6.4

6.4 The three components of velocity in a flow field are given by

$$u = x^2 + y^2 + z^2$$
$$v = xy + yz + z^2$$
$$w = -3xz - z^2/2 + 4$$

(a) Determine the volumetric dilatation rate, and interpret the results. **(b)** Determine an expression for the rotation vector. Is this an irrotational flow field?

(a) Volumetric dilatation rate $= \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}$ (Eq. 6.9)

Thus, for velocity components given

volumetric dilatation rate $= 2x + (x+z) + (-3x - z) = \underline{\underline{0}}$

This result indicates that there is no change in the volume of a fluid element as it moves from one location to another.

(b) From Eqs. 6.12, 6.13, and 6.14 with the velocity components given:

$$\omega_z = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) = \frac{1}{2}(y - 2y) = -\frac{y}{2}$$

$$\omega_x = \frac{1}{2}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right) = \frac{1}{2}\left[0 - (y + 2z)\right] = -\left(\frac{y}{2} + z\right)$$

$$\omega_y = \frac{1}{2}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right) = \frac{1}{2}\left[2z - (-3z)\right] = \frac{5z}{2}$$

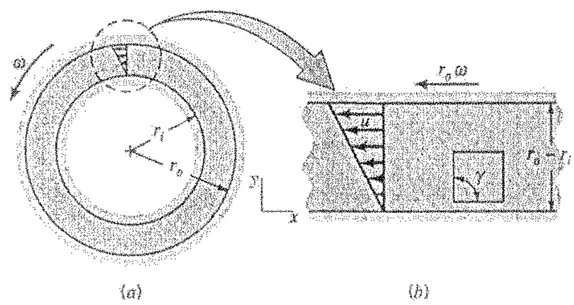
Thus,

$$\vec{\omega} = \underline{\underline{-\left(\frac{y}{2} + z\right)\hat{i} + \frac{5z}{2}\hat{j} - \frac{y}{2}\hat{k}}}$$

Since $\vec{\omega}$ is not zero everywhere the flow field is not irrotational. $\underline{\underline{\text{No.}}}$

6.10

6.10 A viscous fluid is contained in the space between concentric cylinders. The inner wall is fixed, and the outer wall rotates with an angular velocity ω. (See Fig. P6.10a and Video V6.3.) Assume that the velocity distribution in the gap is linear as illustrated in Fig. P6.10b. For the small rectangular element shown in Fig. P6.10b, determine the rate of change of the right angle γ due to the fluid motion. Express your answer in terms of r_0, r_i, and ω.



(a) (b)

■ FIGURE P6.10

From Eq. 6.18

$$\dot{\gamma} = \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}$$

For the linear distribution

$$u = -\frac{r_o \omega}{r_o - r_i}\, y$$

so that

$$\frac{\partial u}{\partial y} = -\frac{r_o \omega}{r_o - r_i}$$

and since $v = 0$

$$\dot{\gamma} = -\frac{r_o \omega}{r_o - r_i}$$

The negative sign indicates that the original right angle is increasing.

6.13

6.13 The velocity components of an incompressible, two-dimensional velocity field are given by the equations

$$u = y^2 - x(1 + x)$$
$$v = y(2x + 1)$$

Show that the flow is irrotational and satisfies conservation of mass.

If the two-dimensional flow is irrotational,

$$\omega_z = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) = 0$$

For the velocity distribution given,

$$\frac{\partial v}{\partial x} = 2y \qquad \frac{\partial u}{\partial y} = 2y$$

Thus,

$$\omega_z = \frac{1}{2}\left(2y - 2y\right) = 0$$

and the <u>flow is irrotational.</u>

To satisfy conservation of mass,

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

Since,

$$\frac{\partial u}{\partial x} = -1 - 2x \qquad \frac{\partial v}{\partial y} = 2x + 1$$

then

$$-1 - 2x + 2x + 1 = 0$$

and

<u>Conservation of mass is satisfied</u>.

6.30

6.30 A fluid with a density of 2000 kg/m^3 flows steadily between two flat plates as shown in Fig. P6.30. The bottom plate is fixed and the top one moves at a constant speed in the x-direction. The velocity is $\mathbf{V} = 0.20\, y\, \hat{\mathbf{i}}$ m/s where y is in meters. The acceleration of gravity is $\mathbf{g} = -9.8\, \hat{\mathbf{j}}$ m/s^2. The only non-zero shear stresses, $\tau_{yx} = \tau_{xy}$, are constant throughout the flow with a value of 5 N/m^2. The normal stress at the origin ($x = y = 0$) is $\sigma_{xx} = -100$ kPa. Use the x- and y-components of the equations of motion (Eqs. 6.50a and b) to determine the normal stress throughout the fluid. Assume that $\sigma_{xx} = \sigma_{yy}$.

