Homework assignment #8 – Solutions
Due on Tuesday 04/26/2016

- Problem 8.4

8.4

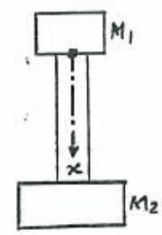
$P = 30000 \text{ N}, \quad \rho = 2 \text{ kg/m}$

$c = \left(\frac{P}{\rho}\right)^{1/2} = \left(30000/2\right)^{1/2} = 122.4745 \text{ m/s}$

$\text{Time taken} = \frac{300}{122.4745} = 2.4495 \text{ s}$

- Problem 8.17

8.17



(a) $u(x,t) = \left(A \cos\frac{\omega x}{c} + B \sin\frac{\omega x}{c}\right)\left(C \cos\omega t + D \sin\omega t\right)$

At $x=0$: $M_1 \frac{\partial^2 u}{\partial t^2} = AE \frac{\partial u}{\partial x}$

$-\omega^2 M_1 A = AE \frac{\omega}{c} B \Rightarrow B = -\left(\frac{\omega M_1 c}{AE}\right) A$

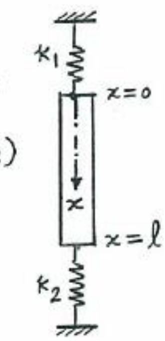
At $x = l$: $M_2 \frac{\partial^2 u}{\partial t^2} = -AE \frac{\partial u}{\partial x}$

$-\omega^2 M_2 \left(A \cos\frac{\omega l}{c} + B \sin\frac{\omega l}{c}\right) = -AE \frac{\omega}{c}\left(-A \sin\frac{\omega l}{c} + B \cos\frac{\omega l}{c}\right)$

i.e. $A\left(-\omega^2 M_2 \cos\frac{\omega l}{c} - AE\frac{\omega}{c} \sin\frac{\omega l}{c}\right) = B\left(-\frac{AE\omega}{c} \cos\frac{\omega l}{c} + \omega^2 M_2 \sin\frac{\omega l}{c}\right)$

i.e. $\omega^2 M_2 \cos\frac{\omega l}{c} + \frac{AE\omega}{c} \sin\frac{\omega l}{c} + \frac{\omega M_1 c}{AE}\left(-\omega^2 M_2 \sin\frac{\omega l}{c} + \frac{AE\omega}{c} \cos\frac{\omega l}{c}\right) = 0$

This is the frequency equation.



(b) $u(x,t) = \left(A \cos\frac{\omega x}{c} + B \sin\frac{\omega x}{c}\right)\left(C \cos\omega t + D \sin\omega t\right)$ --- (E_1)

At $x=0$, $k_1 u = AE \frac{\partial u}{\partial x} \Rightarrow B = \frac{k_1 c}{AE\omega} A$ --- (E_2)

At $x = l$, $k_2 u = -AE \frac{\partial u}{\partial x}$

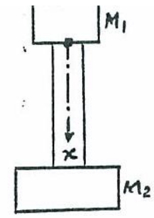
$\Rightarrow k_2\left(A \cos\frac{\omega l}{c} + B \sin\frac{\omega l}{c}\right) = -AE\frac{\omega}{c}\left\{-A \sin\frac{\omega l}{c} + B \cos\frac{\omega l}{c}\right\}$ --- (E_3)

Substituting (E_2) into (E_3), we get

$A\left[\left(k_2 - \frac{k_1 c}{AE\omega}\right)\cos\frac{\omega l}{c} + \left(\frac{k_1 k_2 c}{AE\omega} - \frac{AE\omega}{c}\right)\sin\frac{\omega l}{c}\right] = 0$

- Problem 8.20

8.17 (a) $u(x,t) = \left(\underset{\sim}{A} \cos \frac{\omega x}{c} + \underset{\sim}{B} \sin \frac{\omega x}{c}\right)\left(C \cos \omega t + D \sin \omega t\right)$



At $x=0$: $\quad M_1 \frac{\partial^2 u}{\partial t^2} = AE \frac{\partial u}{\partial x}$

$$-\omega^2 M_1 \underset{\sim}{A} = AE \frac{\omega}{c} \underset{\sim}{B} \quad \Rightarrow \quad \underset{\sim}{B} = -\left(\frac{\omega M_1 c}{AE}\right) \underset{\sim}{A}$$

