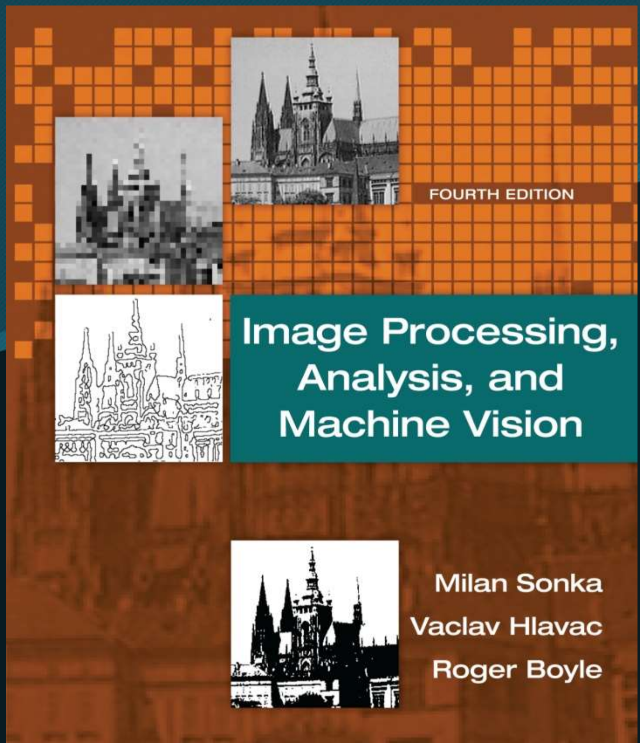
Chapter 3

The image, its mathematical and physical background



CENGAGE Learning

Outline

- Overview
- Linear integral transforms
  - Images as linear systems
  - Introduction to linear integral transforms
  - Fourier transform
  - Sampling and the Shannon constraint
  - Wavelet transform
- Images as stochastic process
- Image formation physics

Overview

- Linearity
  - Linear combination
    - A general linear combination of two vectors x, y can be written as $ax + by$ where a, b are scalars.
  - Consider a mapping $\mathcal{L}$ between two linear spaces.
    - Additive: $\mathcal{L}(x + y) = \mathcal{L}(x) + \mathcal{L}(y)$
    - Homogeneous: $\mathcal{L}(ax) = a\mathcal{L}(x)$ for any scalar a.
    - The mapping $\mathcal{L}$ is linear if it is additive and homogenous.
  - A linear mapping satisfies $\mathcal{L}(ax + by) = a\mathcal{L}(x) + b\mathcal{L}(y)$ for all vectors x, y and scalars a, b.

The Dirac distribution and convolution

- Dirac distribution $\delta(x, y)$

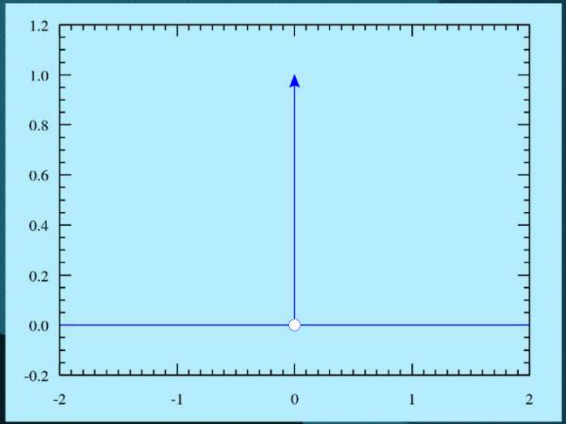
$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \delta(x, y)\mathrm{d}x\mathrm{d}y = 1$$

and $\delta(x, y) = 0$ for all $(x, y) \neq (0,0)$.

- It is used to define the ideal **impulse** in the image plane.

https://zh.wikipedia.org/wiki/%E7%8B%84%E6%8B%89%E5%85%8B%CE%B4%E5%87%BD%E6%95%B0

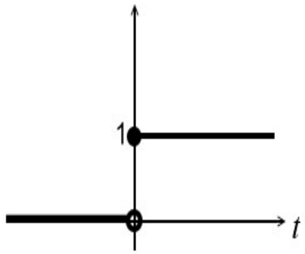
PS. 從純數學的觀點來看，狄拉克δ函數並非嚴格意義上的函數，因為任何在擴展實數線上定義的函數，如果在一個點以外的地方都等於零，其總積分必須為零。δ函數只有在出現在積分以內的時候才有實質的意義。根據這一點，δ函數一般可以當做普通函數一樣使用。



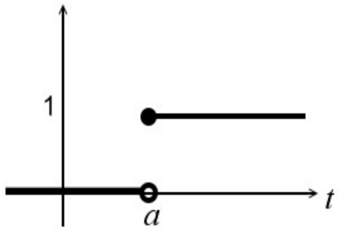
Dirac Delta Function

Heaviside step function:

$$H(t) = \begin{cases} 0 & t < 0 \\ 1 & t \geq 0 \end{cases}$$



$$H(t-a) = \begin{cases} 0 & t < a \\ 1 & t \geq a \end{cases}$$

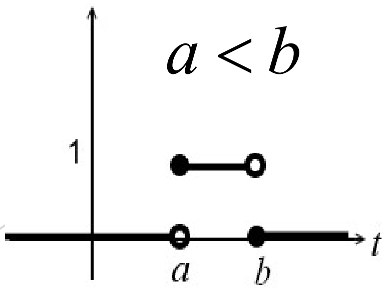


$a \geq 0$

Pulse:

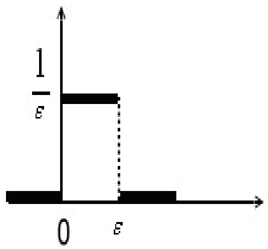
$$H(t-a) - H(t-b) = \begin{cases} 0 & t < a \\ 1 & a \leq t < b \\ 0 & t \geq b \end{cases}$$

$a < b$



Impulse:

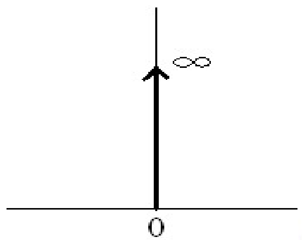
$$\delta_\varepsilon(t) = \frac{1}{\varepsilon}[H(t) - H(t-\varepsilon)]$$



Dirac delta function:

$$\delta(t) = \lim_{\varepsilon \to 0^+} \delta_\varepsilon(t)$$

$$\int_{-\infty}^{\infty} \delta(t)\,dt = 1; \delta(t) = 0, \quad \forall t \neq 0$$



2D delta function:

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \delta(x,y)\,dxdy = 1;$$

$$\delta(x,y) = 0, \quad \forall (x,y) \neq (0,0)$$

The Dirac distribution and convolution

- Sifting (篩選) property of the Dirac distribution.

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\delta(x-a,y-b)\mathrm{d}x\mathrm{d}y = f(a,b)$$

  - It provides the value of the function $f(x,y)$ at the point (a,b).
  - The sifting equation can be used to describe the sampling process of a continuous image function $f(x,y)$.

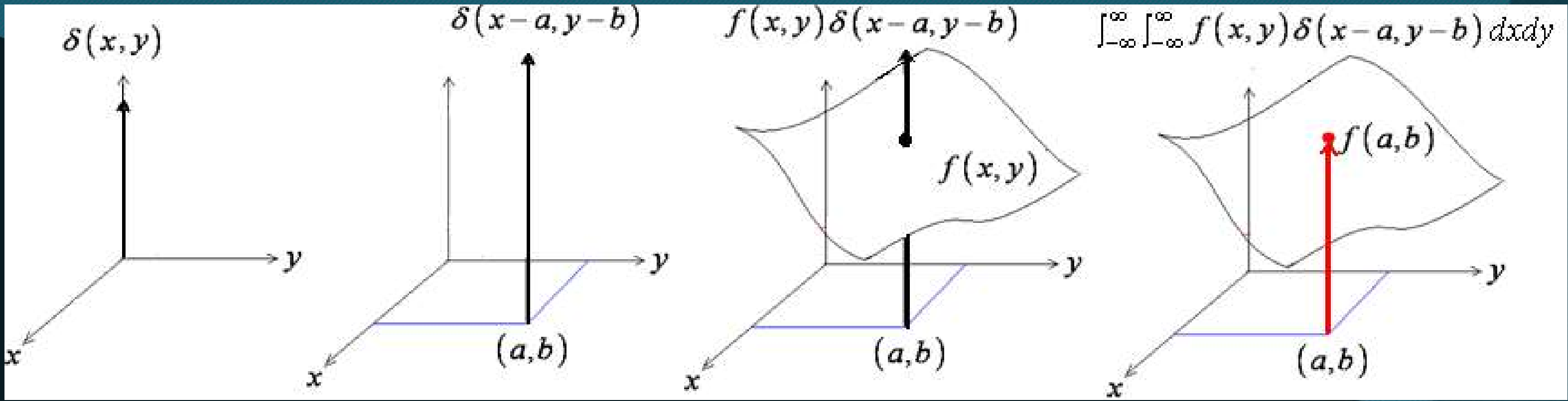


Image representation and image analysis task

- Both representations contain exactly the same information.
  - Human observer v.s. machine recognizer

Figure 1.9: Another representation of Figure 1.8. © R.D. Boyle 2015.

