Матан 11.1.0.

Тройные интегралы

$G \subset \mathbb{R}^3$

$$\iiint_G f(x,y,z)\,dx\,dy\,dz = \iint_{S_{xy}} dx\,dy \int_{\varphi(x,y)}^{\psi(x,y)} f(x,y,z)\,dz =$$

$$= \int_a^b dx \int_{\alpha(x)}^{\beta(x)} dy \int_{\varphi(x,y)}^{\psi(x,y)} f(x,y,z)\,dz$$

Пр. (1) $\underbrace{\int_{-a}^{a} dx \int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} dy}_{\iint_{S_{xy}}} \int_{\sqrt{x^2+y^2}}^{a} f(x,y,z)\,dz$

$z=\sqrt{x^2+y^2}$

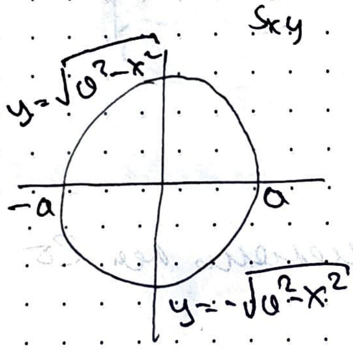
конус

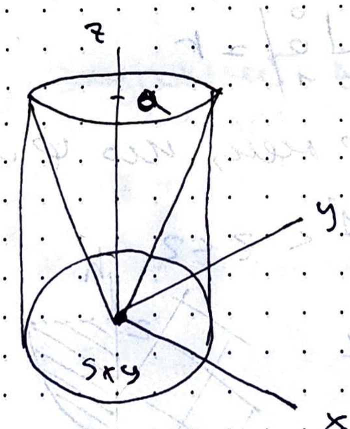
x	y	z	(1)
y	x	z	(2)
x	z	y	(3)
z	y	x	(4)
y	z	x	(5)
z	x	y	(6)

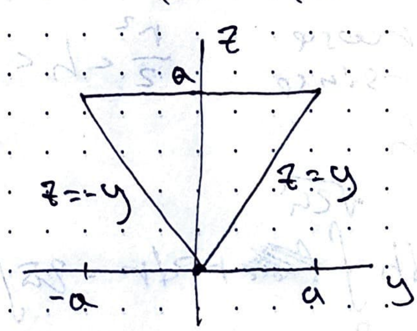
$x \leftrightarrow y$ — взаимозаменяемы в силу симм.

$\Downarrow$

(1) $\leftrightarrow$ (2)
(3) $\leftrightarrow$ (5)
(4) $\leftrightarrow$ (6)







$x^2 = z^2 - y^2$

(2) $\int_{-a}^{a} dy \int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} dx \int_{\sqrt{x^2+y^2}}^{a} f(x,y,z)\,dz$

(5) $\int_{-a}^{a} dy \int_{|y|}^{a} dz \int_{-\sqrt{z^2-y^2}}^{\sqrt{z^2+y^2}} f(x,y,z)\,dx$

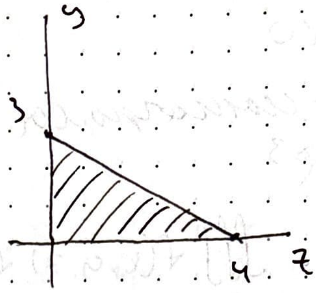
(3) $\int_{-a}^{a} dx \int_{|y|}^{a} dz \int_{-\sqrt{z^2-x^2}}^{\sqrt{z^2-x^2}} f(x,y,z)\,dy$

(4) $\int_{0}^{a} dz \int_{-z}^{z} dy \int_{-\sqrt{z^2-y^2}}^{\sqrt{z^2-y^2}} f(x,y,z)\,dx$

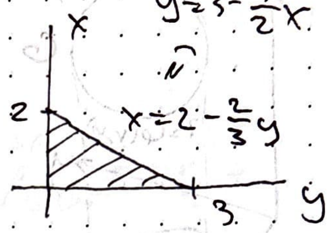
(6) $\int_{0}^{a} dz \int_{-z}^{z} dx \int_{-\sqrt{z^2-x^2}}^{\sqrt{z^2-x^2}} f(x,y,z)\,dy$

Пр. $\underbrace{\int_0^4 dz\int_0^{3-3/4 z} dy}_{S_{yz}} \int_0^{2-2/3 y - z/2} f(x,y,z)\,dx$

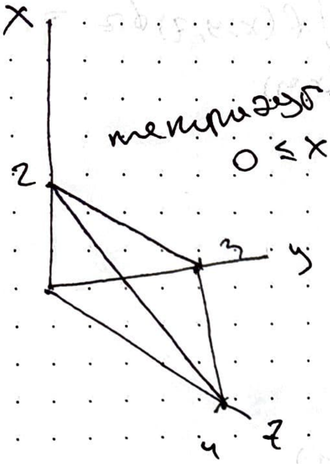
$0 \le y \le 3-\frac{3}{4}z$



$y=3-\frac{3}{2}x$



$x=2-\frac{2}{3}y$



тетраэдр

$0\le x\le 2-\frac{2}{3}y-\frac{z}{2}$

$zyx \to xyz$

$$\int_0^2 dx\int_0^{3-3/2x}dy\int_0^{-2x-4/3y+4} f(x,y,z)\,dz$$

$x=2-\frac{2}{3}y-\frac{z}{2}$

$2x=4-\frac{4}{3}y-z$

$z=-2x-\frac{4}{3}y+4$

Замена переменных в тройном интеграле.

