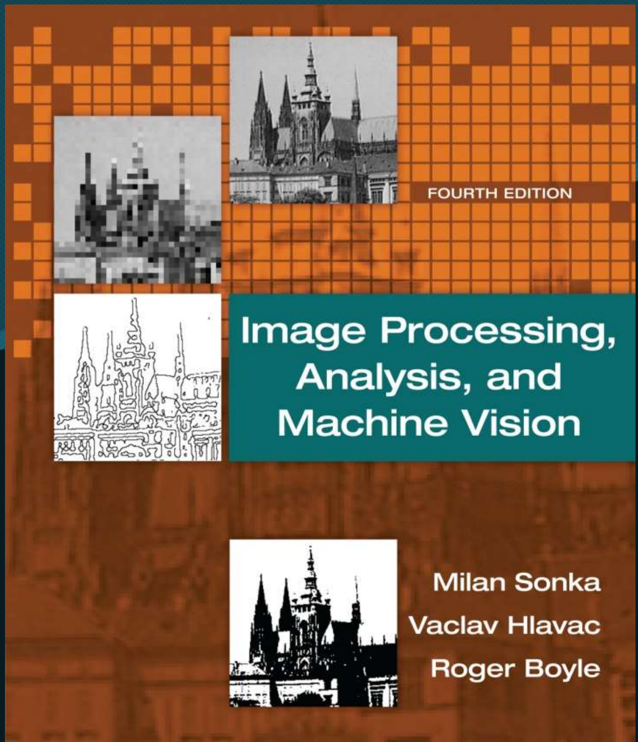
Chapter 3

The image, its mathematical and physical background



Outline

- Overview
- Linear integral transforms
  - Images as linear systems
  - Introduction to linear integral transforms
  - Fourier transform
  - Sampling and the Shannon constraint
  - Wavelet transform
- Images as stochastic process
- Image formation physics

2D Fourier transform

$f(x)$ $F(\xi)$

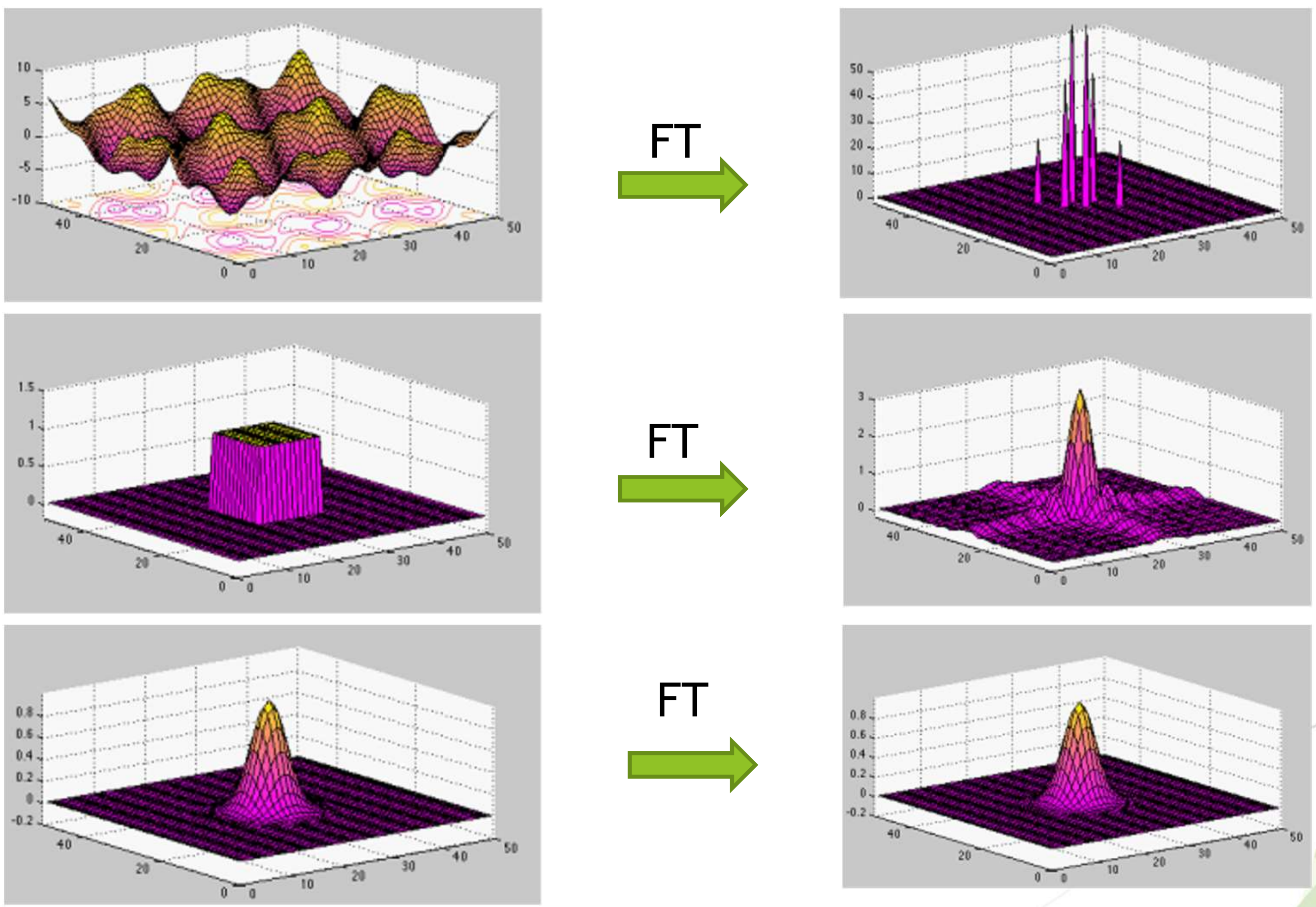


Image representation and image analysis task

- Both representations contain exactly the same information.
  - Human observer v.s. machine recognizer

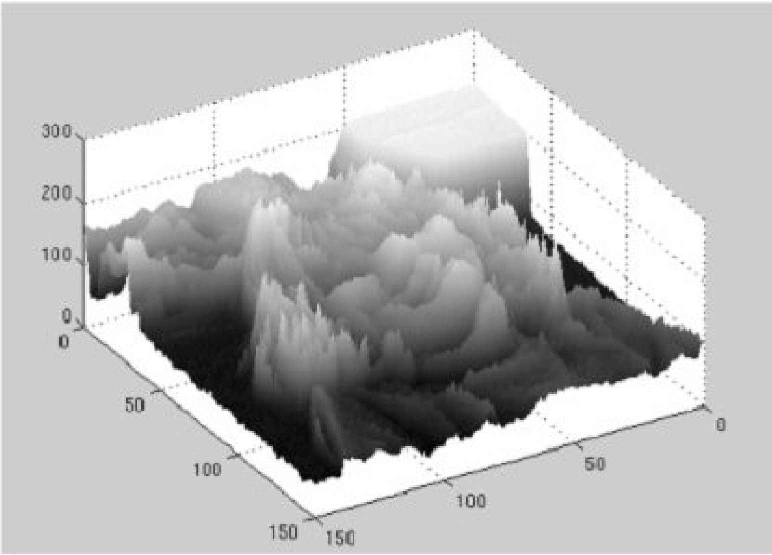
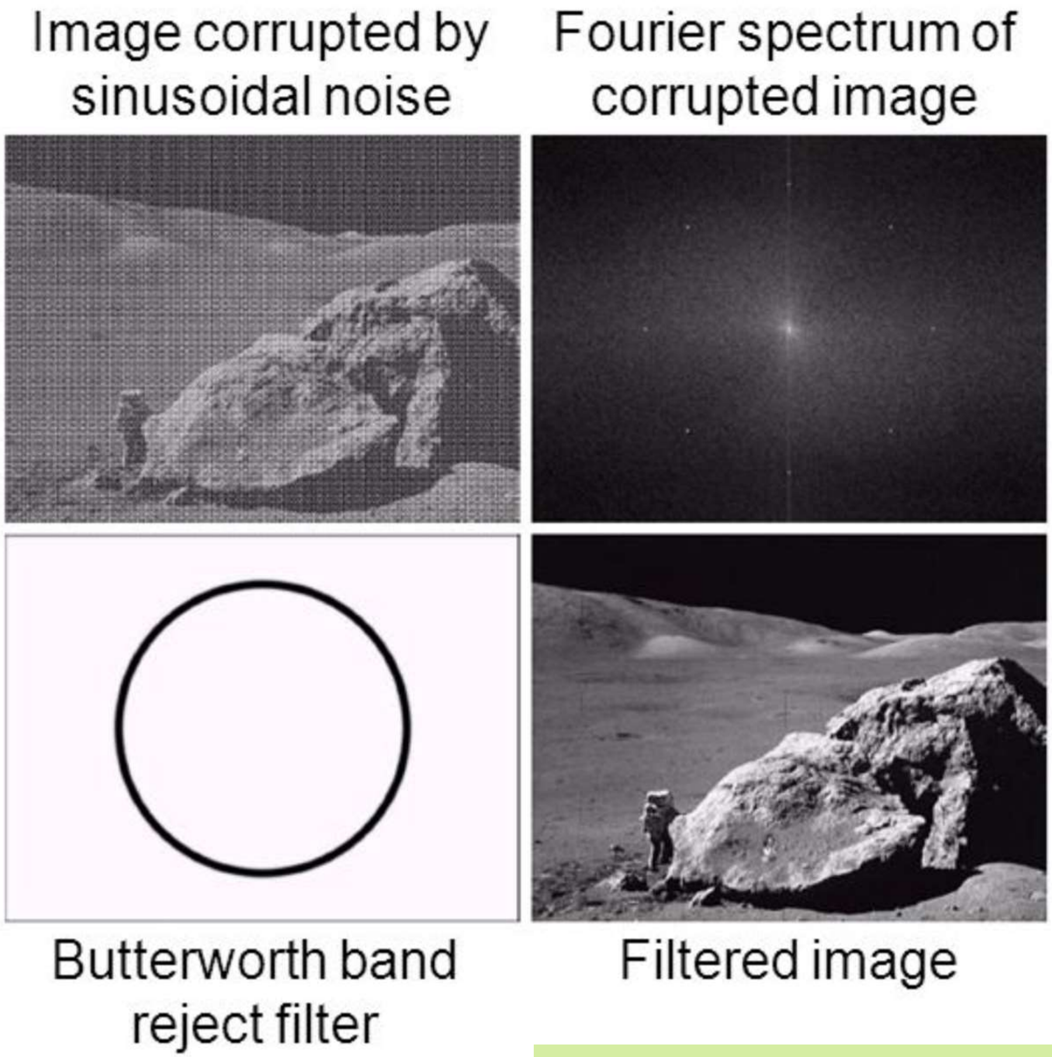


