Problem 5.17

Equations of motion
$$m_1 \ddot{x}_1 + (k_1 + k_2) x_1 - k_2 x_2 = 0$$
$$m_2 \ddot{x}_2 + k_2 x_2 - k_2 x_1 = 0 \qquad (E_1)$$

With $x_i(t) = X_i \cos(\omega t + \phi)$; $i = 1, 2$, Eqs. (E_1) give the frequency equation

$$\begin{vmatrix} -\omega^2 m_1 + k_1 + k_2 & -k_2 \\ -k_2 & -\omega^2 m_2 + k_2 \end{vmatrix} = 0$$

or
$$\omega^4 - \left(\frac{k_1 + k_2}{m_1} + \frac{k_2}{m_2} \right) \omega^2 + \frac{k_1 k_2}{m_1 m_2} = 0 \qquad (E_2)$$

Roots of Eq. (E_2) are

$$\omega_1^2, \omega_2^2 = \frac{k_1 + k_2}{2 m_1} + \frac{k_2}{2 m_2} \mp \sqrt{\frac{1}{4}\left(\frac{k_1 + k_2}{m_1} + \frac{k_2}{m_2} \right)^2 - \frac{k_1 k_2}{m_1 m_2}} \qquad (E_3)$$

If $\vec{X}^{(1)} = \begin{Bmatrix} X_1^{(1)} \\ X_2^{(1)} = r_1 X_1^{(1)} \end{Bmatrix}$ and $\vec{X}^{(2)} = \begin{Bmatrix} X_1^{(2)} \\ X_2^{(2)} = r_2 X_1^{(2)} \end{Bmatrix}$,

$$r_1 = \frac{X_2^{(1)}}{X_1^{(1)}} = \frac{-m_1 \omega_1^2 + k_1 + k_2}{k_2} = \frac{k_2}{-m_2 \omega_1^2 + k_2} \qquad (E_4)$$

$$r_2 = \frac{X_2^{(2)}}{X_1^{(2)}} = \frac{-m_1 \omega_2^2 + k_1 + k_2}{k_2} = \frac{k_2}{-m_2 \omega_2^2 + k_2} \qquad (E_5)$$

General solution of (E_1) is

$$x_1(t) = X_1^{(1)} \cos(\omega_1 t + \phi_1) + X_1^{(2)} \cos(\omega_2 t + \phi_2) \qquad (E_6)$$
$$x_2(t) = r_1 X_1^{(1)} \cos(\omega_1 t + \phi_1) + r_2 X_1^{(2)} \cos(\omega_2 t + \phi_2)$$

where $X_1^{(1)}$, $X_1^{(2)}$, ϕ_1 and ϕ_2 can be found using Eqs. (5.18).

For $m_1 = m$, $m_2 = 2m$, $k_1 = k$ and $k_2 = 2k$, (E_3) gives

$$\omega_1^2 = (2 - \sqrt{3}) \frac{k}{m} \quad , \quad \omega_2^2 = (2 + \sqrt{3}) \frac{k}{m} \qquad (E_7)$$

$$\omega_1^2, \omega_2^2 = \frac{8000}{2} + \frac{6000}{2} \mp \sqrt{\frac{1}{4}\left(\frac{8000}{1} + \frac{6000}{1}\right)^2 - \frac{12 \times 10^6}{1}} = 917.2,\ 13082.8$$

$\omega_1 = 30.2853$ rad/sec , $\omega_2 = 114.3801$ rad/s

Eqs. (E_4) and (E_5) of solution of problem 5.5 give

$$r_1 = \frac{k_2}{-m_2 \omega_1^2 + k_2} = \frac{6000}{-917.2 + 6000} = 1.1805 \ ; \quad \vec{X}^{(1)} = \begin{Bmatrix} 1 \\ 1.1805 \end{Bmatrix} X_1^{(1)}$$

$$r_2 = \frac{k_2}{-m_2 \omega_2^2 + k_2} = \frac{6000}{-13082.8 + 6000} = -0.8471 \ ; \quad \vec{X}^{(2)} = \begin{Bmatrix} 1 \\ -0.8471 \end{Bmatrix} X_1^{(2)}$$

