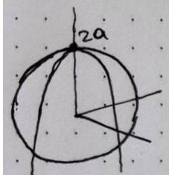
В каком отнош. делит $V$ шара $x^2+y^2+z^2 \le 4az$ пов-ть $x^2+y^2+az=4a^2$, $a>0$

$x^2+y^2+(z-2a)^2=4a^2$

$x'=x,\quad y'=y,\quad z'=z-2a$

шар: $x'^2+y'^2+z'^2 \le 4a^2$ пов-ть: $x^2+y^2+az'=2a^2$ – параболоид

$z'=2a-\frac{x'^2+y'^2}{a}$

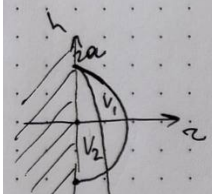



$x'=r\cos\varphi$

$y'=r\sin\varphi$

$z'=h$

шар: $r^2+h^2 \le 4a^2$ пов-ть: $r^2+ah=2a^2$




$V_1+V_2=\frac{4}{3}\pi(2a)^3=\frac{32}{3}\pi a^3$

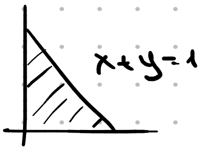
$V_1=\int_0^{2\pi}d\varphi\int -\!\!\shortparallel\!\!-$

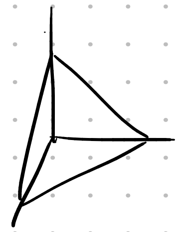
Кр: экстремумы, несобств. интегр., повт. интегралы

$$-V_1=\int_0^{2\pi}d\varphi\int_{-a}^{2a}dh\int_{\sqrt{2a^2-ah}}^{\sqrt{4a^2-h^2}}r\,dr=2\pi\int_{-a}^{2a}dh\left(\frac{4a^2-h^2}{2}-\frac{2a^2-ah}{2}\right)=\pi\int_{-a}^{2a}dh\left(2a^2+ah-h^2\right)=\pi\left(-\frac{h^3}{3}+\frac{ah^2}{2}+2a^2h\right)\Big|_{-a}^{2a}=$$

$$=\pi\left(-\frac{8a^3}{3}-\frac{a^3}{3}+2a^3-\frac{a^3}{2}+6a^3\right)\Rightarrow \underline{V_1=\pi\frac{9}{2}a^3}$$

Пр. $\iiint_G \frac{dx\,dy\,dz}{(1+x+y+z)^3}$ $\quad G:\ x=0,\ y=0,\ z=0,\ x+y+z=1$



$$\int_0^1 dx\int_0^{1-x}dy\int_0^{1-x-y}\frac{d(1+x+y+z)}{(1+x+y+z)^3}=-\frac{1}{2}\int dx\int_0\frac{1}{(1+x+y+z)^2}\Big|_0^{1-x-y}dy=$$

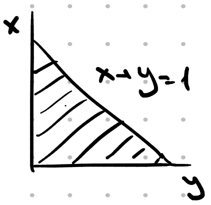
$$=\left[\int\frac{dt}{t^3}=-\frac{1}{2}\frac{1}{t^2}+C\right]=\frac{1}{2}\int_0^1dx\int_0\left(\frac{1}{(1+x+y)^2}-\frac{1}{4}\right)dy=$$

$$=-\frac{1}{2}\int_0^1\left(\frac{1}{1+x+y}\Big|_0^{1-x}+\frac{1}{4}(1-x)\right)dx=-\frac{1}{2}\int_0^1\left(\frac{1}{2}-\frac{1}{1+x}+\frac{1}{4}-\frac{1}{4}x\right)dx=-\frac{1}{2}\left(\frac{1}{2}x-\ln(1+x)+\frac{1}{4}x-\frac{1}{8}x^2\right)\Big|_0^1=$$

$$=\left(\frac{1}{2}-\ln 2+\frac{1}{4}-\frac{1}{8}\right)\left(-\frac{1}{2}\right)=\frac{\ln 2}{2}-\frac{5}{16}$$

Пр. Найти $V$ огр. поверх

$z=x+y\quad z=xy\quad x+y=1\quad x=0\quad y=0$




$xy \lessgtr x+y$

$(x-1)(y-1) \vee 1$

$(1-x)(1-y) \vee 1$

$<$ т.к. $1-x, 1-y \in (0,1)$

$$V=\iint_G(x+y-xy)\,dx\,dy=\int_0^1dx\int_0^{1-x}(x+y-xy)\,dy=\int_0^1dx\left(\frac{y^2}{2}(1-x)+xy\right)\Big|_0^{1-x}=$$

$$=\int_0^1dx\left(\frac{x^2-2x+1}{2}(1-x)+x(1-x)\right)=\int_0^1dx(1-x)\left(\frac{x^2+1}{2}\right)=\frac{1}{2}\int_0^1(x^2-x^3+1-x)\,dx=$$

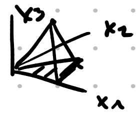
$$=\frac{1}{2}\left(\frac{x^3}{3}-\frac{x^4}{4}+x-\frac{x^2}{2}\right)\Big|_0^1=\frac{1}{2}\left(\frac{1}{3}-\frac{1}{4}+1-\frac{1}{2}\right)=\underline{\frac{7}{24}}$$