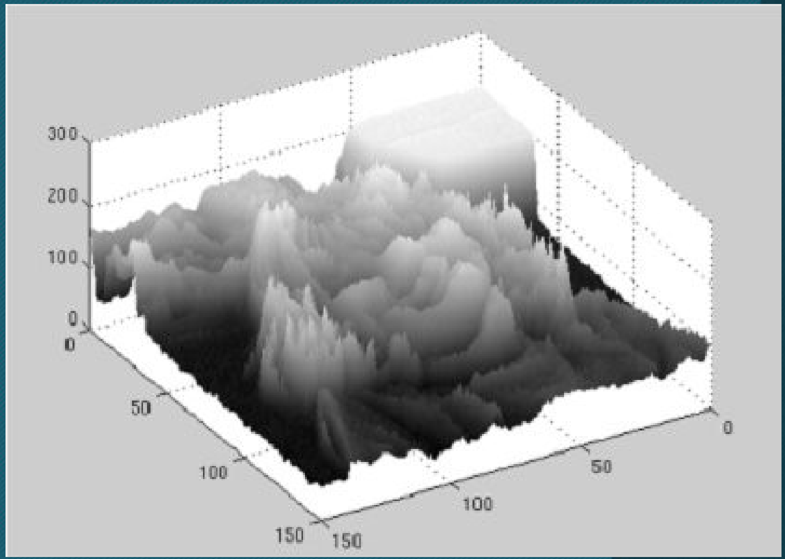


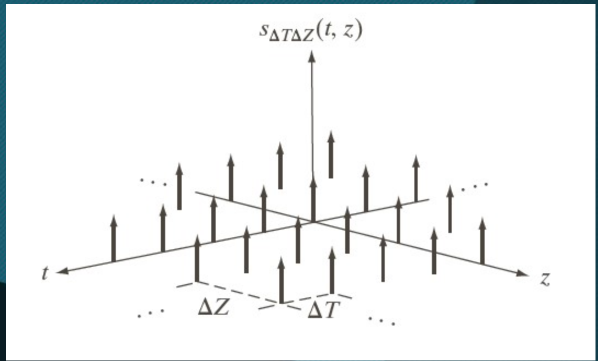
Figure 1.8: An unusual image representation. © R.D. Boyle 2015.

The Dirac distribution and convolution

- Image function

$$f(x,y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(a,b)\delta(a-x, b-y)\mathrm{d}a\mathrm{d}b$$

  - The image can be expressed as a linear combination of Dirac pulses located at the points (a, b) that cover the whole image plane. (You can prove it by yourselves.)



The Dirac distribution and convolution

- Convolution (*)
  - Convolution is a linear operation for image analysis.
  - It is an integral which expresses the amount of overlap of one function $f(t)$ as it is shifted over another $h(t)$.
  - A 1D convolution $f * h$ of functions f, h over a finite range $[0, t]$ is given by

$$(f * h)(t) \equiv \int_0^t f(\tau)h(t-\tau)\mathrm{d}\tau$$

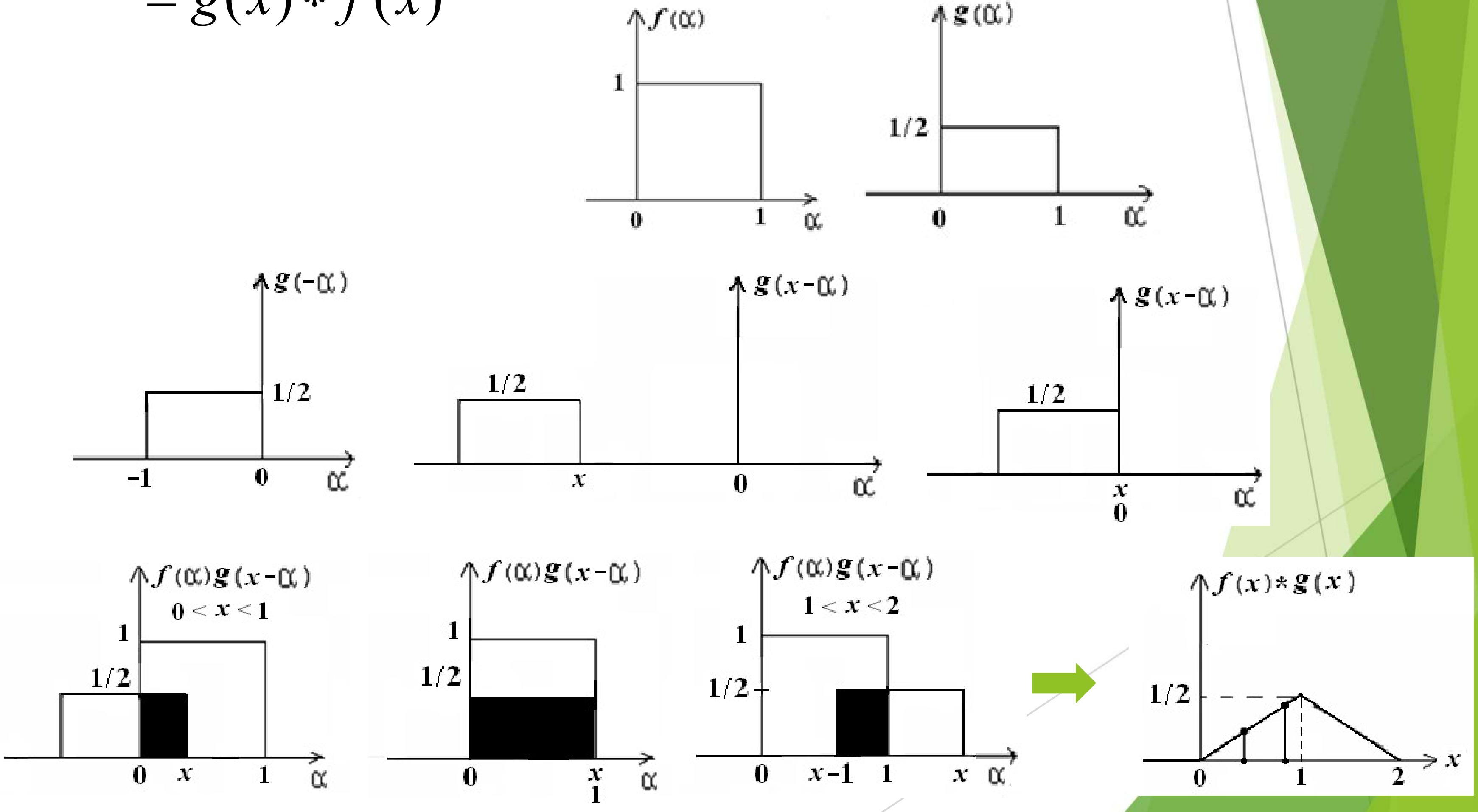
  - To be precise, the convolution integral has bounds $-\infty, \infty$.

$$(f * h)(t) \equiv \int_{-\infty}^{\infty} f(\tau)h(t-\tau)\mathrm{d}\tau = \int_{-\infty}^{\infty} f(t-\tau)h(\tau)\mathrm{d}\tau$$

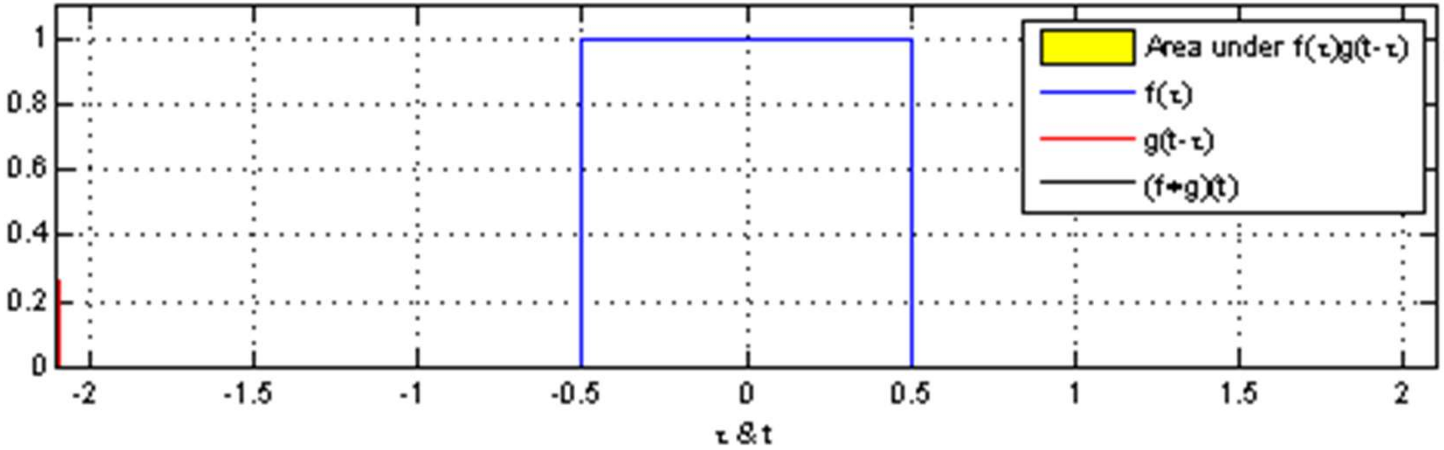
Convolution

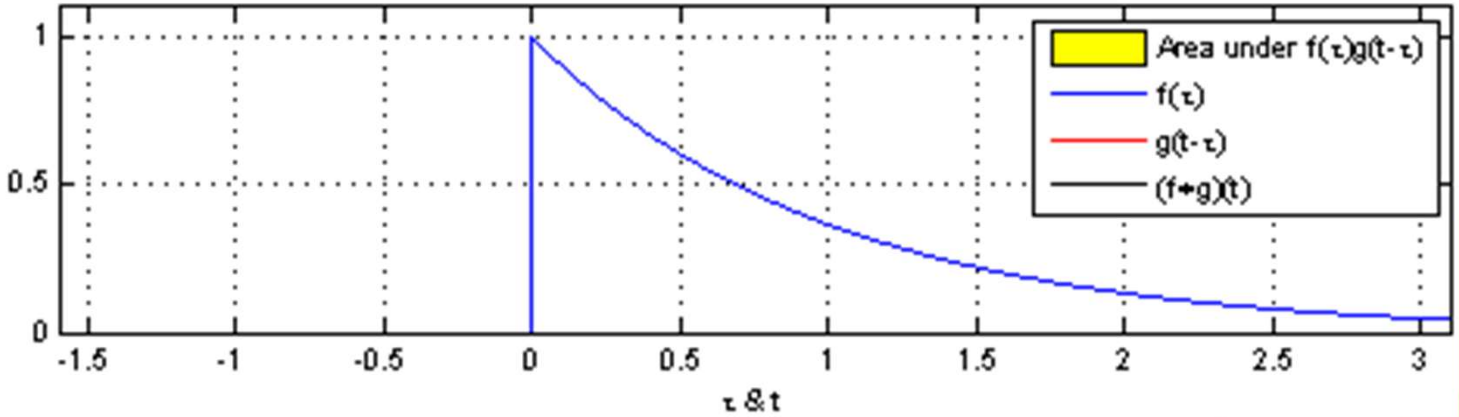
$$h(x) = f(x) * g(x) = \int_{-\infty}^{\infty} f(x-\alpha) g(\alpha) d\alpha = \int_{-\infty}^{\infty} f(\alpha) g(x-\alpha) d\alpha$$