Problem 3: 5.20

5.20 (a) $\omega_1^2 = 917.2$, $\omega_2^2 = 13082.8$, $r_1 = 1.1805$, $r_2 = -0.8471$

$x_1(0) = 0.2$, $x_2(0) = \dot{x}_1(0) = \dot{x}_2(0) = 0$

Eq. (5.18) gives

$$X_1^{(1)} = \frac{1}{(-0.8471 - 1.1805)}\left[(-0.8471)(0.2)\right] = 0.08356$$

$$X_1^{(2)} = \frac{1}{(-0.8471 - 1.1805)}\left[(-1.1805)(0.2)\right] = 0.11644$$

$$\phi_1 = \phi_2 = \tan^{-1}(0) = 0$$

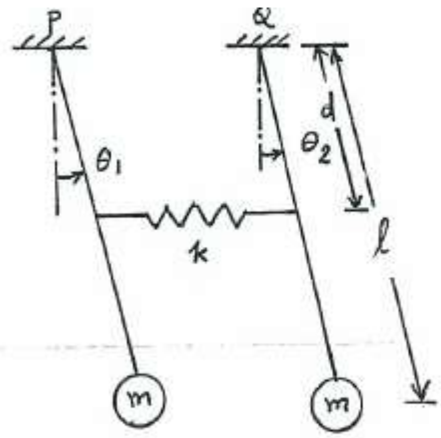
$$x_1(t) = 0.08356 \cos 30.2853t + 0.11644 \cos 114.3801t$$

$$x_2(t) = (1.1805)(0.08356) \cos 30.2853t + (-0.8471)(0.11644) \cos 114.3801t$$

$$= 0.09864 \cos 30.2853t - 0.09864 \cos 114.3801t$$

Problem 4: 5.31

5.31 (a) Equations of motion:



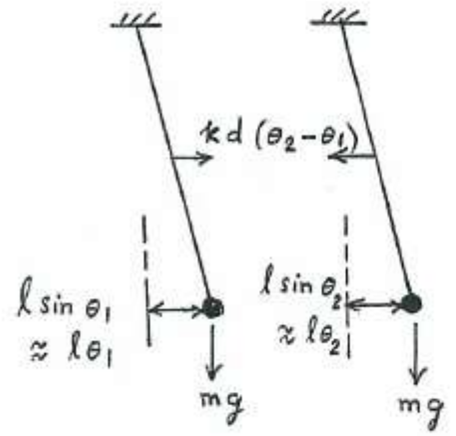
Assume: θ_1, θ_2 are small.

Moment equilibrium equations of the two masses about P and Q:

$$m\ell^2 \ddot{\theta}_1 + mg\ell\theta_1 + k\,d^2(\theta_1-\theta_2) = 0 \qquad (1)$$

$$m\ell^2 \ddot{\theta}_2 + mg\ell\,\theta_2 - k\,d^2(\theta_1-\theta_2) = 0 \qquad (2)$$

(b) Natural frequencies and mode shapes:



Free body diagram

Assume: Harmonic motion with

$$\theta_i(t) = \Theta_i \cos(\omega t - \phi)\,;\ i = 1, 2 \qquad (3)$$

where Θ_1 and Θ_2 are amplitudes of θ_1 and θ_2, respectively, ω is the natural frequency, and ϕ is the phase angle.

Using Eq. (3), Eqs. (1) and (2) can be expressed in matrix form as

$$-\omega^2 m\ell^2 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{Bmatrix} \Theta_1 \\ \Theta_2 \end{Bmatrix} + \begin{bmatrix} mg\ell + kd^2 & -kd^2 \\ -kd^2 & mg\ell + kd^2 \end{bmatrix} \begin{Bmatrix} \Theta_1 \\ \Theta_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \qquad (4)$$