Figure 1.8: An unusual image representation. © *R.D. Boyle 2015.*

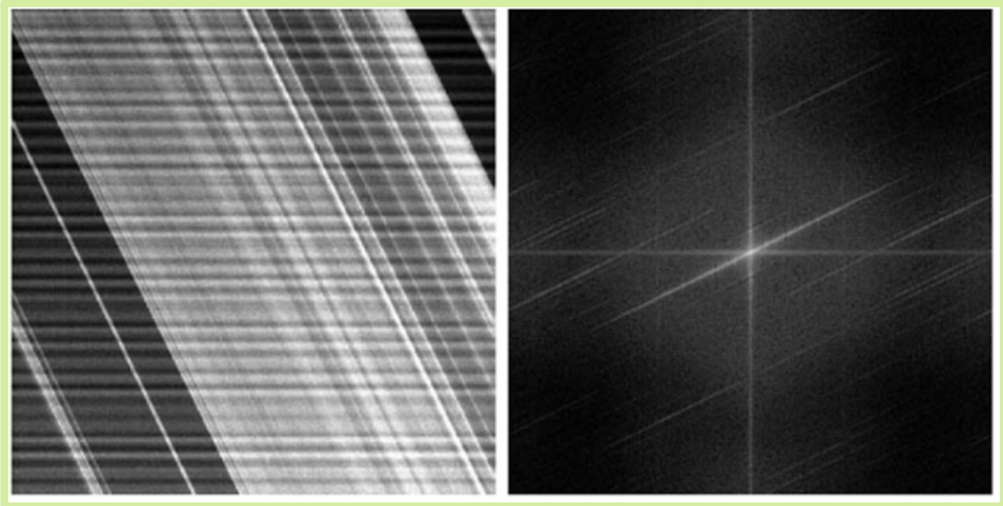
Figure 1.9: Another representation of Figure 1.8. © *R.D. Boyle 2015.*

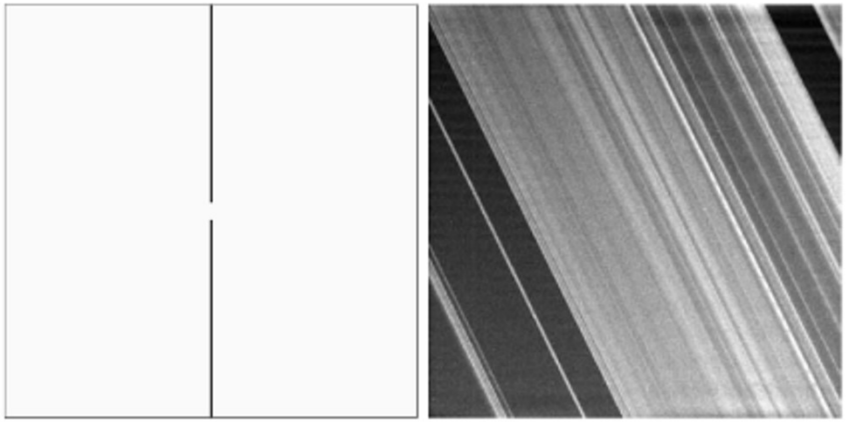
Selective Filtering



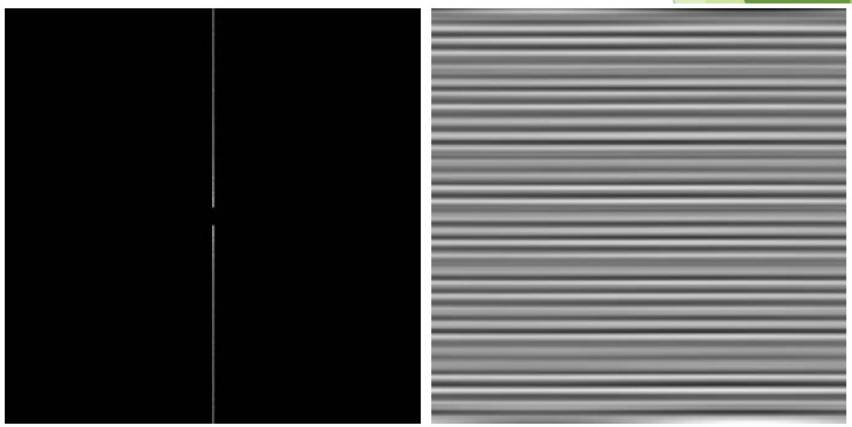
http://slideplayer.com/slide/8267284/

Selective Filtering





Notch reject filters



Notch pass filters

2D Fourier transform

- An image f is a function of two coordinates (x, y) in a plane.
- The 2D Fourier transform for the continuous image f is defined by the integral

$$F(u,v) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)e^{-2\pi i(xu+yv)}\,\mathrm{d}x\mathrm{d}y$$

- The inverse transform is

$$f(x,y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(u,v)e^{2\pi i(xu+y\;\;)}\,\mathrm{d}u\mathrm{d}v$$

  - Parameters (u, v) are called spatial frequencies.
- 2D Fourier transform can be abbreviated to

$$\mathcal{F}\{f(x,y)\} = F(u,v)$$

2D Fourier transform

- Properties of 2D Fourier transform
  - Linearity
$$\mathcal{F}\{af_1(x,y) + bf_2(x,y)\} = aF_1(u,v) + bF_2(u,v)$$
  - Shift of the origin in the image domain
$$\mathcal{F}\{f(x-a, y-b)\} = F(u,v)e^{-2\pi i(au+bv)}$$
  - Shift of the origin in the frequency domain
$$\mathcal{F}\left\{f(x,y)e^{2\pi i(u_0x+v_0y)}\right\} = F(u-u_0, v-v_0)$$
  - If $f(x,y)$ is real-valued then
$$F(-u,-v) = F^*(u,v)$$
    - Note that the image function is always real-valued.
    - If the image function has the property $f(x,y) = f(-x,-y)$ then $F(u,v)$ is a real function.

2D Fourier transform

- Properties of 2D Fourier transform
  - Duality of the convolution
    - Convolution and its Fourier transform are related by
$$\mathcal{F}\{(f * h)(x, y)\} = F(u, v)H(u, v)$$
$$\mathcal{F}\{f(x, y)h(x, y)\} = (F * H)(u, v)$$
    - This is the convolution theorem.

2D Fourier transform

- Discrete 2D Fourier transform

$$F(u,v) = \frac{1}{MN} \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} f(m,n) \exp\left[-2\pi i \left(\frac{mu}{M} + \frac{nv}{N}\right)\right]$$

$$u = 0,1,\dots,M-1, \qquad v = 0,1,\dots,N-1.$$

- The inverse transform is

$$f(m,n) = \sum_{u=0}^{M-1} \sum_{v=0}^{N-1} F(u,v) \exp\left[2\pi i \left(\frac{mu}{M} + \frac{nv}{N}\right)\right]$$

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2D Fourier transform

- Discrete 2D Fourier transform

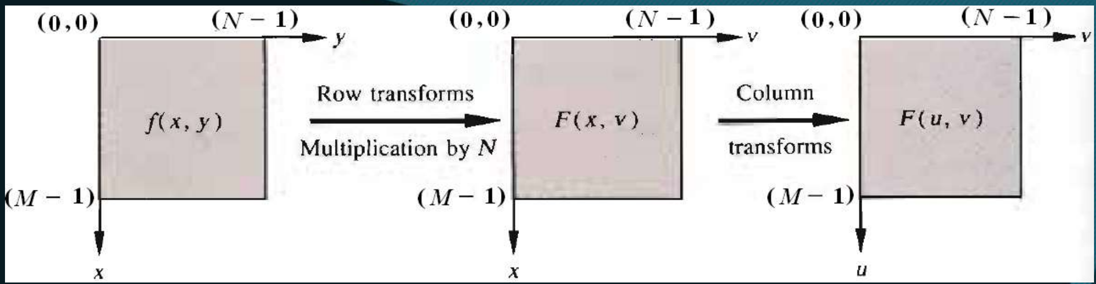
$$F(u,v) = \frac{1}{MN} \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} f(m,n) \exp\left[-2\pi i \left(\frac{mu}{M} + \frac{nv}{N}\right)\right]$$

$$u = 0,1,\dots,M-1, \qquad v = 0,1,\dots,N-1.$$

can be modified to

$$F(u,v) = \frac{1}{M} \sum_{m=0}^{M-1} \left[\frac{1}{N} \sum_{n=0}^{N-1} f(m,n) \exp\left(\frac{-2\pi i n v}{N}\right)\right] \exp\left(\frac{-2\pi i m u}{M}\right)$$

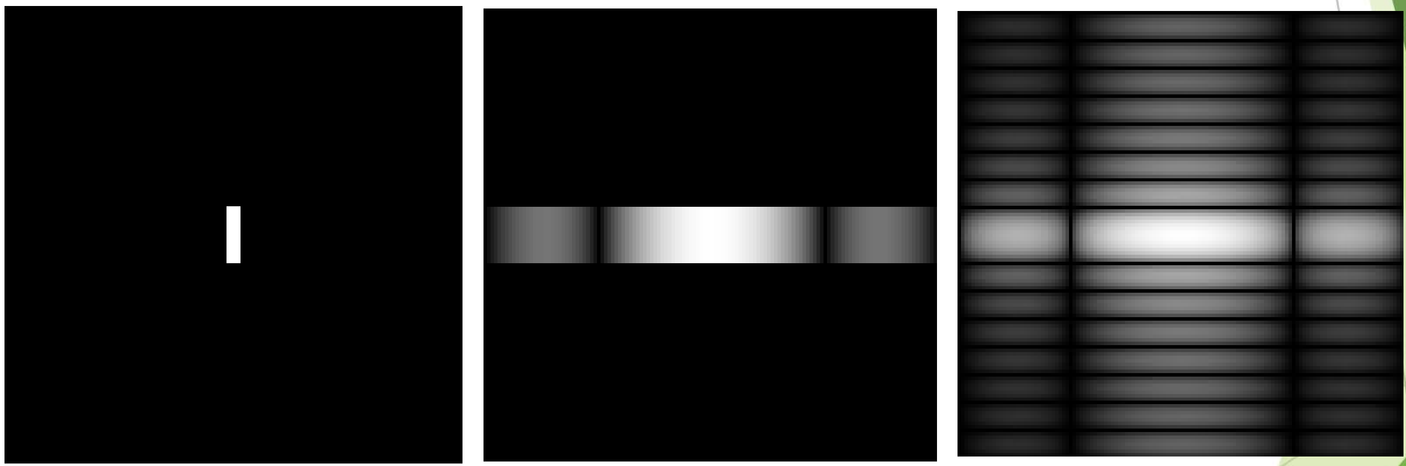
$$u = 0,1,\dots,M-1, \qquad v = 0,1,\dots,N-1.$$



The Two-Dimensional DFT

The 2D DFT $F(u,v)$ can be obtained by

1. taking the 1D DFT of every row of image $f(x,y)$, $F(u,y)$
2. taking the 1D DFT of every column of $F(u,y)$



(a) $f(x,y)$ (b) $F(u,y)$ (c) $F(u,v)$

2D Fourier transform

- Properties of discrete 2D Fourier transform
  - Periodicity (週期性) is an important property of the discrete Fourier transform.

