \int_{\varphi(y)}^{\psi(y)} f(x,y)\,dy$$

Расставить пределы инт. двумя способами по области, огр. линиями

Пр. 1) $y=x^2$ $\quad y+x=2$



$x^2=2-x$

$x^2+x-2=0$

$x=-2; 1$

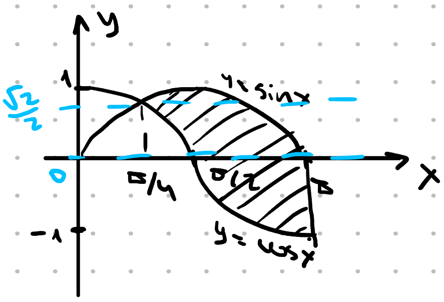
$y=4, 1$

$$\int_{-2}^{1} dx \int_{x^2}^{2-x} f(x,y)\,dy$$

или

$$\int_0^1 dy \int_{-\sqrt{y}}^{\sqrt{y}} f(x,y)\,dx + \int_1^4 dy \int_{-\sqrt{y}}^{2-y} f(x,y)\,dx$$

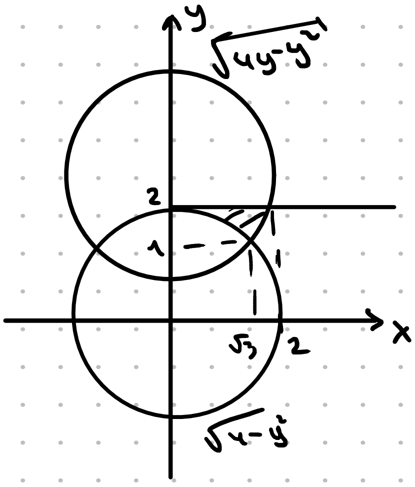
Пр. 2) $y=\sin x$, $y=\cos x$, $x=\pi$ $(x>0)$



$$\int_{\pi/4}^{\pi} dx \int_{\cos x}^{\sin x} f(x,y)\,dy$$

$$\int_{-1}^{0} dy \int_{\arccos y}^{\pi} f(x,y)\,dx + \int_{0}^{\sqrt{2}/2} dy \int_{\arccos y}^{\pi-\arcsin y} f(x,y)\,dx + \int_{\sqrt{2}/2}^{1} dy \int_{\arcsin y}^{\pi-\arcsin y} f(x,y)\,dx$$

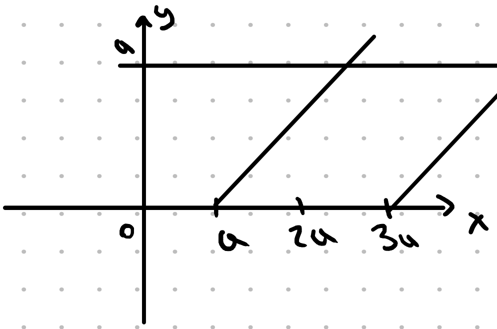
Пр. 3) $x=\sqrt{4-y^2}$ $x=\sqrt{4y-y^2}$ $y=2$



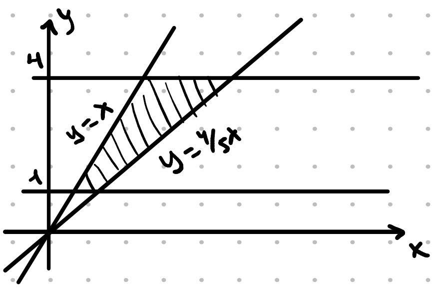
$$\int_{1}^{2} dy \int_{\sqrt{4-y^2}}^{\sqrt{4y-y^2}} f(x,y)\,dx$$

$$\int_{0}^{\sqrt{3}} dx \int_{\sqrt{4-x^2}}^{2} f(x,y)\,dy + \int_{\sqrt{3}}^{2} dx \int_{2-\sqrt{4-x^2}}^{2} f(x,y)\,dy$$

Пр. 4) $\iint\limits_G (x^2+y^2)\,dx\,dy$ G огр. линиями $y=0$ $y=a$ $y=x-a$ $y=x-3a$, $a>0$



Пр. 5) $\iint\limits_G \sqrt{x-y}\,dx\,dy$ $G=\left\{\frac{4}{5}x \le y \le x,\ 1\le y\le 4\right\}$



