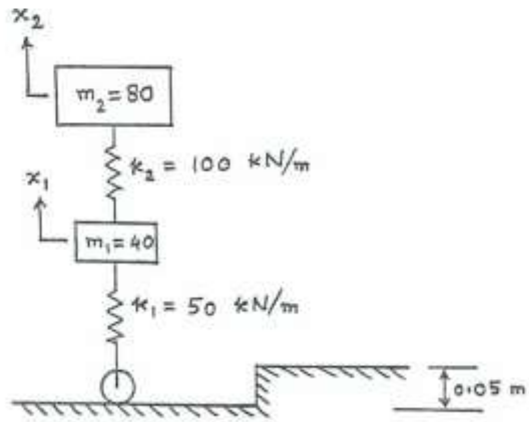
ME4189 Spring 2016. HW6: solutions

Problem 6.95

6.95

$$[k] = \begin{bmatrix} 150 & -100 \\ -100 & 100 \end{bmatrix} (10^3) \ \text{N/m}$$

$$[m] = \begin{bmatrix} 40 & 0 \\ 0 & 80 \end{bmatrix} \ \text{kg}$$



Natural frequencies are given by (see Eq. (3) in the solution of Problem 5.5):

$$\omega_{1,2}^2 = \frac{k_1 + k_2}{2\,m_1} + \frac{k_2}{2\,m_2} \pm \left\{ \frac{1}{4} \left(\frac{k_1 + k_2}{m_1} + \frac{k_2}{m_2} \right)^2 - \frac{k_1\,k_2}{m_1\,m_2} \right\}^{\frac{1}{2}}$$

$$= \left[\frac{150}{80} + \frac{100}{160} \pm \left\{ \frac{1}{4} \left(\frac{150}{40} + \frac{100}{80} \right)^2 - \frac{(100)\,(50)}{(40)\,(80)} \right\}^{\frac{1}{2}} \right] (10^3)$$

$$= 334.9365,\; 4665.1$$

$$\omega_1 = 18.3013 \text{ rad/sec} \;\; ; \;\; \omega_2 = 68.3015 \text{ rad/sec}$$

Mode shapes are defined by Eqs. (4) and (5) in the solution of Problem 5.1:

$$\frac{X_2^{(1)}}{X_1^{(1)}} = \frac{k_2}{-\,m_2\,\omega_1^2 + k_2} = \frac{100\,(10^3)}{-\,(80)\,(334.9365) + (100)\,(10^3)} = 1.3660$$

$$\vec{X}^{(1)} = a \begin{Bmatrix} 1.0 \\ 1.366 \end{Bmatrix}$$

where a is a constant.

$$\frac{X_2^{(2)}}{X_1^{(2)}} = \frac{k_2}{-\,m_2\,\omega_2^2 + k_2} = \frac{(100)\,(10^3)}{-\,(80)\,(4665.1) + (100)\,(10^3)} = -\,0.3660$$

$$\vec{X}^{(2)} = b \begin{Bmatrix} 1.0 \\ -0.366 \end{Bmatrix}$$

where b is a constant.

Orthogonalization of modes:

$$\vec{X}^{(1)T}\,[m]\,\vec{X}^{(1)} = a^2\,(1.0 \quad 1.366) \begin{bmatrix} 40 & 0 \\ 0 & 80 \end{bmatrix} \begin{Bmatrix} 1.0 \\ 1.366 \end{Bmatrix} = 189.2765\,a^2 = 1$$

$$a = 0.07269$$

$$\vec{X}^{(2)T}\,[m]\,\vec{X}^{(2)} = b^2\,(1.0 \quad -0.366) \begin{bmatrix} 40 & 0 \\ 0 & 80 \end{bmatrix} \begin{Bmatrix} 1.0 \\ -0.366 \end{Bmatrix} = 50.7165\,b^2 = 1$$

$$b = 0.14042$$

Modal matrix:

$$[X] = \begin{bmatrix} 0.07269 & 0.14042 \\ 0.09929 & -0.05139 \end{bmatrix}$$

Due to the elevation of 0.05 m, spring k_1 and hence m_1 will be subjected to additional compression of k_1 (0.05) = 2500 N.

$$\vec{F}(t) = \begin{Bmatrix} 2500 \\ 0 \end{Bmatrix} \text{ N}$$

Equation (6.111) gives:

$$\vec{Q}(t) = [X]^T \vec{F}(t) = \begin{Bmatrix} 181.725 \\ 351.05 \end{Bmatrix}$$