$$= g(x) * f(x)$$



Convolution





https://en.wikipedia.org/wiki/Convolution

The Dirac distribution and convolution

- Let f, g, h be functions and a a scalar constant. Convolution satisfies the following properties

$$f * h = h * f$$
$$f * (g * h) = (f * g) * h$$
$$f * (g + h) = (f * g) + (f * h)$$
$$a(f * g) = (af) * g = f * (ag)$$

- Taking the derivative of a convolution gives

$$\frac{\mathrm{d}}{\mathrm{d}x}(f * h) = \frac{\mathrm{d}f}{\mathrm{d}x} * h = f * \frac{\mathrm{d}h}{\mathrm{d}x}$$

The Dirac distribution and convolution

- Convolution of 2D functions f and h.

$$(f * h)(x, y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(a, b) h(x-a, y-b) \mathrm{d}a \mathrm{d}b$$

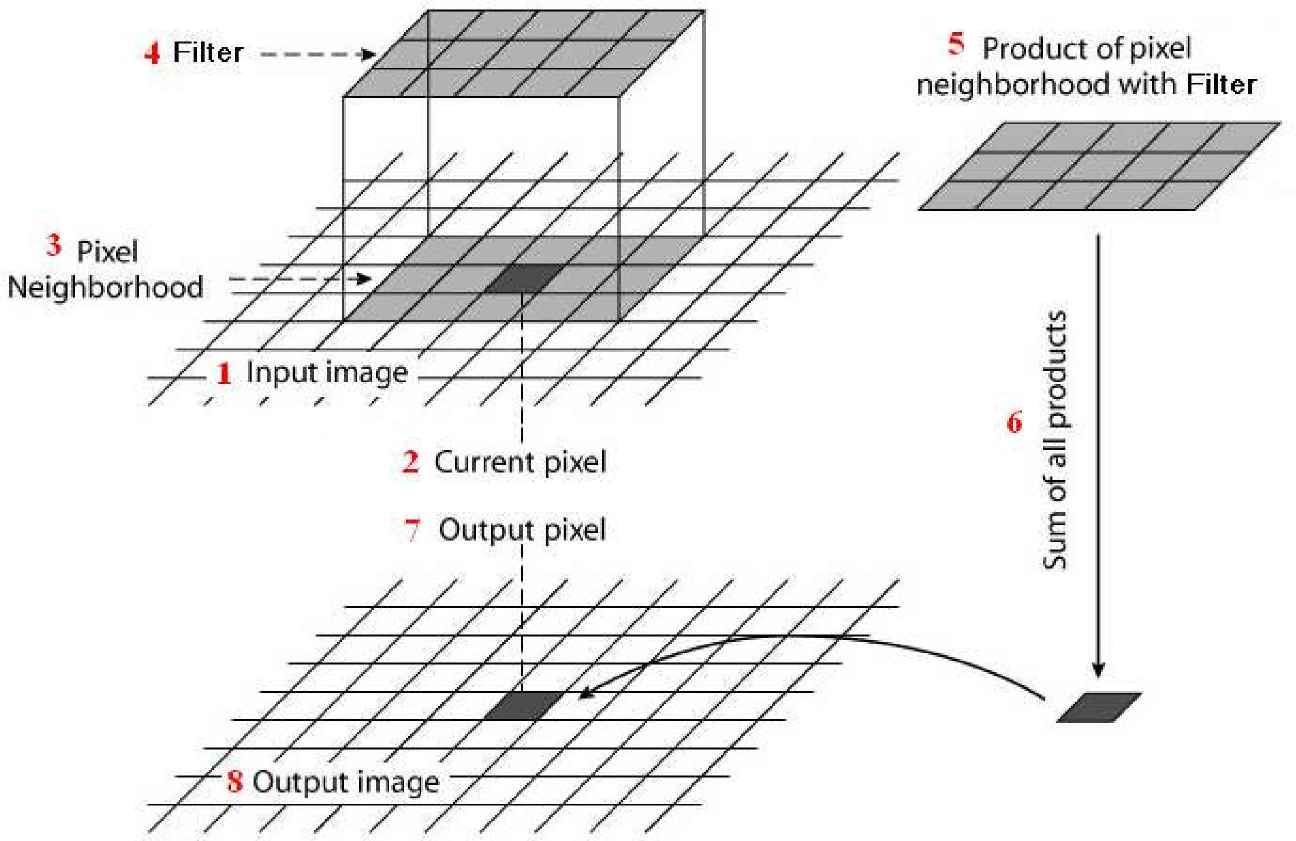
$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x-a, y-b) h(a, b) \mathrm{d}a \mathrm{d}b$$

$$= (h * f)(x, y)$$

$$(f * h)(x, y) = (h * f)(x, y)$$

The Dirac distribution and convolution

- Discrete convolution



Spatial Correlation and Convolution

- Two linear spatial filters: Correlation VS Convolution
  - A 2D example

$$w(x,y) \star f(x,y) = \sum_{s=-a}^{a} \sum_{t=-b}^{b} w(s,t) f(x+s, y+t)$$

$$w(x,y) \bigstar f(x,y) = \sum_{s=-a}^{a} \sum_{t=-b}^{b} w(s,t) f(x-s, y-t)$$

Origin $f(x,y)$

$$\begin{matrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{matrix} \qquad w(x,y) \; \begin{matrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{matrix}$$

(a)

Padded f

$$\begin{matrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{matrix}$$

(b)

FIGURE 3.30 Correlation (middle row) and convolution (last row) of a 2-D filter with a 2-D discrete, unit impulse. The 0s are shown in gray to simplify visual analysis.

Spatial Correlation and Convolution

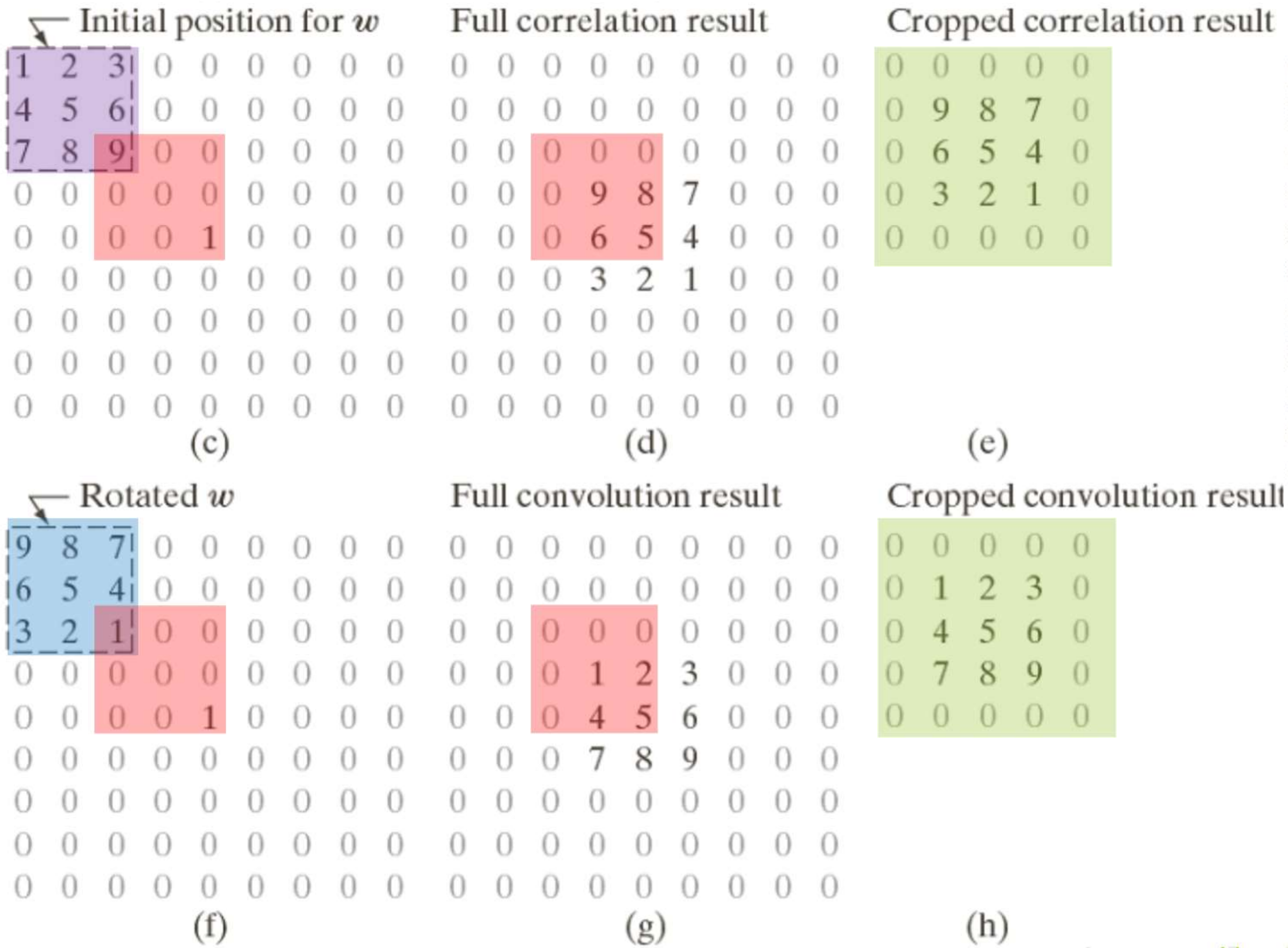


FIGURE 3.30 Correlation (middle row) and convolution (last row) of a 2-D filter with a 2-D discrete, unit impulse. The 0s are shown in gray to simplify visual analysis.

