A property verification of node-weight equalization focusing on cycles of a graph

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***Abstract*—Utilization of cycle-based network management has contributed to realization of distributed control in an efficient, highly-scalable, and reliable manner. As seen in load-balancing problems, uniform assignment of weights on nodes is held as one of the important functions within a domain. Iterative computations based on cycle structure of a graph have strong intuition that they converge to the optimal solution. However, it has not been proved that the method guarantees to equalize all the weights for a given network topology. This paper provides a proof of the property based on graph theoretical nature as well as a way to shorten convergence time for the node-weight equalization problem.**

I. Introduction

Information networks are becoming larger and complicated, which requires a lot of the systems to be controlled in a distributed fashion. Use of cycles in a distributed network domain assures a simplified, resilient, and scalable operation against various kinds of problems. In our previous work [1], a theory that focuses on cycles, so called “tie-sets” in a mesh network is proposed. In [2], it is demonstrated by simulation results that distribution of electricity based on cycle structure leads to the global optimization. The cycle (tie-set) based fault-tolerance also enables faster and stable recovery [3].

This paper deals with the convergence issue demonstrated in [2] that realizes the global optimization by conducting distributed algorithms based on tie-sets. The contribution of this paper is providing the proof of the property based on graph theoretical nature that underlies a topology as well as a way to shorten convergence time for the node-weight equalization problem.

II. Definition

For a given bi-connected and undirected graph $G = (V, E)$ with a set of vertices $V = \{v_1, v_2, ..., v_n\}$ and a set of edges $E = \{e_1, e_2, ..., e_m\}$, let $W(v_i)$ be a given real number, called ”weight” assigned to each node v_i. A closed chain that consists of nodes and edges is defined as ”cycle” C_i. A set of nodes that belongs to a cycle C_i is expressed by $V(C_i)$. Let $\overline{W_i}$ be the average value of weights of nodes that belong to C_i. In this paper, averaging all the weights in a cycle C_i from $W(v_i)$ to $\overline{W_i}$ is referred to as equalization of $W(v_i)$. Node-Weight Equalization Problem is a problem where cycle-based iterations of averaging the weights lead to the global averaging. The issue to be analyzed in this problem is whether all the weights in G can converge on $\overline{W}$ or not by repeating the procedure that picks C_i and averages its weights. As G is bi-connected and simple graph, each node belongs to more than one cycle. Therefore, any cycle has more than one adjacent cycle unless G itself is a cycle. In this paper, ”adjacent” is defined as having at more than one shared node with other cycles.

The following is a proposition of the problem:

For a given bi-connected and undirected graph $G = (V, E)$ with real number $W(v)$ assigned to each node $v \in V$and an arbitrary positive real number ϵ, iterations of cycle equalization allow average deviation of G to be less than ϵ . Node-weight equalization problem is the same problem where the deviation of G becomes 0.

III. Previous Research

Some simulations using the method in [2] have been conducted in node-weight equalization problem. One of the experiments is to examine the convergence of weight by randomly selecting cycle and averaging them step by step. Using a given simulation graph with 200 nodes and 390 edges, the experiment results that the overall weights gradually converge on the global average value as the number of cycle equalizations increases.

Another experiment is to verify effectiveness of distributed control method based on cycle. In the experiment, we examine the maximum number of steps after the average deviation becomes less than $\epsilon(= 1 \times 10^{-6})$. In other words, it is the maximum times of equalization in a cycle when the global optimization is completed. With ranging from 100 to 1000 nodes, even though the number of nodes increases, the number of optimizations conducted in each cycle does not change. This result indicates that distributed control based on cycle is suitable for a large-scale network as the processing load of each cycle does not increase.

IV. Procedure of Node-Weight Equalization

A. STEP1:The case of two cycles

A sequence of cycles $C_1, C_2, \cdots C_R$ has following characteristics.

1) For any r $(1 < r \leq R)$,
$V(C_1) \cup V(C_2) \cup ... \cup V(C_{r-1}) - V(C_r) \neq \emptyset$,
$(V(C_1) \cup V(C_2) \cup ... \cup V(C_{r-1})) \cap V(C_r) \neq \emptyset$,
$V(C_r) - V(C_1) \cup V(C_2) \cup ... \cup V(C_{r-1}) \neq \emptyset$,
2) $V(C_1) \cup V(C_2) \cup ... \cup V(C_R) = V$

The sequence of cycles exists in G because of a characteristic of bi-connected graph. According to an order of the sequence, cycle equalization is conducted. First, we consider the case that alternately equalizes first picked-up cycles C_1 and C_2. In this case the following equations can be obtained.