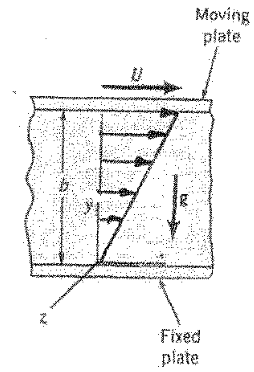


Figure P6.30

$$\rho g_x + \frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} = \rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right) \quad \textbf{(6.50a)}$$

$$\rho g_y + \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \tau_{zy}}{\partial z} = \rho\left(\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z}\right) \quad \textbf{(6.50b)}$$

For the given flow: $u = 0.20y\ \frac{m}{s}$, $v = 0$, $w = 0$, $\tau_{xy} = \tau_{yx} = 5\ N/m^2$, $\tau_{zx} = \tau_{zy} = 0$, $g_x = g_z = 0$, and $g_y = -9.8\ m/s^2$

Thus, $\frac{\partial \tau_{yx}}{\partial y} = \frac{\partial \tau_{zx}}{\partial z} = 0$, $\frac{\partial u}{\partial t} = 0$, $\frac{\partial u}{\partial x} = \frac{\partial u}{\partial z} = 0$, and $v\frac{\partial u}{\partial y} = 0$ so that Eq. 6.50a becomes simply

$\frac{\partial \sigma_{xx}}{\partial x} = 0$, or $\sigma_{xx} = f(y) + C$, where C = constant and $f(y)$ is an unknown function of y.

Similarly, since $\sigma_{yy} = \sigma_{xx}$, Eq. 6.50b becomes:

$$\rho g_y + \frac{\partial \sigma_{xx}}{\partial y} = 0 \quad \text{or}$$

$$\frac{\partial \sigma_{xx}}{\partial y} = -\rho g_y = -2000\frac{kg}{m^3}\left(-9.8\frac{m}{s^2}\right) = 19.6\times10^3\ \frac{kg}{m^2 s^2}$$

But

$$\frac{\partial \sigma_{xx}}{\partial y} = \frac{df}{dy} \text{ so that } \frac{df}{dy} = 19.6\times10^3\ \frac{kg}{m^2s^2} \text{ or } f = 19.6\times10^3\, y\ \frac{kg\ m}{m^2s^2} + \text{const.}$$

$$= 19.6\times10^3 y\ \frac{N}{m^2} + \text{const.}$$

Hence, since $\sigma_{xx} = f + C$

$\sigma_{xx} = 19.6\, y$ kPa + constant. But $\sigma_{xx} = -100$ kPa at $y = x = 0$ so that

$\sigma_{xx} = \underline{(19.6y - 100)\text{ kPa}}$, where $y \sim m$.

Note: This is the standard hydrostatic pressure distribution.

6.31

6.31 A fluid with a density of 2 slug/ft^3 flows steadily between two stationary flat plates as shown in Fig. P6.31. The velocity is $V = 0.5\,[1 - (y/h)^2]\,\hat{i}$ ft/s where y and h are in feet. The only non-zero shear stresses, $\tau_{yx} = \tau_{xy}$, are given by $\tau_{yx} = -4.0\,y$ lb/ft^2 and the acceleration of gravity is negligible. The normal stress at the origin ($x = y = 0$) is $\sigma_{xx} = -10$ lb/ft^2. Use the x- and y-components of the equations of motion (Eqs. 6.50a and b) to determine the normal stress throughout the fluid. Assume that $\sigma_{xx} = \sigma_{yy}$.

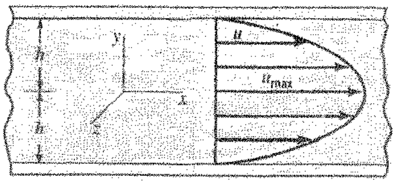


Figure P6.31

$$\rho g_x + \frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} = \rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right) \qquad \textbf{(6.50a)}$$

$$\rho g_y + \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \tau_{zy}}{\partial z} = \rho\left(\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z}\right) \qquad \textbf{(6.50b)}$$

For the given flow, $u = 0.5[1-(y/h)^2]$, $v = w = 0$, $\tau_{yx} = \tau_{xy} = -4y$ lb/ft^2,

$\tau_{zx} = \tau_{zy} = 0$, $g_x = g_y = 0$.

