HW 3 solutions

Pb. 4.8

1 Problem 4.8

1.1 Decomposition of $y(t)$ into Fourier series

$y(t)$ is the following function of time:

$$\begin{aligned} y(t) &= \frac{Y}{\tau}t \\ y(t+\tau) &= y(t) \end{aligned} \tag{1}$$

Because j is periodic with a period τ,

$$y(t) = \frac{a_0}{2} + \sum_{j=1}^{\infty} a_j \cos(j\omega_0 t) + \sum_{j=1}^{\infty} b_j \sin(j\omega_0 t) \tag{2}$$

where $\omega_0 = \frac{2\pi}{\tau}$. The Fourier series coefficient of $y(t)$, a_j ($j = 0,\ldots,+\infty$) and b_j ($j = 1,\ldots,+\infty$), are given by:

$$\begin{aligned} a_j &= \frac{2}{\tau}\int_0^{\tau} y(t)\cos(j\omega_0 t)dt \\ &= \frac{2}{\tau}\int_0^{\tau} \frac{Y}{\tau} t\cos(j\omega_0 t)dt \\ &= \frac{2Y}{\tau^2}\int_0^{\tau} t\cos(j\omega_0 t)dt \end{aligned} \tag{3}$$

For $j = 0$:

$$\begin{aligned} a_0 &= \frac{2Y}{\tau^2}\int_0^{\tau} tdt \\ &= \frac{2Y}{\tau^2}\frac{\tau^2}{2} \\ &= Y \end{aligned} \tag{4}$$

For $j \neq 0$, integration by part (let $u = t$, $v' = \cos(j\omega_0 t)$)

$$\begin{aligned} a_j &= \frac{2Y}{\tau^2}\Big[\frac{t}{j\omega_0}\sin(j\omega_0 t)\Big|_0^{\tau} - \int_0^{\tau}\frac{1}{j\omega_0}\sin(j\omega_0 t)dt\Big] \\ &= \frac{2Y}{\tau^2}\Big[0 + \Big(\frac{1}{j\omega_0}\Big)^2\cos(j\omega_0 t)\Big|_0^{\tau}\Big] \\ &= \frac{2Y}{\tau^2}\Big(\frac{\tau}{j2\pi}\Big)^2\Big[\cos(j2\pi) - \cos 0\Big] \\ &= 0 \end{aligned} \tag{5}$$

$$\begin{aligned} b_j &= \frac{2}{\tau}\int_0^{\tau} y(t)\sin(j\omega_0 t)dt \\ &= \frac{2}{\tau}\int_0^{\tau} \frac{Y}{\tau} t\sin(j\omega_0 t)dt \\ &= \frac{2Y}{\tau^2}\int_0^{\tau} t\sin(j\omega_0 t)dt \end{aligned} \tag{6}$$

Integration by part (let $u = t$, $v' = \sin(j\omega_0 t)$)

$$\begin{aligned} b_j &= \frac{2Y}{\tau^2}\Big[-\frac{t}{j\omega_0}\cos(j\omega_0 t)\Big|_0^\tau + \int_0^\tau \frac{1}{j\omega_0}\cos(j\omega_0 t)dt\Big] \\ &= \frac{2Y}{\tau^2}\Big[-\frac{\tau}{j\omega_0}\cos(2\pi j) + \Big(\frac{1}{j\omega_0}\Big)^2 \sin(j\omega_0 t)\Big|_0^\tau\Big] \\ &= \frac{2Y}{\tau^2}\Big[-\frac{\tau^2}{j2\pi} + 0\Big] \\ &= -\frac{Y}{j\pi} \end{aligned} \tag{7}$$

such that:

$$y(t) = \frac{Y}{2} - \sum_{j=1}^{\infty} \frac{Y}{j\pi}\sin(j\omega_0 t) \tag{8}$$

which can be written as a complex Fourier series:

$$y(t) = \frac{Y}{2} + \sum_{j=1}^{\infty} \frac{iY}{2j\pi} e^{ij\omega_0 t} + \sum_{j=-\infty}^{-1} \frac{-iY}{2j\pi} e^{-ij\omega_0 t} \tag{9}$$