Linear integral transforms

- Images as linear system
  - A linear operator $\mathcal{L}$ has the property

$$\mathcal{L}\{af_1 + bf_2\} = a\mathcal{L}\{f_1\} + b\mathcal{L}\{f_2\}$$

  - An image f can be expressed as a linear combination of point spread function represented by Dirac pulses δ.
    - Assume that the input image f is given by

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(a,b)\delta(a-x,b-y)\mathrm{d}a\mathrm{d}b = f(x,y)$$

  - The response g of the linear system to the input image f is given by

$$g(x,y) = \mathcal{L}\{f(x,y)\} = \ldots = (f * h)(x,y)$$

Linear integral transforms

- Images as linear system

$$g(x,y) = \mathcal{L}\{f(x,y)\} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(a,b)\mathcal{L}\{\delta(x-a,y-b)\}\mathrm{d}a\mathrm{d}b$$

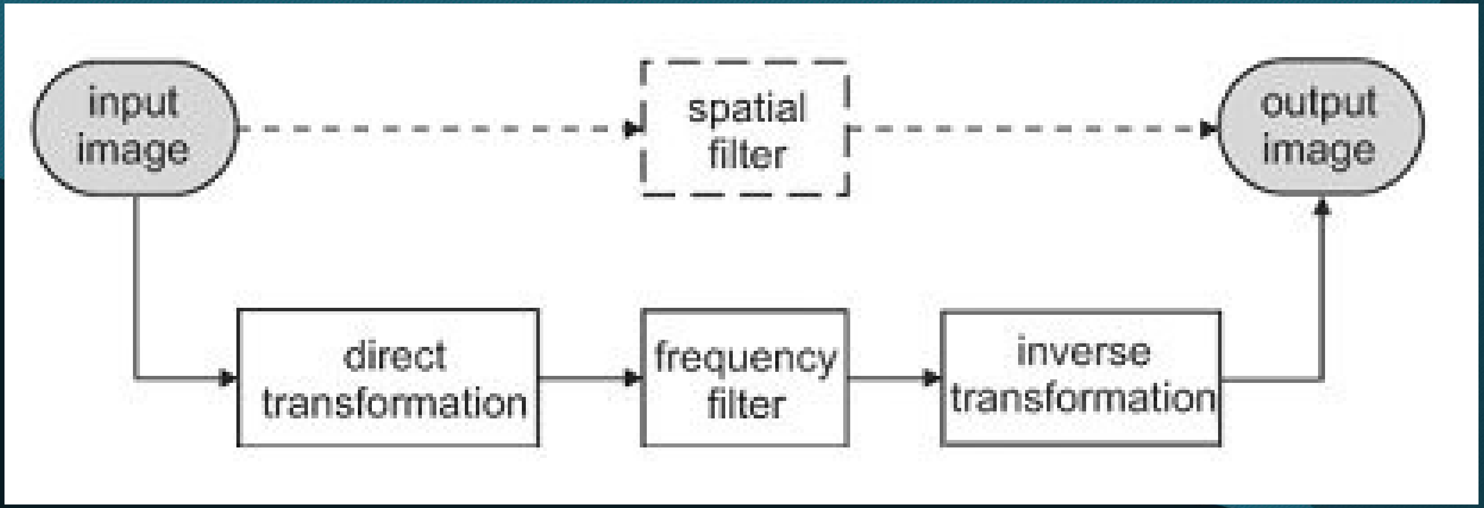
$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(a,b)h(x-a,y-b)\mathrm{d}a\mathrm{d}b = (f*h)(x,y)$$

where h is the impulse response of the linear system $\mathcal{L}$.

- The output of the linear system $\mathcal{L}$ is expressed as the convolution of the input image f with an impulse response h of the linear system $\mathcal{L}$.

Introduction to linear integral transforms

- Image filtering
  - An application of a linear integral transform in image processing
  - Filtering can be performed in either spatial or frequency domains.
  - There is one-to-one mapping between the spatial and frequency domains.
  - For linear operations, these two ways should provide equivalent results.

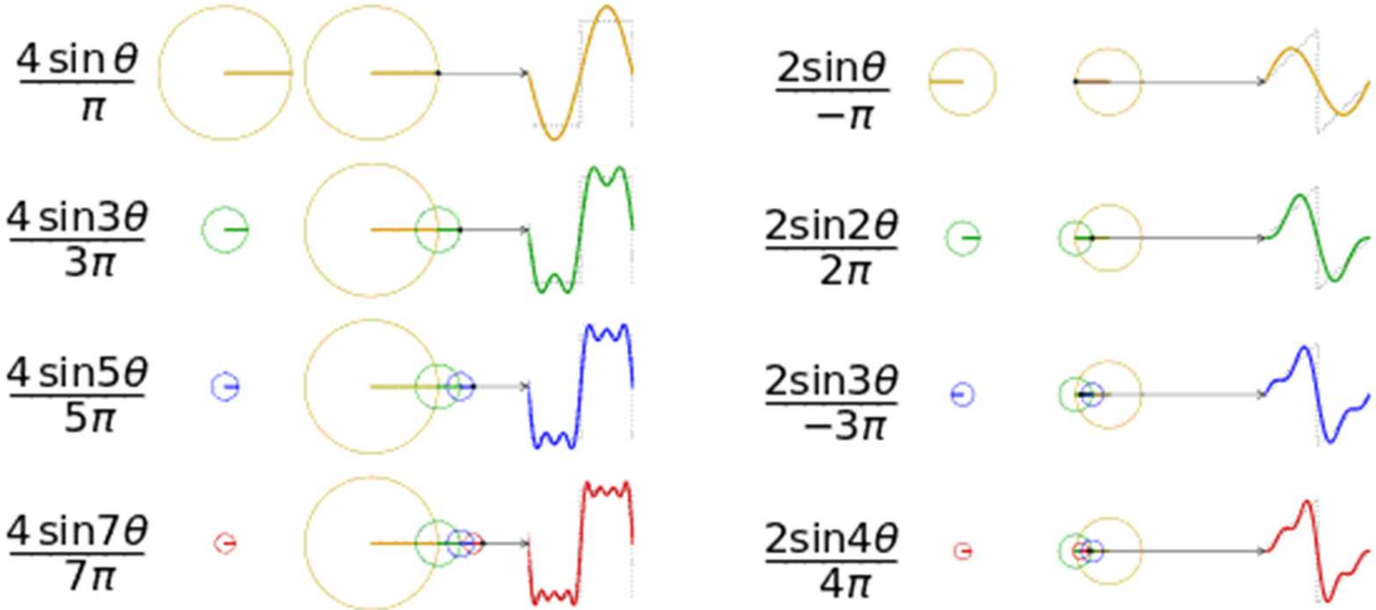


Fourier transform

In the first frames of the animation, a function *f* is resolved into Fourier series: a linear combination of sines and cosines (in blue). The component frequencies of these sines and cosines spread across the frequency spectrum, are represented as peaks in the frequency domain (actually Dirac delta functions, shown in the last frames of the animation). The frequency domain representation of the function, $\hat{f}$, is the collection of these peaks at the frequencies that appear in this resolution of the function.

