Master 2 Financial Economics – Université Paris 1 Panthéon-Sorbonne

Empirical Application 1

Financial Econometrics

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I. EMPIRICAL APPLICATION 1

Question 1: Argument your choice of series by referring to a financial issue you are interested in

The FRED's database is well known by all economists and beyond for its large number of series and its great historical depth. Having at our disposal such a wealth of data, we decided for the upcoming empirical applications to investigate the relationship between three economic variables:

i. 5 years Breakeven Inflation rate (Breakeven5Y)
ii. S&P500 Stock Index (SP500)
iii. WTI Spot Crude Oil Price (WTI)

The summary of the chosen variables can be found in Table 1. The idea behind this choice is to understand how the main U.S. stock index (the S&P500 accounts for roughly 75% of the total market capitalization) and oil prices (WTI is the main oil produced in the U.S.) contribute to forming inflation expectations, using the FRED's 5-years breakeven inflation expectations measure.

The existing literature has already explored the many domestic surveys of consumers to understand the underlying properties of inflation expectations and identified idiosyncratic characteristics that may be determinant to predict future inflation. In our study, we rather focus on the view of professional market participants that should have a better understanding of economic mechanisms and thus have a more accurate forecast. As the number of variables is limited, we decided to go for two control variables, which are the S&P500 Stock Index, which will account for the general state of the economy, and the WTI Spot crude oil price, which we think is highly yet unconsciously linked to the notion of increases in prices.

Question 2: Analysis of the dynamics of each series

A. Justify the choice of the observation period and of the frequency

In order to identify any trend, drift or seasonal variations in the series chosen, we gather as much historical data as available in the FRED's Database. The frequency of our time series is monthly, starting in November 2011 and ending in August 2021. Our sample is gapless within the observation period. We chose monthly data as it is the finest degree of time granularity available to us. We also expect this monthly data to be the best way to identify possible trends, drifts and seasonal variations. Figure 1 displays the time plot of the untransformed variables.

B. Transform the series (log) and decompose each series separately and analyze different components

Step 1: Log-transformation of the S&P500 and WTI variables:

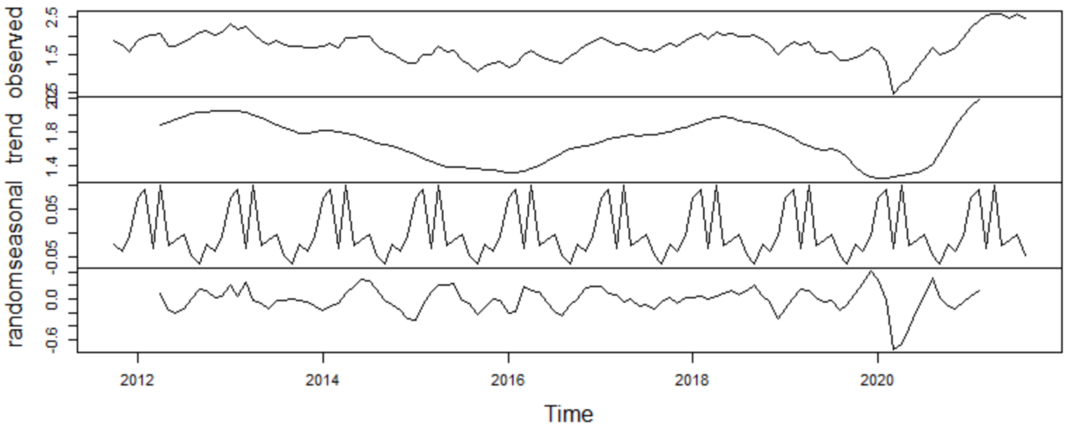
As the *Breakeven5Y* variable is already in percentages, we do not log-transform it. However, we do perform the log-transformation for the *SP500* and *WTI* variables, as they account for an index and dollars respectively. The new variables are then *ln(SP500)* and *ln(WTI)* Figure 2 displays the time plot of the transformed variables.

Step 2: decomposition of the log-transformed series of S&P500 and WTI, and of 5Y breakeven inflation rate

Once the variables are log-transformed, we perform a decomposition of the time series to analyze their different components. Figure 3, figure 4 and figure 5 show this decomposition for the *Breakeven5Y*, *SP500* and *WTI* variables respectively.

