Homework Set 3

Due date: February 17, 2016 after class

Problem 1: Consider a rod of diameter D = 4 mm with thermal conductivity k = 50 W/m·K, and length 2L where L = 100 mm. The rod is perfectly insulated over one portion of its length $-L \le x \le 0$, and experiences convection with a fluid (T_∞ = 20°C, h = 500 W/m²·K) over the other portion of its length $0 \le x \le L$. A constant heat flux of q"(x = -L) = 500 k W/m² is introduced at one end while the other end ($x = L$) is separated from a heat source operating at T = 100 °C by an interfacial thermal contact resistance $R''_{t,c} = 10 \times 10^{-3}$ m²·K/W. Determine the following:

a) the temperature at x = 0 and plot the temperature as a function of x from $-L \le x \le L$ using a plotting program (e.g., MatLAB)

b) The total heat dissipated to the environment via convective heat transfer.

From conversation of energy: $\dot{E}_{\text{in}} = \dot{E}_{\text{out}}$

$$-k\left.\frac{\partial T}{\partial x}\right|_{x=-L} = q'' = -k\left.\frac{\partial T}{\partial x}\right|_{x=0}$$

$$-k\left.\frac{\partial T}{\partial x}\right|_{x=L} = \frac{T(L)-T_3}{R''_{t,c}}$$

Energy balance for the uninsulated rod:

$$q_x - q_{x+\Delta x} - A_S h\left(T-T_\infty\right) = 0$$

$$A_C = \frac{\pi D^2}{4} \text{ and } A_S = \pi D \Delta x$$

$$q_{x+\Delta x} = q_x + \frac{\partial q_x}{\partial x}\Delta x + H.O.T \Rightarrow q_{x+\Delta x} - q_x = \frac{\partial q_x}{\partial x}\Delta x$$

$$-\frac{\partial q_x}{\partial x}\Delta x - A_S h\left(T-T_\infty\right) = 0 = -\frac{\partial q_x}{\partial x}\Delta x - \pi D h \Delta x\left(T-T_\infty\right) \Rightarrow -\frac{\partial q_x}{\partial x} - \pi D h\left(T-T_\infty\right) = 0$$

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$$q_x = -\frac{\pi D^2}{4}k\frac{dT}{dx} \Rightarrow \frac{\pi D^2}{4}\frac{\partial}{\partial x}\left[k\frac{dT}{dx}\right] - \pi D h\left(T-T_\infty\right)$$

$$\frac{d^2T}{dx^2} - \frac{4h}{Dk}\left(T-T_\infty\right) = 0$$

Let $\theta = T - T_\infty, m^2 = \dfrac{4h}{Dk}$

$$\frac{d^2\theta}{dx^2} - m^2\theta = 0 \Rightarrow \theta(x) = C_1\sinh\left(mx\right) + C_2\cosh\left(mx\right)$$

Note this can also be represented as: $\dfrac{d^2\theta}{dx^2} - m^2\theta = 0 \Rightarrow \theta(x) = D_1\exp\left(-mx\right) + D_2\exp\left(mx\right)$

which results in different coefficients but the same temperature profile and answers

$$q''_{x=-L} = q''_{x=0} = -k\left.\frac{dT}{dx}\right|_{x=0} = -k\left.\frac{d\theta}{dx}\right|_{x=0} = q''_{\text{given}} = 500{,}000\ \text{W/m}^2 \Rightarrow \left.\frac{d\theta}{dx}\right|_{x=0} = -10{,}000\ \frac{\text{K}}{\text{m}}$$

$$\left.\frac{d\theta}{dx}\right|_{x=0} = -10{,}000\ \frac{\text{K}}{\text{m}} = \text{m}C_1 \cancel{\cosh(0)}^{1} + \text{m}C_2 \cancel{\sinh(0)}^{0} \Rightarrow C_1 = \frac{-10{,}000}{m} = \text{-100}$$

$$\theta(L) = \frac{-10{,}000}{m}\sinh(mL) + C_2\cosh(mL) \text{ and } \left.\frac{d\theta}{dx}\right|_{x=L} = -10{,}000\cosh(mL) + mC_2\sinh(mL)$$

$$-k\left.\frac{d\theta}{dx}\right|_{x=L} = \frac{\theta(x=L) + T_\infty - T_3}{R''_c}$$

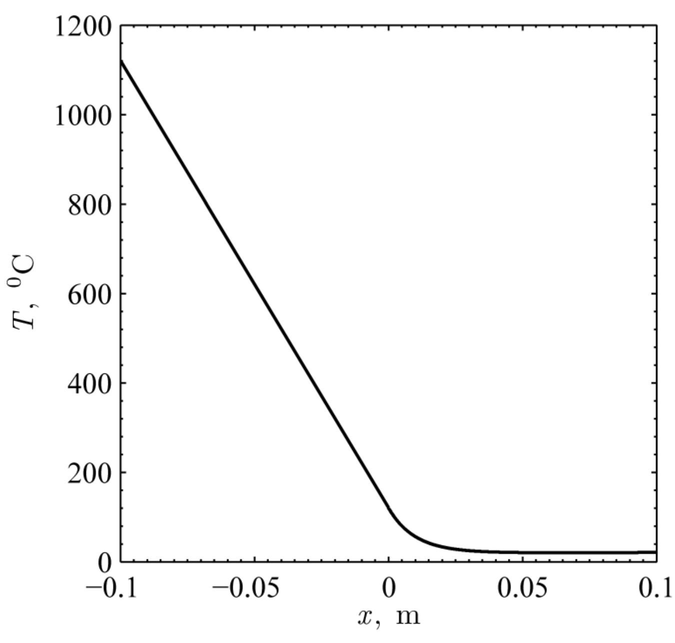
$$-kR''_c\left[-10{,}000\cosh(mL) + mC_2\sinh(mL)\right] = \frac{-10{,}000}{m}\sinh(mL) + C_2\cosh(mL) - T_3 + T_\infty$$

$$C_2 = \frac{\frac{-10{,}000}{m}\sinh(mL) - T_3 + T_\infty - 10{,}000kR''_c\cosh(mL)}{-mkR''_c\sinh(mL) - \cosh(mL)} = 100$$