https://en.wikipedia.org/wiki/Fourier_transform

Fourier series



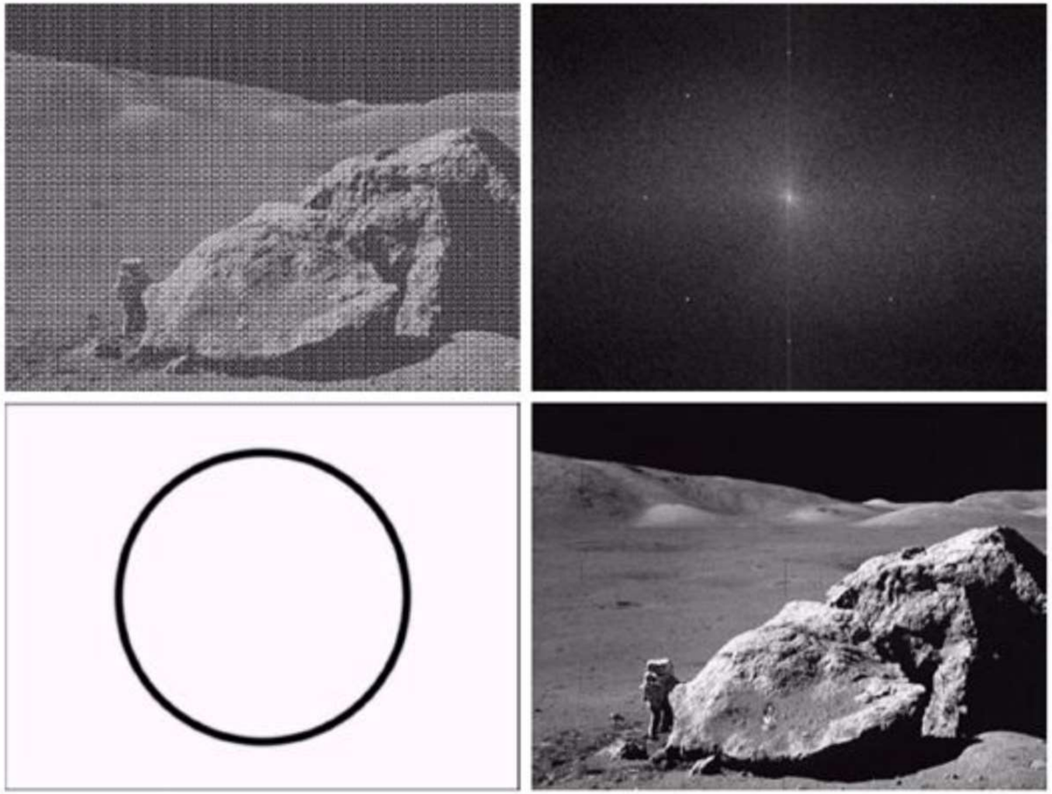
https://en.wikipedia.org/wiki/Fourier_series

Fourier series visualisation with d3.js.
https://bl.ocks.org/jinroh/7524988

Selective Filtering

Image corrupted by sinusoidal noise

Fourier spectrum of corrupted image



Butterworth band reject filter

Filtered image

http://slideplayer.com/slide/8267284/

1D Fourier transform

- Fourier transform
  - Developed by the French mathematician Joseph Fourier
  - The 1D Fourier transform $\mathcal{F}$ transforms a function $f(t)$ into a frequency domain representation

$$\mathcal{F}\{f(t)\} = F(\xi\omega)$$

  where ξ is a frequency and $2\pi\xi$ is an angular frequency.

  - Let i be the usual imaginary (虛數) unit, the continuous Fourier transform $\mathcal{F}$ is given by

$$\mathcal{F}\{f(t)\} = F(\xi) = \int_{-\infty}^{\infty} f(t)e^{-2\pi i\xi t}\,\mathrm{d}t$$

  - The inverse Fourier transform $\mathcal{F}^{-1}$ is given by

$$\mathcal{F}^{-1}\{F(\xi)\} = f(t) = \int_{-\infty}^{\infty} F(\xi)e^{2\pi i\xi t}\,\mathrm{d}\xi$$

Ps. $e^{i\omega} = \cos\omega + i\sin\omega$

https://en.wikipedia.org/wiki/Euler%27s_formula

https://zh.wikipedia.org/wiki/%E6%AC%A7%E6%8B%89%E5%85%AC%E5%BC%8F

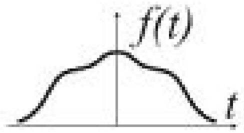
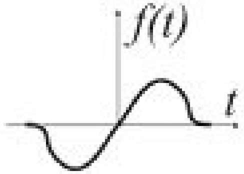
1D Fourier transform

- Fourier transform
  - The conditions for the existence of the Fourier spectrum of a function f are
    - $\int_{-\infty}^{\infty} |f(t)|\, \mathrm{d}t < \infty$
    - f can have only a finite number of discontinuities in any finite interval.
  - The Fourier transform always exists for images as they are bounded and have a finite number of discontinuities.

數學上確保傅立葉轉換可以收斂的條件是：
(1)絕對可積分，即 $\int_{-\infty}^{\infty} |f(t)|\, \mathrm{d}t < \infty$
(2)任意有限時間區間內，$f(t)$極值(包括極大與極小)的個數有限。
(3)任意有限時間區間內，$f(t)$不連續點的個數有限且這些不連續點也必須為有限值。
https://zh.wikibooks.org/zh-tw/%E8%A8%8A%E8%99%9F%E8%88%87%E7%B3%BB%E7%B5%B1/%E5%82%85%E7%AB%8B%E8%91%89%E9%A0%BB%E8%AD%9C%E7%9A%84%E7%89%B9%E6%80%A7

1D Fourier transform

- The Fourier transform exhibits predictable **symmetries**.
- Notation of **even**, **odd**, and **conjugate (共軛) symmetric function**

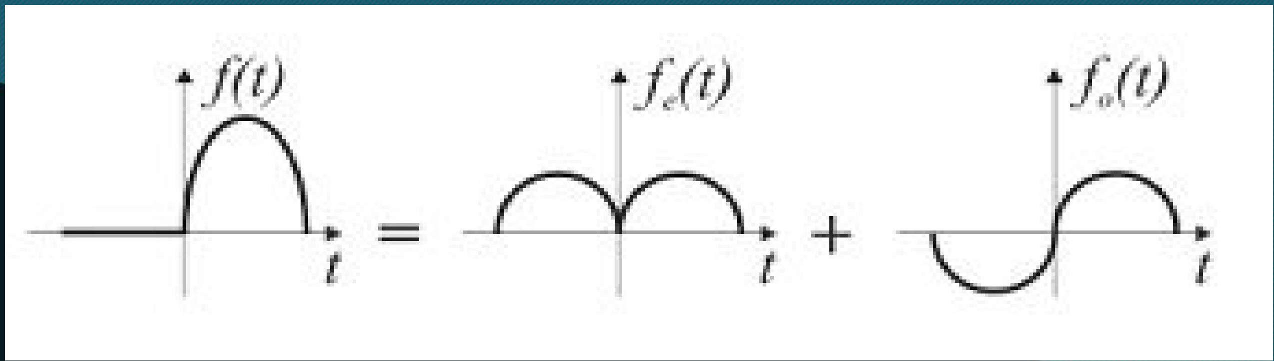
Even	$f(t) = f(-t)$	
Odd	$f(t) = -f(-t)$	
Conjugate symmetric	$f(\xi) = f^*(-\xi)$	$f(\ \ 5) = 4+7i$ $f(-5) = 4-7i$

1D Fourier transform

- Any 1D function $f(t)$ shape can always be decomposed into its even $f_e(t)$ and odd $f_o(t)$ parts.

$$f(t) = f_e(t) + f_o(t)$$

$$f_e(t) = \frac{f(t) + f(-t)}{2} \qquad f_o(t) = \frac{f(t) - f(-t)}{2}$$



1D Fourier transform

- The symmetries of the Fourier transform and its values

real $f(t)$	values of $F(\xi)$	symmetry of $F(\xi)$
general	complex	conjugate symmetric
even	only real	even
odd	only imaginary	odd

$$f(t) = f_e(t) + f_o(t) \longrightarrow F(\xi) = \mathrm{Re}\{F(\xi)\} + i\mathrm{Im}\{F(\xi)\}$$

$$\mathcal{F}\{f_e(t)\} = \mathrm{Re}\{F(\xi)\}$$

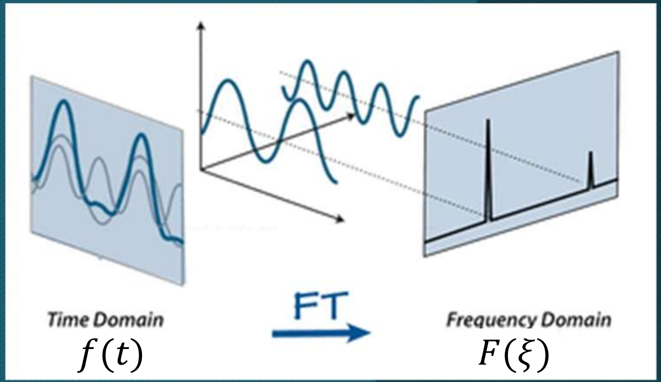
$$\mathcal{F}\{f_o(t)\} = i\mathrm{Im}\{F(\xi)\}$$

Let $\mathrm{Re}(c)$ denote the real part of a complex number c and its imaginary part $\mathrm{Im}(c)$.

1D Fourier transform

- The properties of the Fourier transform

Property	$f(t)$	$F(\xi)$
Linearity	$a f_1(t) + b f_2(t)$	$a\, F_1(\xi) + b\, F_2(\xi)$
Duality	$F(t)$	$f(-\xi)$
Convolution	$(f * g)(t)$	$F(\xi)\, G(\xi)$
Product	$f(t)\, g(t)$	$(F * G)(\xi)$
Time shift	$f(t - t_0)$	$e^{-2\pi i \xi t_0}\, F(\xi)$
Frequency shift	$e^{2\pi i \xi_0 t} f(t)$	$F(\xi - \xi_0)$
Differentiation	$\frac{\mathrm{d} f(t)}{\mathrm{d}t}$	$2\pi i \xi F(\xi)$
Multiplication by t	$t\, f(t)$	$\frac{i}{2\pi} \frac{\mathrm{d}F(\xi)}{\mathrm{d}\xi}$
Time scaling	$f(a\, t)$	$\frac{1}{\lvert a \rvert} F\,(\xi / a)$



http://mriquestions.com/fourier-transform-ft.html

1D Fourier transform

- Some other properties of the Fourier transform
  - The DC (direct current) offset is $F(0)$, and

$$F(0) = \int_{-\infty}^{\infty} f(t)\,\mathrm{d}t$$

$$F(\xi) = \int_{-\infty}^{\infty} f(t) e^{-2\pi i \xi t}\,\mathrm{d}t$$

  - The value of $f(0)$ is the area under the frequency spectrum $F(\xi)$.

