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SOLUTIONS FOR THE MODIFIED NEWTONIAN DYNAMICS FIELD EQUATION[1]

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ABSTRACT

We describe general properties of the solutions for the modified Newtonian dynamics (MOND) field equation suggested by Bekenstein and Milgrom. We also describe a numerical scheme for solving this equation for axisymmetric mass configurations and present numerical results.

Among the general properties we discuss are the type of boundary conditions which determine the solution uniquely, the similarity relations between the solutions from which one can obtain important analytic results, the radial behavior of the terms in the asymptotic multipole expansion of the solutions, and the solution by a perturbation method. We use the last to obtain the analytic perturbation solution for an arbitrary, low-acceleration, mass distribution in a constant acceleration field. A clear manifestation of a breakdown of the strong equivalence principle is evident in such systems.

The problems for which we give examples of numerical solutions are as follows: (1) The force law between two point masses (of obvious use in analyzing binary-galaxy dynamics). (2) Rotation curves of various model disk galaxies. We found that rotational velocities calculated from the exact theory are practically identical to those obtained from the algebraic phenomenological equation for MOND. (3) The field of a point mass in a constant external acceleration field. This solution is needed for the understanding of the trajectories of the long-period comets in the field of the Sun which is embedded in the field of the galaxy.

Subject headings: cosmology — stars: stellar dynamics

I. INTRODUCTION

In a recent paper Bekenstein and Milgrom (1984, hereafter Paper I) suggest a Lagrangian formulation for the modified Newtonian dynamics (MOND). MOND invloves a departure from Newtonian dynamics in the limit of small accelerations, and it was put forth (Milgrom 1983*a*, *b*, *c*) as an alternative to the hidden mass hypothesis. Milgrom used a phenomenological formulation of MOND appropriate for the description of test particle motion in the mean gravitational field of various bodies (such as galaxies, clusters of stars, clusters of galaxies, etc.). The phenomenological formulation cannot be applied, however, to an arbitrary N-body gravitating system without encountering some conceptual problems. These maladies have found a cure in the Lagrangian formulation (which essentially reduces to its archetype when test particle motion is considered). The Lagrangian theory is the one we now use for MOND.

The Lagrangian theory is a nonlinear differential equation for the nonrelativistic gravitational potential φ produced by a mass density distribution ρ. Being of a nonlinear nature the field equation does not, in general, lend itself to analytic solution. With the exception of systems of one-dimensional symmetry (and perhaps a few other simple configurations) the solution of the field equation demands a numerical approach. One of the purposes of the present paper is to describe a numerical scheme (§ III) and present examples of numerical solutions for some problems of interest in astronomy (§ IV). Still, much can be said about the solutions of certain problems without actually solving those. The second section of the paper is devoted to the discussion of some general properties. Section V summarizes the paper.

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II. GENERAL PROPERTIES OF THE SOLUTIONS

Various considerations had led us to propose in Paper I the following field equation for the nonrelativistic gravitational potential produced by a density distribution $\rho(\boldsymbol{r})$:

$$\nabla \cdot [\mu(|\nabla\varphi|/a_o)\nabla\varphi] = 4\pi G\rho(\boldsymbol{r}) \, . \tag{1}$$

Here G is the gravitational constant and $\mu(x)$ satisfies $\mu(x \ll 1) \approx x$, $\mu(x \gg 1) \approx 1$.

We shall be interested in solutions with two types of asymptotic behavior at infinity, viz., $\nabla\varphi \rightarrow 0$ (representing ideally isolated systems) and $\nabla\varphi \rightarrow -\boldsymbol{g}_\infty$, where $\boldsymbol{g}_\infty$ is a constant acceleration (corresponding to a subsystem in an external field of a larger system). We now discuss some general properties of the solutions, some of which are of help in setting the numerical scheme and solving it.

All the results of this paper can be carried, mutatis mutandis, to the case where the MOND theory involves a potential $\varphi = \Sigma\varphi_i$ with each of the φ_i satisfying an equation like equation (1) [with different forms of $\mu(x)$ and different values of G]

a) Uniqueness of the Solutions: The Boundary Value Problem

Each of the asymptotic limits mentioned above determine the solution φ of equation (1) uniquely (up to an additive constant) for any distribution $\rho(\boldsymbol{r})$, provided $I(x) \equiv x\mu(x)$ is an increasing function of x [we assume all along that $\mu(x)$ is an increasing function and thus so must $I(x)$ be].

Consider first the case where boundary conditions are given on the surface, Σ, of a finite volume, V within which the density $\rho(\boldsymbol{r})$ is given (as will be the case in the numerical scheme). We ask ourselves which boundary conditions on Σ determine the solution in V. Let $\varphi_1(\boldsymbol{r})$, $\varphi_2(\boldsymbol{r})$ be solutions of equation (1) in V, such that at any point on Σ either φ or $\mu(|\nabla\varphi|/a_0)\partial_n\varphi$ is given ($\partial_n\varphi$ is the component of $\nabla\varphi$ normal to Σ). The following integral thus vanishes:

$$\int_\Sigma (\varphi_1 - \varphi_2)[\mu(\nabla\varphi_1/a_0)\nabla\varphi_1 - \mu(\nabla\varphi_2/a_0)\nabla\varphi_2] \cdot d\boldsymbol{\sigma} = 0 \ . \quad (2)$$

Writing the integral as a volume integral of the divergence of the integrand vector in equation (2), we have

$$\int_V d^3r(\nabla\varphi_1 - \nabla\varphi_2) \cdot [\mu(\nabla\varphi_1/a_0)\nabla\varphi_1 - \mu(\nabla\varphi_2/a_0)\nabla\varphi_2] + \int_V d^3r(\varphi_1 - \varphi_2)\{\nabla \cdot [\mu(\nabla\varphi_1/a_0)\nabla\varphi_1] - \nabla \cdot [\mu(\nabla\varphi_2/a_0)\nabla\varphi_2]\} = 0 \ . \quad (3)$$

The second integral vanishes because both terms in brackets are equal to $4\pi G\rho(\boldsymbol{r})$ since φ_1 and φ_2 are solutions of equation (1). Now, for any two pairs of parallel vectors $\boldsymbol{k}_1 \| \boldsymbol{0}_1$ and $\boldsymbol{k}_2 \| \boldsymbol{0}_2$ such that $|\boldsymbol{k}_1| > |\boldsymbol{k}_2|$ and $|\boldsymbol{0}_1| > |\boldsymbol{0}_2|$ one has $(\boldsymbol{k}_2 - \boldsymbol{k}_2) \cdot (\boldsymbol{0}_1 - \boldsymbol{0}_2) > 0$. Applying this inequality to the vector pairs $\nabla\varphi_1 \| \mu(\nabla\varphi_1/a_0)\nabla\varphi_1$ and $\nabla\varphi_2 \| \mu(\nabla\varphi_2/a_0)\nabla\varphi_2$ we get that the integrand in the first integral in equation (3) is positive [if $I(x) = x\mu(x)$ is increasing] unless $\nabla\varphi_1 = \nabla\varphi_2$. Thus equation (3) can be satisfied only if $\nabla\varphi_1 = \nabla\varphi_2$ everywhere in V. *Equation (1) thus determines its solution uniquely, within a finite volume, under a combination of a Dirichlet boundary condition* (φ *given*) *and a generalization of the Newman condition* [$\mu(\nabla\varphi/a_0)\partial_n \varphi$ *given*]. It is this Newman-like condition that we use in the numerical code on the boundary of the problem region.