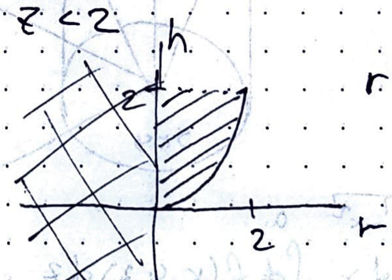
Цилиндрические коорд.

$$\begin{cases} x=r\cos\varphi \\ y=r\sin\varphi \\ z=h \end{cases} \qquad \frac{D(x,y,z)}{D(r,\varphi,h)}=\begin{vmatrix} \dots & \dots & 0 \\ \dots & \dots & 0 \\ 0 & 0 & 1 \end{vmatrix}=r$$

$0\le\varphi\le 2\pi$ если φ нет, то φ меньше чем 2π

Пр. $\iiint_G (x^2+y^2)\,dx\,dy\,dz$, $\frac{x^2+y^2}{2}<z<2$

$$\begin{cases} x=r\cos\varphi \\ y=r\sin\varphi \\ z=h \end{cases} \qquad \frac{r^2}{2}<h<2$$

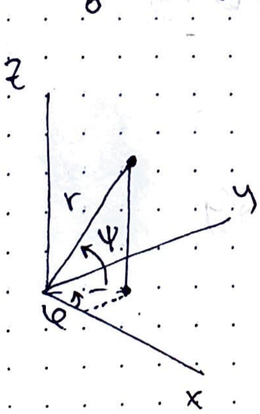


$r=\sqrt{2h}$

$$\int_0^{2\pi}d\varphi\int_0^{2}dh\int_0^{\sqrt{2h}} r^3\,dr = 2\pi\int_0^2 dh =$$

$$=2\pi\int_0^2 dh\,\frac{4h^2}{4}=2\pi\frac{h^3}{3}\Big|_0^2=\frac{16\pi}{3}$$

Сферические коорд.



$-\frac{\pi}{2}\le\psi\le\frac{\pi}{2}$ ~~ψ - долгота~~ ψ - широта

$-\pi\le\varphi\le\pi$ (или $0\le\varphi\le2\pi$) φ - долгота

$$\frac{D(x,y,z)}{D(r,\varphi,\psi)}=r^2\cos\psi$$

$$\begin{cases} x=r\cos\psi\cos\varphi \\ y=r\cos\psi\sin\varphi \\ z=r\sin\psi \end{cases} \qquad x^2+y^2+z^2=r^2$$

Пр. $\iiint\limits_G \sqrt{x^2+y^2+z^2}\,dx\,dy\,dz$; $G: x^2+y^2+z^2 < z$ — сфера

$r^2 = r\sin\psi$

$$\begin{cases} x = r\cos\psi\cos\varphi \\ y = r\cos\psi\sin\varphi \\ z = r\sin\psi \end{cases} \Rightarrow \quad 0 < r < \sin\psi,\quad 0 < \varphi < 2\pi,\quad 0 < \psi < \pi/2$$

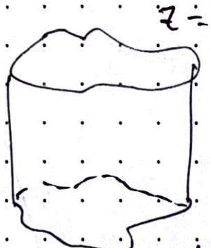
$$\int_0^{2\pi} d\varphi \int_0^{\pi/2} d\psi \int_0^{\sin\psi} r^2\cos\psi \cdot r\,dr = 2\pi\int_0^{\pi/2} d\psi \int_0^{\sin\psi} r^3 dr = 2\pi\int_0^{\pi/2} d\psi \cdot \frac{\sin^4\psi}{4}\cdot\cos\psi =$$

$$= \begin{bmatrix} u = \sin\psi \\ du = \cos\psi\,d\psi \end{bmatrix} = 2\pi\int_0^1 \frac{u^4}{4}du = 2\pi\cdot\frac{1}{20} = \frac{\pi}{10}$$

Ответ.

$$V_G = \iiint\limits_G dx\,dy\,dz = \iint\limits_{G_0} (\psi(x,y) - \varphi(x,y))\,dx\,dy$$

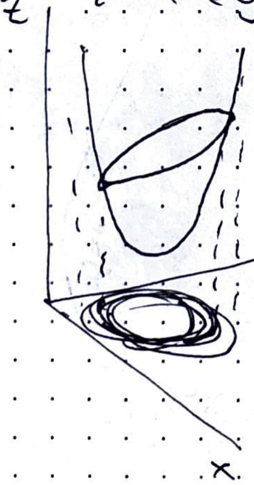
$z = \psi(x,y)$



$z = \varphi(x,y)$

Пр. Найти V заключ. пар. вали
(ограничен. поверх. при пересечении)