Figure 3

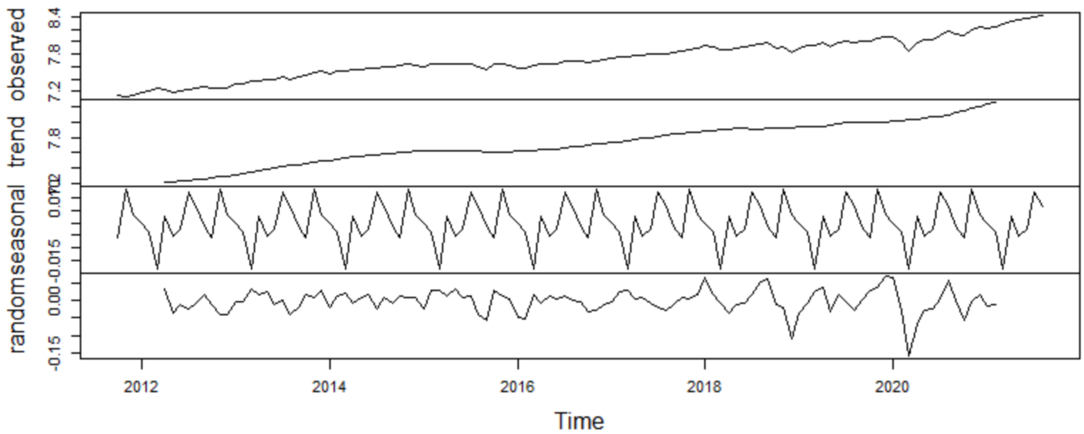
Decomposition of the untransformed Inflation Expectations *variable*



The *Inflation Expectations* variable exhibits a cubic trend curve and a strong seasonal component.

Figure 4

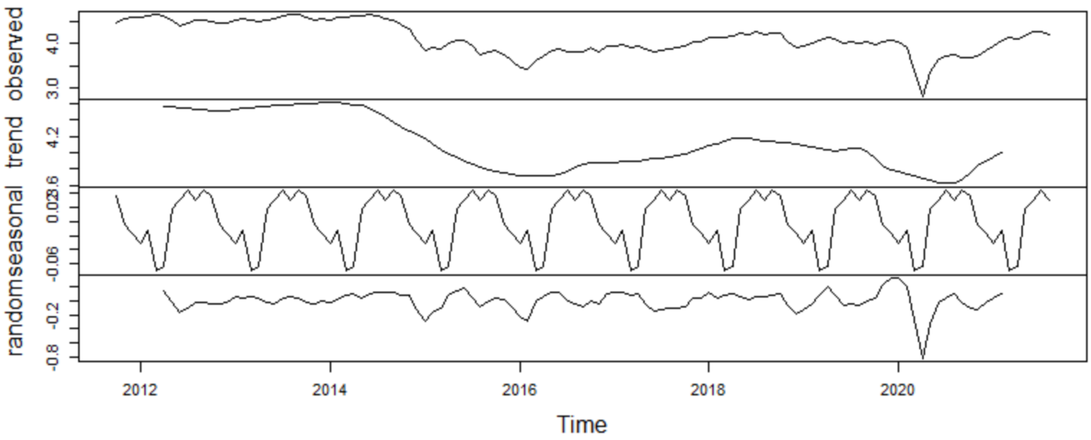
Decomposition of the log-transformed SP500 *variable*



The *ln(SP500)* variable exhibits a linear upward trend as well as a strong seasonal component.

Figure 5

Decomposition of the log-transformed WTI *variable*



The *ln(WTI)* variable exhibits also strong seasonal component but no clear trend.

C. Is there a deterministic trend (drift)?

In the previous section's figures, we established that our series exhibit different kind of trends and have seasonal components, leading to possible non-stationarity of our data. To corroborate our visual hypotheses, we conduct a series of tests to identify formally the existence of a drift and the trend in each series. By regressing each variable (namely *ln(SP500)*, *ln(WTI)* and *Breakeven5Y*) on their time trend, with an intercept, we find :

For the *Breakeven5Y* variable:

R-Code 1

Drift and trend regression for the Inflation Expectations *variable*

```
Call:
tslm(formula = Breakeven5Y ~ trend)

Residuals:
     Min       1Q   Median       3Q      Max 
-1.22545 -0.21196 -0.00169  0.22617  0.85926 

Coefficients:
              Estimate Std. Error t value Pr(>|t|)    
(Intercept)  1.7497693  0.0676160  25.878   <2e-16 ***
trend       -0.0003365  0.0009780  -0.344    0.731    
---
Signif. codes:  0 ‘***’ 0.001 ‘**’ 0.01 ‘*’ 0.05 ‘.’ 0.1 ‘ ’ 1

Residual standard error: 0.3665 on 117 degrees of freedom
Multiple R-squared:  0.001011,	Adjusted R-squared:  -0.007528 
F-statistic: 0.1184 on 1 and 117 DF,  p-value: 0.7314
```

The above results suggest that the value for the intercept (drift) is statistically significant yet it is not for the time trend. Therefore, we can conclude that the variable *Breakeven5Y* does have a drift but no time trend.