We now come back to the case of boundary condition at infinity. The asymptotic behavior of φ at $r \to \infty$ is given in Paper I for both $\nabla\varphi \to 0$ and $\nabla\varphi \to -g_\infty$. Using these results, we see that the integral in equation (2) vanishes here too, and hence $\nabla\varphi$ is again determined uniquely. We will assume in what follows that $I(x)$ is increasing. It is easy to show that this is also a necessary condition for the uniqueness of the solutions. We cannot prove that a solution always exists under those boundary conditions, but we deem it a very plausible conjecture strongly supported by the discussion of the numerical solution in § II.

b) An Equivalent Formulation of the Field Equation

The field equation can be written in another form as two first-order equations in the components of the vector field $\boldsymbol{u} = \mu(\nabla\varphi/a_0)\nabla\varphi$. Because $I(x)$ is monotonic, we can invert this relation and write $\nabla\varphi = v(u/a_0)\boldsymbol{u}$, where $v(y) = I^{-1}(y)/y$. We can now write

$$\nabla \cdot \boldsymbol{u} = 4\pi G\rho \ , \quad (4a)$$

$$\nabla \times [v(u/a_0)\boldsymbol{u}] = 0 \ , \quad (4b)$$

where the first equation follows from equation (1) and the second expresses the fact that $v(u/a_0)\boldsymbol{u}$ is a gradient field. The Newman-like boundary condition dictates the value of u_n, the component of $\boldsymbol{u}$ perpendicular to the boundary surface. For various reasons which will be discussed below, we found this formulation the best suited for the numerical scheme.

c) Jump Conditions

Across a thin layer of mass of finite surface density σ (such as we will usually assume a galactic disk to be), $\nabla\varphi$ (or $\boldsymbol{u}$) satisfy the following jump conditions: u_n jumps by $4\pi G\sigma$, and the two components of $\nabla\varphi$ [or of $v(u/a_0)\boldsymbol{u}$] parallel to the surface are continuous as the layer is crossed. If the mass system possesses a reflection symmetry about the plane tangent to the thin layer at point $\boldsymbol{r}$, $u_n = \pm 2\pi G\sigma$ just ouside the layer at $\boldsymbol{r}$ (u_n points away from the layer on both sides). We make extensive use of this relation in computing the field of disk galaxies.

d) Similarity Relations

Two similar density distributions $\rho_2(\boldsymbol{r}) = \alpha^{-1}\rho_1(\alpha^{-1}\boldsymbol{r})$ (α being an arbitrary positive constant) give rise to similar potentials $\varphi_2(\boldsymbol{r}) = \alpha\varphi_1(\alpha^{-1}\boldsymbol{r})$, for each of the two kinds of boundary asymptotic behavior at infinity. The ratio of total masses in the two systems is $M_2/M_1 = \alpha^2$. The similarity relation follows from the fact that the field equation (1) can be written in a dimensionless form:

$$\nabla_\lambda \cdot [\mu(\nabla_\lambda \psi)\nabla_\lambda \psi] = 4\pi\zeta(\boldsymbol{\lambda}) \ . \quad (5)$$

Here the unit of mass is taken to be the total mass M of the system, $\boldsymbol{\lambda}$ is $\boldsymbol{r}$ in units of the transition radius $r_t \equiv (MG/a_0)^{1/2}$, ψ is the potential in units of $\varphi_0 \equiv (MGa_0)^{1/2}$, ζ is the density in units of $\rho_0 = M/r_t^3 = M^{-1/2}(a_0/G)^{3/2}$ (ζ is normalized so that $\int d^3\lambda\zeta(\lambda) = 1$), and ∇_λ is the gradient with respect to $\boldsymbol{\lambda}$($\nabla_\lambda \psi$ is $\nabla\varphi$ in units of a_0). The scaling law follows from the dependence of the units chosen on M—the only system dependent parameter which appears in the definition of the units. Examples of the use of this relation are given in § IV in connection with the two-body force and the field of a point mass in a constant external field. This scaling law, as well as the one discussed below, holds also for the original MOND formulation (Milgrom 1983*a*) which has the same dimensional structure as the field equation. The scaling laws have been put to use in studies of galaxy rotation curves (Milgrom 1983*b*) and isothermal spheres (Milgrom 1984).

e) Low Acceleration Systems

In many instances we have $|\nabla\varphi| \ll a_0$ (or $|\nabla_\lambda \psi| \ll 1$) everywhere in the system. It follows immediately that the resulting approximate field equation [obtained by setting $\mu(x) = x$], i.e., $a_0^{-1}\nabla \cdot (|\nabla\varphi|\nabla\varphi) = 4\pi G\rho$, enjoys a two-parameter scaling invariance. Two density distributions ρ_1, ρ_2 which are related by $\rho_2(\boldsymbol{r}) = \beta\rho_1(\alpha^{-1}\boldsymbol{r})$ produce solutions φ_1 and φ_2, respectively, which are related by $\varphi_2(\boldsymbol{r}) = \alpha^{3/2}\beta^{1/2}\varphi_1(\alpha^{-1}\boldsymbol{r})$ [if the boundary fields at infinity are related by $\nabla\varphi_2(\infty) = (\alpha\beta)^{1/2}\nabla\varphi_1(\infty)$].

This scaling relation has many important consequences. For example, it follows from it that the expression for the force between two masses $m_1 \geq m_2 (m_1 + m_2 = M)$ must be of the form $h(m_2/m_1)M^{3/2}R^{-1}(Ga_0)^{1/2}$, when the distance R between the two masses becomes very large (the dependence of the force on the mass ratio has to be determined numerically; see § IV. Another immediate result is that the shape of the rotation curve of a low surface density (LSD) galaxy depends only on the form of the mass distribution and not, say, on the total mass (see Milgrom 1983*b*). For example, all LSD exponential disks have rotation curves of the same shape. This statement is always true in Newtonian dynamics but not in MOND where the *shape* of the rotation curve depends, in general, on the total mass and scale of the galaxy as well as on the mass distribution. Some implications of the scaling laws for LSD isothermal spheres are discussed in Milgrom (1984), and others, related to negative "hidden mass" are used in Milgrom (1985).

The scaling laws are, of course, of great help in numerical calculations.

f) *Perturbation Solutions*

If the solution $\nabla\varphi_0(\boldsymbol{r})$ is known for a density distribution $\rho_0(\boldsymbol{r})$ we can linearize the field equation in the increment $\epsilon\nabla\delta(\boldsymbol{r})$ in $\nabla\varphi_0$ which is produced by a change $\epsilon\hat{\rho}(\boldsymbol{r})$ in ρ when ϵ is very small. Expanding in ϵ and taking the first-order equation we get

$$\nabla \cdot [\mu_0(1 + L_0 \boldsymbol{e}_g \otimes \boldsymbol{e}_g)\nabla\delta] = 4\pi G\hat{\rho}(\boldsymbol{r}) . \quad (6)$$

As before, μ_0 and L_0 which are functions of the field strength, are calculated for $\boldsymbol{g}_0(\boldsymbol{r}) = -\nabla\varphi_0$ and $\boldsymbol{e}_g$ is a unit vector in the direction of $\boldsymbol{g}_0$. They are all $\boldsymbol{r}$ dependent. The increment potential δ thus satisfies an equation analogous to the electrostatic field equation in a medium with a dielectric constant $\mu_0(1 + L_0 \boldsymbol{e}_g \otimes \boldsymbol{e}_g)$ which is inhomogeneous and anisotropic.