- $P = V(C_1) - V(C_2)$
- $Q = V(C_1) \cap V(C_2)$
- $R = V(C_2) - V(C_1)$
- $u = |P|, n = |Q|, w = |R|$
- $\overline{W} = \sum_{v \in P \cup Q \cup R} W(v)/(u + n + w)$

In alternate cycle equalization, weights of P, Q and R move as seen as indicated in the left side of Fig.1. Here, we consider cycle equalization of Ⅲ→ Ⅳ and define the following.

- Weights of elements of P and $Q \cup R$ before cycle equalization are a and b $(a > b)$, respectively.
- Weight, after cycle equalization, of elements of $Q \cup R$ is a'.
- Average deviation of $P \cup Q \cup R$ before cycle equalization is X.
- Average deviation of $P \cup Q \cup R$ after cycle equalization is X'.

Then, the following equations are derived.

$$\begin{aligned}
\overline{W} &= \frac{au + b(n+w)}{u+n+w} \\
a' &= \frac{au + bn}{u+n} \\
a - \overline{W} &= \frac{(a-b)(n+w)}{u+n+w} \\
\overline{W} - b &= \frac{(a-b)u}{u+n+w} \\
a' - \overline{W} &= \frac{(a-b)uw}{(u+n)(u+n+w)} \\
X &= \frac{2(a-b)u(n+w)}{(u+n+w)^2} \\
X' &= \frac{2(a-b)uw}{(u+n+w)^2} = \frac{w}{n+w} X \qquad (1)
\end{aligned}$$

Equation 1 shows that X' is less than or equal to X. Cycle equalization of Ⅳ → Ⅲ produces the same result in a similar way. Therefore, if cycle equalization of C_1 and C_2 is alternately iterated, the average deviation converges to 0. However, in most cases, infinite cycle equalization is required to let the average deviation 0. Therefore, the procedure terminates when the average deviation is less than $\epsilon^{(2)}$. That $\epsilon^{(2)}$ is a positive real number. The procedure above is called "quasi-cycle equalization of quasi-cycle $C_1 \cup C_2$ within margin of error $\epsilon^{(2)}$. Suppose that $\epsilon^{(r)}$ is the margin of error from $C_1 \cup C_2 \cup C_3 \cup \cdots \cup C_r$ as is the case with $\epsilon^{(2)}$; thereby, $\epsilon = \epsilon^{(R)}$.

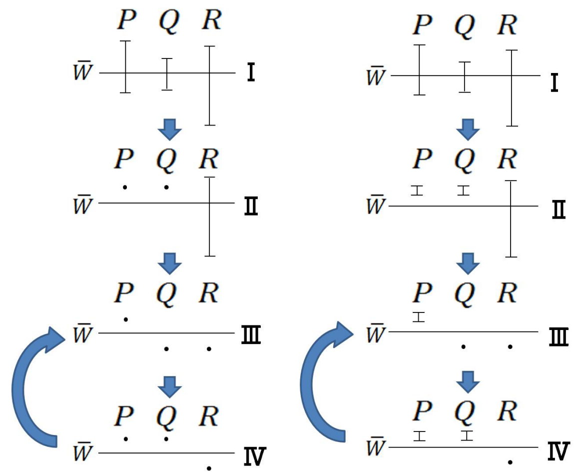

Fig. 1. Transition of weights by cycle equalization.

B. STEP2:The case of more than three cycles

Next, let us consider conducting the equalization of quasi-cycle $C_1 \cup C_2 \cup C_3$ by alternatively conducting the equalization of quasi-cycle $C_1 \cup C_2$ within error $\epsilon^{(2)}$ and equalization C_3. P, Q, R, u, n, w and $\overline{W}$ are defined similarly as the case of two-cycles convergence in STEP 1. The weights move I → Ⅱ → Ⅲ → Ⅳ → Ⅲ → Ⅳ ... as indicated in the right side of Fig.1.