Пр   Симплекс $\Pi_n = \{0 \le x_n \le x_{n-1} \le \ldots \le x_2 \le x_1 \le a\}$

$$\int\limits_{\Pi_n} x_1 x_2 \ldots x_n \, dx_1 dx_2 \ldots dx_n =$$

$n=1$ – отрезок

$n=2$ – 

$n=3$ – 

$$= \int_0^a x_1 dx_1 \int_0^{x_1} x_2 dx_2 \int_0^{x_2} x_3 dx_3 \ldots \int_0^{x_{n-1}} x_n dx_n$$

$$I_1 = \int_0^a x_1 dx_1 = a^2/2$$

$$I_2 = \int_0^a x_1 dx_1 \int_0^{x_1} x_2 dx_2 = \int_0^a x_1 dx_1 \cdot \frac{x_1^2}{2} = \int_0^a \frac{x_1^3}{2} = \frac{a^4}{2\cdot 4}$$

$\vdots$

$$I_n(a) = \frac{a^{2n}}{(2n)!!} \leftarrow 2\cdot 4\cdot 6 \ldots 2n$$

Д-во по индукции: $I_n(a) = \dfrac{a^{2n}}{(2n)!!}$

$n = 1,2$ – проверено

$$I_{n+1}(a) = \int_0^a x_1 dx_1 \underbrace{\int_0^{x_1} x_2 dx_2 \int_0^{x_2} x_3 dx_3 \ldots \int_0^{x_{n-1}} x_n dx_n \int_0^{x_n} x_{n+1} dx_{n+1}}_{I_n(x_1)}$$

8.34 не решен

$\Rightarrow$ по предполож. инд.

$$I_{n+1}(a) = \int_0^a x_1 \cdot \frac{x_1^{2n}}{(2n)!!} = \int_0^a \frac{x_1^{2n+1}}{(2n)!!} dx_1 = \frac{a^{2n+2}}{(2n+2)!!}$$ – ч.т.д.

Задание 2.

Криволин. инт. 1-го рода

$$\int\limits_\gamma f(x,y)\,ds = \int_a^b f(x(t), y(t)) \cdot \sqrt{x_t'^2 + y_t'^2}\, dt$$

$\gamma$ – гладкая кривая

$x = x(t)$, $y = y(t)$  $t \in [a, b]$

$f(x,y)$ – непр. на $\gamma$

Пр. $\int\limits_\gamma (x^{4/3} + y^{4/3})\, ds$

$$\int\limits_\gamma (x^{4/3} + y^{4/3})\, ds$$

$\gamma$: $x = a\cos^3 t$, $y = a\sin^3 t$, $0 \le t \le 2\pi$ — астроида

$x_t' = -3a\cos^2 t \cdot \sin t$

$y_t' = 3a\sin^2 t \cos t$

$$\sqrt{x_t'^2 + y_t'^2} = \sqrt{9a^2(\cos^4 t \sin^2 t + \sin^4 t \cos^2 t)} = 3a|\cos t \sin t| =$$

$$= \frac{3a}{2}|\sin 2t|$$

$$I = 4\int_0^{\pi/2} (a^{4/3}\cos^4 t + a^{4/3}\sin^4 t) \cdot \frac{3a}{2}\sin 2t\, dt = 4a^{7/3}\int_0^{\pi/2} \frac{1+\cos^2 2t}{2} \cdot \frac{3}{2}\sin 2t\, dt =$$

$$= \left[\begin{matrix} \cos 2t = u \\ -\sin 2t = \frac{du}{2} \end{matrix}\right] = 3a^{7/3}\int_{-1}^{1} (1+u^2)\frac{du}{2} = 3a^{7/3}\int_0^1 (1+u^2)\,du = 3a^{7/3}\left(u + \frac{u^3}{3}\right)\Big|_0^1 =$$

$$= 4a^{7/3}$$

Криволин. интегр. 2-го рода

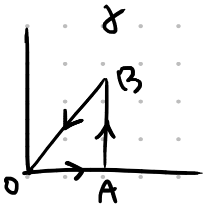
$\gamma$ = ориент. кус. гладкая кривая

$$\int_\gamma P(x,y)\,dx + Q(x,y)\,dy = \pm\int_a^b \left(P(x(t),y(t))\,x'_t + Q(x(t),y(t))\,y'_t\right)dt$$

$P, Q$ - непр. на $\gamma$ $\qquad$ $\oplus$, если $t \uparrow\uparrow$ напр. обхода

$\ominus$, если $t \uparrow\downarrow$ напр. обхода

Пр. $\int_\gamma (3x^2-y)\,dx + (1-2x)\,dy$



$\overline{OA}$: $x=t,\ y=0$ $\quad t\in[0,1]$ $\oplus$

$$\int_{\overline{OA}} = \int_0^1 3t^2\,dt = t^3\Big|_0^1 = 1$$

$\overline{AB}$ $x=1$ $y=t$ $t\in[0,1]$

$\oplus$ $\int_{\overline{AB}} = -\int dy = -1$

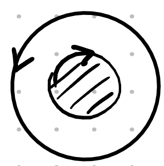
$\overline{BO}$ $x=t,\ y=t,\ t\in[0,1]$

$$\int_{\overline{BO}} = -\int_0^1 (3t^2-t)\,dt + (1-2t)\,dt = -\int_0^1 (3t^2-3t+1)\,dt = -\left(t^3-\frac{3}{2}t^2+t\right)\Big|_0^1 = -1/2$$

## Ф-ла Грина

$G$ - область $\quad$ вкл., ориентированная $\partial G \pm$ при обходе внутр. слева

$P, Q \in C^1(\overline{G})$



$$\int_{\partial G^+} (P\,dx + Q\,dy) = \iint_G \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)dx\,dy$$

$$\int_\gamma P\,dx + Q\,dy = -\iint_G dx\,dy = -\frac{1}{2}$$