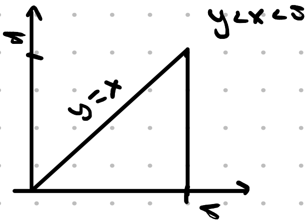
$$\int_{1}^{4} dy \int_{y}^{4/5y} \sqrt{x-y}\,dx = \int_{1}^{4} dy\, \frac{2}{3}(x-y)^{3/2}\Big|_{y}^{4/5y} = \int_{1}^{4} \frac{2}{3}\left(\frac{5}{4}y-y\right)^{3/2} dy =$$

$$= \int_{1}^{4} \frac{2}{3}\cdot\frac{1}{8}\cdot y^{3/2}\,dy = \frac{2}{3}\cdot\frac{1}{8}\cdot\frac{2}{5}\cdot y^{5/2}\Big|_{1}^{4} = \frac{2}{3}\cdot\frac{1}{8}\cdot\frac{2}{5}(32-1) = \frac{31}{30}$$

Ответ: $\frac{31}{30}$

Вычислим повт. инт., поменяв порядок инт.

$$\int_0^\pi dy \int_y^\pi \frac{\sin x}{x} dx = \int_0^\pi dx \int_0^x \frac{\sin x}{x} dy = \int_0^\pi \frac{\sin x}{x} dx = \int_0^\pi \sin x\, dx = 2$$



Замечание: $\int_a^b dx \int_c^d f(x)g(y)dy = \int_a^b f(x)dx \int_c^d g(y)dy$

Замена переменных в двойном инт.

$G_{x,y} \leftrightarrow D_{u,v}$ биективно

$x = x(u,v)$
$y = y(u,v)$

якобиан.

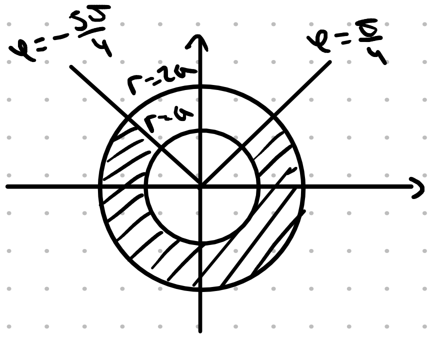
$$\iint_G f(x,y)\,dx\,dy = \iint_D f(x(u,v), y(u,v)) \cdot \left|\frac{D(x,y)}{D(u,v)}\right| du\,dv$$

Полярные коорд.:

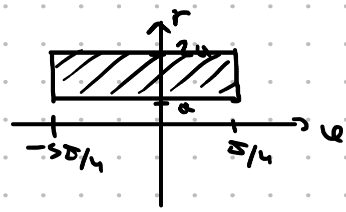
$x = r\cos\varphi$
$y = r\sin\varphi$
$\frac{D(x,y)}{D(r,\varphi)} = r$

Пр. $\iint_G f(x,y)\,dx\,dy$ Перейти к полярным коорд.

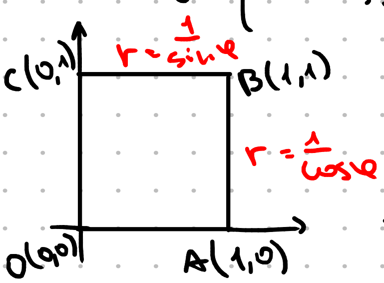
$G = \{a^2 < x^2+y^2 < 4a^2,\ y < |x|\}$

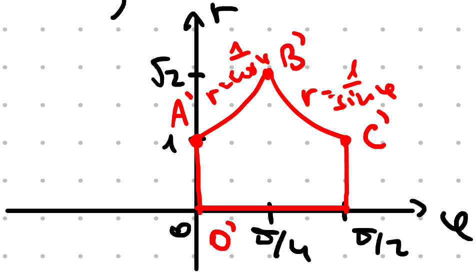


$$\int_{-5\pi/4}^{\pi/4} d\varphi \int_a^{2a} f(r\cos\varphi, r\sin\varphi)\,dr = \int_a^{2a} dr \int_{-5\pi/4}^{\pi/4} f(r\cos\varphi, r\sin\varphi)\,d\varphi$$



Пр. $G = \{0 \le x, y \le 1\}$





$$\int_0^{\pi/4} d\varphi \int_0^{\frac{1}{\cos\varphi}} r f(r\cos\varphi, r\sin\varphi)\,dr + \int_{\pi/4}^{\pi/2} d\varphi \int_0^{\frac{1}{\sin\varphi}} r f(r\cos\varphi, r\sin\varphi)\,dr =$$

$x = r\cos\varphi \quad y = r\sin\varphi$

A: $r=1 \quad \varphi=0$

B: $r=\sqrt{2} \quad \varphi=\pi/4$

C: $r=1 \quad \varphi=\pi/2$

O: $r=0 \quad \varphi$ не опр.

$$= \int_0^1 r\,dr \int_0^{\pi/2} f(r\cos\varphi, r\sin\varphi)\,d\varphi + \int_1^{\sqrt{2}} r\,dr \int_{\arccos\frac{1}{r}}^{\arcsin\frac{1}{r}} f(r\cos\varphi, r\sin\varphi)\,d\varphi$$