Frequency equation:

$$\begin{vmatrix} -\omega^2 m\ell^2 + mg\ell + kd^2 & -kd^2 \\ -kd^2 & -\omega^2 m\ell^2 + mg\ell + kd^2 \end{vmatrix} = 0$$

or

$$\omega^4 - \omega^2\left(\frac{2g}{\ell} + \frac{2kd^2}{m\ell^2}\right) + \left(\frac{g^2}{\ell^2} + \frac{2gkd^2}{m\ell^3}\right) = 0 \qquad (5)$$

Solution of Eq. (5) gives

$$\omega_1^2 = \frac{g}{\ell}\,, \qquad \omega_2^2 = \frac{g}{\ell} + \frac{2kd^2}{m\ell^2} \qquad (6)$$

By substituting for ω_1^2 and ω_2^2 into Eq. (4), we obtain

$$\left(\frac{\Theta_2}{\Theta_1}\right)^{(1)} = 1 \quad \text{or} \quad \begin{Bmatrix} \Theta_1 \\ \Theta_2 \end{Bmatrix}^{(1)} = \begin{Bmatrix} 1 \\ 1 \end{Bmatrix} \Theta_1^{(1)}$$

and

$$\left(\frac{\Theta_2}{\Theta_1}\right)^{(2)} = -1 \quad \text{or} \quad \begin{Bmatrix} \Theta_1 \\ \Theta_2 \end{Bmatrix}^{(2)} = \begin{Bmatrix} 1 \\ -1 \end{Bmatrix} \Theta_1^{(2)}$$

Thus the motion of the masses in the two modes is given by

$$\vec{\theta}^{(1)}(t) = \begin{Bmatrix} \theta_1^{(1)}(t) \\ \theta_2^{(1)}(t) \end{Bmatrix} = \Theta_1^{(1)} \begin{Bmatrix} 1 \\ 1 \end{Bmatrix} \cos(\omega_1 t + \phi_1) \tag{7}$$

$$\vec{\theta}^{(2)}(t) = \begin{Bmatrix} \theta_1^{(2)}(t) \\ \theta_2^{(2)}(t) \end{Bmatrix} = \Theta_1^{(2)} \begin{Bmatrix} 1 \\ -1 \end{Bmatrix} \cos(\omega_2 t + \phi_2) \tag{8}$$

(c) Free vibration response:

Using linear superposition of natural modes, the free vibration response of the system is given by

$$\vec{\theta}(t) = c_1\, \vec{\theta}^{(1)}(t) + c_2\, \vec{\theta}^{(2)}(t) \tag{9}$$

By choosing $c_1 = c_2 = 1$, with no loss of generality, Eqs. (7) to (9) lead to

$$\theta_1(t) = \Theta_1^{(1)} \cos(\omega_1 t + \phi_1) + \Theta_1^{(2)} \cos(\omega_2 t + \phi_2) \tag{10}$$

$$\theta_2(t) = \Theta_1^{(1)} \cos(\omega_1 t + \phi_1) - \Theta_1^{(2)} \cos(\omega_2 t + \phi_2) \tag{11}$$

where $\Theta_1^{(1)}$, ϕ_1, $\Theta_1^{(2)}$ and ϕ_2 are constants to be determined from the initial conditions. When $\theta_1(0) = a$, $\theta_2(0) = 0$, $\dot{\theta}_1(0) = 0$ and $\dot{\theta}_2(0) = 0$, Eqs. (10) and (11) yield

$$\left.\begin{aligned} a &= \Theta_1^{(1)} \cos\phi_1 + \Theta_1^{(2)} \cos\phi_2 \\ 0 &= \Theta_1^{(1)} \cos\phi_1 - \Theta_1^{(2)} \cos\phi_2 \\ 0 &= -\omega_1 \Theta_1^{(1)} \sin\phi_1 - \omega_2 \Theta_1^{(2)} \sin\phi_2 \\ 0 &= -\omega_1 \Theta_1^{(1)} \sin\phi_1 + \omega_2 \Theta_1^{(2)} \sin\phi_2 \end{aligned}\right\} \tag{12}$$