1.2 Equation of motion

Applying Newton's 2nd law:

$$m\ddot{x} + c\dot{x} + kx = c\dot{y} + ky \tag{10}$$

1.3 Solution

The steady-state solution is written as a complex Fourier series:

$$x(t) = \sum_{j=-\infty}^{\infty} X_j e^{ij\omega_0 t} \tag{11}$$

Value of X_0:

$$kX_0 = k\frac{Y}{2} \tag{12}$$

such that $X_0 = \frac{Y}{2}$.

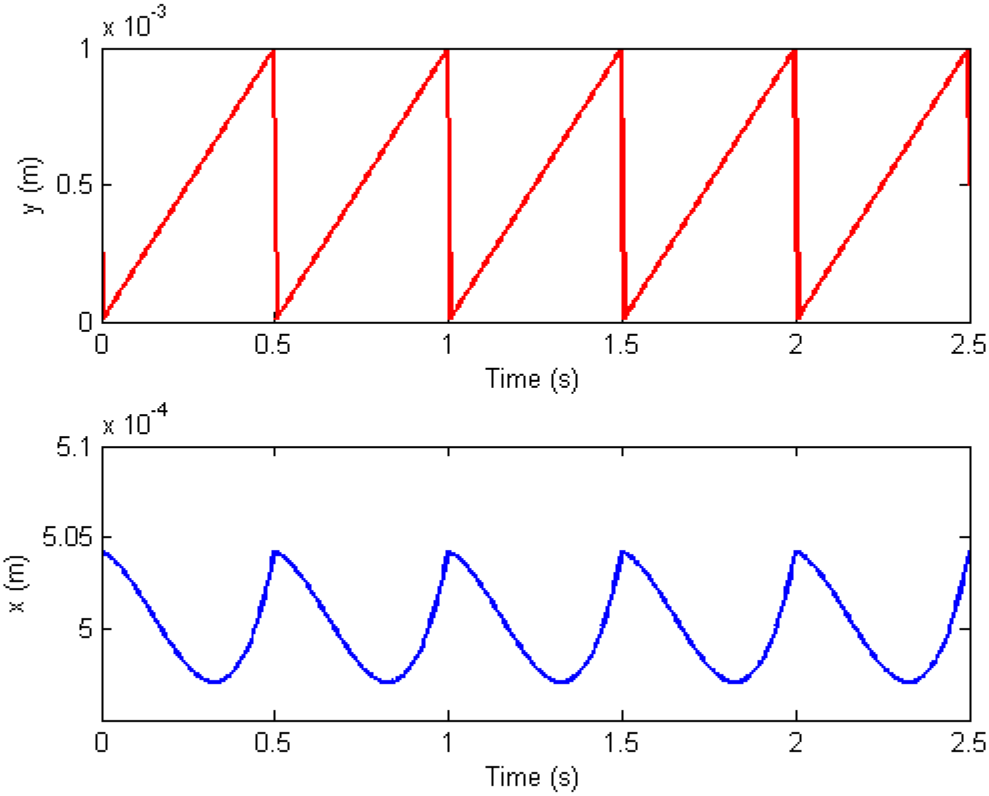
Value of X_j (for $j > 0$)

$$\Big[-m(j\omega_0)^2 + Cij\omega_0 + k\Big]X_j = \frac{-iY}{2j\pi}\Big[cij\omega_0 + k\Big] \tag{13}$$

$$X_j = \frac{-iY}{2j\pi}\frac{cij\omega_0 + k}{k - m(j\omega_0)^2 + Cij\omega_0} \tag{14}$$

$$X_j = \frac{-iY}{2j\pi}\frac{2i\xi\frac{j\omega_0}{\omega_n} + 1}{1 - \Big(\frac{j\omega_0}{\omega_n}\Big)^2 + 2i\xi\frac{j\omega_0}{\omega_n}} \tag{15}$$