For example, when the unperturbed system is spherically symmetric equation (6) becomes (using spherical coordinates r, θ, ϕ)

$$r^{-2}\frac{\partial}{\partial r}\left[r^2\mu_0(1 + L_0)\frac{\partial\delta}{\partial r}\right] - \mu_0\, r^{-2}\mathscr{L}^2\delta = 4\pi G\hat{\rho}(r, \theta, \phi) . \quad (7)$$

Here μ_0 and L_0 are known functions of r and $-r^{-2}\mathscr{L}^2$ is the angular part of the Laplacian operator ($\mathscr{L}^2$ is the square of the quantum mechanical angular momentum operator).

In many problems we find that although $\epsilon|\nabla\delta| \ll |\nabla\varphi_0|$ almost everywhere, there are small isolated regions where this inequality is not achieved. Such is the case, for example, near a point where $\nabla\varphi_0 = 0$ or near a point mass contribution to $\hat{\rho}$. The perturbation expansion and solution are then valid only outside these regions.

g) *Pertubation on a Constant Acceleration Field*

An important and very useful instance of solution by linearization is the case where $\boldsymbol{g}_0$ is a constant vector all over space. In this case $\mu_0 \equiv \mu(g_0/a_0)$ and $L_0 \equiv L(g_0/a_0)$ are constant numbers and $\boldsymbol{e}_g$ is a constant unit vector which we chose in the z-direction to obtain from equation (6):

$$\nabla^2\delta + L_0\,\partial^2\delta/\partial z^2 = 4\pi\mu_0^{-1}G\hat{\rho}(\boldsymbol{r}) . \quad (8)$$

In the coordinates $x' \equiv x, y' \equiv y, z' \equiv (1 + L_0)^{-1/2}z$ equation (8) reads

$$\nabla'^2\delta(\boldsymbol{r}') = 4\pi\mu_0^{-1}G\rho'(\boldsymbol{r}') , \quad (9)$$

with $\rho'(\boldsymbol{r}') = \hat{\rho}(\boldsymbol{r})$.

Equation (9) is readily solved for an arbitrary density distribution (remembering, of course, the requirement $|\nabla\delta| \ll g_0$ necessary for eq. [9] to be a good approximation) since it is identical to the Poisson equation. We get

$$\delta(\boldsymbol{r}') = \mu_0^{-1}G\int d^3R'\rho'(\boldsymbol{R}')|\boldsymbol{r}' - \boldsymbol{R}'|^{-1} , \quad (10)$$

or in terms of the true space coordinates

$$\delta(\boldsymbol{r}) = \mu_0^{-1}G(1 + L_0)^{-1/2}\int d^3R\hat{\rho}(\boldsymbol{R}) \times [(x - X)^2 + (y - Y)^2 + (z - Z)^2/(1 + L_0)]^{-1/2} \quad (11)$$

We can immediately draw some general conclusions relevant to such systems which involve small perturbations on a constant acceleration field.

1. A generalization of Birkhoff's theorem holds, i.e., the field generated by a spherical system at a distance r from the center of symmetry is determined only by the density distribution within this radius, and thus vanishes in a central cavity in a spherical system. (Note that this field is neither radial nor spherically symmetric.) This result follows from the fact that equation (9) is linear and that a spherical shell produces a vanishing field inside it because the corresponding $\rho'(\boldsymbol{r}')$ is an ellipsoidal shell contracted along one axis from a spherical shell and that is known to have a vanishing *Newtonian* gravitational field inside.

2. Equation (9) tells us that gravity within the system, as given by the potential $\delta(\boldsymbol{r})$, is semi-Newtonian with two important deviations: (*a*) the effective strength of gravity is enhanced by a constant factor $1/\mu(g_0/a_0)$ which roughly equals a_0/g_0 for $g_0 \ll a_0$ and can thus be very large. (*b*) There is a distortion connected with the relation between z' and z. The distortion factor $(1 + L_0)^{1/2}$ is never larger than 2, but the net effect is very important. These deviations from the Newtonian solution are direct manifestations of a breakdown of the strong equivalence principle.

We now concentrate on one example in more detail. Consider an arbitrary self-gravitating, low surface density system with density distribution $\hat{\rho}(\boldsymbol{r})$, isotropic pressure and no bulk motion, seated in a dominant constant external acceleration field. An example may be a low surface density gas cloud or a two-body–relaxed star cluster in the field of a galaxy. In addition to the Poisson equation (9) in $\boldsymbol{r}'$, the system satisfies the equation of hydrostatic equilibrium $\nabla p = -\hat{\rho}\nabla\varphi$ or, equivalently, $\nabla' p = -\rho'\nabla'\varphi$. According to a well-known theorem, the system must then be spherical *in the coordinates* $\boldsymbol{r}'$. In other words, all equipotential surfaces are of the form $|r'| =$ constant or $x^2 + y^2 + z^2/(1 + L_0) =$ constant. Thus all the equipotential and equidensity surfaces are prolate ellipsoids of revolution with the (longer) axis along the direction of the external field. The nonrotating system with isotropic velocity distribution which, in the absence of the external field would be spherical, is now elliptical with a constant axes ratio of $(1 + L_0)^{1/2}$. The maximal ellipticity which can be produced in this way is 0.29 (assuming $L_0 \leq 1$). In the solar neighborhood where the galactic field satisfies $g_0/a_0 \approx 1$ (see Milgrom 1983*b*) L_0 is smaller than 1, and the eccentricity produced is smaller than the maximum value which can be achieved in the outskirts of the galaxy.

The effect discussed here is very different from the tidal elongation of a nonrotating system with isotropic pressure because (1) It works at full strength in a constant field for which there are no tidal effects; (2) it acts everywhere within the affected system (not just beyond some "tidal" radius) and produces a constant eccentricity; (3) the elongation axis lies in the direction of the external acceleration field and not along one of the principal axes of the tidal potential field.

h) *Multipole Expansion*

We can expand the potential field φ of an arbitrary mass distribution in spherical harmonics:

$$\varphi(r, \theta, \phi) = \sum_{l,m,i} A^i_{lm}(r)Y^i_{lm}(\theta, \phi) , \quad (12)$$

where Y^i_{lm} is the ϕ-even (i = e) or ϕ-odd (i = o) spherical harmonic of order l, $m(Y^e_{lm} \propto \cos m\phi$, and $Y^o_{lm} \propto \sin m\phi)$. In general, this expansion is quite useless because of the nonlinearity of the field equation. There are, however, two cases where the multipole expansion is of use.

1. Asymptotic behavior of the multipole coefficients at large distances from an arbitrary bound system. As $r \to \infty$, the

spherically symmetric term $\varphi_0(r) = (MGa_0)^{1/2} \ln (r)$ becomes dominant over the aspherical terms, and we can linearize the field equation in the aspherical contribution and use the perturbative equation (7). Noting that asymptotically $L_0 \approx 1$ and $\mu_0 \propto r^{-1}$ and remembering that $\mathscr{L}^2 Y^i_{lm} = l(l+1)Y^i_{lm}$, we find that in the limit $r \to \infty$, A^i_{lm} satisfies

$$r \frac{d}{dr}\left(r \frac{dA^i_{lm}}{dr}\right) = \frac{l(l+1)}{2} A^i_{lm} \quad l \neq 0 \ , \tag{13}$$

and thus has the asymptotic behavior $A^i_{lm}(r) \to a^i_{lm}\, r^{\pm \xi_l}$, where $\xi_l = [l(l+1)/2]^{1/2}$.The minus sign is taken since we require $\nabla\varphi \xrightarrow{r\to\infty} 0$. Thus, unlike the lth multiple in Newtonian theory which decays as $r^{-(l+1)}$, in MOND it decays slower. It is possible to choose the origin so that the $l = 1$ terms disappear from the *asymptotic* power multipole expansion (we cannot in general make the $l = 1$ term vanish, but we can make A_1 decrease faster than any power of r).