Thus, all of the terms in Eq. 6.50a are zero except

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} = 0 \quad \text{or} \quad \frac{\partial \sigma_{xx}}{\partial x} = -\frac{\partial \tau_{yx}}{\partial y} = -\left(-4\,\frac{\text{lb}}{\text{ft}^3}\right) = 4\,\frac{\text{lb}}{\text{ft}^3}$$

Integration gives

$$\sigma_{xx} = 4x\,\frac{\text{lb}}{\text{ft}^2} + f(y), \text{ where } f(y) \text{ is a function of } y \qquad (1)$$

Similarly, Eq. 6.50b reduces to

$$\frac{\partial \sigma_{yy}}{\partial y} = 0 \quad \text{or since } \sigma_{xx} = \sigma_{yy},$$

$$\frac{\partial \sigma_{xx}}{\partial y} = 0$$

Hence, from Eq. (1),

$$\frac{\partial \sigma_{xx}}{\partial y} = \frac{df}{dy} = 0 \quad \text{or } f = \text{constant} = C$$

Thus,

$\sigma_{xx} = 4x + C$, but $\sigma_{xx} = -10$ lb/ft^2 at $x = y = 0$

Hence, $C = -10$ lb/ft^2 or

$$\underline{\underline{\sigma_{xx} = (4x - 10)\,\frac{\text{lb}}{\text{ft}^2}, \text{ where } x \sim \text{ft}}}$$

6.32

6.32 Given the streamfunction for a flow as $\psi = 4x^2 - 4y^2$, show that the Bernoulli equation can be applied between any two points in the flow field.

For the Bernoulli equation to be applied between any two points in the flow field (as opposed to only points along a streamline), the flow must be irrotational. That is, $\nabla \times \vec{V} = 0$, which for two-dimensional flow can be written as

(1) $$\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = 0$$

For the given flow,

$$u = \frac{\partial \psi}{\partial y} = -8y \quad \text{and} \quad v = -\frac{\partial \psi}{\partial x} = -8x$$

Thus,

$$\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = \frac{\partial}{\partial x}(-8x) - \frac{\partial}{\partial y}(-8y) = -8 + 8 = 0$$

Hence, Eq. (1) is satisfied, the flow is irrotational, and the Bernoulli equation can be applied between any two points.

6.95

6.95 Two immiscible, incompressible, viscous fluids having the same densities but different viscosities are contained between two infinite, horizontal, parallel plates (Fig. P6.95. The bottom plate is fixed and the upper plate moves with a constant velocity U. Determine the velocity at the interface. Express your answer in terms of U, μ_1, and μ_2. The motion of the fluid is caused entirely by the movement of the upper plate; that is, there is no pressure gradient in the x direction. The fluid velocity and shearing stress is continuous across the interface between the two fluids. Assume laminar flow.

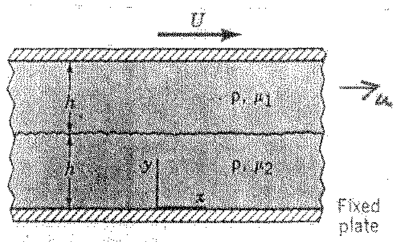


FIGURE P6.95

For the specified conditions, $v=0$, $w=0$, $\frac{\partial p}{\partial x}=0$, and $g_x=0$, so that the x-component of the Navier-Stokes equations (Eq. 6.127a) for either the upper or lower layer reduces to

$$\frac{d^2u}{dy^2}=0 \qquad (1)$$

Integration of Eq. (1) yields

$$u = Ay + B$$

which gives the velocity distribution in either layer. In the upper layer at $y=2h$, $u=U$ so that

$$B_1 = U - A_1(2h)$$

where the subscript 1 refers to the upper layer. For the lower layer at $y=0$, $u=0$ so that

$$B_2 = 0$$

where the subscript 2 refers to the lower layer. Thus,

$$u_1 = A_1(y-2h)+U$$

and

$$u_2 = A_2\, y$$

At $y=h$, $u_1=u_2$ so that

$$A_1(h-2h)+U = A_2 h$$

or

$$A_2 = -A_1 + \frac{U}{h} \qquad (2)$$

(con't)

6.95 (con't)

