HW #5 - Solutions

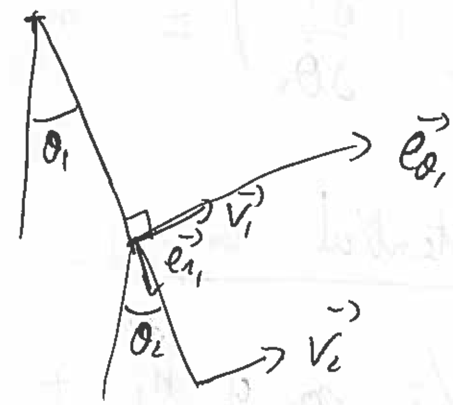
6.42.

Kinetic energy:

$$T = \frac{1}{2} m_1 V_1^2 + \frac{1}{2} m_2 V_2^2 + \frac{1}{2} m_3 V_3^2$$

$$V_1 = l_1 \dot{\theta}_1$$

$$\vec{V}_2 = \left(V_1 + V_2 \cos(\theta_2 - \theta_1)\right) \vec{e}_{\theta_1} + -V_2 \sin(\theta_2 - \theta_1)\, \vec{e}_{r_1}$$



$$\Rightarrow V_2^2 = \left(V_1 + V_2 \cos(\theta_2 - \theta_1)\right)^2 + V_2^2 \sin^2(\theta_2 - \theta_1)^2$$

$$= \left(l_1 \dot{\theta}_1 + l_2 \dot{\theta}_2 \underbrace{\cos(\theta_2 - \theta_1)}_{\simeq 1 - \frac{(\theta_2-\theta_1)^2}{2}}\right)^2 + (l_2 \dot{\theta}_2)^2 \underbrace{\sin^2(\theta_2 - \theta_1)}_{\sim (\theta_2-\theta_1)^2}$$

small neglect

$$= \left(l_1 \dot{\theta}_1 + l_2 \dot{\theta}_2\right)^2$$

Similarly $V_3^2 = \left(l_1 \dot{\theta}_1 + l_2 \dot{\theta}_2 + l_3 \dot{\theta}_3\right)^2$

$$\Rightarrow T = \frac{1}{2} m_1 l_1 \dot{\theta}_1^2 + \frac{1}{2} m_2 \left(l_1 \dot{\theta}_1 + l_2 \dot{\theta}_2\right)^2 + \frac{1}{2} m_3 \left(l_1 \dot{\theta}_1 + l_2 \dot{\theta}_2 + l_3 \dot{\theta}_3\right)^2$$

$$\Rightarrow \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\theta}_1}\right) = (m_1 + m_2 + m_3)\, l_1^2\, \ddot{\theta}_1 + (m_2 + m_3)\, l_1 l_2\, \ddot{\theta}_2 + m_3\, l_1 l_3\, \ddot{\theta}_3$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\theta}_2}\right) = (m_2 + m_3)\, l_2^2\, \ddot{\theta}_2 + (m_2 + m_3)\, l_1 l_2\, \ddot{\theta}_1 + m_3\, l_2 l_3\, \ddot{\theta}_3$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\theta}_3}\right) = m_3\, l_3\, \ddot{\theta}_3 + m_3\, l_1 l_3\, \ddot{\theta}_1 + m_3\, l_2 l_3\, \ddot{\theta}_2$$

$$\frac{\partial T}{\partial \theta_1} = \frac{\partial T}{\partial \theta_2} = \frac{\partial T}{\partial \theta_3} = 0$$

<u>Potential energy</u>

$$V = m_1 g h_1 + m_2 g h_2 + m_3 g h_3$$
$$= m_1 g l_1 (1 - \cos\theta_1) + m_2 g \big(l_1 (1-\cos\theta_1) + l_2 (1-\cos\theta_2)\big)$$
$$+ m_3 g \big(l_1 (1-\cos\theta_1) + l_2 (1-\cos\theta_2) + l_3 (1-\cos\theta_3)\big)$$

$$\Rightarrow \frac{\partial V}{\partial \theta_1} = m_1 g l_1 \sin\theta_1 + m_2 g l_2 \sin\theta_2 + m_3 g l_3 \sin\theta_3$$

if $\theta_1, \theta_2, \theta_3$ small

$$\simeq g\,(m_1 l_1 \theta_1 + m_2 l_2 \theta_2 + m_3 l_3 \theta_3)$$

$$\frac{\partial V}{\partial \theta_2} = m_2 g l_2 \sin\theta_2 + m_3 g l_3 \sin\theta_3$$
$$\simeq m_2 g l_2 \theta_2 + m_3 g l_3 \theta_3$$

$$\frac{\partial V}{\partial \theta_3} = m_3\, g\, l_3 \sin\theta_3 = m_3\, g\, l_3\, \theta_3$$