$$f(0) = \int_{-\infty}^{\infty} F(\xi)\,\mathrm{d}\xi$$

$$f(t) = \int_{-\infty}^{\infty} F(\xi) e^{2\pi i \xi t}\,\mathrm{d}\xi$$

  - Parceval's theorem (帕塞瓦爾定理)

$$\int_{-\infty}^{\infty} |f(t)|^2\,\mathrm{d}t = \int_{-\infty}^{\infty} |F(\xi)|^2\,\mathrm{d}\xi$$

https://zh.wikipedia.org/wiki/%E5%B8%95%E5%A1%9E%E7%93%A6%E5%B0%94%E5%AE%9A%E7%90%86

1D Fourier transform

- Some examples

real $f(t)$	values of $F(\xi)$	symmetry of $F(\xi)$
general	complex	conjugate symmetric
even	only real	even
odd	only imaginary	odd

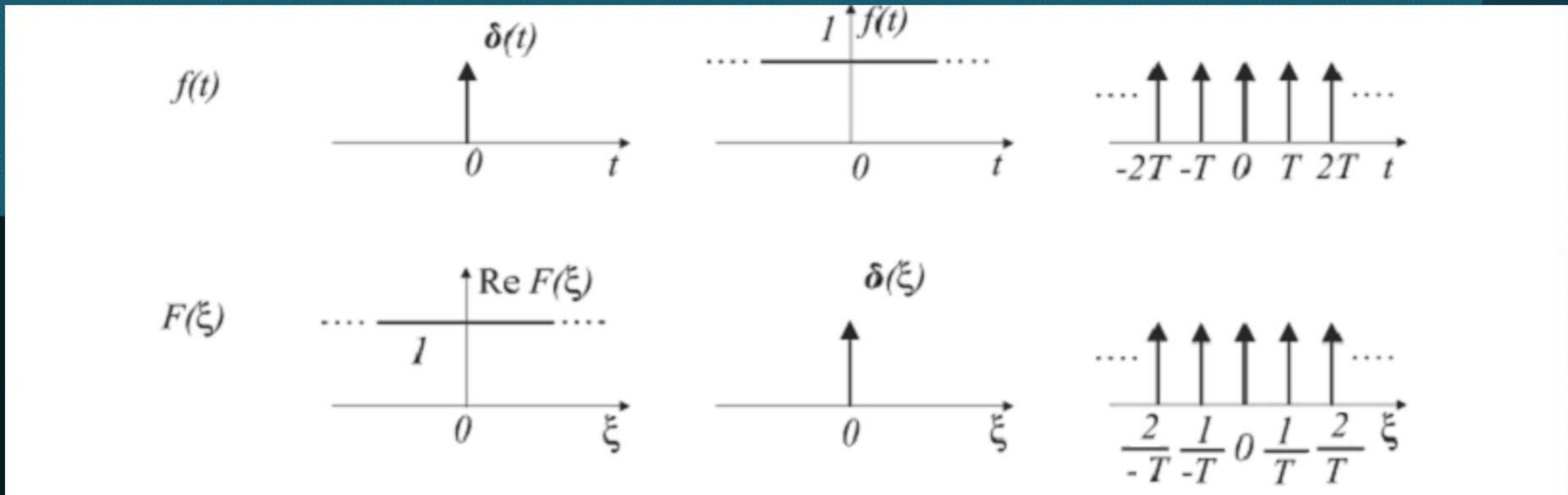


Figure 3.3: 1D Fourier transform of the Dirac pulse, constant value and infinite sequence of Dirac pulses. © *Cengage Learning 2015.*

real $f(t)$	values of $F(\xi)$	symmetry of $F(\xi)$
general	complex	conjugate symmetric
even	only real	even
odd	only imaginary	odd

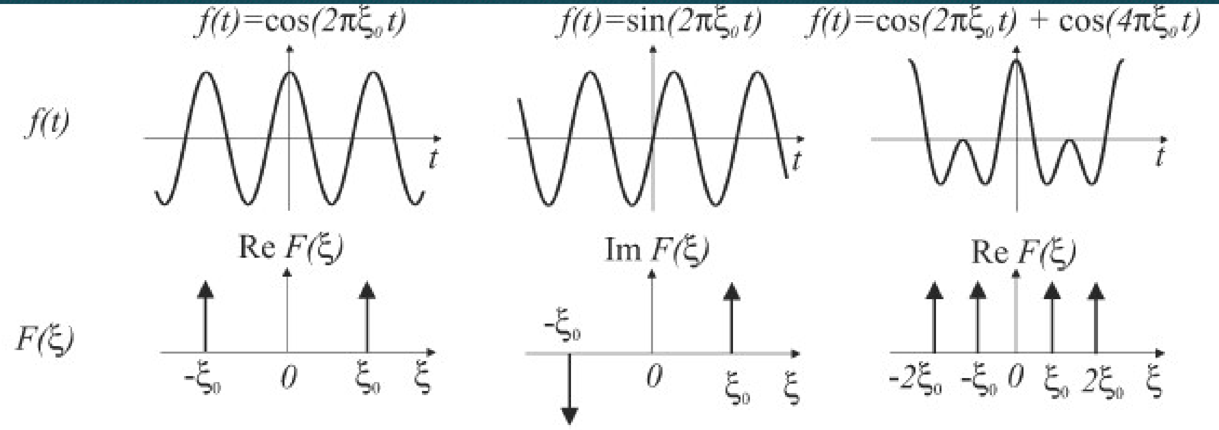


Figure 3.4: 1D Fourier transform of the sine, cosine, and sum of two different cosines. © *Cengage Learning 2015.*

real $f(t)$	values of $F(\xi)$	symmetry of $F(\xi)$
general	complex	conjugate symmetric
even	only real	even
odd	only imaginary	odd

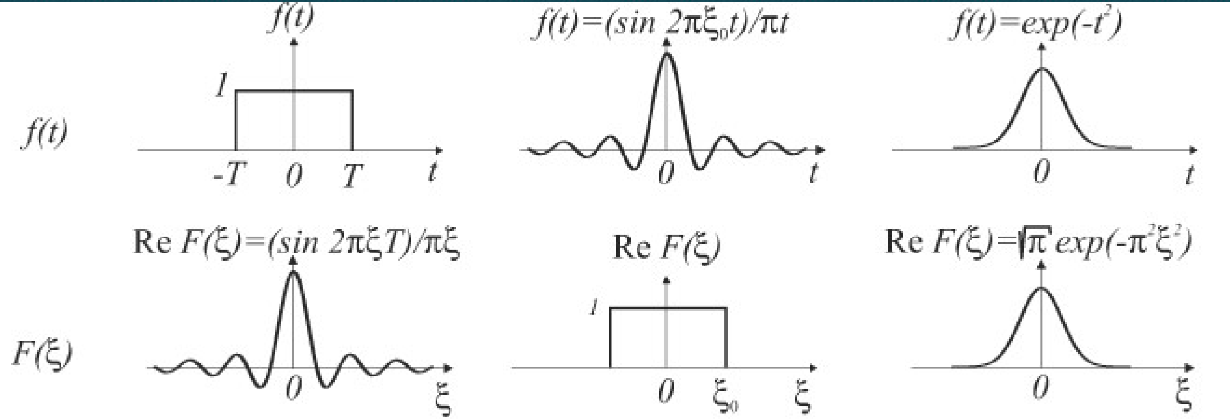


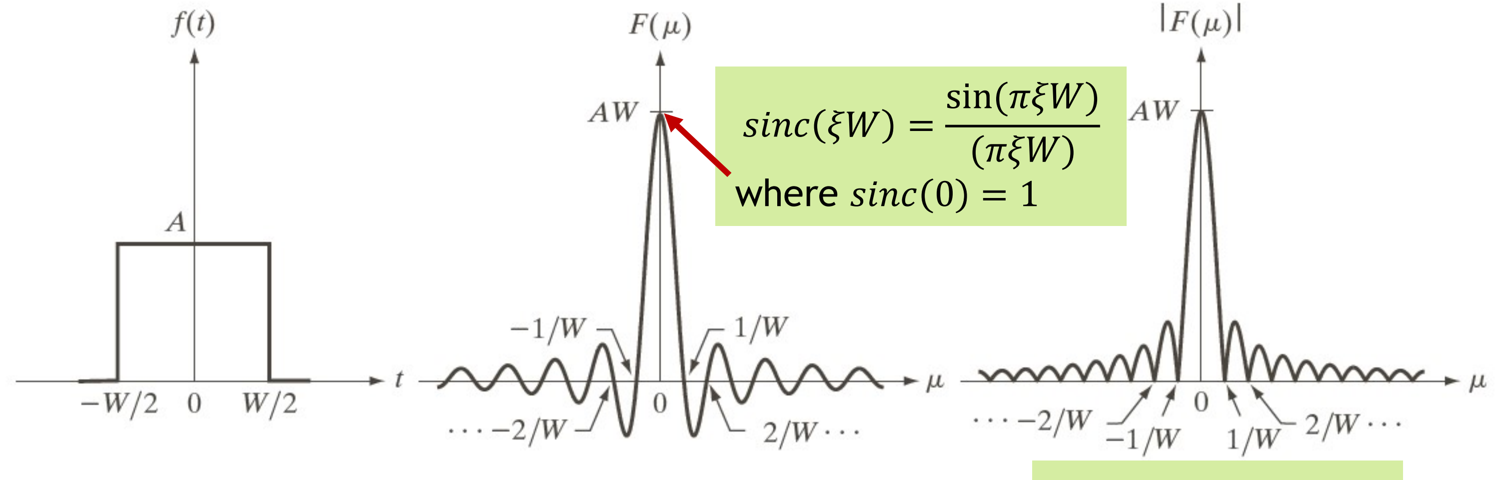
Figure 3.5: 1D Fourier transform of the idealized rectangular pulse of length $2T$ in the time domain gives the spectrum $(2\cos 2\pi\xi T)/\xi$. Symmetrically, the idealized rectangular spectrum corresponds to an input signal of the form $(2\cos 2\pi\xi_0 t)/t$. The right column shows that a Gaussian pulse has the same form as its Fourier spectrum. © *Cengage Learning 2015.*