It is difficult to analyze the equalization of quasi-cycle $C_1 \cup C_2$ and equalization of C_3. The reason is that $W(v)$ in $P \cup Q$ cannot be determined as an unambiguous value because the equalization of $C_1 \cup C_2$ has margin of error $\epsilon^{(2)}$. Therefore, we should consider the possible maximum average deviation of $C_1 \cup C_2 \cup C_3$ when the average deviation cannot decrease any more by iterative equalization of C_3 and $C_1 \cup C_2$. When the average deviation cannot decrease any more, the average deviation of $C_1 \cup C_2$ is within $\epsilon^{(2)}$ after conducting equalization of C_3.

Fig.2 shows an example of distribution of $W(v)$ where the average deviation of $C_1 \cup C_2 \cup C_3$ can take the maximum values. The characteristics are as follows:

1) A distribution of nodes is Ⅲ
2) $u = n + w, n = 1$
3) Each node weight $W(v)$ in P has the same value.

We can see from the above characteristics that it maximizes the difference between each $W(v)$ and $\overline{W}$ in $C_1 \cup C_2 \cup C_3$, and the average deviation takes the maximum value $\epsilon^{(3)}$. Since $\epsilon^{(3)}$ is the worst value, if a large enough number of times of equalizations of $C_1 \cup C_2$ and C_3 are repeated, a determinate margin of error can be within $\epsilon^{(3)}$ eventually.

The following result was obtained from the analysis: $\epsilon^{(r+1)} = \{a_2^2/4(a_2 - 1)\}\epsilon^{(r)}$. That a_2 is a number of nodes of $C_1 \cup C_2$. Therefore, the quasi-cycle $C_1 \cup C_2 \cup C_3$ within error $(4/3)\epsilon^{(r)}$ can be equalized. $C_1 \cup C_2 \cup ... \cup C_R$ within the margin of error $\epsilon^{(2)} \prod_{k=3}^{R} [\{(a_2 - 2)2^{k-3} + 2\}^2 / \{(a_2 - 2)2^{(k-1)} + 4\}]$ can be also equalized in a similar way. Thus, if

Algorithm 1 The procedure of convergence on $\epsilon^{(R)}$

i:=1;
$\epsilon' := \epsilon^{(R)} / \prod_{k=3}^{R}; [\{(a_2-2)2^{k-1}+4\}/\{(a_2-2)2^{k-3}+2\}^2]$
while $i \le R$ **do**
  Converge C_i
  if average deviation of $\bigcup_{k=1}^{i} C_k < \epsilon'$ **then**
    $i := i+1;$
    $\epsilon' := \epsilon'[\{(a_2-2)2^{i-3}+2\}^2/\{(a_2-2)2^{i-1}+4\}];$
  else
    $i := i-1;$
    $\epsilon' := \epsilon'[\{(a_2-2)2^{i-1}+4\}/\{(a_2-2)2^{i-3}+2\}^2];$
  end if
end while

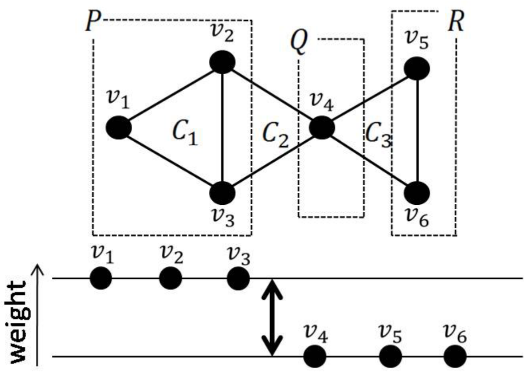

Fig. 2. Distribution of weights where deviation takes maximum value.

$$\epsilon^{(2)} = \epsilon \prod_{k=3}^{R} \frac{(a_2-2)2^{k-1}+4}{\{(a_2-2)2^{k-3}+2\}^2} \tag{2}$$

is given, the average deviation of the graph can be less than $\epsilon^{(R)}(=\epsilon)$. The above procedure is described in Algorithm 1.

V. Analysis on Speed of Convergence

We are now able to see that any bi-connected graph has a way to converge weights of its nodes. According to equation 2, $\epsilon^{(2)}$ depends on the number of cycles of whole graph, R, and the number of nodes of $C_1 \cup C_2$, a_2. Fig.3 shows how much $\epsilon^{(r)}$ increases by increasing r and a_2. As seen in Fig.3, $\epsilon^{(r)}$ increases exponentially according to increases of a_2 and especially r. Therefore, it is required to set $\epsilon^{(2)}$ as a quite small number if the number of cycles R becomes large. In other words, a number of times of cycle equalizations to converge is needed. Since the simulation in which a cycle is picked up randomly results in the overall convergence of $W(v)$ in G, there could be a selection method of cycle contributes to reduce the average deviation quickly.