Eqs. (12) can be solved for $\Theta_1^{(1)}$, ϕ_1, $\Theta_1^{(2)}$ and ϕ_2 to obtain

$$\left.\begin{aligned} \theta_1(t) &= a \cos \frac{\omega_2 - \omega_1}{2} t \cdot \cos \frac{\omega_2 + \omega_1}{2} t \\ \theta_2(t) &= a \sin \frac{\omega_2 - \omega_1}{2} t \cdot \sin \frac{\omega_2 + \omega_1}{2} t \end{aligned}\right\} \tag{13}$$

(d) conditions for beating:

When $\frac{2k d^2}{m\ell^2} << \frac{g}{\ell}$ or $k << \frac{mg\ell}{2d^2}$, (14)

the two frequency components in Eqs. (13), namely, $\frac{\omega_2-\omega_1}{2}$ and $\frac{\omega_2+\omega_1}{2}$, can be approximated as

$$\Omega_1 = \frac{\omega_2-\omega_1}{2} \simeq \frac{k}{2m}\frac{d^2}{\sqrt{g\ell^3}} \tag{15}$$

and

$$\Omega_2 = \frac{\omega_2+\omega_1}{2} \simeq \sqrt{\frac{g}{\ell}} + \frac{k}{2m}\frac{d^2}{\sqrt{g\ell^3}} \tag{16}$$

This implies that the motions of the pendulums are given by

$$\left.\begin{aligned} \theta_1(t) &\simeq a\cos\Omega_1 t\cdot\cos\Omega_2 t \\ \theta_2(t) &\simeq a\sin\Omega_1 t\cdot\sin\Omega_2 t \end{aligned}\right\} \tag{17}$$

This motion, Eqs. (17), denotes beating phenomenon.

Problem 5.61

(5.61) Equations of motion: $m_1 \ddot{x}_1 + (k_1+k_2)x_1 - k_2 x_2 = F_{10} \cos\omega t = Re\left(F_{10}\, e^{i\omega t}\right)$

$$m_2 \ddot{x}_2 + (k_2+k_3)x_2 - k_2 x_1 = F_{20} \cos\omega t = Re\left(F_{20}\, e^{i\omega t}\right)$$

Assuming $x_j(t) = X_j\, e^{i\omega t}$; $j = 1,2$ along with $F_j(t) = F_{j0}\, e^{i\omega t}$; $j=1,2$, the equations of motion can be expressed as

$$(-\omega^2 m_1 + k_1 + k_2)\, X_1 - k_2 X_2 = F_{10}$$

$$-k_2 X_1 + (-\omega^2 m_2 + k_2 + k_3)\, X_2 = F_{20}$$

i.e. $[Z(i\omega)]\, \vec{X} = \vec{F_0}$ ---- (E_1)

where $Z_{11}(i\omega) = -\omega^2 m_1 + k_1 + k_2$, $Z_{12}(i\omega) = Z_{21}(i\omega) = -k_2$,

$Z_{22}(i\omega) = -\omega^2 m_2 + k_2 + k_3$,

$\vec{X} = \begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix}$, $\vec{F_0} = \begin{Bmatrix} F_{10} \\ F_{20} \end{Bmatrix}$

Solution of (E_1) can be expressed, using Eqs. (5.35), as

$$X_1 = \frac{(-\omega^2 m_2 + k_2 + k_3)\, F_{10} + k_2 F_{20}}{(-\omega^2 m_1 + k_1 + k_2)(-\omega^2 m_2 + k_2 + k_3) - k_2^2} \qquad \text{---- } (E_2)$$

$$X_2 = \frac{k_2 F_{10} + (-\omega^2 m_1 + k_1 + k_2)\, F_{20}}{(-\omega^2 m_1 + k_1 + k_2)(-\omega^2 m_2 + k_2 + k_3) - k_2^2} \qquad \text{---- } (E_3)$$

Since X_1 and X_2 are real (since there is no damping), the final solution is given by

$$x_1(t) = X_1 \cos\omega t$$

$$x_2(t) = X_2 \cos\omega t$$

where X_1 and X_2 are given by (E_2) and (E_3).