  - A periodic transform F is derived and a periodic function f defined

$$F(u,-v) = F(u,N-v), \qquad f(-m,n) = f(M-m,n),$$
$$F(-u,v) = F(M-u,v), \qquad f(m,-n) = f(m,N-n),$$
$$F(aM+u,bN+v) = F(u,v), \qquad f(aM+m,bN+n) = f(m,n),$$

where a and b are integers.

2D Fourier transform

- The outcome of the 2D Fourier transform is a complex-valued 2D spectrum.
- For easier visualization, the range of values is usually decreased by applying a monotonic function.
  - For examples, $\sqrt{|F(u,v)|}$ or $\log|F(u,v)|$
- It is also useful to visualize a centered spectrum with the origin of the coordinate system (0,0) in the middle of the spectrum.

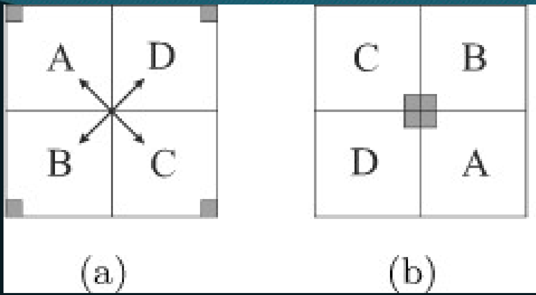
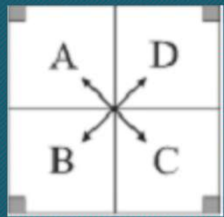
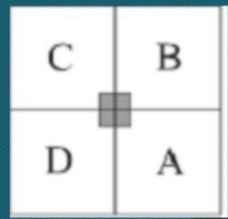


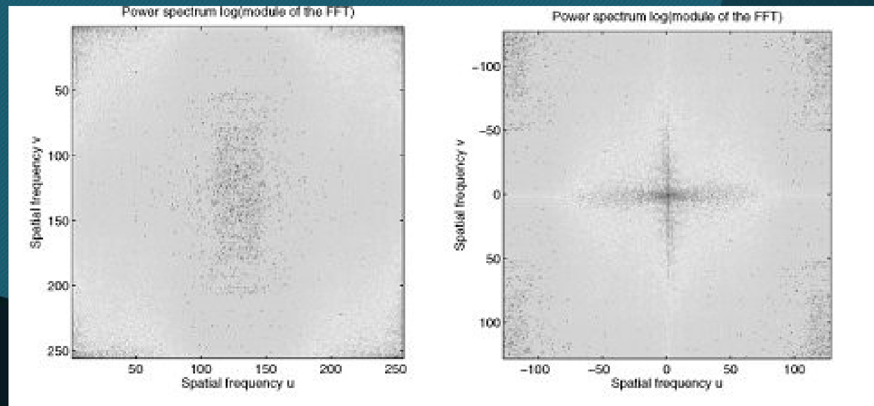
Figure 3.6: Centering of the 2D Fourier spectrum places the low frequencies around the coordinates origin. (a) Original spectrum. (b) Centered spectrum with the low frequencies in the middle. © *Cengage Learning 2015.*

2D Fourier transform

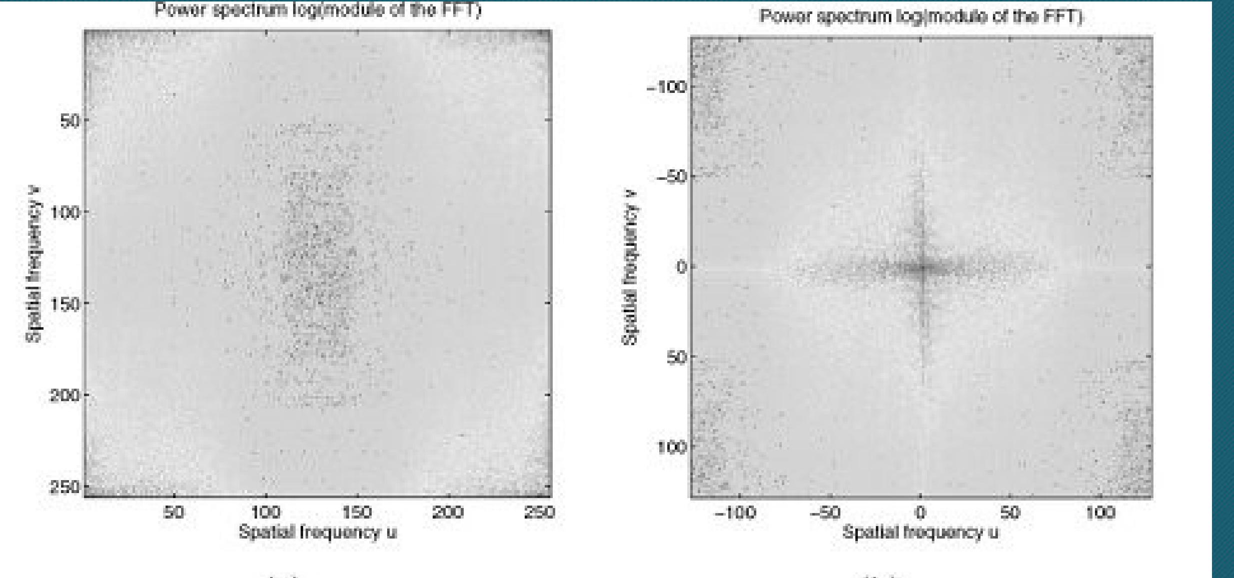
- An example of 2D Fourier transform





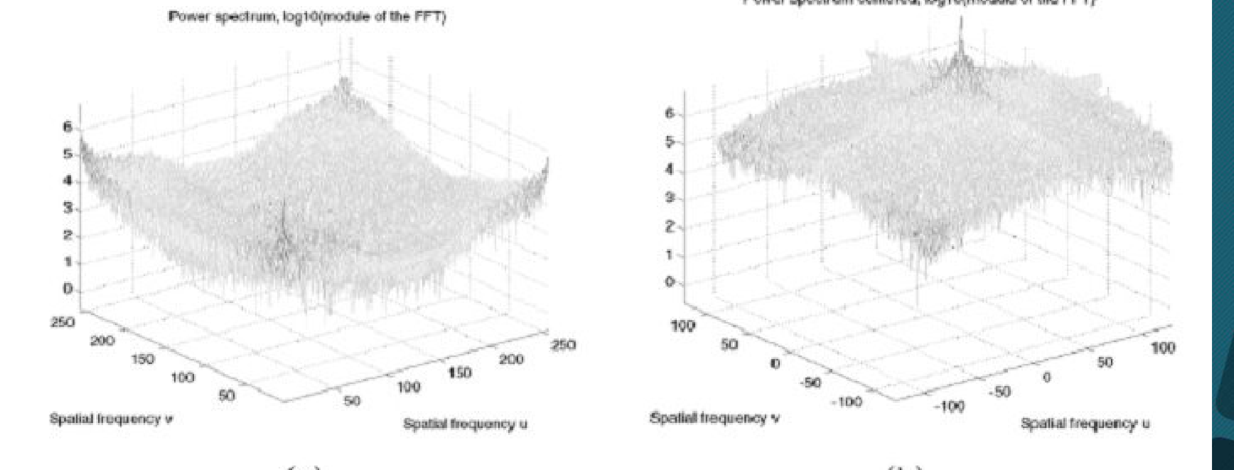


(a) (b)



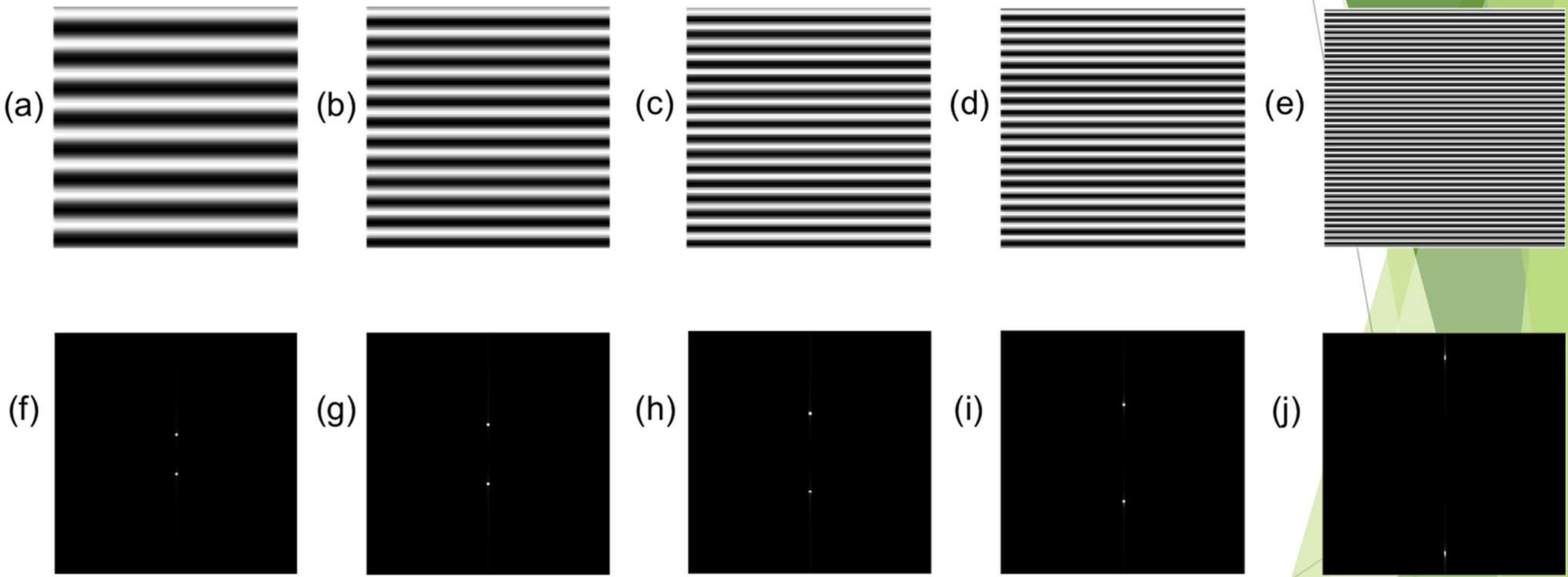
(a) (b)