Here, we show characteristics to speed up the reduction of average deviation of graph. In particular, we consider which cycle contributes to reduce the average deviation quickly.

A. *Classification of Cycles*

Any cycle C_i can be classified into following four types.

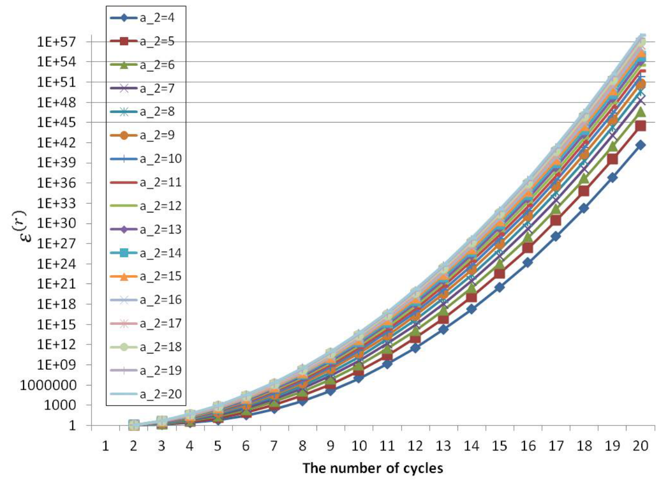

Fig. 3. Increase in margin of error.

TypeA : C_i whose $W(v_i)$ is equal to or higher than $\overline{W}$ and has at least one v_i, where $W(v_i) > \overline{W}$
TypeB : C_i whose $W(v_i)$ is equal to or lower than $\overline{W}$ and has at least one v_i, where $W(v_i) < \overline{W}$
TypeC : C_i whose $W(v_i)$ is equal to $\overline{W}$
TypeD : C_i that has at least one v_i and v_j, where $W(v_i) > \overline{W}$ and $W(v_j) < \overline{W}$, respectively

When a cycle C_i of Type A, B, and C is equalized, the following equation 3 is derived.

$$|\overline{W} - \overline{W}_i| = \frac{\sum_{v_i \in V(C_i)} |W(v_i) - \overline{W}|}{|V(C_i)|} \tag{3}$$

When type a cycle C_i of Type D is equalized, the following equation 4 is derived.

$$|\overline{W} - \overline{W}_i| < \frac{\sum_{v_i \in V(C_i)} |W(v_i) - \overline{W}|}{|V(C_i)|} \tag{4}$$

The left side of equations 3 and 4 denotes the average of difference between $\overline{W}$ and each $W(v_i)$ after equalization in each case, respectively. In addition, the right side of them denotes the average of the difference before equalization. According to equations 3 and 4, when a cycle of Type A, B, C is equalized, the average deviation of whole graph remains static. In addition, when a cycle of Type D is equalized, the average deviation reduces. Therefore, to reduce the average deviation of whole graph, a cycle of Type D should be equalized unless a cycle of Type D does not exist in graph.

In the case that type D does not exist in graph, let W_a and W_b be, respectively, maximum value and minimum value of weight in whole graph, and unless each weight are the same value, $W_b < \overline{W} < W_a$ holds. This inequality guarantees that at least one cycle from Type A and one cycle from Type B exist in graph when graph does not have a cycle of Type D.

As G includes a cycle of Type A and Type B (or, Type D) as indicated in Fig.4, there must be some nodes where different types of cycle adjacent with each other.

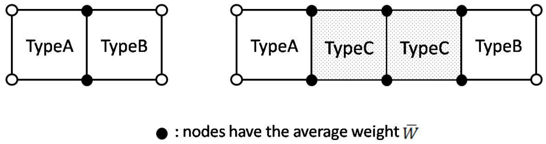

Fig. 4. Nodes where different types are adjacent.