And for $j < 0$, $X_j = X^*_{-j}$


x 10^-3
1
0.5
0
y (m)
0
0.5
1
1.5
2
2.5
Time (s)
x 10^-4
5.1
5.05
5
x (m)
0
0.5
1
1.5
2
2.5
Time (s)

Pb. 4.36

4.36 (a) Unit impulse response function for undamped case:

Use $\zeta = 0$ and $\omega_d = \omega_n$ in Eq. (4.25):

$$x(t) = \frac{1}{m\,\omega_n} \sin \omega_n t \qquad (E.1)$$

(b) Unit impulse response function for underdamped case: Eq. (4.25):

$$x(t) = \frac{1}{m\,\omega_d}\, e^{-\zeta \omega_n t} \sin \omega_d t \qquad (E.2)$$

(c) Unit impulse response function for critically damped case:

Free vibration response of a critically damped system is given by Eq. (2.80):

$$x(t) = \left\{ x_0 + (\dot{x}_0 + \omega_n x_0)\, t \right\} e^{-\omega_n t} \qquad (E.3)$$

Using the initial conditions $x_0 = 0$ and $\dot{x}_0 = \frac{1}{m}$, Eq. (E.3) gives

$$x(t) = \frac{t}{m}\, e^{-\omega_n t} \qquad (E.4)$$

(d) Unit impulse response function for an over damped case:

Free vibration response of an overdamped system is given by Eq. (2.81):

$$x(t) = C_1\, e^{\left(-\zeta + \sqrt{\zeta^2 - 1}\right)\omega_n t} + C_2\, e^{\left(-\zeta - \sqrt{\zeta^2-1}\right)\omega_n t} \qquad (E.5)$$

where

$$C_1 = \frac{x_0\, \omega_n \left(\zeta + \sqrt{\zeta^2 - 1}\right) \ \dot{x}_0}{2\,\omega_n \sqrt{\zeta^2 - 1}} \; ; \quad C_2 = \frac{-x_0\, \omega_n \left(\zeta - \sqrt{\zeta^2-1}\right) - \dot{x}_0}{2\,\omega_n \sqrt{\zeta^2 - 1}}$$

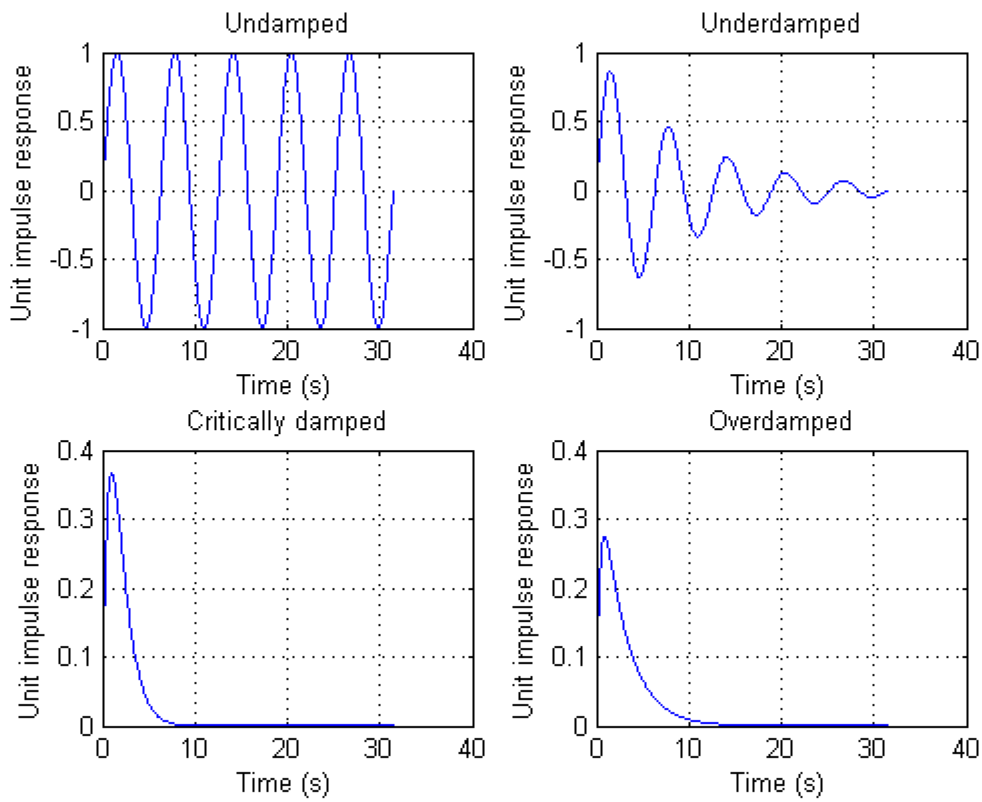
For the initial conditions $x_0 = 0$ and $\dot{x}_0 = \frac{1}{m}$,