The Fourier Transform of Functions of One Continuous Variable

- The Fourier transform of the function in Figure 4.4 (a)

$$F(\xi) = \int_{-\infty}^{\infty} f(t)e^{-2\pi i\xi t}\,dt = \int_{-W/2}^{W/2} Ae^{-2\pi i\xi t}\,dt = \frac{-A}{2\pi i\xi}\left[e^{-2\pi i\xi t}\right]_{-W/2}^{W/2}$$

$$= \frac{-A}{2\pi i\xi}\left[e^{-\pi i\xi W} - e^{\pi i\xi W}\right] = \frac{A}{2i\pi\xi}\left[e^{i\pi\xi W} - e^{-i\pi\xi W}\right] = AW\frac{\sin(\pi\xi W)}{(\pi\xi W)}$$



$$sinc(\xi W) = \frac{\sin(\pi\xi W)}{(\pi\xi W)}$$

where $sinc(0) = 1$

Fourier spectrum

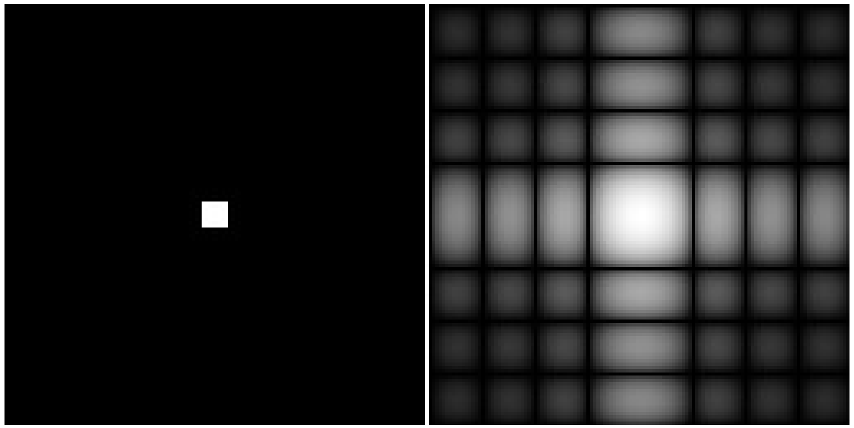
a b c

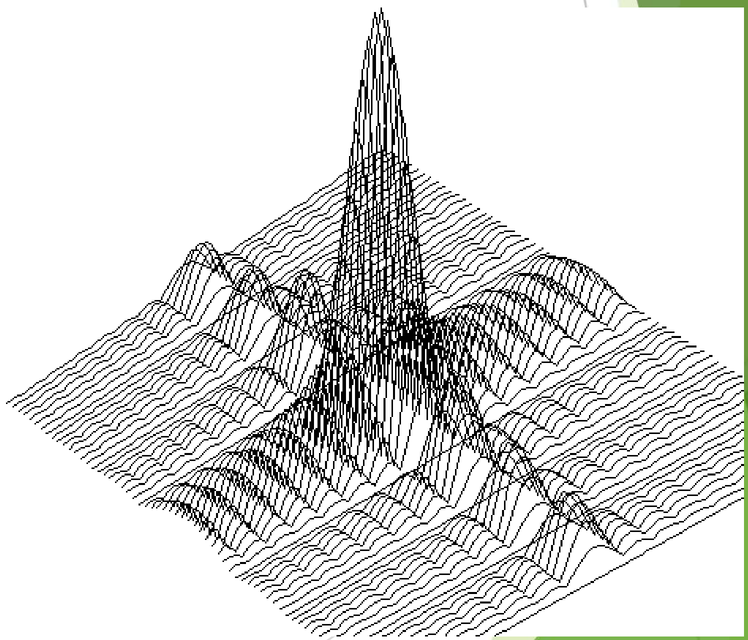
FIGURE 4.4 (a) A simple function; (b) its Fourier transform; and (c) the spectrum. All functions extend to infinity in both directions.

$$\sin\omega = \frac{1}{2i}(e^{i\omega} - e^{-i\omega})$$

$$\int e^{ax}dx = \frac{1}{a}e^{ax} + C$$

The 2D Sinc function





1D Fourier transform

- Let $\mathrm{Re}(c)$ denote the real part of a complex number c and its imaginary part $\mathrm{Im}(c)$.
- The formulas describing four function spectrum definitions are as follows:
  - Complex spectrum $F(\xi) = \mathrm{Re}\big(F(\xi)\big) + i\mathrm{Im}\big(F(\xi)\big)$
  - Amplitude spectrum $|F(\xi)| = \sqrt{\mathrm{Re}\big(F^2(\xi)\big) + \mathrm{Im}\big(F^2(\xi)\big)}$
  - Phase spectrum $\phi(\xi) = \arctan\left(\frac{\mathrm{Im}\big(F(\xi)\big)}{\mathrm{Re}\big(F(\xi)\big)}\right)$, if defined.
  - Power spectrum $P(\xi) = |F(\xi)|^2 = \mathrm{Re}\big(F(\xi)\big)^2 + \mathrm{Im}\big(F(\xi)\big)^2$

1D Fourier transform

- Discrete Fourier Transform (DFT)
  - Given discrete signals $f(n), n = 0,1,..,N-1$
  - The discrete Fourier transform

$$F(k) = \frac{1}{N}\sum_{n=0}^{N-1} f(n)\exp(-2\pi i\frac{nk}{N})$$

  - Inverse Fourier transform

$$f(n) = \sum_{k=0}^{N-1} F(k)\exp(2\pi i\frac{nk}{N})$$

  - The spectrum $F(k)$ is periodically extended with period N.

2D Fourier transform

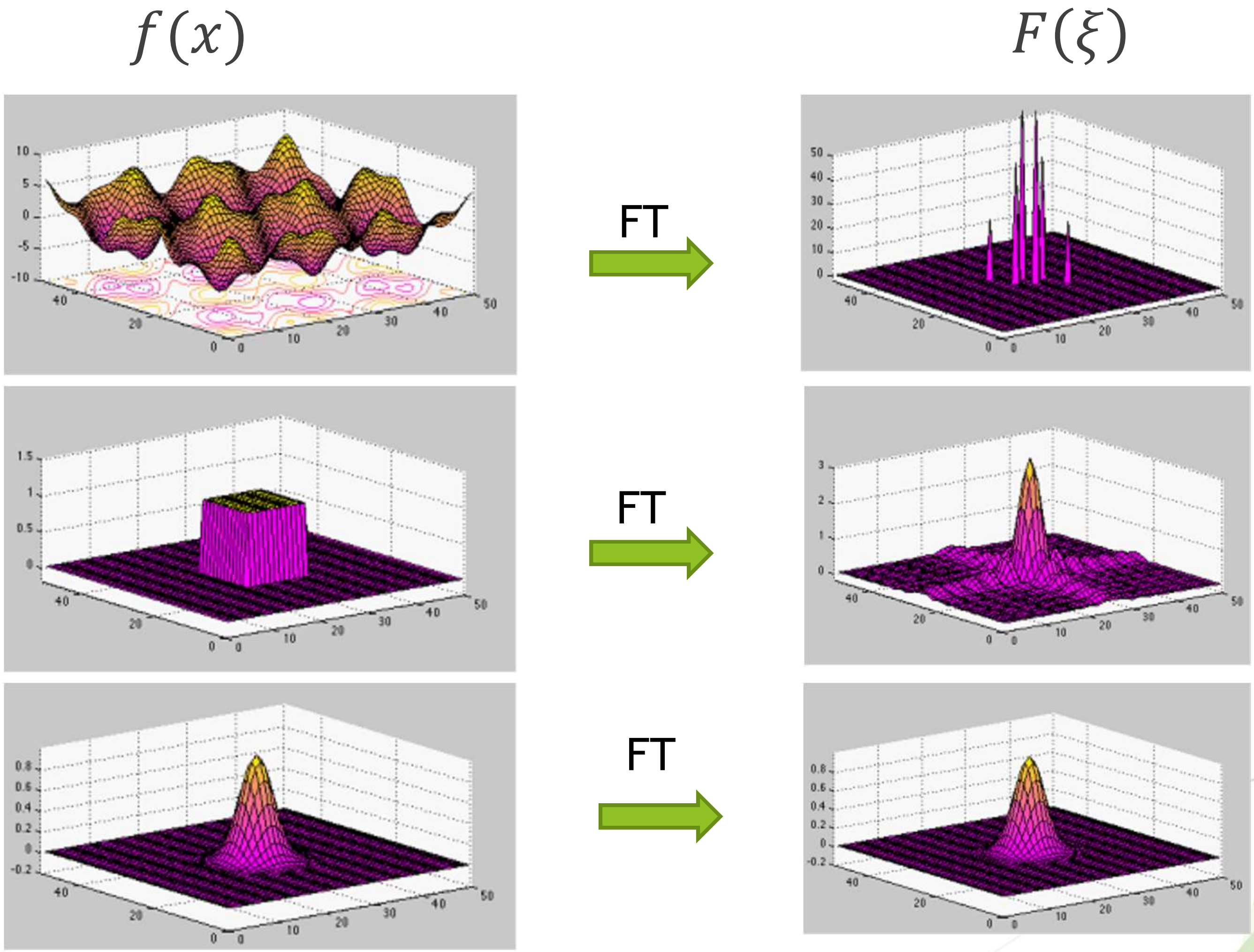


Image representation and image analysis task

- Both representations contain exactly the same information.
  - Human observer v.s. machine recognizer