$\Rightarrow$ Lagrange's equation

$$[m]\{\ddot{\vec{\theta}}\} + [k]\{\vec{\theta}\} = \{\vec{0}\}$$

$$\{\vec{\theta}\} = \begin{Bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{Bmatrix}$$

$$[m] = \begin{bmatrix} (m_1+m_2+m_3)\, l_1^2 & (m_2+m_3)\, l_1 l_2 & m_3 l_1 l_3 \\ (m_2+m_3)\, l_1 l_2 & (m_2+m_3)\, l_2^2 & m_3 l_2 l_3 \\ m_3 l_1 l_3 & m_3 l_2 l_3 & m_3 l_3 \end{bmatrix}$$

$$[k] = \begin{bmatrix} (m_1+m_2+m_3)\, l_1 g & 0 & 0 \\ 0 & (m_2+m_3)\, g\, l_2 & 0 \\ 0 & 0 & m_3\, g\, l_3 \end{bmatrix}$$

$\boxed{6.58}$

$$[m] = m\ell^2 \begin{bmatrix} 3 & 2 & 1 \\ 2 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix} \qquad [k] = mg\ell \begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\det([k] - \omega^2 [m]) = 0$$

$$\begin{vmatrix} 3mg\ell - 3m\ell^2\omega^2 & -2m\ell^2\omega^2 & -m\ell^2\omega^2 \\ -2m\ell^2\omega^2 & 2mg\ell - 2m\ell^2\omega^2 & -m\ell^2\omega^2 \\ -m\ell^2\omega^2 & -m\ell^2\omega^2 & mg\ell - m\ell^2\omega^2 \end{vmatrix} = 0$$

divide by $mg\ell$, let $\alpha = \frac{\ell}{g}\omega^2$

$$\begin{vmatrix} 3(1-\alpha) & -2\alpha & -\alpha \\ -2\alpha & 2(1-\alpha) & -\alpha \\ -\alpha & -\alpha & 1-\alpha \end{vmatrix} = 0$$

$$-\alpha^3 + 9\alpha^2 - 18\alpha + 6 = 0$$

$$\alpha^3 - 9\alpha^2 + 18\alpha - 6 = 0$$

$\Rightarrow$ use MATLAB roots

$\alpha_1 = 0.4158$

$\alpha_2 = 2.2943 \quad \Rightarrow$

$\alpha_3 = 6.29$

$\omega_1 = \sqrt{\alpha_1}\sqrt{\frac{g}{\ell}} = 0.6448\sqrt{\frac{g}{\ell}}$

$\omega_2 = 1.5147\sqrt{\frac{g}{\ell}}$

$\omega_3 = 2.5078\sqrt{\frac{g}{\ell}}$

Mode shapes $\quad \{\vec{X}^{(i)}\} = \begin{Bmatrix} X_1^{(i)} \\ X_2^{(i)} \\ X_3^{(i)} \end{Bmatrix}$

$$\begin{bmatrix} 3(1-\alpha_i) & -2\alpha_i & -\alpha_i \\ -2\alpha_i & 2(1-\alpha_i) & -\alpha_i \\ -\alpha_i & -\alpha_i & 1-\alpha_i \end{bmatrix} \begin{Bmatrix} X_1^{(i)} \\ X_2^{(i)} \\ X_3^{(i)} \end{Bmatrix} = \{\vec{0}\}$$

$$\Rightarrow \begin{bmatrix} -2\alpha_i & -\alpha_i \\ 2(1-\alpha_i) & -\alpha_i \end{bmatrix} \begin{Bmatrix} X_2^{(i)} \\ X_3^{(i)} \end{Bmatrix} = \begin{Bmatrix} -3(1-\alpha_i) \\ +2\alpha_i \end{Bmatrix} X_1^{(i)}$$