In axisymmetric systems, such as those we wish to solve numerically, only $m = 0$ terms contribute in equation (12) when we take the origin on the symmetry axis. The expansion then becomes one in Legendre polynomials of $\cos\theta$. If, in addition, the system possesses a plane of reflection symmetry which is perpendicular to the azimuthal axis, all odd l terms disappear when the origin is in that plane. Note that the asymptotic multipole coefficients a^i_{lm} cannot, in general, be written in terms of multipoles of the density distribution.

2. A bound system which is spherically symmetric except for a small aspherical perturbation. We expand the density in this system

$$\rho(r, \theta, \phi) = \rho_o(r) + \epsilon \sum_{l \geq 1mi} \rho^i_{lm}(r) Y^i_{lm}(\theta, \phi) \ . \tag{14}$$

By assumption, the change in the acceleration field due to aspherical terms is very small compared with the field produced by $\rho_0(r)$. (As discussed above, there may be small volumes where the perturbation is larger, such as at the origin where the unperturbed acceleration vanishes.) The linearized form equation (7) of the field equation is now valid almost everywhere, and thus, substituting the expansions (12) and (14) in it we get for the radial functions:

$$\frac{d}{dr}\left[r^2 \mu_0 (1 + L_0) \frac{dA^i_{lm}}{dr}\right] - \mu_0 l(l+1)A^i_{lm} = 4\pi\epsilon G r^2 \rho^i_{lm}(r) \qquad (l \neq 0\,) . \tag{15}$$

Here μ_0 and L_0 are μ and L calculated for the radial field and are thus known functions of r after the spherical problem is solved. Equation (15) can be solved for A^i_{lm}. As we saw before, the solution which vanishes at infinity behaves asymptotically as $A^i_{lm} \xrightarrow{r\to\infty} a^i_{lm} r^{-\xi_l}$. Now we can express a^i_{lm} in terms of the density multipole $\rho^i_{lm}(r)$ in the following way. Let $B^i_{lm}(r)$ be the solution of the *homogeneous* part of equation (15) which approaches r^{ξ_l} at large distances (we saw before that there is such a solution). Multiply equation (15) by $B^i_{lm}(r)$ and integrate from 0 to some large r where the asymptotic form is valid. Integrating the first term twice by parts, making use of the fact that the B^i_{lm} values when substituted for A^i_{lm} make the left-hand side of equation (15) vanish and assuming that B^i_{lm} does not diverge too rapidly at the origin, we get

$$a^i_{lm} = -\epsilon\pi G[2/l(l+1)]^{1/2} r_t^{-1} \int B^i_{lm}(r) \rho^i_{lm}(r) r^2 dr \ , \tag{16}$$

where the transition radius of the system, r_t, is defined following equation (5).

III. THE NUMERICAL SCHEME

Our purpose is to solve the dimensionless first-order two-equation system

$$\nabla_\lambda \cdot \boldsymbol{U} = 4\pi\zeta(\boldsymbol{\lambda}) \ , \tag{17a}$$

$$\nabla_\lambda \times [\nu(U)\boldsymbol{U}] = 0 \ , \tag{17b}$$

where $\boldsymbol{U} = \mu(\nabla\varphi/a_0)\nabla\varphi/a_0$ is $\boldsymbol{u}$ in units of a_0. We confine ourselves to systems with axisymmetry, work with cylindrical coordinates $\boldsymbol{\lambda} = (r, z, \phi)$, and take z along the symmetry axis. In accordance with the choice of units for the dimensionless formulation, the total mass of the system is normalized to 1, and distances are understood to be in units of the transition radius $r_t = (MG/a_0)^{1/2}$.

a) *The Discrete Net*

Equations (17*a*)–(17*b*) are solved on a discrete net in the finite volume $r_{\min} \leq r \leq r_{\max}$, $z_{\min} \leq z \leq z_{\max}$. The lower boundary $r_{\min}$ is usually taken as zero. If the system is symmetric to reflection about a plane perpendicular to z, we take this plane to be at $z_{\min} = 0$; otherwise, $z_{\min} < 0$. The upper boundaries $r_{\max}$ and $z_{\max}$ are taken much larger than both the transition radius ($\lambda_t = 1$ in the dimensionless variables) and the scale of the mass distribution (the same is true for $|z_{\min}|$ when $z_{\min} \neq 0$).

The volume is divided into (r, z) rectangular cells defined by a division of the r domain into N_r segments $r_i \leq r \leq r_{i+1}$, with $r_i = r_{\min}$; $r_{Nr+1} = r_{\max}$, and $\delta r_i = r_{i+1} - r_i = (1 + \epsilon_r)\,\delta r_{i-1}$ forming a geometric series, and a similar division of the z domain into N_z segments.

b) *Mass Distribution*

The mass distribution is situated around $z = 0$, $r = 0$. We allow four types of contributions to the mass distribution (1) Point masses situated on the z-axis (e.g., when calculating the force between two point masses or the field of a point mass in a constant mass). If possible, we arrange that the center of mass of these point masses be at the origin, to decrease the dipole contribution to the asymptotic field. (2) A thin layer of given surface density in the plane $z = 0$. This is allowed only when the system has a reflection symmetry about the $z = 0$ plane. Such a layer represents a galactic disk and is treated in a particularly simple fashion, not as a mass contributing to the right-hand side of the divergence equation (17a), but as a boundary condition (see below). (3) A spherical mass centered at the origin and made up of a finite number of homogeneous spheres or spherical shells. (4) Any other mass distributed in the cells at will. Types 1, 3, and 4 contribute to the mass m_{ij}, in cell (i, j); type 2 does not. The total mass is normalized to be 1. Thus, if m_d is the disk mass, we have $m_d + \sum_{i,j} m_{i,j} = 1$.

c) *Centering the Discrete Quantities*

The independent variables which are the r and z components of $\boldsymbol{U}$(U^r and U^z, respectively) are defined one component on each of the cell boundaries (Fig. 1). The discretization amounts to assuming that U^k is constant across that cell boundary. Only the component perpendicular to the boundary is defined on that boundary (see Fig. 1). Thus $U^r_{i,j}$ is defined on the r_i boundary and $U^z_{i,j}$ on the z_j boundary of cell (i, j). The total number of unknowns is thus $2N_r N_z - N_r - N_z$ like the

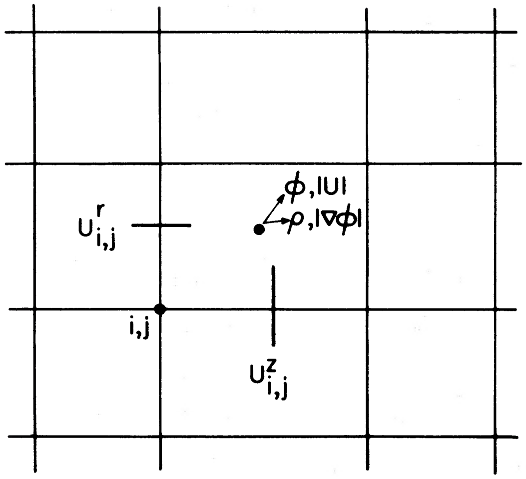

FIG. 1.—Locations where the different discrete variables are defined

number of internal cell boundaries (the values of the perpendicular component on boundaries of the problem region are dictated as boundary conditions).