Power spectrum displayed as height in a 3D mesh



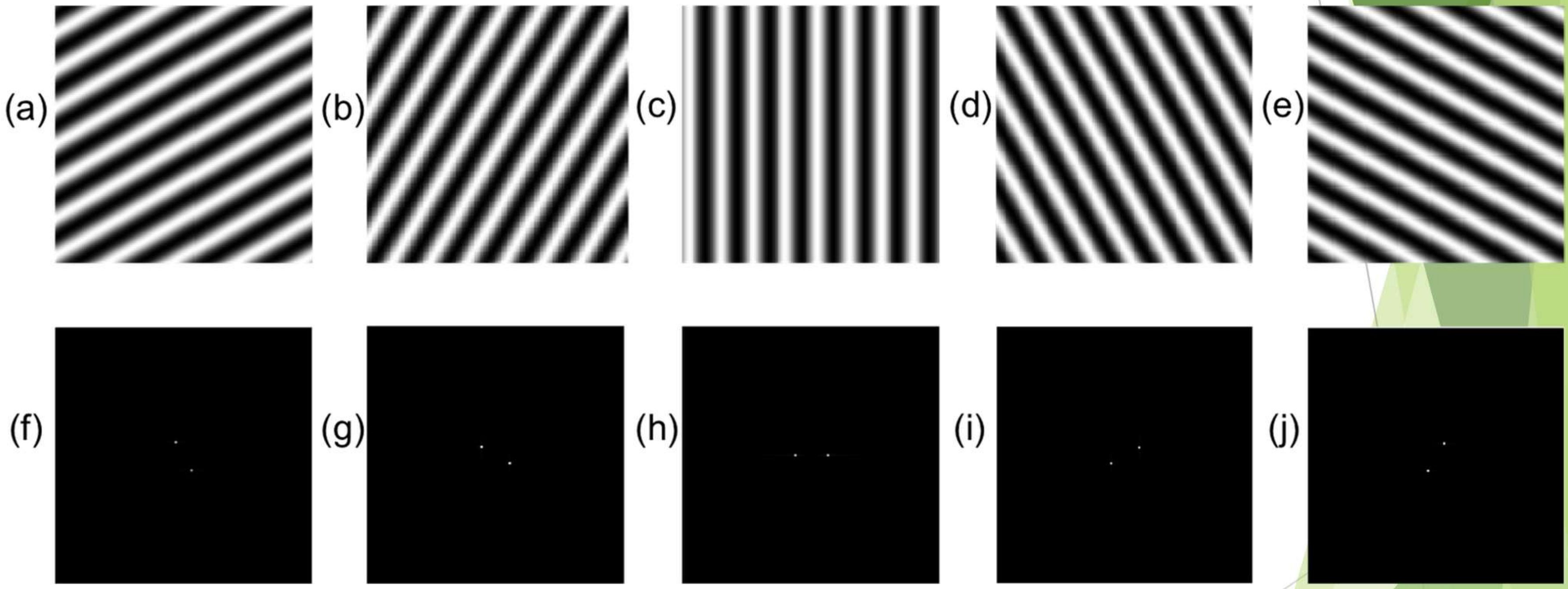
(a) (b)

2D Fourier transform



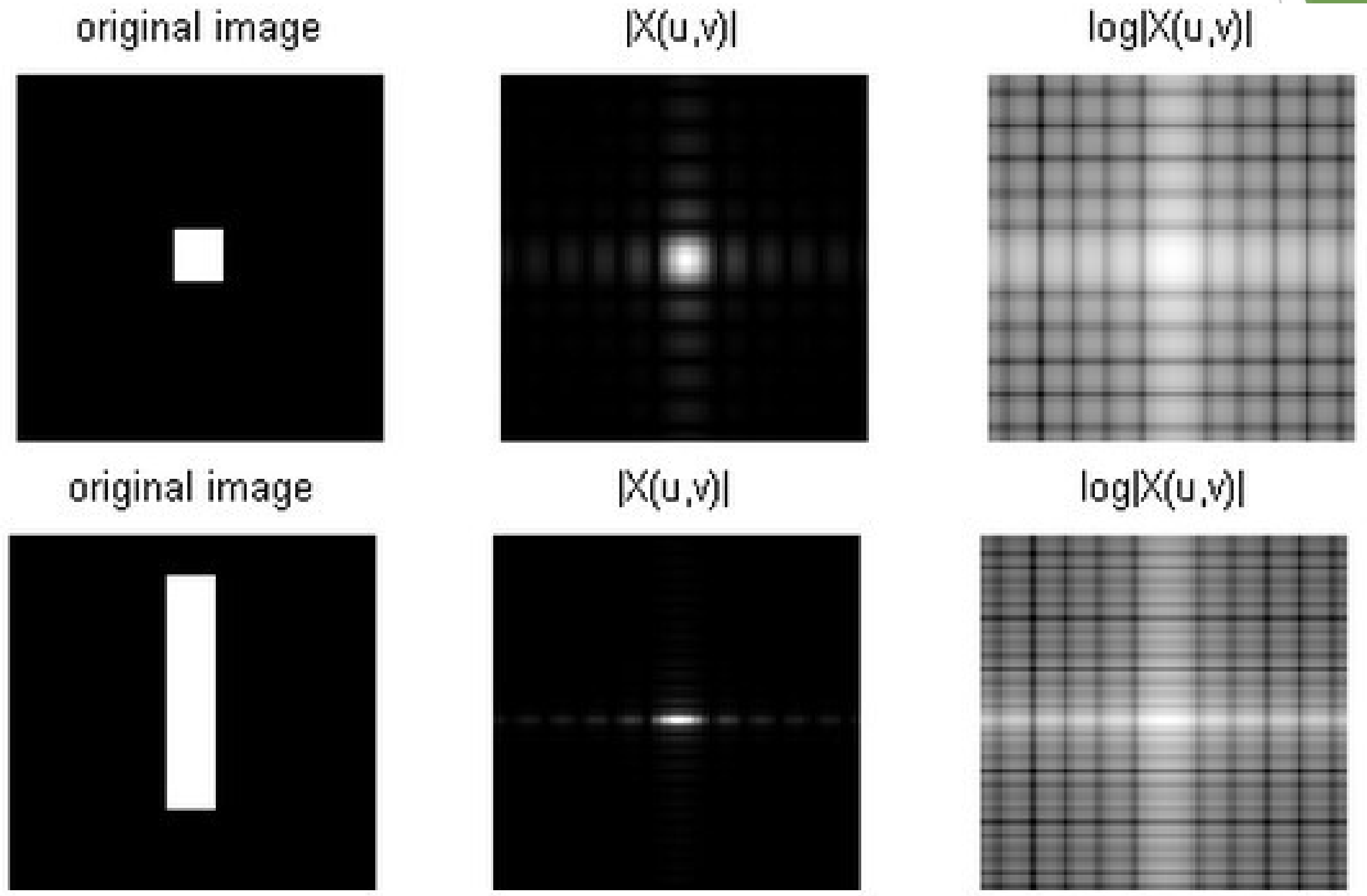
http://kirbycheng.blogspot.tw/2011_07_01_archive.html

2D Fourier transform



http://kirbycheng.blogspot.tw/2011_07_01_archive.html

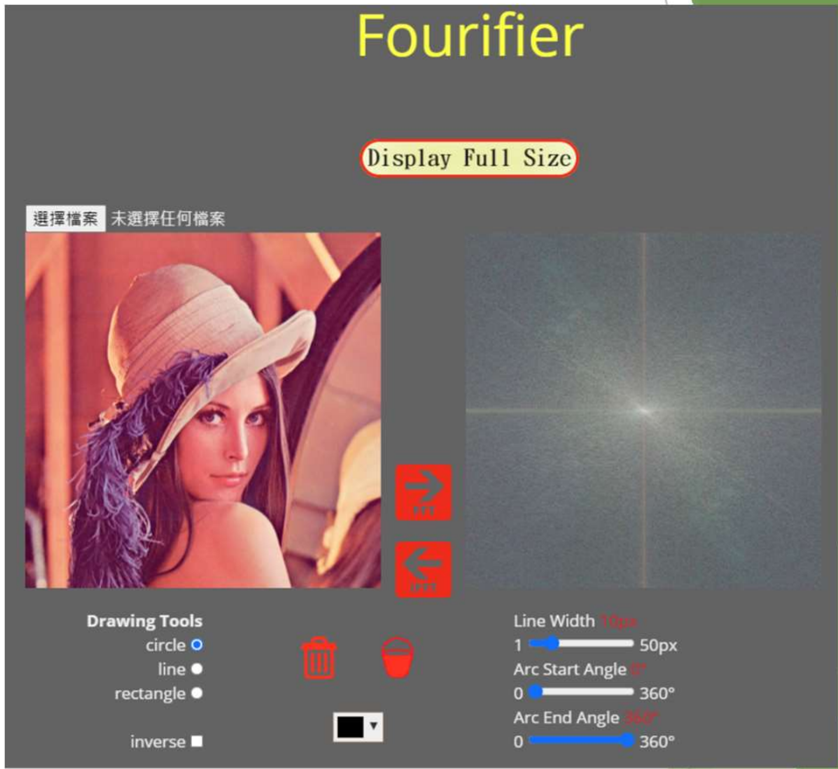
2D Fourier transform



http://dsp.stackexchange.com/questions/18670/2d-fourier-transform-most-people-cant-explain-this