$\Rightarrow$ solve using MATLAB for each modal vector

$$\Rightarrow \begin{Bmatrix} X_2^{(i)} \\ X_3^{(i)} \end{Bmatrix} = \begin{Bmatrix} 1.2921 \\ 1.6312 \end{Bmatrix} X_1^{(1)} \qquad \{\vec{X}^{(1)}\} = \begin{Bmatrix} 1 \\ 1.2921 \\ 1.6321 \end{Bmatrix} X_1^{(1)}$$

$$\{\vec{X}^{(2)}\} = \begin{Bmatrix} 1 \\ 0.3529 \\ -2.2981 \end{Bmatrix} X_1^{(2)}$$

$$\{\vec{X}^{(3)}\} = \begin{Bmatrix} 1 \\ -1.645 \\ 0.7669 \end{Bmatrix} X_1^{(3)}$$

Mass-normalization of modal vectors

$$\{\vec{X}^{(1)}\} [m] \{\vec{X}^{(1)}\} = (ml^2)\; 21.463\; (X_1^{(1)})^2 = 1$$

$$\Rightarrow X_1^{(1)} = \frac{1}{\sqrt{21.463}\sqrt{ml^2}}$$

$$\{\vec{X}^{(1)}\} = \begin{Bmatrix} 0.2149 \\ 0.2777 \\ 0.3506 \end{Bmatrix} \frac{1}{\sqrt{ml^2}}$$

Similarly : $\{\vec{X}^{(2)}\}^T [m] \{\vec{X}^{(2)}\} = 3.9228\, X_1^{(2)\,2}\, ml^2$

$$\{\vec{X}^{(2)}\} = \begin{Bmatrix} 0.5049 \\ 0.1782 \\ -1.2108 \end{Bmatrix} \frac{1}{\sqrt{ml^2}}$$

$$\{\vec{X}^{(3)}\}^T [m] \{\vec{X}^{(3)}\} = 1.4309 \left(X_1^{(3)}\right)^2 ml^2$$

$$\Rightarrow \{\vec{X}^{(3)}\} = \begin{Bmatrix} 0.8360 \\ -1.3752 \\ 0.6411 \end{Bmatrix} \frac{1}{\sqrt{ml^2}}$$

$\Rightarrow$ Modal matrix:

$$[X] = \frac{1}{\sqrt{ml^2}} \begin{bmatrix} 0.2149 & 0.5049 & 0.8360 \\ 0.2777 & 0.1782 & -1.3752 \\ 0.3506 & -1.2108 & 0.6411 \end{bmatrix}$$

Compute:

$$\{\vec{q}(0)\} = [X]^T [m] \{\vec{\theta}_0\} = [X]^T [m] \begin{Bmatrix} 0 \\ 0 \\ 1 \end{Bmatrix} \theta_{30}$$

$$= \begin{Bmatrix} 0.8433 \\ -0.5277 \\ 0.1019 \end{Bmatrix} \frac{\theta_{30}}{\sqrt{ml^2}}$$

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$$\{\dot{\vec{q}}(0)\} = [X]^T [m] \{\dot{\vec{\theta}}_0\} = \{\vec{0}\}$$

$$\Rightarrow \{\vec{q}(t)\} = \begin{Bmatrix} 0.8433 \dfrac{\theta_{30}}{\sqrt{ml^2}} \cos\omega_1 t \\ -0.5277 \dfrac{\theta_{30}}{\sqrt{ml^2}} \cos\omega_2 t \\ 0.1019 \dfrac{\theta_{30}}{\sqrt{ml^2}} \cos\omega_3 t \end{Bmatrix}$$

$$\{\vec{\theta}(t)\} = [X]\{\vec{q}(t)\}$$

$$\begin{Bmatrix} \theta_1(t) \\ \theta_2(t) \\ \theta_3(t) \end{Bmatrix} = \begin{Bmatrix} 0.1813 \cos\omega_1 t - 0.2664 \cos\omega_2 t + 0.0852 \cos\omega_2 t \\ 0.2342 \cos\omega_1 t - 0.0940 \cos\omega_2 t - 0.1401 \cos\omega_2 t \\ 0.2957 \cos\omega_1 t + 0.6389 \cos\omega_2 t + 0.0653 \cos\omega_3 t \end{Bmatrix}$$