Solution is given by (without initial conditions) Eq. (6.114):

$$q_i(t) = \frac{1}{\omega_i} \int_0^t Q_i(\tau) \sin \omega_i (t - \tau) \, d\tau \; ; \; i = 1, 2$$

$$\text{Since} \quad \int_{\tau=0}^{t} \sin \Omega (t - \tau) \, d\tau = - \int_{\tau'=t-\tau=t}^{\tau'=t-\tau=0} \sin \Omega \tau' \, d\tau' = \frac{1}{\Omega} (1 - \cos \Omega t)$$

we find

$$q_1(t) = \frac{181.725}{(18.3013^2)} (1 - \cos 18.3013 \, t) = 0.5426 \, (1 - \cos 18.3013 \, t)$$

$$q_2(t) = \frac{351.05}{(68.3015^2)} (1 - \cos 68.3015 \, t) = 0.07525 \, (1 - \cos 68.3015 \, t)$$

Response of the masses can be found from Eq. (6.104):

$$\vec{x}(t) = [X] \vec{q}(t) = \begin{bmatrix} 0.07269 & 0.14042 \\ 0.09929 & -0.05139 \end{bmatrix} \begin{Bmatrix} 0.5426 \, (1 - \cos 18.3013 \, t) \\ 0.07525 \, (1 - \cos 68.3015 \, t) \end{Bmatrix}$$

$$= \begin{Bmatrix} 0.03944 \, (1 - \cos 18.3013 \, t) + 0.01057 \, (1 - \cos 68.3015 \, t) \\ 0.05387 \, (1 - \cos 18.3013 \, t - 0.00387 \, (1 - \cos 68.3015 \, t) \end{Bmatrix}$$

Note:

This problem can also be solved by specifying the initial conditions as

$$\vec{x}(0) = \begin{Bmatrix} 0.05 \\ 0.05 \end{Bmatrix} \text{ m} \quad ; \quad \dot{\vec{x}}(0) = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

and solving the free vibration problem.

Problem 6.99

(6.99) Equations of motion $\quad [m]\ \ddot{\vec{x}} + [c]\ \dot{\vec{x}} + [k]\ \vec{x} = \vec{F} \quad \text{---}(E.1)$

where

$$[m] = \begin{bmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{bmatrix},\ [c] = \begin{bmatrix} c_1+c_2 & -c_2 & 0 \\ -c_2 & c_2+c_3 & -c_3 \\ 0 & -c_3 & c_3+c_4 \end{bmatrix},$$

$$[k] = \begin{bmatrix} k_1+k_2 & -k_2 & 0 \\ -k_2 & k_2+k_3 & -k_3 \\ 0 & -k_3 & k_3+k_4 \end{bmatrix},\quad \vec{x} = \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix},\quad \vec{F} = \begin{Bmatrix} F_1 = F_0 \cos\omega t \\ F_2 = 0 \\ F_3 = 0 \end{Bmatrix}$$

Since $F_j(t) = \text{Re}\left[F_{j0}\, e^{i\omega t}\right]$ with $F_{10} = F_0$, and $F_{20} = F_{30} = 0$, we assume $x_j(t) = X_j\, e^{i\omega t}$; $j = 1,2,3$. Then (E.1) becomes

$$\left[Z_{rs}(i\omega)\right] \vec{X} = \vec{F_0} \quad \text{---}(E.2)$$

where

$$Z_{11}(i\omega) = -m_1\omega^2 + (c_1+c_2)\, i\omega + (k_1+k_2) = -\omega^2 + 2i\omega + 200$$

$$Z_{12}(i\omega) = Z_{21}(i\omega) = -c_2\, i\omega - k_2 = -i\omega - 100$$

$$Z_{13}(i\omega) = Z_{31}(i\omega) = 0$$

$$Z_{22}(i\omega) = -m_2\omega^2 + (c_2+c_3)\, i\omega + (k_2+k_3) = -\omega^2 + 2i\omega + 200 \quad \text{---}(E.3)$$

$$Z_{23}(i\omega) = Z_{32}(i\omega) = -c_3\, i\omega - k_3 = -i\omega - 100$$

$$Z_{33}(i\omega) = -m_3\omega^2 + (c_3+c_4)\, i\omega + (k_3+k_4) = -\omega^2 + 2i\omega + 200$$

$$
\begin{aligned}
(2i + 199)\ X_1 - (i+100)\ X_2 + (0)\ X_3 &= 10 \\
-(i+100)\ X_1 + (2i+199)\ X_2 - (i+100)\ X_3 &= 0 \qquad \text{--- (E.4)} \\
(0)\ X_1 - (i+100)\ X_2 + (2i+199)\ X_3 &= 0
\end{aligned}
$$