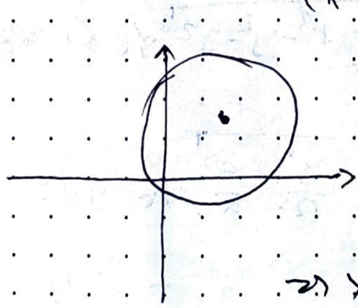
$z = x^2+y^2$ и $z = x+y$



$$\iint\limits_{G_0} (x+y-x^2-y^2)\,dx\,dy$$

G_0: $x^2+y^2 < x+y$

$(x-1/2)^2 + (y-1/2)^2 < 1/2$



$$\begin{cases} x = 1/2 + r\cos\varphi \\ y = 1/2 + r\sin\varphi \end{cases} \quad r = \frac{1}{\sqrt2}$$

$0 < \varphi < 2\pi$

$\Rightarrow x+y-x^2-y^2 = \frac12 - (x-1/2)^2 - (y-1/2)^2$

$$\int_0^{2\pi} d\varphi \int_0^{1/\sqrt2} r(1/2 - r^2)\,dr = 2\pi\left(\frac{r^2}{4} - \frac{r^4}{4}\right)\Big|_0^{1/\sqrt2} = \frac{\pi}{2}\left(\frac12 - \frac14\right) = \frac{\pi}{8}$$

Ответ.

Пр. $\iiint\limits_G \sqrt{1 - \frac{x^2}{a^2} - \frac{y^2}{b^2} - \frac{z^2}{c^2}}\,dx\,dy\,dz$ $\qquad \frac{D(x,y,z)}{D(r,\varphi,\psi)} = abc\,r^2\cos\psi$

$G: \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} < 1$ — эллипс

$$\begin{cases} x = a r\cos\psi\cos\varphi \\ y = b r\cos\psi\sin\varphi \\ z = c r\sin\psi \end{cases}$$

$0 < \varphi < 2\pi$

$-\pi/2 < \psi < \pi/2$

$0 < r < 1$

$\left[\begin{matrix} r = \sin t \\ dr = \cos t \end{matrix}\right]$

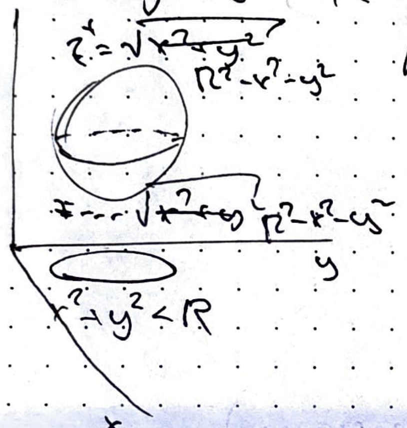
$$\int_0^{2\pi} d\varphi \int_{-\pi/2}^{\pi/2} d\psi \int_0^1 abc\, r^2 \cos\psi \sqrt{1-r^2}\, dr = 2\pi abc \int_{-\pi/2}^{\pi/2} \cos\psi\, d\psi \int_0^1 r^2\sqrt{1-r^2}\, dr =$$

$$= 2\pi abc \cdot 2 \cdot \int_0^{\pi/2} \sin^2 t \cos t \cos t\, dt = \pi abc \int_0^{\pi/2} \sin^2 2t\, dt =$$

$$= \pi abc \int_0^{\pi/2} \frac{1-\cos 4t}{2}\, dt = \frac{\pi^2}{4} abc \to \text{Ответ: } \frac{\pi^2}{4} abc$$

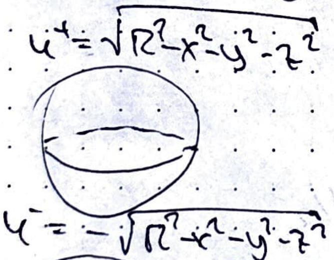
Пр. Объем трехмерного шара V_3

$x^2+y^2+z^2 < R^2$



$$V_3 = \iint\limits_{x^2+y^2<R^2} (z^+ - z^-)\, dx\, dy$$

Объем четырехмерного шара V_4



$$V_4 = \iiint\limits_{x^2+y^2+z^2<R^2} (u^+ - u^-)\, dx\, dy\, dz =$$

$$= \iiint\limits_{x^2+y^2+z^2<R^2} 2\sqrt{R^2-x^2-y^2-z^2}\, dx\, dy\, dz =$$

$$= 2R \iiint\limits_{\frac{x^2}{R^2}+\frac{y^2}{R^2}+\frac{z^2}{R^2}<1} \sqrt{1-\frac{x^2}{R^2}-\frac{y^2}{R^2}-\frac{z^2}{R^2}}\, dx\, dy\, dz = 2R\frac{\pi^2}{4}R^3 = \frac{\pi^2}{2}R^4$$

$a=b=c=R$

$\Rightarrow V_4 = \frac{\pi^2}{2}R^4$ объем четырех. шара

$V_1 = 2R$

$V_2 = \pi R^2$

$V_3 = 4/3\, \pi R^3$ шар - гамма-ф-ция

$V_4 = \pi^2/2\, R^4$

$\vdots$