Programs of 2D Fourier transform

- http://www.ejectamenta.com/Imaging-Experiments/fourierimagefiltering.html
- https://www.ejectamenta.com/Fourifier-fullscreen/
- https://github.com/Lung-Yu/ImageToolBox/
- http://www.jcrystal.com/products/ftlse/index.htm



Sampling and the Shannon constraint

- A continuous image function $f(x, y)$ can be sampled using a discrete grid of sampling points in the plane.
- The ideal sampling $s(x, y)$ in the regular grid can be represented using a collection of Dirac distributions δ.

$$s(x,y) = \sum_{j=1}^{M} \sum_{k=1}^{N} \delta(x - j\Delta x, y - k\Delta y)$$

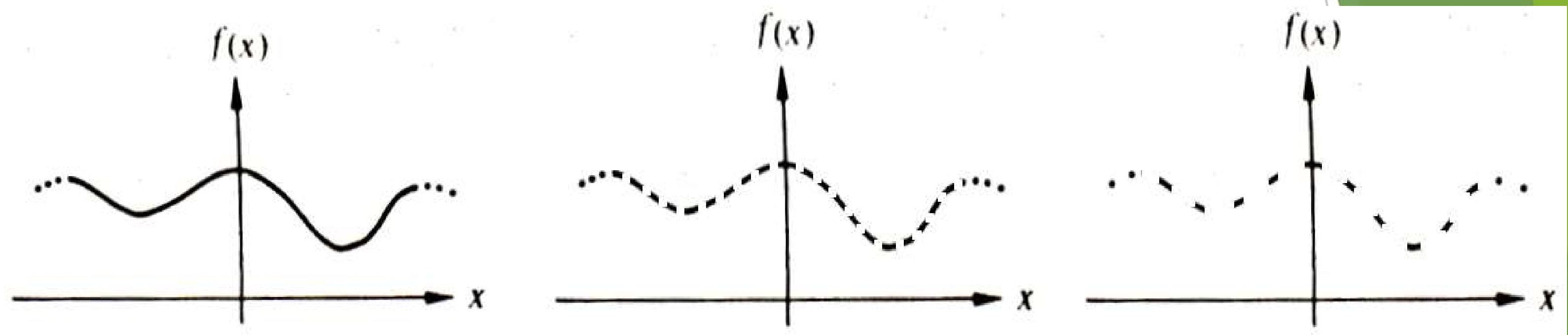
where Δx, Δy are called sampling intervals.

- The sampled image $f_s(x, y)$ can be obtained by

$$f_s(x,y) = f(x,y)s(x,y) = f(x,y) \sum_{j=1}^{M} \sum_{k=1}^{N} \delta(x - j\Delta x, y - k\Delta y)$$

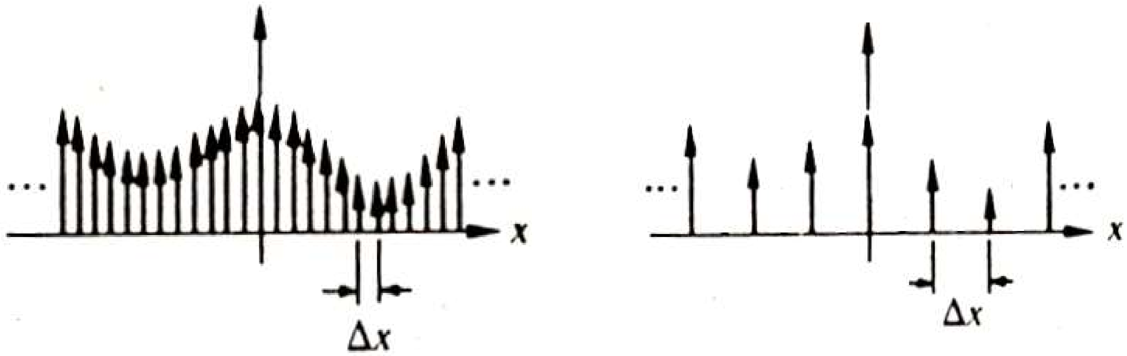
$$f(x,y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(a,b)\delta(a - x, b - y)\mathrm{d}a\mathrm{d}b$$

Sampling theory



Continuous function

Discrete function

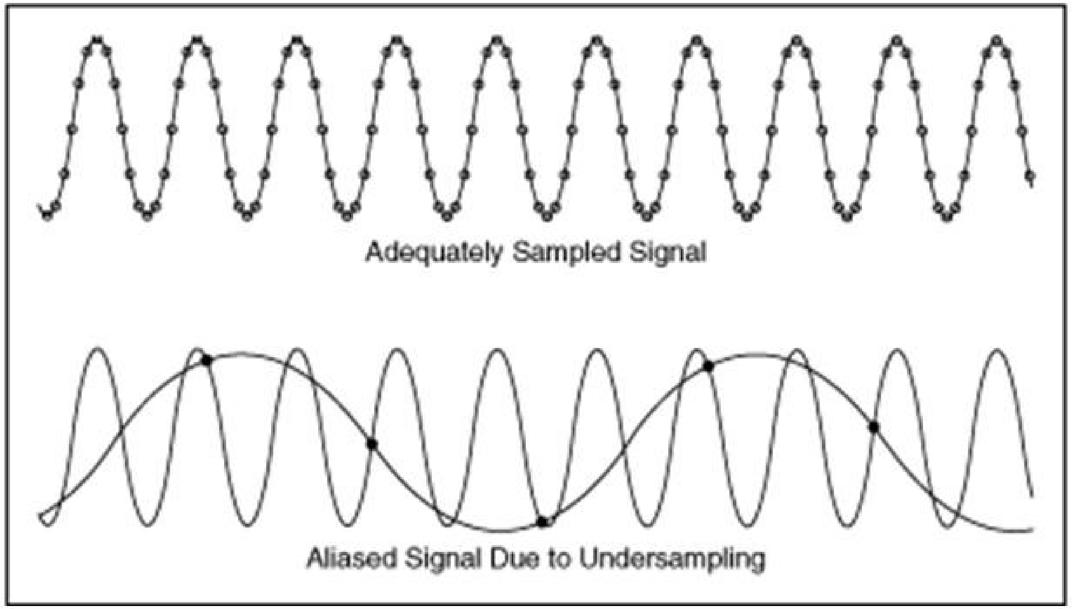


Dense sampling

Sparse sampling

How many samples should be taken so that no information is lost in the sampling process?

Aliasing

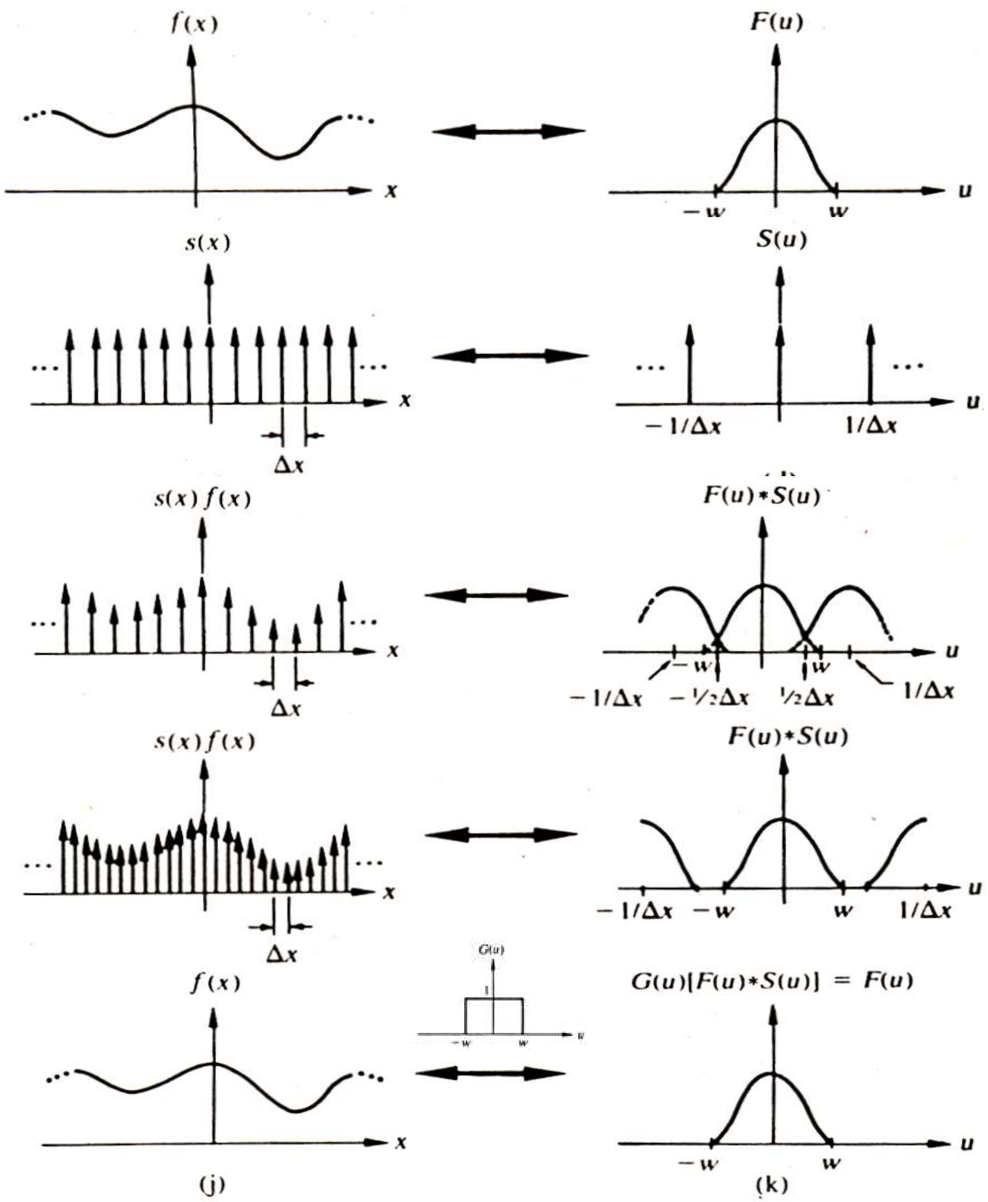


http://www.ni.com/white-paper/10669/en/

Spatial domain Frequency domain

$f(x)$:
band-limited function

$s(x)$:
Sampling function



$$w < \frac{1}{2\Delta x} \Rightarrow \Delta x < \frac{1}{2w}$$

Sampling and the Shannon constraint

- Band-limited signal
  - Assume the maximal frequency of a signal is f_m, then the signal is band-limited.
  - The Fourier transform $\mathcal{F}$ of the band-limited signal is zero outside a certain interval $|f| > f_m$.
  - The spectra will be repeated as follows.