Solution of (E.4) can be expressed as

$$X_j = \frac{\Delta_j}{\Delta} \; ; \quad j = 1, 2, 3 \qquad \text{--- (E.5)}$$

where

$$\Delta_1 = \begin{vmatrix} 10 & -(i+100) & 0 \\ 0 & (2i+199) & -(i+100) \\ 0 & -(i+100) & (2i+199) \end{vmatrix} = 295\,980 + 5960\ i$$

$$\Delta_2 = \begin{vmatrix} (2i+199) & 10 & 0 \\ -(i+100) & 0 & -(i+100) \\ 0 & 0 & (2i+199) \end{vmatrix} = 198\,980 + 3990\ i$$

$$\Delta_3 = \begin{vmatrix} (2i+199) & -(i+100) & 10 \\ -(i+100) & (2i+199) & 0 \\ 0 & -(i+100) & 0 \end{vmatrix} = 99\,990 + 2000\ i$$

$$\Delta = \begin{vmatrix} (2i+199) & -(i+100) & 0 \\ -(i+100) & (2i+199) & -(i+100) \\ 0 & -(i+100) & (2i+199) \end{vmatrix} = 118\,002\ i + 3\,899\,409$$

Using (E.5), we get

$$X_1 = \frac{87639.682}{1154\,850.392 + 11\,685.754\ i} \; ; \quad \begin{aligned} &\text{Amplitude} = 0.0758845 \text{ m} \\ &\text{Phase angle} = 0.5798^\circ \end{aligned}$$

$$X_2 = \frac{39\,608.961}{776\,375.228 + 7921.396\ i} \; ; \quad \begin{aligned} &\text{Amplitude} = 0.0510152 \text{ m} \\ &\text{Phase angle} = 0.5845^\circ \end{aligned}$$

$$X_3 = \frac{10\,002.000}{390\,137.914 + 4000.202\ i} \; ; \quad \begin{aligned} &\text{Amplitude} = 0.0256357 \text{ m} \\ &\text{Phase angle} = 0.5874^\circ \end{aligned}$$

Thus the steady state responses are:

$$x_1(t) = 0.0758845 \ \cos(\omega t + 0.5798^\circ) \ \text{m}$$

$$x_2(t) = 0.0510152 \ \cos(\omega t + 0.5845^\circ) \ \text{m}$$

$$x_3(t) = 0.0256357 \ \cos(\omega t + 0.5874^\circ) \ \text{m}$$

Problem 5.61

5.61 Equations of motion: $m_1 \ddot{x}_1 + (k_1+k_2)\,x_1 - k_2 x_2 = F_{10}\cos\omega t = Re\,(F_{10}\, e^{i\omega t})$

$m_2 \ddot{x}_2 + (k_2+k_3)\,x_2 - k_2 x_1 = F_{20}\cos\omega t = Re\,(F_{20}\, e^{i\omega t})$

Assuming $x_j(t) = X_j\, e^{i\omega t}$; $j = 1,2$ alongwith $F_j(t) = F_{j0}\, e^{i\omega t}$; $j=1,2$, the equations of motion can be expressed as

$$(-\omega^2 m_1 + k_1 + k_2)\,X_1 - k_2 X_2 = F_{10}$$

$$-k_2 X_1 + (-\omega^2 m_2 + k_2 + k_3)\,X_2 = F_{20}$$

i.e. $$[Z(i\omega)]\,\vec{X} = \vec{F_0} \qquad \text{---- } (E_1)$$

where $Z_{11}(i\omega) = -\omega^2 m_1 + k_1 + k_2$, $Z_{12}(i\omega) = Z_{21}(i\omega) = -k_2$,

$Z_{22}(i\omega) = -\omega^2 m_2 + k_2 + k_3$,

$$\vec{X} = \begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix} , \quad \vec{F_0} = \begin{Bmatrix} F_{10} \\ F_{20} \end{Bmatrix}$$

Solution of (E_1) can be expressed, using Eqs. (5.35), as

$$X_1 = \frac{(-\omega^2 m_2 + k_2 + k_3)\,F_{10} + k_2 F_{20}}{(-\omega^2 m_1 + k_1 + k_2)(-\omega^2 m_2 + k_2 + k_3) - k_2^2} \qquad \text{---- } (E_2)$$

$$X_2 = \frac{k_2 F_{10} + (-\omega^2 m_1 + k_1 + k_2)\,F_{20}}{(-\omega^2 m_1 + k_1 + k_2)(-\omega^2 m_2 + k_2 + k_3) - k_2^2} \qquad \text{---- } (E_3)$$

Since X_1 and X_2 are real (since there is no damping), the final solution is given by

$$x_1(t) = X_1 \cos\omega t$$

$$x_2(t) = X_2 \cos\omega t$$

where X_1 and X_2 are given by (E_2) and (E_3).