C_1 and C_2 become

$$C_1 = \frac{1}{2 m \omega_n \sqrt{\zeta^2 - 1}} \quad ; \quad C_2 = - \frac{1}{2 m \omega_n \sqrt{\zeta^2 - 1}}$$

and hence Eq. (E.5) yields

$$x(t) = \frac{-1}{2 m \omega_n \sqrt{\zeta^2 - 1}} \left\{ e^{-\left(\zeta + \sqrt{\zeta^2 - 1}\right) \omega_n t} - e^{-\left(\zeta - \sqrt{\zeta^2 - 1}\right) \omega_n t} \right\} \tag{E.6}$$



Pb. 4.17

4.17 From Eq. (4.31), $x(t) = \frac{1}{m\omega_d}\int_0^t F_0\, e^{-\alpha\tau}\, e^{-\zeta\omega_n(t-\tau)} \sin\omega_d(t-\tau)\, d\tau$

$$= \frac{F_0}{m\omega_d} e^{-\zeta\omega_n t}\int_0^t e^{-(\alpha-\zeta\omega_n)\tau}\sin\omega_d(t-\tau)\, d\tau$$

$$x(t) = \frac{F_0 e^{-\zeta\omega_n t}}{m\omega_d}\int_0^t e^{-(\alpha-\zeta\omega_n)\tau}\cdot\left\{\sin\omega_d t\cdot\cos\omega_d\tau - \cos\omega_d t\cdot\sin\omega_d\tau\right\} d\tau$$

$$= \frac{F_0 e^{-\zeta\omega_n t}}{m\omega_d}\sin\omega_d t\left[\frac{e^{-(\alpha-\zeta\omega_n)\tau}}{(\alpha-\zeta\omega_n)^2+\omega_d^2}\left\{-(\alpha-\zeta\omega_n)\cos\omega_d\tau + \omega_d\sin\omega_d\tau\right\}\right]_0^t$$

$$- \frac{F_0 e^{-\zeta\omega_n t}}{m\omega_d}\cos\omega_d t\left[\frac{e^{-(\alpha-\zeta\omega_n)\tau}}{(\alpha-\zeta\omega_n)^2+\omega_d^2}\left\{-(\alpha-\zeta\omega_n)\sin\omega_d\tau - \omega_d\cos\omega_d\tau\right\}\right]_0^t$$

$$= \frac{F_0 e^{-\zeta\omega_n t}}{m\omega_d\left[(\alpha-\zeta\omega_n)^2+\omega_d^2\right]}\left\{\omega_d\, e^{-(\alpha-\zeta\omega_n)t} + (\alpha-\zeta\omega_n)\sin\omega_d t - \omega_d\cos\omega_d t\right\}$$

$$= \frac{F_0 e^{-\alpha t}}{m\left[(\alpha-\zeta\omega_n)^2+\omega_d^2\right]} + \frac{F_0 e^{-\zeta\omega_n t}}{m\omega_d\left[(\alpha-\zeta\omega_n)^2+\omega_d^2\right]}\sin(\omega_d t - \phi)$$

where $\phi = \tan^{-1}\left(\frac{\omega_d}{\alpha-\zeta\omega_n}\right)$.