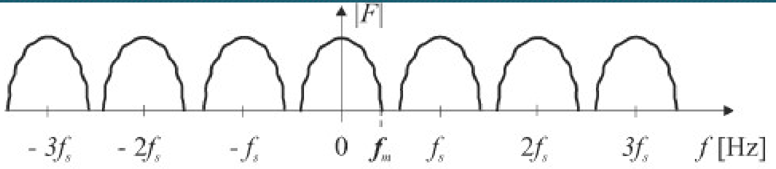


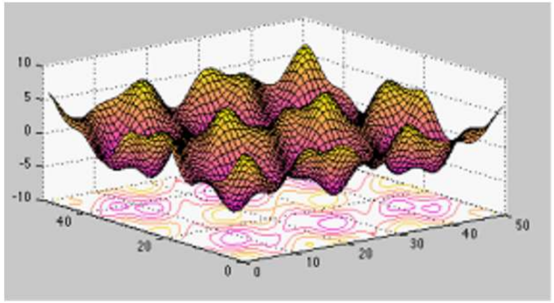
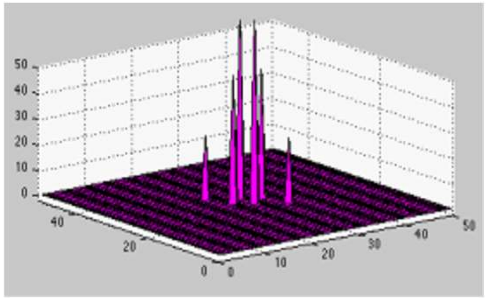
Figure 3.10: Repeated spectra of the 1D signal due to sampling. Non-overlapped case when $f_s \geq 2f_m$. *© Cengage Learning 2015.*

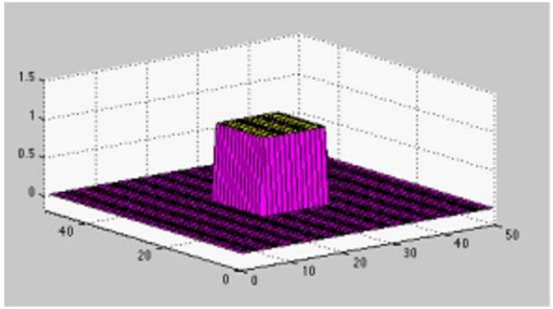
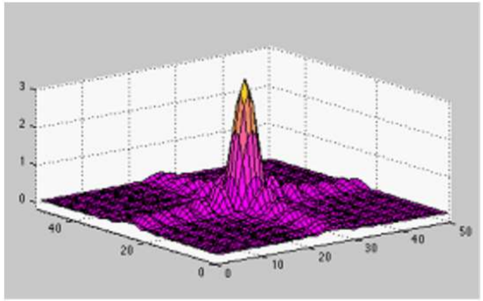
- In the case of 2D image, band-limited means that

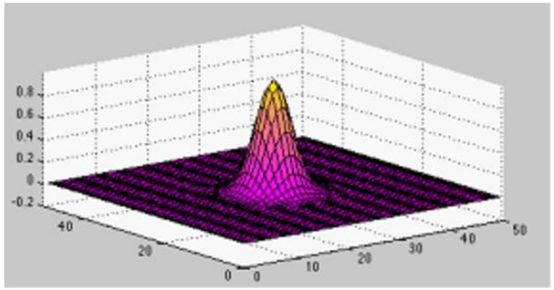
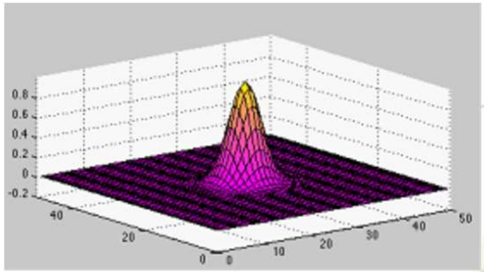
the spectrum $F(u, v) = 0$ for $|u| > U, |v| > V$, where U, V are maximal frequencies.

Which one is band-limited function?

$f(x)$ $F(\xi)$

(A)

FT

(D)

(B)

FT

(E)

(C)

FT

(F)

Sampling and the Shannon constraint

- Periodic repetition of the Fourier transform result $F(u,v)$ may under certain conditions cause distortion of the image, which is called aliasing (效果失真或混疊).
- This happens when individual digitized components $F(u,v)$ overlap.
- Overlapping of the periodically repeated results of the Fourier transform $F(u,v)$ of an image with band-limited spectrum can be prevented of the sampling interval is chosen such that

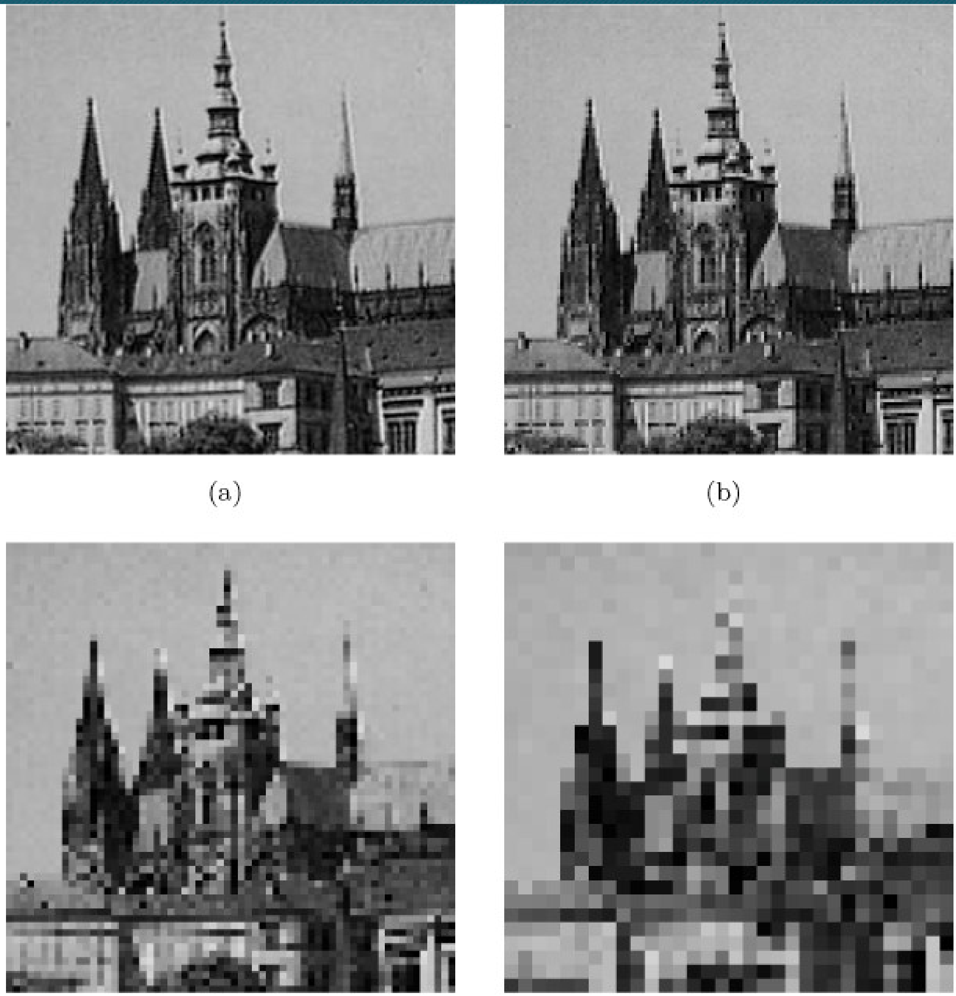
$$\Delta x < \frac{1}{2U}, \qquad \Delta y < \frac{1}{2V}$$

where U, V are maximal frequencies.

- This is Shannon's sampling theorem (known for signal processing theory).
- It has a simple physical interpretation in image analysis:

  The sample interval should be chosen such that it is less than half of the smallest interesting detail in the image.

Sampling and the Shannon constraint



(a) (b) (c) (d)

Figure 3.11: Digitizing. (a) 256 × 256. (b) 128 × 128. (c) 64 × 64. (d) 32 × 32. Images have been enlarged to the same size to illustrate the loss of detail. © *Cengage Learning 2015.*