The values of $|U|$, $\nu(U)$, ρ, φ, and $|\nabla\varphi|$ are defined at the center of the cells.

d) *Boundary Conditions*

We always work with the Newman-like boundary condition; i.e., U^r is dictated at $r_{\min}$, and $r_{\max}$ and U^z at $z_{\min}$ and $z_{\max}$. At $r = r_{\min} = 0$, we put $U^r = 0$. If the system possesses a $\pm z$ symmetry we distinguish between two cases when dictating boundary conditions at $z = z_{\min} = 0$. When no disk (thin layer at $z = 0$) is present, we put $U^z = 0$ at $z = 0$. When a disk of surface density $\sigma(r)$ is present, we take $U^z(r) = 2\pi G\sigma(r)$ at $z = 0$ (see discussion of the jump conditions in § II) Thus the disk does not contribute to ζ in equation (17a). The field equations are solved outside the disk which then affects the result only as a boundary condition. When the system is not symmetric about $z = 0$, $z_{\min} \neq 0$ is a large-distance boundary and is treated like the $r = r_{\max}$ and $z = z_{\max}$ boundaries. These boundaries are treated in the following fashion. In paper I we give the asymptotic behavior of $\nabla\varphi$ (easily written as one for $\boldsymbol{U}$) for both $\nabla\varphi \to 0$ and $\nabla\varphi \to -\boldsymbol{g}_\infty$. We take the dominant term in the asymptotic behavior as the exact value of U^n (the component perpendicular to the boundary). In fact, for $g_\infty = 0$ the asymptotic form of $\boldsymbol{U}$ is just the Newtonian field $\nabla\varphi_{\mathrm{N}}$. When $g_\infty \neq 0$, the asymptotic form of U is readily obtained from a Newtonian solution (see below), so in both cases it is easily calculated. This is another advantage of using $\boldsymbol{U}$ as the dependent variable. The discrete value of U^n is taken so that U^n times the area of the cell boundary equals the exact flux of $\boldsymbol{U}$ through the boundary as calculated from the analytic expression. This definition, together with the boundary conditions on the thin layer at $z = 0$ if it exists, ensures that the boundary condition satisfies Gauss's theorem implied by equation (17a) exactly; i.e.,

$$\int U^n d\sigma = 4\pi \sum_{i,j} m_{i,j}\,, \tag{18}$$

or in discrete form, $\Sigma U^n_k A_k = 4\pi \sum_{i,j} m_{i,j}$ where A_k is the area of the boundary element and the sum is over all the problem boundary elements. As we proved in § II, U^n on the problem boundary determines the solution in the problem volume uniquely.

e) *The Discretized Field Equations*

For the discretized version of the divergence equation (17a) we use the integral of that equation over the volume of each of the cells (i, j). Using Gauss' theorem gives $\int_{i,j} U^n d\sigma = 4\pi m_{i,j}$, or in discretized form $\Sigma U^n_k A_k = 4\pi m_{ij}$. The k sum is over the four cell boundaries, A_k being the respective areas. The number of equations we get in this way is $N_r N_z$, but only $N_r N_z - 1$ of them are independent because the sum of all of them is satisfied automatically (from eq. [18]). The exact form of the equation written for the cell i, j is

$$(U^z_{i,j+1} - U^z_{i,j})\pi(r^2_{i+1} - r^2_i) + 2\pi\delta z_j(r_{i+1}U^r_{i+1,j} - r_i U^r_{i,j}) = 4\pi m_{ij}\,. \tag{19}$$

To discretize the curl equation (17b) we first integrate it over the rectangular surface defined by the centers of the four cells around the vertex (r_i, z_j). The integration gives $\oint \nu(U)\boldsymbol{U}\cdot d\boldsymbol{l} = 0$ where the contour of integration is around the boundary of the rectangle. We discretize the integral and obtain one which involves the four U components on the cell boundaries radiating from r_i, z_j (see Fig. 2). We get

$$\eta_1 U^r_{i,j} - \eta_2 U^r_{i,j-1} + \eta_3 U^z_{i-1,j} - \eta_4 U^z_{i,j} = 0\,. \tag{20}$$

Here η_k are the discrete representations of νdl. For example, η_1, which goes with $U^r_{i,j}$ is given by $\eta_1 = (\nu_{i-1,j}\delta r_{i-1} + \nu_{i,j}\delta r_i)/2$. Here $\nu_{i,j}$ is the value of ν calculated for the value of $|U|$ at the center of cell i, j. Since ν is a function of $|U|$, the circulation equations (20) are non-linear. The number of such independent equations equals the number of internal vertices, i.e., $(N_z - 1)(N_r - 1)$. The number of independent equations thus total $2N_r N_z - N_r - N_z$, which equals the number of unknowns.

The discretization of the field equations described by equations (19)–(20) ensures that the discretized Gauss' theorem will hold *exactly* for an arbitrary volume made up of cells, and that the circulation of $\nu(U)\boldsymbol{U}$ around any discretized closed loop vanishes *exactly*.

f) *Solving the Difference Equations*

We found the following scheme to be very efficient:

First, we find a *discrete* $\boldsymbol{U}$ field which satisfies the divergence (flux) equations (eq. [19]) exactly and which has the same

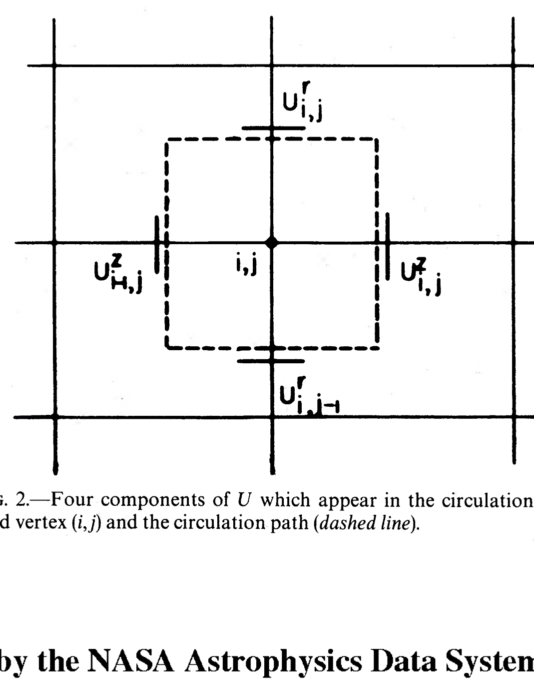

FIG. 2.—Four components of U which appear in the circulation equation around vertex (i,j) and the circulation path (*dashed line*).

asymptotic behavior as $\boldsymbol{U}$ but which does not necessarily satisfy the curl (circulation) equations (eq. [20]). In particular, the Newtonian acceleration field $\boldsymbol{U}_N \equiv \nabla\varphi_N$ satisfies these requirements for the case $g_\infty = 0$. If we integrate the components of $\boldsymbol{U}_N$ on a cell boundary and divide by the area of the boundary, we get discretized values of $\boldsymbol{U}_N$ which satisfy equation (19) *exactly*. We also noted before that $\boldsymbol{U} \to \boldsymbol{U}_N$ asymptotically.