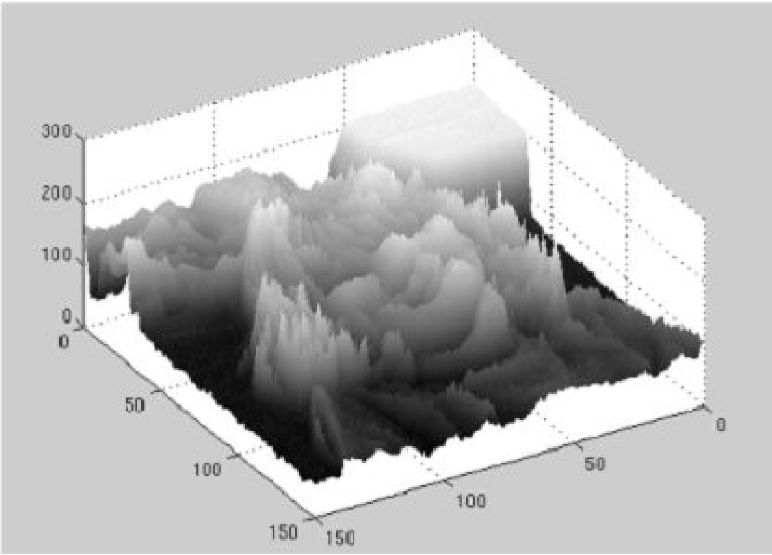
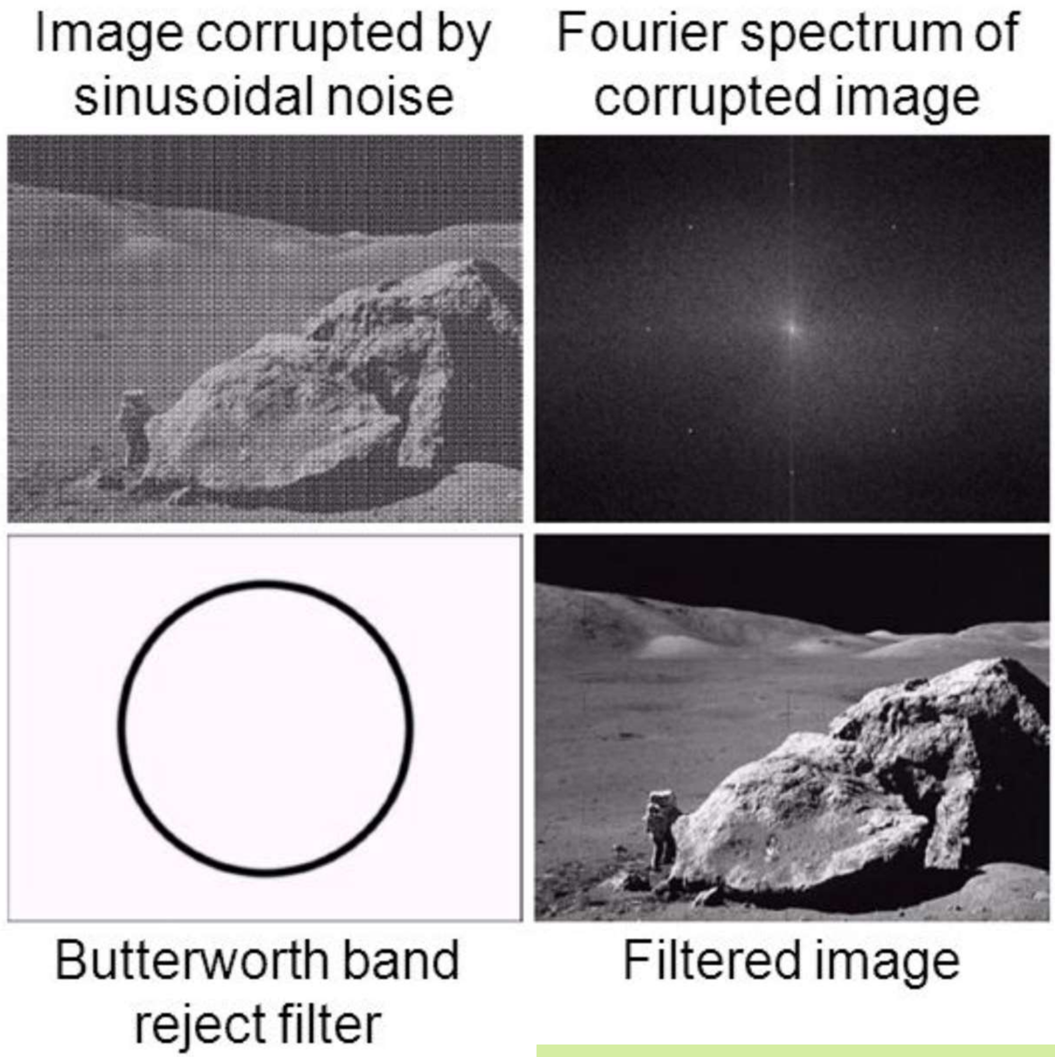


Figure 1.8: An unusual image representation. © *R.D. Boyle 2015.*

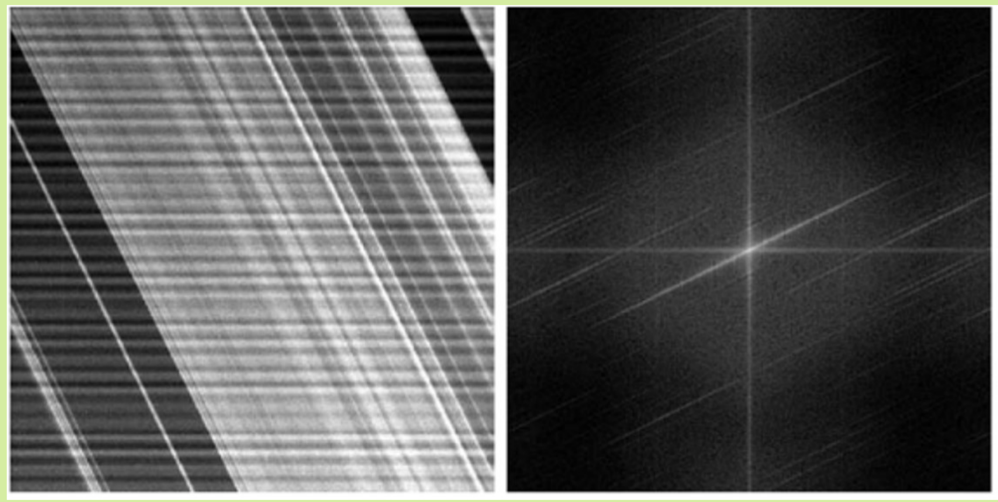
Figure 1.9: Another representation of Figure 1.8. © *R.D. Boyle 2015.*

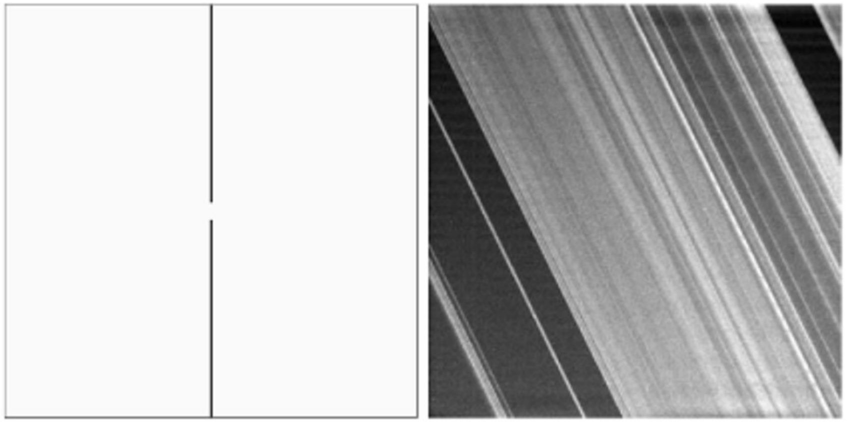
Selective Filtering



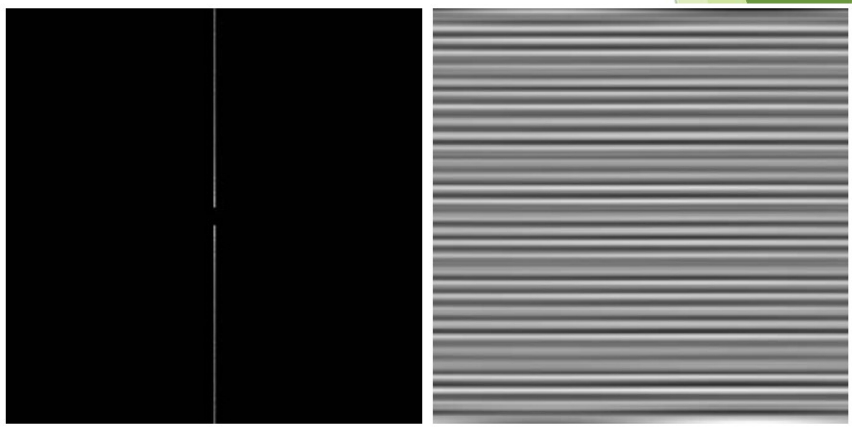
http://slideplayer.com/slide/8267284/

Selective Filtering





Notch reject filters



Notch pass filters