At $x=\ell$: $\quad M_2 \frac{\partial^2 u}{\partial t^2} = -AE \frac{\partial u}{\partial x}$

$$-\omega^2 M_2 \left(\underset{\sim}{A} \cos \frac{\omega \ell}{c} + \underset{\sim}{B} \sin \frac{\omega \ell}{c}\right) = -AE \frac{\omega}{c}\left(-\underset{\sim}{A} \sin \frac{\omega \ell}{c} + \underset{\sim}{B} \cos \frac{\omega \ell}{c}\right)$$

i.e. $\quad \underset{\sim}{A}\left(-\omega^2 M_2 \cos \frac{\omega \ell}{c} - AE \frac{\omega}{c} \sin \frac{\omega \ell}{c}\right) = \underset{\sim}{B}\left(-\frac{AE\omega}{c} \cos \frac{\omega \ell}{c} + \omega^2 M_2 \sin \frac{\omega \ell}{c}\right)$

i.e. $\quad \omega^2 M_2 \cos \frac{\omega \ell}{c} + \frac{AE\omega}{c} \sin \frac{\omega \ell}{c} + \frac{\omega M_1 c}{AE}\left(-\omega^2 M_2 \sin \frac{\omega \ell}{c} + \frac{AE\omega}{c} \cos \frac{\omega \ell}{c}\right) = 0$

This is the frequency equation.

2.17

(b) $u(x,t) = \left(A \cos\frac{\omega x}{c} + B \sin\frac{\omega x}{c}\right)\left(C \cos\omega t + D \sin\omega t\right)$ (E_1)

At $x=0$, $k_1 u = AE\frac{\partial u}{\partial x} \Rightarrow B = \frac{k_1 c}{AE\omega} A$ (E_2)

At $x=L$ $k_2 u = -AE\frac{\partial u}{\partial x}$

$$\Rightarrow k_2\left(A\cos\frac{\omega L}{c} + B\sin\frac{\omega L}{c}\right) = -AE\frac{\omega}{c}\left(-A\sin\frac{\omega L}{c} + B\cos\frac{\omega L}{c}\right) \quad (E_3)$$

Substituting (E_2) into (E_3), we obtain:

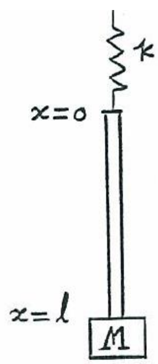
$$A\left[(k_2+k_1)\cos\frac{\omega L}{c} + \left(\frac{k_1 k_2 c}{AE\omega} - \frac{AE\omega}{c}\right)\sin\frac{\omega L}{c}\right] = 0$$

Hence the frequency equation is given by:

$$(k_2+k_1)\cos\frac{\omega L}{c} + \left(\frac{k_1 k_2 c}{AE\omega} - \frac{AE\omega}{c}\right)\sin\frac{\omega L}{c} = 0$$

$$\Rightarrow \tan\frac{\omega L}{c} = \left(\frac{-(k_1+k_2)\,AE\,\omega c}{k_1 k_2 c^2 - A^2E^2\omega^2}\right)$$

(c) $u(x,t) = \left(\tilde{A} \cos \frac{\omega x}{c} + \tilde{B} \sin \frac{\omega x}{c}\right)\left(C \cos \omega t + D \sin \omega t\right)$ --- (E_1)



At $x=0$, $\quad k\, u = AE \frac{\partial u}{\partial x} \quad \Rightarrow \quad \tilde{B} = \frac{k c}{AE\omega} \tilde{A}$ --- (E_2)

At $x=l$, $\quad AE \frac{\partial u}{\partial x} = -M \frac{\partial^2 u}{\partial t^2}$

$$\Rightarrow AE\left(-\tilde{A} \frac{\omega}{c} \sin \frac{\omega l}{c} + \tilde{B} \frac{\omega}{c} \cos \frac{\omega l}{c}\right)$$

$$= M\left(\tilde{A} \cos \frac{\omega l}{c} + \tilde{B} \sin \frac{\omega l}{c}\right) \omega^2 \quad \text{--- } (E_3)$$

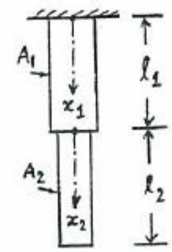
Substituting (E_2) into (E_3), we get

$$\frac{AE\omega}{c} \tilde{A}\left(-\sin \frac{\omega l}{c} + \frac{kc}{AE\omega} \cos \frac{\omega l}{c}\right) = M\omega^2 \tilde{A}\left(\cos \frac{\omega l}{c} + \frac{kc}{AE\omega} \sin \frac{\omega l}{c}\right)$$

This gives the frequency equation

$$\tan \frac{\omega l}{c} = \left\{ \frac{AE\omega c\,(k - M\omega^2)}{A^2E^2\omega^2 - M\omega^2 k c^2} \right\}$$

8.20 Set up two coordinates x_1 and x_2 as shown.



$$u_1(x_1,t) = \left(\tilde{A}_1 \cos \frac{\omega x_1}{c_1} + \tilde{B}_1 \sin \frac{\omega x_1}{c_1}\right)\left(C \cos \omega t + D \sin \omega t\right)$$

$$u_2(x_2,t) = \left(\tilde{A}_2 \cos \frac{\omega x_2}{c_2} + \tilde{B}_2 \sin \frac{\omega x_2}{c_2}\right)\left(C \cos \omega t + D \sin \omega t\right)$$