Now, we use an iteration scheme which changes only the circulation of $\nu(U)\boldsymbol{U}$ but does not change at all the total discretized flux of $\boldsymbol{U}$ into any of the cells. This is done in the following way. We go over all the *internal* vertices of the mesh (those not falling on the problem boundary) one by one and change at each of them all four components of $\boldsymbol{U}$ defined on the boundaries radiating from that vertex (see Fig. 2). Suppose P_{ij} is the value of the circulation (left-hand side of eq. [20]) around vertex i, j after the nth iteration. We change the four components $U^r_{i,j}, U^r_{i,j-1}, U^z_{i,j}, U^z_{i-1,j}$ around this vertex by $\delta U^r_{i,j}$, $\delta U^r_{i,j-1}$, $\delta U^z_{i,j}$, and $\delta U^z_{i-1,j}$, respectively, so that (*a*) the U fluxes into all four cells bordering with the vertex remain exactly the same. This requirement imposes *three* independent linear constraints on the δU's:

$$\begin{aligned} r_i \delta z_j \delta U^r_{ij} + S_i \delta U^z_{i,j} &= 0 , \\ r_i \delta z_{j-1} \delta U^r_{i,j-1} - S_i \delta U^z_{i,j} &= 0 , \\ r_i \delta z_j \delta U^r_{i,j} - S_{i-1} \delta U^z_{i-1,j} &= 0 . \end{aligned} \tag{21}$$

The fourth $(r_i \delta z_{j-1} \delta U^r_{i,j-1} + S_{i-1} \delta U^z_{i-1,j} = 0)$ follows from them. In these equations $S_i \equiv (r^2_{i+1} - r^2_i)/2$. (*b*) We require that the new circulation *using the old values of* η_i, vanishes; i.e.,

$$\eta_1 \delta U^r_{i,j} - \eta_2 \delta U^r_{i,j-1} + \eta_3 \delta U^z_{i-1,j} - \eta_4 \delta U^z_{i,j} = -P_{ij} . \tag{22}$$

These four equations are readily solved for each vertex at a time. Iterations are needed because nullifying the curl at a vertex destroys the nullification which was achieved earlier for neighboring vertices and also because the values of η_i change as $\boldsymbol{U}$ changes, and the circulation equation which was satisfied with the old values of η is not satisfied with the new ones. Nevertheless, the convergence to a discretized field $\boldsymbol{U}$ which satisfies equation (20) at all vertices is achieved very fast when we use a relaxation method. By this method the changes δU at each step are not taken as the solutions of equations (21)–(22) but rather as κ times those δU, where κ is a relaxation parameter $1 \leq \kappa < 2$. (See e.g., the discussion in Adler and Piran 1984 on the role of relaxation in solving an equation similar to ours but with a different scheme). A good value of κ is found by trial and error, but even for $\kappa = 1$ (no relaxation) convergence is achieved in a reasonable number of iterations.

It is important to note that while iterating we do not change at all the values of U on the boundary of the problem region and the value of the left-hand side of the flux equation (19). Since both had the correct values for the initial conditions we start with, the result of the iteration procedure is the desired solution. This efficient scheme is made possible by the choice of the components of $\boldsymbol{U}$ as a dependent variables, using φ say, or the components of $\nabla\varphi$ leads to a scheme which does not lend itself to such a simple numerical solution. Note that all the information about the mass distribution is contained, in our scheme, in the boundary and initial conditions we take and is not involved in the iteration procedure.

g) Initial Conditions

There remains to be said a little more about setting the initial conditions. We have already said that when $\boldsymbol{g}_\infty = 0$ The Newtonian solution $\nabla\varphi_N$ may serve as an initial and a boundary condition for $\boldsymbol{U}$. When $\boldsymbol{g}_\infty \neq 0$, we proceed as follows. Let $\zeta(\boldsymbol{\lambda})$ be the dimensionless density of the problem; define as before the coordinates $\boldsymbol{\lambda}' = (r', z') \equiv (r, z/[1 + L_0]^{1/2})$. Solve the Newtonian problem $\nabla'_\lambda \varphi_N(\boldsymbol{\lambda}') = 4\pi\zeta[\boldsymbol{\lambda}(\boldsymbol{\lambda}')]$. Then take $U^r_0(\boldsymbol{\lambda}) = \partial\varphi_N/\partial r'(\boldsymbol{\lambda}')$; $U^z_0(\boldsymbol{\lambda}) = (1 + L_0)^{1/2} \;\; \partial\varphi_N/\partial z'(\boldsymbol{\lambda}') - {}_0\mu(\gamma_0)$. It is quite easy to check that $\boldsymbol{U}_0$ satisfies the divergence equation and has the same asymptotic behavior as that of the desired solution $\boldsymbol{U}$. Applying the curl iteration procedure to $\boldsymbol{U}_0$ yields $\boldsymbol{U}$.

How do we get the Newtonian solution which constitutes the initial conditions? The discretized initial value of the normal component of $\boldsymbol{U}$ on a cell boundary is calculated by integrating the exact Newtonian flux over the boundary and dividing by the area of the boundary. It is straightforward to obtain an analytic expression of the contribution to $\boldsymbol{U}$ from point masses on the z-axis or a set of homogeneous spheres or shells. The calculation of the contribution of a thin disk deserves a little elaboration.

The discretized disk in our scheme is made up of a set of concentric coplanar thin rings each of constant surface density between radii r_i and r_{i+1}; $i = 1, N_r$. Using results from Toomre (1963), we can write the Newtonian potential of a thin ring of unit surface density and radii $a < b$ as

$$\varphi_N(z, r) = \int_0^\infty S(k) J_0(kr) e^{-kz} dk \quad (z \geq 0) , \tag{23}$$

where

$$S(k) = -2\pi G \int_a^b J_0(kx) x dx , \tag{24}$$

and $J_0(x)$ is the Bessel function of order zero. We need flux integrals over cell boundary of the form $2\pi r \int_{z_1}^{z_2} \partial\varphi_N/\partial r \, dz$ and $2\pi \int_{r_1}^{r_2} r \partial\varphi_N/\partial z \, dr$. Using various formulae for Bessel functions and their integrals from Gradshteyn and Ryzhik (1965), we find that the triple flux integrals can be written in terms of single integrals of the form

$$Q(r, R, z) = 2\pi \int_0^\infty k^{-1} e^{-kz} J_1(kr) J_1(kR) dk . \tag{25}$$

This integral in turn can be written in a form better suited for numerical evaluation. Using an expression for $J_1(kr)J_1(kR)$ from Magnus and Oberhettinger (1943, p. 29) and integrating over k, we get

$$Q(r, R, z) = 2Rr \int_0^{\pi/2} \mathrm{Re}\,[f(\theta)] d\theta , \tag{26}$$

where $f(\theta) = [(z^2 + 2\cos\theta B)^{1/2} - z]^2/B^2$ and $B = Re^{i\theta} + re^{-i\theta}$.

h) Miscellaneous Remarks

1. The solution of the discrete problem depends only on the total mass placed within each cell and not on where exactly in the cell this mass is. Thus, for example, if we solve a problem with point masses on the z-axis, the solution will effectively smear out each point mass in the cell in which it falls. The only way to calculate the field of a well localized mass is to make the mesh cell which contains it as small as necessary.

2. The force between two point masses is calculated as a surface integral over a surface which surrounds any one of the two masses using formulae in Paper I. We have checked that the calculated value of the force on a mass does not depend on the integration surface we chose around that mass.