Since the velocity distribution is linear in each layer the shearing stress

$$\tau_{yx} = \mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) = \mu \frac{du}{dy}$$

is constant throughout each layer. For the upper layer

$$\tau_1 = \mu_1 A_1$$

and for the lower layer

$$\tau_2 = \mu_2 A_2$$

At the interface $\tau_1 = \tau_2$ so that

$$\mu_1 A_1 = \mu_2 A_2$$

or

$$\frac{A_1}{A_2} = \frac{\mu_2}{\mu_1} \qquad (3)$$

Substitution of Eq. (3) into Eq. (2) yields

$$A_2 = -\frac{\mu_2}{\mu_1} A_2 + \frac{U}{h}$$

or

$$A_2 = \frac{U/h}{1 + \mu_2/\mu_1}$$

Thus, velocity at the interface is

$$u_2\,(y=h) = A_2\, h = \underline{\underline{\frac{U}{1 + \frac{\mu_2}{\mu_1}}}}$$

6.108

6.108 An incompressible Newtonian fluid flows steadily between two infinitely long, concentric cylinders as shown in Fig. P6.108. The outer cylinder is fixed, but the inner cylinder moves with a longitudinal velocity V_0 as shown. The pressure gradient in the axial direction is $-\Delta p/\ell$. For what value of V_0 will the drag on the inner cylinder be zero? Assume that the flow is laminar, axisymmetric, and fully developed.

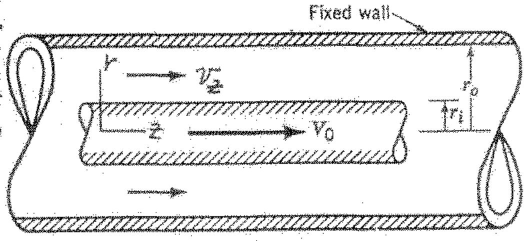


FIGURE P6.108

Equation 6.147, which was developed for flow in circular tubes, applies in the annular region. Thus,

$$v_z = \frac{1}{4\mu}\left(\frac{\partial p}{\partial z}\right) r^2 + c_1 \ln r + c_2 \qquad (1)$$

With boundary conditions, $r = r_o$, $v_z = 0$, and $r = r_i$, $v_z = V_o$, it follows that:

$$0 = \frac{1}{4\mu}\left(\frac{\partial p}{\partial z}\right) r_o^2 + c_1 \ln r_o + c_2 \qquad (2)$$

$$V_o = \frac{1}{4\mu}\left(\frac{\partial p}{\partial z}\right) r_i^2 + c_1 \ln r_i + c_2 \qquad (3)$$

Subtract Eq. (2) from Eq. (3) to obtain

$$V_o = \frac{1}{4\mu}\left(\frac{\partial p}{\partial z}\right)\left(r_i^2 - r_o^2\right) + c_1 \ln \frac{r_i}{r_o}$$

so that

$$c_1 = \frac{V_o - \frac{1}{4\mu}\left(\frac{\partial p}{\partial z}\right)\left(r_i^2 - r_o^2\right)}{\ln \frac{r_i}{r_o}}$$

The drag on the inner cylinder will be zero if

$$\left(\tau_{rz}\right)_{r=r_i} = 0$$

Since,

$$\tau_{rz} = \mu\left(\frac{\partial v_r}{\partial z} + \frac{\partial v_z}{\partial r}\right) \qquad (\text{Eq. } 6.126f)$$

and with $v_r = 0$, it follows that

$$\tau_{rz} = \mu \frac{\partial v_z}{\partial r}$$

(con't)

6.108 (Con't)

Differentiate Eq. (1) with respect to r to obtain

$$\frac{\partial v_z}{\partial r} = \frac{1}{2\mu}\left(\frac{\partial p}{\partial z}\right)r + \frac{c_1}{r}$$

so that at $r = r_i$

$$\left(\tau_{rz}\right)_{r=r_i} = \mu\left[\frac{1}{2\mu}\left(\frac{\partial p}{\partial z}\right)r_i + \frac{V_0 - \frac{1}{4\mu}\left(\frac{\partial p}{\partial z}\right)(r_i^2 - r_0^2)}{r_i \ln \frac{r_i}{r_0}}\right]$$

Thus, in order for the drag to be zero,

$$\frac{1}{2\mu}\left(\frac{\partial p}{\partial z}\right)r_i + \frac{V_0 - \frac{1}{4\mu}\left(\frac{\partial p}{\partial z}\right)(r_i^2 - r_0^2)}{r_i \ln \frac{r_i}{r_0}} = 0$$

or

$$\underline{\underline{V_0 = -\frac{1}{4\mu}\left(\frac{\partial p}{\partial z}\right)\left[2 r_i^2 \ln \frac{r_i}{r_0} - (r_i^2 - r_0^2)\right]}}$$