$u_1(0,t) = 0 \quad \Rightarrow \quad \tilde{A}_1 = 0$

$u_1(l_1,t) = u_2(0,t) \quad \Rightarrow \quad \tilde{B}_1 \sin \frac{\omega l_1}{c_1} = \tilde{A}_2$

$A_1E_1 \frac{\partial u_1}{\partial x_1}(l_1,t) = A_2E_2 \frac{\partial u_2}{\partial x_2}(0,t) =$ tensile force same in both areas

i.e. $\quad A_1E_1 \tilde{B}_1 \frac{\omega}{c_1} \cos \frac{\omega l_1}{c_1} = A_2E_2 \frac{\omega}{c_2} \tilde{B}_2 \quad \Rightarrow \quad \tilde{B}_2 = \frac{A_1E_1c_2}{A_2E_2c_1} \cos \frac{\omega l_1}{c_1} \cdot \tilde{B}_1$

$$u_2(x_2,t) = \tilde{B}_1\left(\sin \frac{\omega l_1}{c_1} \cos \frac{\omega x_2}{c_2} + \frac{A_1E_1c_2}{A_2E_2c_1} \cos \frac{\omega l_1}{c_1} \sin \frac{\omega x_2}{c_2}\right)\left(C \cos \omega t + D \sin \omega t\right)$$

$$\frac{\partial u_2}{\partial x_2}(l_2,t) = 0 \Rightarrow \tilde{B}_1 \frac{\omega}{c_2}\left\{-\sin \frac{\omega l_1}{c_1} \cdot \sin \frac{\omega l_2}{c_2} + \frac{A_1E_1c_2}{A_2E_2c_1} \cdot \cos \frac{\omega l_1}{c_1} \cdot \cos \frac{\omega l_2}{c_2}\right\} = 0$$

$\therefore$ Frequency equation is $\quad \tan \frac{\omega l_1}{c_1} \cdot \tan \frac{\omega l_2}{c_2} = \frac{A_1E_1c_2}{A_2E_2c_1}$

- Problem 8.31

8.31 Boundary conditions are:

At $x=0$, fixed end $\Rightarrow$ $W(0)=0$ --- (E_1) , $\frac{dW}{dx}(0)=0$ --- (E_2)

At $x=l$, free end $\Rightarrow$ $\frac{d^2W}{dx^2}(l)=0$ --- (E_3) , $\frac{d^3W}{dx^3}(0)=0$ --- (E_4)

The deflection (normal) function is given by

$$W(x) = C_1 \cos\beta x + C_2 \sin\beta x + C_3 \cosh\beta x + C_4 \sinh\beta x \qquad (E_5)$$

from which

$$\frac{dW}{dx}(x) = \beta\left[-C_1 \sin\beta x + C_2 \cos\beta x + C_3 \sinh\beta x + C_4 \cosh\beta x\right] \qquad (E_6)$$

Eqs. (E_1) and (E_5) give $\quad C_1 + C_3 = 0 \qquad (E_7)$

Eqs. (E_2) and (E_6) yield $\quad \beta(C_2 + C_4) = 0 \qquad (E_8)$

Use of (E_7) and (E_8) in (E_5) leads to

$$W(x) = C_1(\cos\beta x - \cosh\beta x) + C_2(\sin\beta x - \sinh\beta x) \qquad (E_9)$$

Use of Eqs. (E_3), (E_4) and (E_9) yields

$$C_1(\cos\beta l + \cosh\beta l) + C_2(\sin\beta l + \sinh\beta l) = 0 \qquad (E_{10})$$

$$C_1(\sin\beta l - \sinh\beta l) - C_2(\cos\beta l + \cosh\beta l) = 0 \qquad (E_{11})$$

The frequency equation can be obtained by setting the coefficient matrix in (E_{10}) and (E_{11}) to zero as

$$\begin{vmatrix} \cos\beta l + \cosh\beta l & \sin\beta l + \sinh\beta l \\ \sin\beta l - \sinh\beta l & -\cos\beta l - \cosh\beta l \end{vmatrix} = 0 \qquad (E_{12})$$

Upon simplification, Eq. (E_{12}) yields the frequency equation

$$\cos\beta l \cdot \cosh\beta l = -1 \qquad (E_{13})$$

First four roots of (E_{13}) are given by

$\beta_1 l = 1.875104$, $\beta_2 l = 4.694091$, $\beta_3 l = 7.854757$, $\beta_4 l = 10.995541$.

For each problem, you need to show your work and circle your final answer. Please staple your homework.