IV. SOME NUMERICAL RESULTS

Numerical solution of the field equation is the only means by which we have been able to approach many problems. The following are some examples of interest:

1) The law of the gravitational force between two masses;

2) The rotation curves of disk galaxies;

3) The field produced by a point mass in a constant external field, such as the field of the Sun with the solar neighborhood galactic field as a boundary condition. The dynamics of the long period comets, for example, depend strongly on these results.

We now discuss each of these problems in detail and give results of numerical calculations.

a) The Two-Body Force

We are interested in the gravitational force $F(m_1, m_2, r)$ between two point masses m_1 and $m_2 \le m_1$ at a distance r from each other. We can deduce the following facts about F from the field equation.

1. The dependence of F on one of the variables can be eliminated using the scaling law discussed in § II. We can write, for example, $F = m_1 m_2 Gr^{-2} f(m_2/m_1, r/r_t)$, where $r_t \equiv (MG/a_0)^{1/2}$ and $M \equiv m_1 + m_2$, and we only need find the dependence of the dimensionless function f on its two dimensionless variables, the mass ratio $q \equiv m_2/m_1$ and $\lambda = r/t_t$.

2. In the limit $\lambda \ll 1$ the Newtonian expression for the force approximately holds, so $f(q, \lambda \ll 1) \approx 1$.

3. When $q \ll 1$, we may consider m_2 a test particle in the field $g_1(r)$ of m_1 so that $F = m_2 g_1(r)$ and $g_1(r) = a_0 I^{-1}(m_1 G/a_o r^2)$ [the field g of a point mass m is given by $\mu(g/a_0)g = mGr^{-2}$] from which we get for f in this limit $f(q \ll 1, \lambda) \approx \lambda^2 I^{-1}(\lambda^{-2})$.

4. In the limit $\lambda \gg 1$, we are in the low acceleration limit and can use the additional scaling relation discussed in § II (although for point masses the acceleration is never small very near the masses, we may consider the two masses to be of a size, l, very small compared with their separation r, yet large enough ($l \gg r_t$) so we can assume $g \ll a_0$ everywhere, without changing the force law). Thus $F \approx h(q) M a_0 \lambda^{-1}$ or $f(q, \lambda \gg 1) \approx A(q)\lambda$. From the above form of f for $q \ll 1$, we find $A(q) \xrightarrow{q \to 0} 1$. The function $A(q)$ was determined numerically and found to be of order 1 for all values of $q[A(1) \approx 0.8$, $A(0.25) \approx 0.85$, $A(0.125) \approx 0.88]$. The limit $\lambda \gg 1$ is applicable, for example, to binary galaxies for which the separation satisfies $r \gg r_t$ in most cases ($r_t \approx 10$ kpc for the Milky Way). Thus one does not err by more than 20% if one uses for the force between two galaxies the expression $F = m_1 m_2 (m_1 + m_2)^{-1/2} (Ga_0)^{1/2} r^{-1}$.

We have computed $f(q, \lambda)$ for many values of q and λ [with $\mu(x) = x/(1 + x)$]. Since for $\lambda \ll 1$ f is independent of q and we found that for $\lambda \gg 1$ f depends only weakly on q (we have not been able to show this analytically yet), it is not surprising to find that $f(q, \lambda)$ is only weakly dependent on q for all values of λ. In Figure 3 we show $f(q, \lambda)$ for $q = 1$ and for $q = \frac{1}{8}$. For the grid we used we estimate the maximum error in F to be $\sim 5\%$ (comparing computed results for the Newtonian force with known values, and so on).

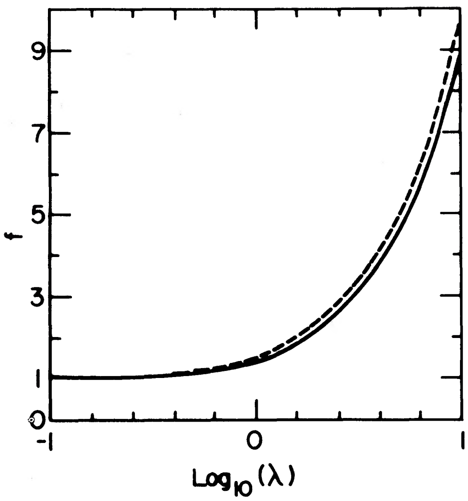

FIG. 3.—Dimensionless force between two masses as a function of the dimensionless distance λ, calculated for $q = 1$ (*solid line*) and for $q = \frac{1}{8}$ (*dashed line*).

b) Rotation Curves of Disk Galaxies

An extensive study of rotation curves according to MOND was presented in Milgrom (1983*b*). Those were calculated using the phenomenological equation for the acceleration field of a galaxy. Obtaining the rotation curves from the Lagrangian theory requires numerical computations. We have calculated such curves for galaxies with different disk surface density laws, disk to bulge mass ratios, dimensionless disk scale lengths, using a few forms for $\mu(x)$, and so on.

The main lesson we have learned from these calculations is that the rotation curve obtained from the Lagrangian theory is very nearly the same as that obtained from the archetype formula. For reasons that we only partly understand, the relative difference in velocities derived in these two ways is in general much smaller than, but practically never larger than 5%. This result is of great practical importance since it allows us to calculate rotation curves safely from the phenomenological equation which is algebraic, straightforward to use, and orders of magnitude quicker.

We present in Figure 4 the results for a few models. For each galaxy model (all those given in Fig. 4 consist of an exponential disk and an homogeneous spherical bulge) we use the numerical code to solve for the acceleration in the disk's plane in two ways: one using some form of $\mu(x)$ with MOND, and another time obtaining the Newtonian field $\boldsymbol{g}_N$ with our numerical code and then obtaining the MOND acceleration $\boldsymbol{g}$ via $\mu(g/a_0)\boldsymbol{g} = \boldsymbol{g}_N$. Both results are obtained with the same code using identical mesh and galaxy mass discretization, to minimize differences due to numerical details. On the scale used in Figue 4 the curves obtained in these two ways are practically indistinguishable.

c) A Point Mass in a Constant External Field

This problem involves just one dimensionless parameter, viz, the (dimensionless) value of the external field $\gamma_\infty(\equiv g_\infty/a_0)$. An important problem for which such a solution is needed is the dynamics of the long-period comet cloud proposed by Oort (see Milgrom 1983*a*). One is after the solution $\psi(\boldsymbol{\lambda}, \gamma_\infty)$ of the dimensionless field equation, with a source $\zeta(\boldsymbol{\lambda}) = \delta^3(\boldsymbol{\lambda})$, satisfying $\nabla_\lambda \psi \xrightarrow{\lambda \to \infty} -\gamma_\infty \boldsymbol{e}_z$ ($\boldsymbol{e}_z$ is a unit vector in the z-direction). We are, in fact, interested not in ψ itself but in the internal field which describes the acceleration of a test particle relative to the point mass (which itself falls in the external field with acceleration γ_∞). Thus define the potential $\eta(\boldsymbol{\lambda}, \gamma_\infty)$ such that $\nabla_\lambda \eta = \nabla_\lambda \psi + \gamma_\infty \boldsymbol{e}_z$. From results given in Paper I we have $\eta \xrightarrow{\lambda \to \infty} \mu_0^{-1}[\lambda_x^2 + \lambda_y^2 + \lambda_z^2/(1 + L_0)]^{-1/2} + C$, where $\mu_0 =$