In case that a cycle of Type A and a cycle of Type B are adjacent, weights of their shared nodes must be $\overline{W}$ because of each cycles characteristics. In addition, in case that the shared nodes do not exist between a cycle of Type A and a cycle of Type B (even though they exist in G), a cycle of Type C is connected between a cycle of Type A and a cycle of Type B. In this case, there is at least one cycle of Type C that connects cycles from Type A and Type B because each cycle has adjacent cycle. In other words, a cycle of Type A and Type B are adjacent with each other whose share nodes are $\overline{W}$ as indicated in left side of Fig.4, or there is at least one part where a cycle of Type C is connected between cycles of Type A and Type B as indicated in right side of Fig.4. In case of the left side of Fig.4, cycle equalization of either Type A or Type B makes another one become Type D. In case of the right side of Fig.4 cycle equalization of Type A or Type B makes adjacent cycle of Type C become a cycle which is the same type of equalized cycle. Iterations of the cycle equalization make cycles of Type C disappeared, and thus a cycle of Type A and a cycle of Type B become adjacent as left of Fig. 4. Hence, in case that a cycle of Type D does not exist in G, it can be produced by iterative cycle equalization. As a result, as long as the weights in a whole graph is not the same. a cycle of Type D exists or can be produced, which leads to reduction of the average deviation of entire graph.

B. Reduction by Equalization of Cycle Type D

Let V_i^s be a set of nodes within C_i of Type D that have a smaller weight than $\overline{W}$, and V_i^l be a set of nodes within C_i that have a larger weight than $\overline{W}$, respectively. In this case, A is defined as a total value of difference between $W(v)$ of V_i^s and $\overline{W}$, and B as a total value of difference between $W(v)$ of V_i^s and $\overline{W}$ as follows:

$$A = \sum_{v \in V_i^s} |W(v) - \overline{W}| \tag{5}$$

$$B = \sum_{v \in V_i^l} |W(v) - \overline{W}| \tag{6}$$

By these definitions of 5 and 6, the average deviation before equalizing the cycle of Type D is denoted as $(A+B)/|V(C_i)|$, and the average deviation of the cycle after equalization as $|A-B|/|V(C_i)|$. Therefore, the delta of the average deviations before and after equalization is as follows:

$$\frac{A+B}{|V(C_i)|} - \frac{A-B}{|V(C_i)|} = \begin{cases} \frac{2A}{|V(C_i)|}(A < B) \\ \frac{2B}{|V(C_i)|}(A > B) \end{cases} \tag{7}$$

Shortly, the average deviation within a cycle decreases in the amount of $2A/|V(C_i)|$ or $2B/|V(C_i)|$ (the smaller one) when equazlizing C_i of Type D. Therefore, the average deviation of an entire graph by equalizing a cycle of Type D decreases in the following amount.

$$\frac{2 \times \min\{\sum_{v \in V_i^s} |W(v) - \overline{W}|, \sum_{v \in V_i^l} |W(v) - \overline{W}|\}}{|V|} \tag{8}$$

Although it has been discussed that the average deviation decreases by equalization of cycles of Type D, it is not proved that the deviation decreases close to 0. However, any anti-example where the average deviation draws close to a value other than 0 has not been found so that convergence on 0 in arbitrary bi-connected graph is considered possible.

C. Selection Method of Cycle

On top of the property of classified cycles, we also consider the equalization method in detail. Type D has an advantage to quickly reduce the average deviation. Therefore, cycles of Type D should be first picked up to make convergence faster. Otherwise, it should be considered to produce cycles of Type D as well. However, to classify cycles into Type A,B,C, or D, it is required to calculate the global knowledge of the average value $\overline{W}$ of an entire graph and examine the magnitude relation between every $W(v)$ and $\overline{W}$ when selecting a cycle. This procedure makes the computation times increased. In addition, this process of calculating the average of all the weights is not suitable for distributed computation as it need a centralized system.

Without calculating $\overline{W}$, a precision classification of cycles is needed to effectively converge all the weights. One way to realize the fast convergence is to pick up a cycle in which the delta between the maximum weight and the minimum weight is large. This can be done in a distributed manner as each cycle communicates with adjacent cycles to exchange their delta of weights, which leads to a global solution.

VI. Conclusion

In this paper, we proved the equalization of weights on nodes that is practically demonstrated with the distributed control based on cycle structure. As a result of proof, we substantiate that there is a way to equalize all the weights based on cycles in any bi-connected graph. As a future work, we will conduct more precise analysis on the characteristics of the convergence. This property will also be related to demand response as independent monitoring in each cycle to optimize the its condition leads to global optimization.

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