For $-L \le x \le 0$: $q''_{-L} = -k\left.\frac{dT}{dx}\right|_{x=-L} = -k\frac{T(0) - T(x)}{L} \Rightarrow T(x) = T(0) + \frac{Lq''_{-L}}{k}$

For $0 \le x \le L$: $T(x) = T_\infty + C_1\sinh(mx) + C_2\cosh(mx)$

$T(x=0) = 120°C$



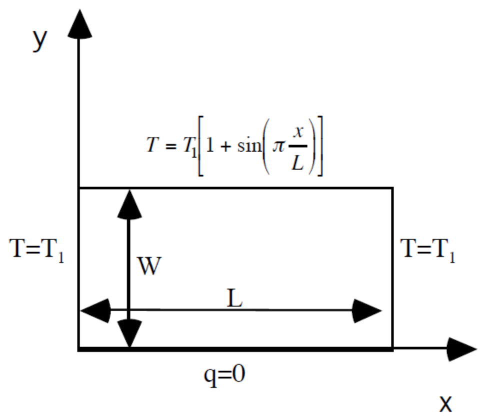
$$q_{\text{conv}} = \int_0^L h\theta(x)\,dx = \int_0^L \pi Dh\left[C_1\sinh(mx) + C_2\cosh(mx)\right]dx = \frac{h\pi D}{m}\left[C_1\cosh(mx) + C_2\sinh(mx)\right]_0^L$$

$$q_{\text{conv}} = \frac{h\pi D}{m}\left[C_1\left(\cosh(mL) - 1\right) + C_2\sinh(mL)\right] = 6.38\ \text{W}$$

$$q_{\text{conv}} = \frac{\pi}{4}\text{D}^2\left[q''_{x=0} + \frac{T_3 - T(x=L)}{R''_c}\right] = 6.38\text{ W}$$

Problem 2: Analytically solve for the temperature distribution in a rectangle with $L \times W$ where the left and right surfaces are maintained at a constant temperature, T_1, the bottom surface is well-insulated, and the top side temperature is given by

$$T = T_1\left[1 + \sin\left(\pi\frac{x}{L}\right)\right]$$



$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0 \Rightarrow \frac{\partial^2 \theta^*}{\partial x^{*,2}} + \frac{\partial^2 \theta^*}{\partial y^{*,2}}$$

$$\theta^* = \frac{T - T_1}{T_1}, x^* = \frac{x}{L}, y^* = \frac{y}{L}, W^* = \frac{W}{L}$$

Boundary Conditions:

$\theta_1^* = 0$ for $x^* = 0$ and 1

$\left.\frac{\partial \theta^*}{\partial y^*}\right|_{y^*=0} = 0$ and $\theta^* = \sin(\pi x^*)$ at $y^* = W^*$

$$\theta^* = X(x)Y(y) \Rightarrow Y\frac{\partial^2 X}{\partial x^2} + X\frac{\partial^2 Y}{\partial y^2} = 0$$

$$\frac{\partial^2 X}{X\partial x^2} = -\frac{\partial^2 Y}{Y\partial y^2} = -\lambda^2$$

$$\frac{\partial^2 Y}{\partial y^2} - \lambda^2 Y = 0 \Rightarrow Y = C_1\sinh(\lambda y^*) + C_2\cosh(\lambda y^*)$$

$$\frac{\partial^2 X}{\partial x^2} - \lambda^2 X = 0 \Rightarrow X = C_3\sin(\lambda x^*) + C_4\cos(\lambda x^*)$$

$$\theta^* = \left[C_1\sinh(\lambda y^*) + C_2\cosh(\lambda y^*)\right]\left[C_3\sin(\lambda x^*) + C_4\cos(\lambda x^*)\right]$$

$$0=\left[C_1\sinh(\lambda y)+C_2\cosh(\lambda y^*)\right]\left[C_3\sin(0)+C_4\cancel{\cos(0)}^{1}\right]\Rightarrow C_4=0$$

$$0=\left[\lambda C_1\cosh(0)+\lambda C_2\sinh(0)\right]C_3\sin(\lambda y^*)\Rightarrow C_1=0$$

$$\theta^*=C_5\cosh(\lambda y^*)\sin(\lambda x^*)$$

$$0=C_5\cosh(\lambda y^*)\sin(\lambda)=\lambda=0,\pi,\ldots,n\pi\quad \text{for n = 1,2,3, ... ,}\infty$$

$$\theta^*=\sum_{n=1}^{\infty}C_n\cosh(n\pi y^*)\sin(n\pi x^*)$$

$$\sin(\pi x^*)=\sum_{n=1}^{\infty}C_n\cosh(n\pi W^*)\sin(n\pi x^*)\Rightarrow C_n=0\ \text{when n>1}$$

$$C_1=\frac{1}{\cosh(\pi)}$$

$$\theta^*=\frac{\cosh(\pi y^*)}{\cosh(\pi W^*)}\sin(\pi x^*)$$