Basics of Filtering in the Frequency Domain

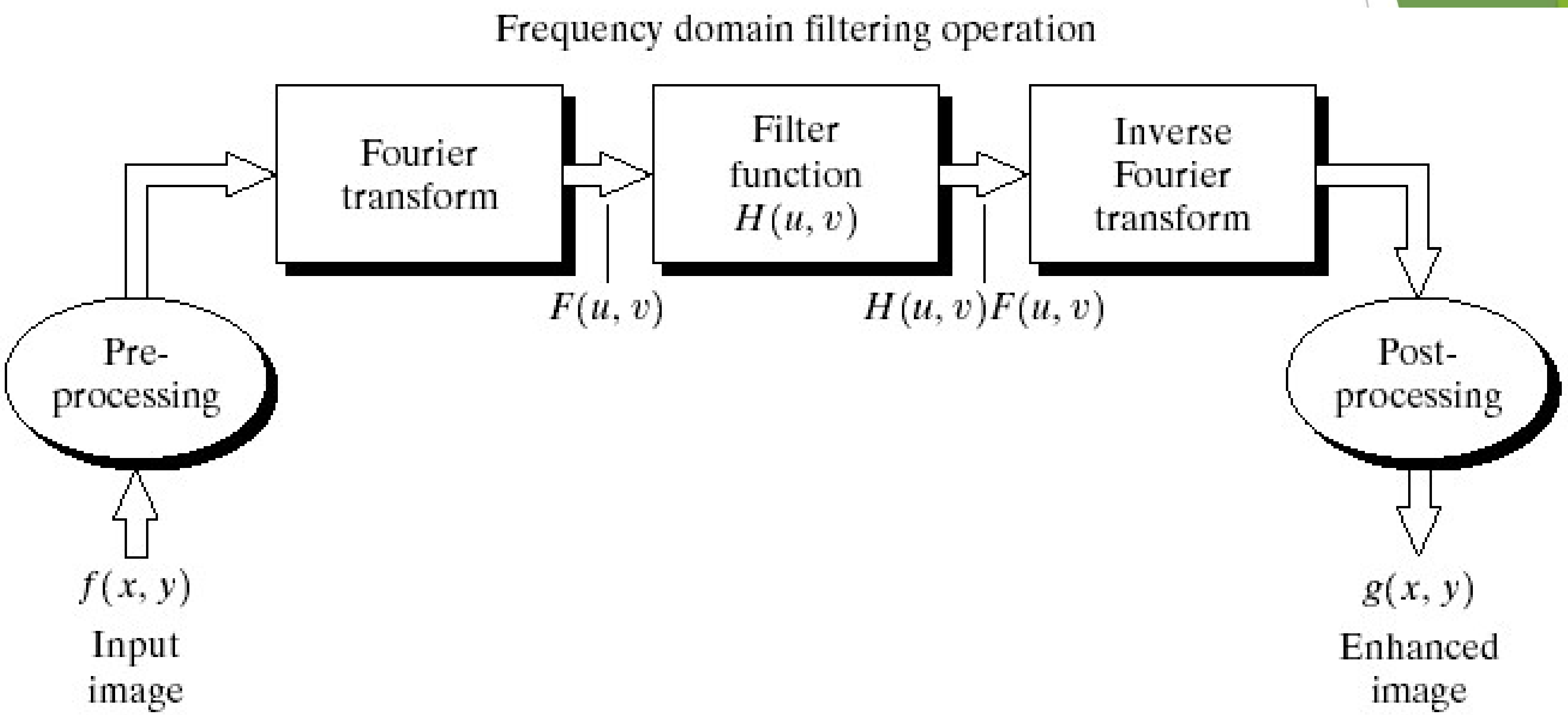
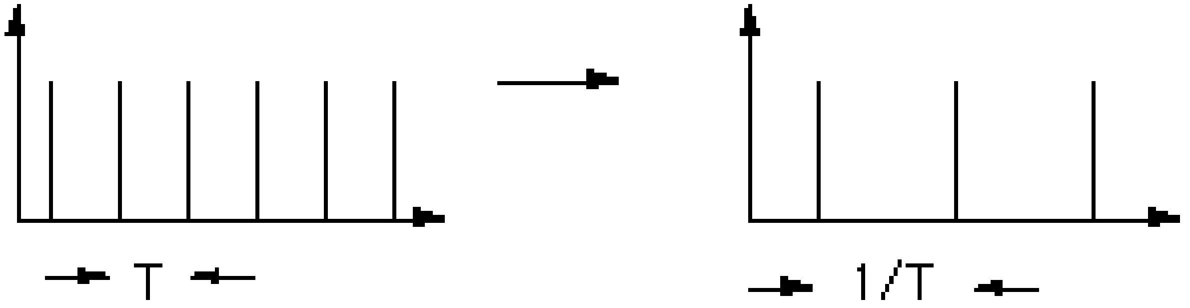


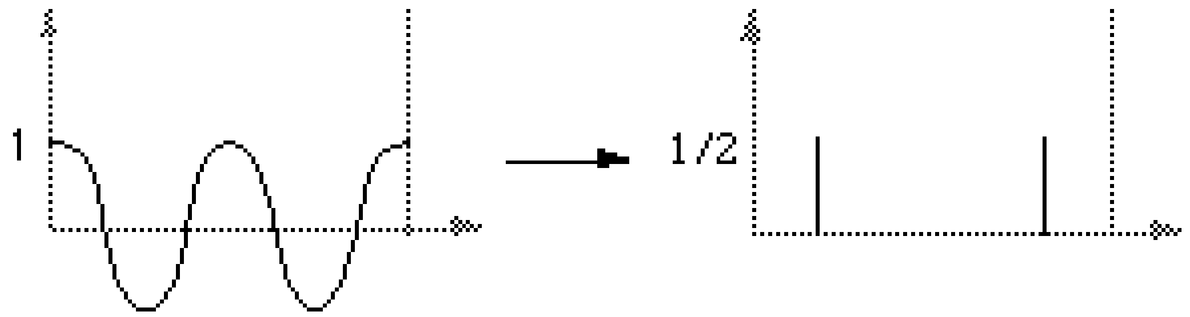
FIGURE 4.5 Basic steps for filtering in the frequency domain.

The One-Dimensional Fourier Transform Some Examples

- The transform of an infinite train of delta functions spaced by T is an infinite train of delta functions spaced by 1/T.

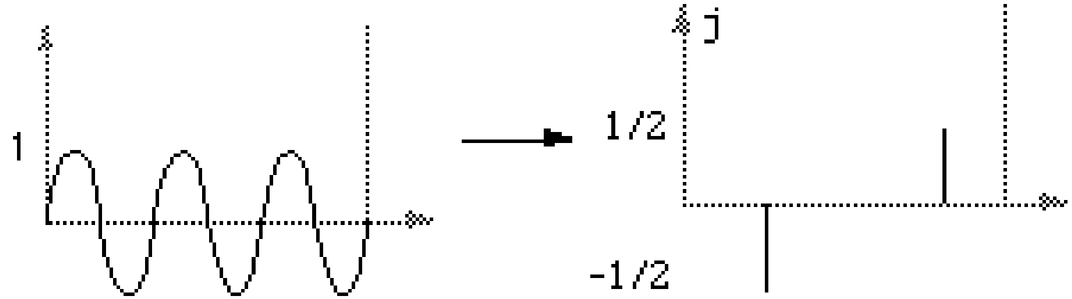


- The transform of a cosine function is a positive delta at the appropriate positive and negative frequency.

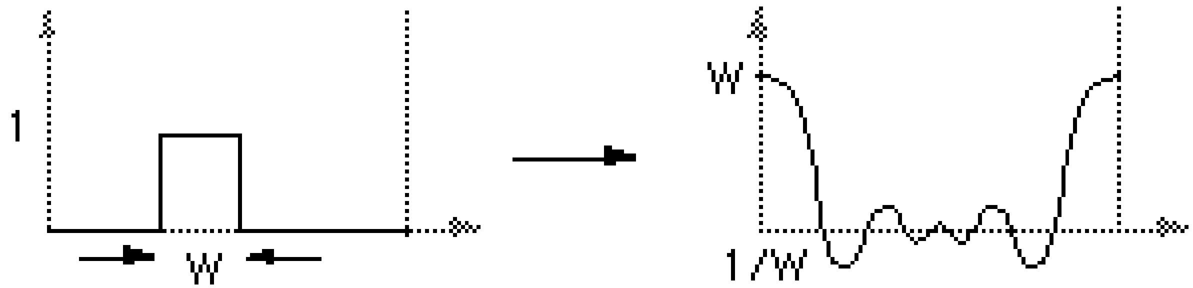


The One-Dimensional Fourier Transform
Some Examples

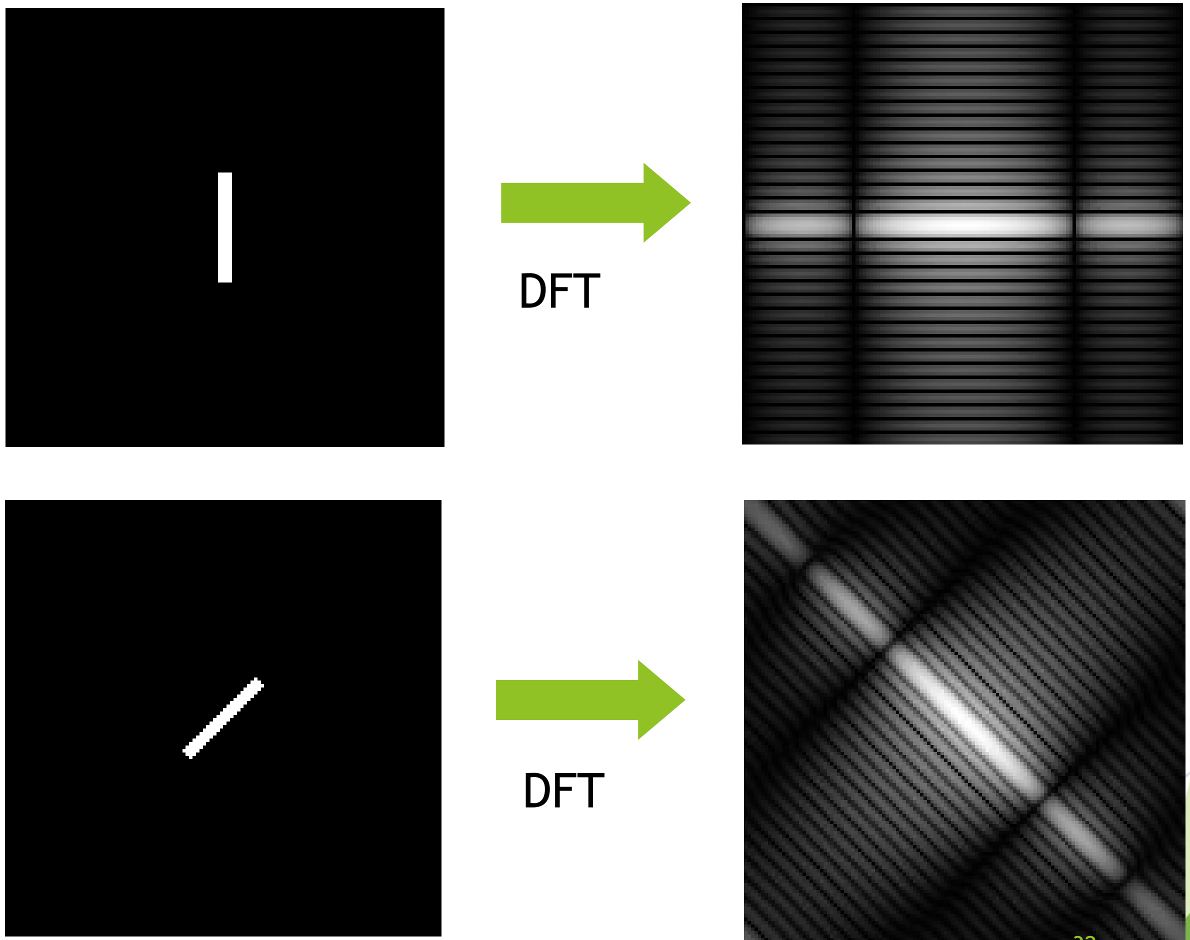
- The transform of a sin function is a negative complex delta function at the appropriate positive frequency and a negative complex delta at the appropriate negative frequency.