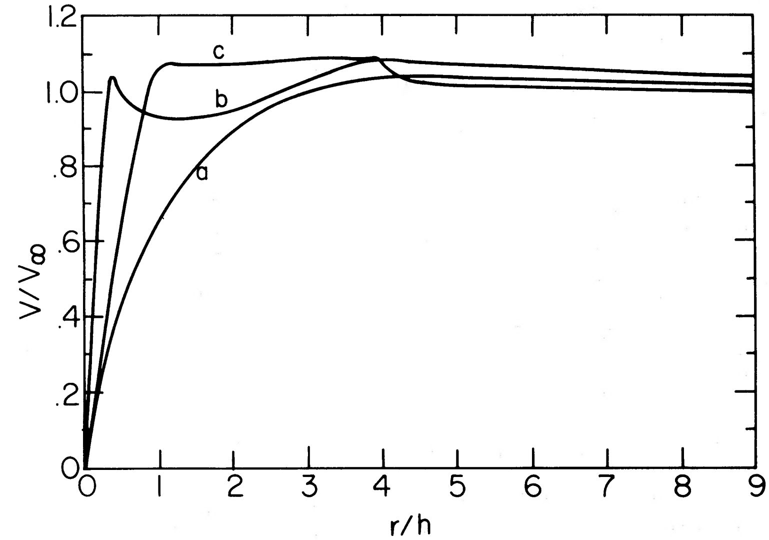

FIG. 4.—Rotation curves for disk galaxies (exponential disk of scale h with a homogeneous spherical bulge of radius r_s and fractional mass α_s) computed from the field equation. V is given in units of V_∞ and r in units of h. (*a*) Very low surface density pure disk ($\xi \equiv V_\infty^2/h \ll 1$) with $\mu = x/(1+x)$. (*b*) Disk and sphere with $\xi = 2$, $\alpha_s = 0.2$, $r_s/h = 0.4$. The disk is cut off beyond $4h$, and $\mu(x) = 1$ for $x > 1$ and $\mu(x) = x$ for $x \le 1$. (*c*) As in (*b*) with $\xi = 1$, $\alpha_s = 0.5$, $r_s/h = 1$, no cutoff in the disk, and $\mu = x/(1+x)$.

$\mu(\gamma_\infty)$, $L_0 = L(\gamma_\infty)$. The asymptotic form is a good approximation as long as $\lambda^{-2} \ll \mu_0 \gamma_\infty$. Thus the asymptotic equipotential surfaces are ellipsoids of revolution with their major axis in the direction of the external field. We find by numerical calculations that, in general, the field is not reflection symmetric about the $z = 0$ plane. For example, we find for $\gamma_\infty = 1$ a difference of $\sim 10\%$ in the potentials at $\lambda_z = 1$ and $\lambda_z = -1$ for $\lambda_x = \lambda_y = 0$. Thus the potential field which the comets in the Oort cloud see is axisymmetric about the galactic radius of the Sun but somewhat asymmetric to reflection in a plane perpendicular to the radius.

When we are very near the point mass, i.e., when $\lambda \to 0$, η is very nearly Newtonian, and we can write $\eta(\lambda, \gamma_\infty) \overset{\lambda \to 0}{\approx} -\lambda^{-1} - \alpha(\gamma_\infty)$. [Written in dimensional form this relation takes the form $\varphi(r, \gamma_\infty) \overset{r \to 0}{\approx} -MGr^{-1} - (MGa_0)^{1/2}\alpha(\gamma_\infty)$]. The $\boldsymbol{r}$-independent parameter $\alpha(\gamma_\infty)$ is not a free constant which can be subtracted from the potential (as we normalized η to vanish at infinity) but is a meaurable quantity. We have $\alpha(\gamma_\infty) \xrightarrow{\gamma_\infty \to \infty} 0$ and $\alpha(\gamma_\infty) \xrightarrow{\gamma_\infty \to 0} \infty$ (since $\gamma_\infty = 0$ corresponds to no external field, and η then diverges logarithmically at infinity).

Thus, the effect of MOND on our analysis of test particle motion near a point mass in an external field is to introduce a constant term in the potential energy. The Newtonian expression for the energy, per unit mass, of a particle $\epsilon_N = v^2/2 - \lambda^{-1}$ (defined such that it vanishes for a particle at rest at infinity) is to be replaced near the point mass by $\epsilon = v^2/2 - \lambda^{-1} - \alpha(\gamma_\infty)$. Thus, for example, a long-period comet which is observed near the Sun is bound if it has $v^2/2 - \lambda^{-1} - \alpha(\gamma_\infty) < 0$ [or in dimensional form $V^2/2 - M_\odot G r^{-1} - (M_\odot G a_0)^{1/2}\alpha(\gamma_\infty) < 0$].

For various reasons, computing α is a great strain on the code. We determined approximate values of α numerically for a few values of γ_∞ and find, for example [with $\mu(x) = x/(1+x)$], $\alpha(0.5) \approx 2.0$, $\alpha(1) \approx 1.5$. Thus if we adopt $\gamma_\infty = 1$ for the galactic field at the Sun's position (see Milgrom 1983*b*) we get that a marginally bound comet will appear to have a *positive* Newtonian energy (per unit mass) near the Sun equal to $\sim 1.5(M_\odot G a_0)^{1/2} \approx 0.25(a_0/2 \times 10^{-8}\ \mathrm{cm\ s^{-2}})^{1/2}\ \mathrm{km^2\ s^{-2}}$.

V. SUMMARY

Exact solutions of the MOND Lagrangian theory have, so far, been found only for simple systems. To answer many of the questions which arise in connection with applications of MOND we have developed a numerical code to solve the field equation for source distributions which are axisymmetric. Most of the problems of interest in astronomy are axisymmetric.

A second avenue we have followed involves the extraction of general properties of the solutions from the field equation itself. In addition to those discussed in Paper I, we describe in the present paper the following results.

1. We demonstrate that the boundary values of interest at infinity determine the solutions of the field equation uniquely. We show that for a surface surrounding a finite volume it is the normal component of the "displacement" vector $\boldsymbol{u} = \mu(\nabla\varphi/a_0)\nabla\varphi$ which has to be given on the surface (which involves both components of $\nabla\varphi$). This finding is of particular importance in setting a numerical code.

2. Similarity laws which relate the solutions of related mass distributions yield many useful results concerning the explicit dependence of the solutions on various systems parameters. These similarity laws are particularly powerful for systems with accelerators much smaller than a_0. As is always the case, the existence of similarity relation allow us great savings in computing.

3. We deduce the asymptotic dependence on radius of the different angular multipoles of the potential for an arbitrary bound system.

4. We present a perturbation formalism by which one can solve the field equation approximately when the mass distribution involves a small perturbation on a configuration for which the solution is known. The unperturbed system can be one with one-dimensional symmetry for which the field equation can be solved exactly. For example, when we consider a small perturbation on the approximately constant field of a mother system the perturbation equation can be solved exactly. This

approximation is a good description of many systems of interest such as low surface density open clusters, very wide binaries, and dwarf spheroidals in the field of a parent galaxy as well as any gravitating system in a laboratory in the Earth's field.

The dynamics of such systems demonstrate explicitly the breakdown of the strong equivalence principle in that clear-cut effects of the presence of the *constant* external field are manifested in the internal structure and dynamics of the subsystem.

We have also presented analytic and numerical results for three problems, viz., the dependence of the two-body force on the two masses and the distance between them, a point mass in a constant external field, and the rotation curves of disk galaxies.

This research is a direct outgrowth of work I did with Jacob Bekenstein on the MOND Lagrangian theory. I thank him for many discussions we had on various subjects described in this paper. I also thank Tzvi Piran for discussions on various concepts connected with the numerical code.

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