- The transform of a square pulse is a sinc function.

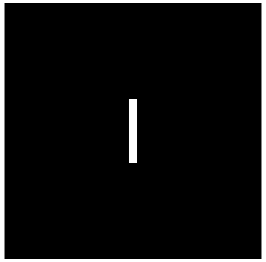


The Property of Two-Dimensional DFT
Rotation

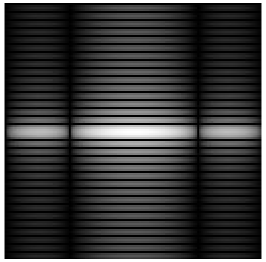


The Property of Two-Dimensional DFT
Linear Combination

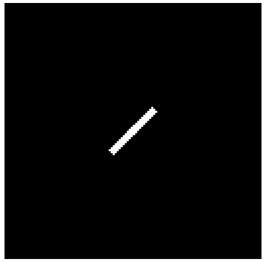
A



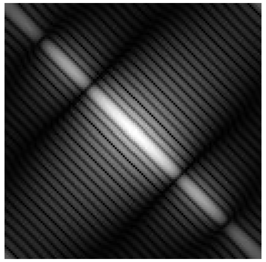
DFT



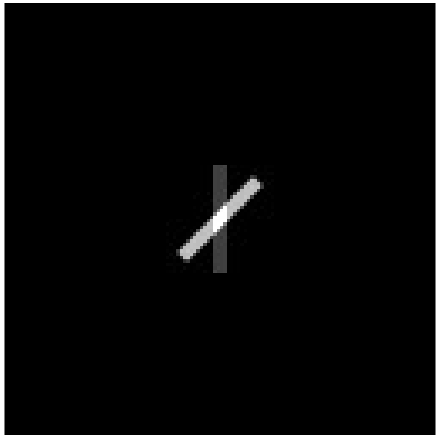
B



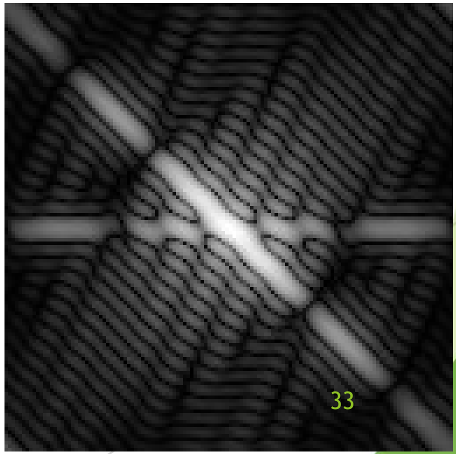
DFT



0.25 * A
+ 0.75 * B

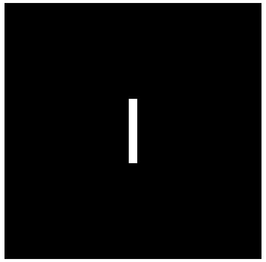


DFT

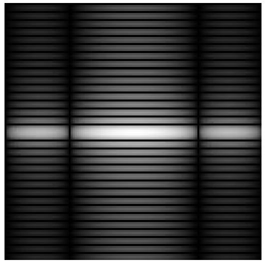


The Property of Two-Dimensional DFT Expansion

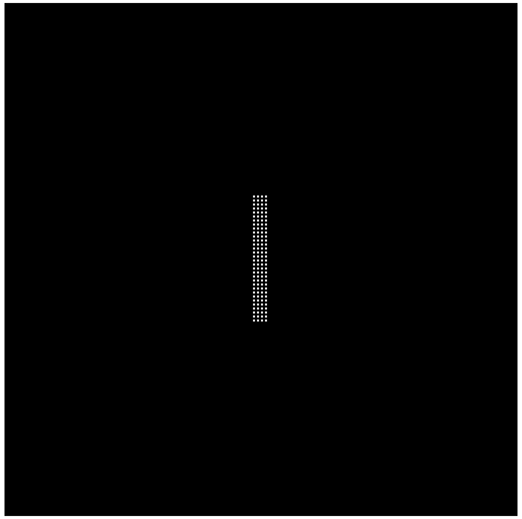
A



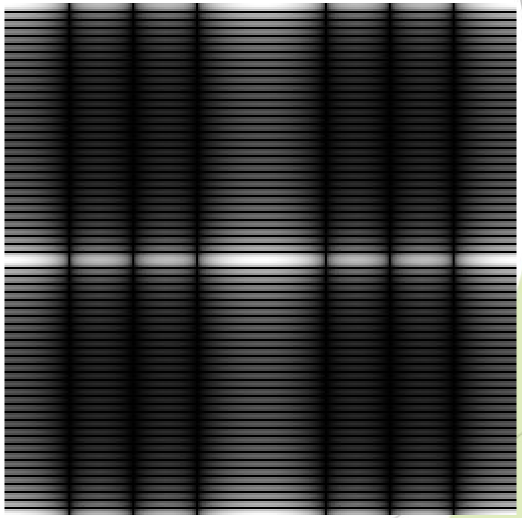
DFT



B



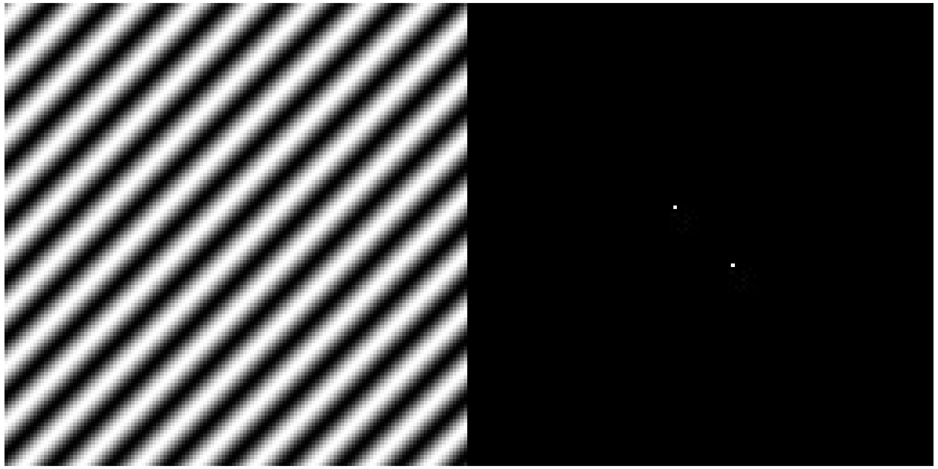
DFT



Expanding the original image by a factor of *n* (*n*=2), filling the empty new values with zeros, results in the same DFT.

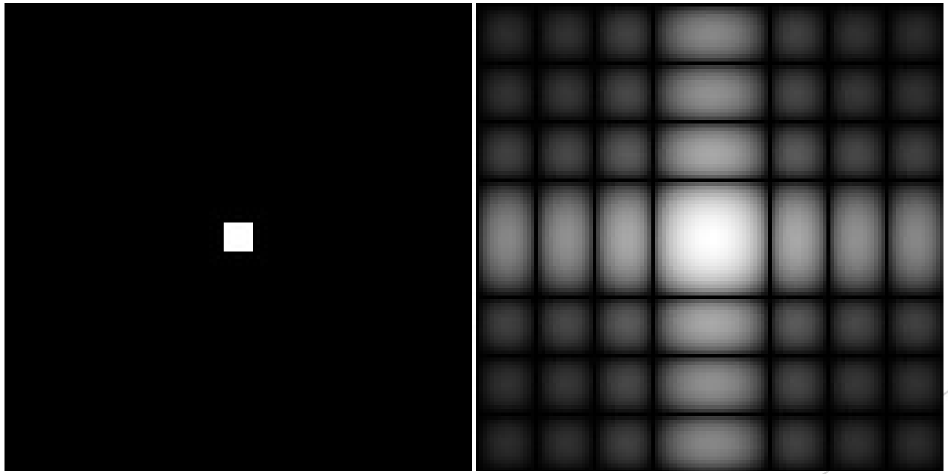
Two-Dimensional DFT with Different Functions

Sine wave



Its DFT

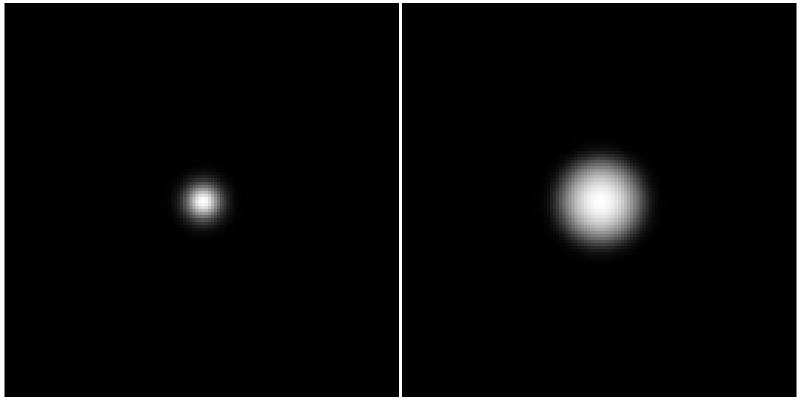
Rectangle



Its DFT

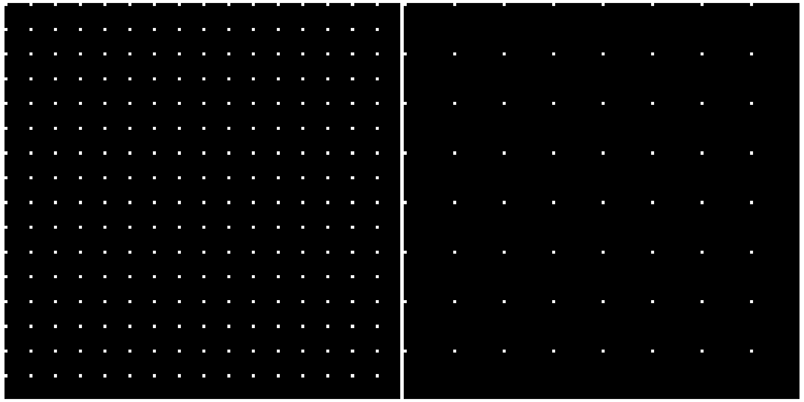
Two-Dimensional DFT with Different Functions

2D Gaussian function



Its DFT

Impulses



Its DFT