For the *ln(SP500)* variable :

R-Code 2

Drift and trend regression for the log Stock Index *variable*

```
Call:
tslm(formula = lSP500 ~ trend)

Residuals:
      Min        1Q    Median        3Q       Max 
-0.240750 -0.042467 -0.007759  0.044616  0.167695 

Coefficients:
             Estimate Std. Error t value Pr(>|t|)    
(Intercept) 7.1915181  0.0121400  592.38   <2e-16 ***
trend       0.0088878  0.0001756   50.62   <2e-16 ***
---
Signif. codes:  0 ‘***’ 0.001 ‘**’ 0.01 ‘*’ 0.05 ‘.’ 0.1 ‘ ’ 1

Residual standard error: 0.0658 on 117 degrees of freedom
Multiple R-squared:  0.9563,	Adjusted R-squared:  0.956 
F-statistic:  2562 on 1 and 117 DF,  p-value: < 2.2e-16
```

The above results suggest statistically significant values for both the intercept (drift) and the time trend. Therefore, we can confirm that the variable *ln(SP500)* has a drift and a time trend.

For the *ln(WTI)* variable :

R-Code 3

Drift and trend regression for the log WTI *variable*

```
Call:
tslm(formula = lWTI ~ trend)

Residuals:
     Min       1Q   Median       3Q      Max 
-1.04269 -0.18426  0.08262  0.18986  0.53200 

Coefficients:
              Estimate Std. Error t value Pr(>|t|)    
(Intercept)  4.5194693  0.0525523  85.999  < 2e-16 ***
trend       -0.0065087  0.0007601  -8.563  5.1e-14 ***
---
Signif. codes:  0 ‘***’ 0.001 ‘**’ 0.01 ‘*’ 0.05 ‘.’ 0.1 ‘ ’ 1

Residual standard error: 0.2848 on 117 degrees of freedom
Multiple R-squared:  0.3852,	Adjusted R-squared:   0.38 
F-statistic: 73.32 on 1 and 117 DF,  p-value: 5.097e-14
```

The above results also suggest statistically significant values for both the intercept (drift) and the time trend. Therefore, we can confirm that the variable *ln(WTI)* has a drift and a time trend.

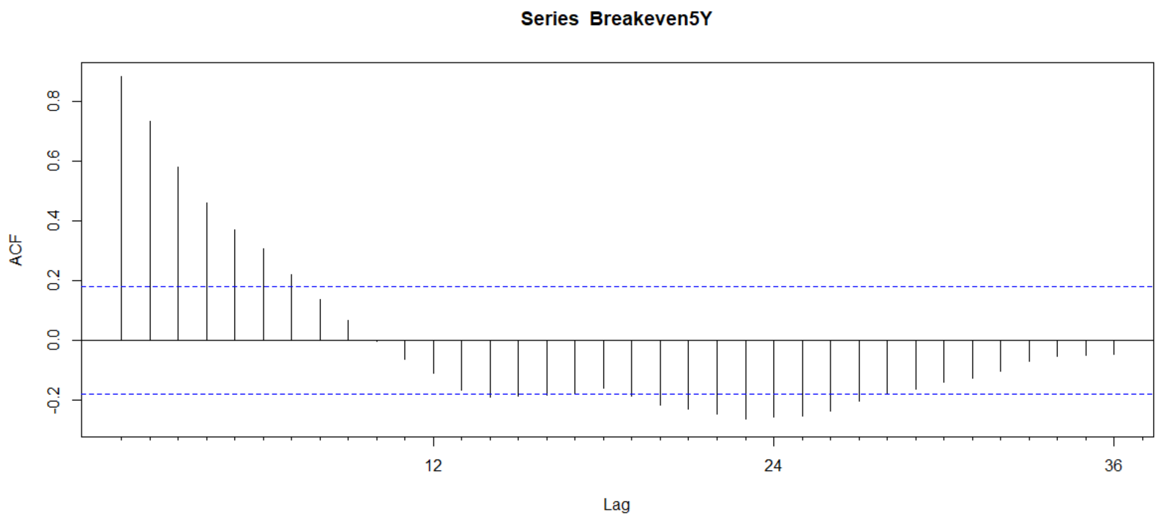
D. Are there seasonal variations?

As we have visually seen in Figures 3, 4 and 5, the chosen time series seem to exhibit a seasonal component. However, we suspect a case of "magnifying glass" effect since the axis scale has been lowered enough to visually exhibit the seasonality variations but that does not imply they are statistically significant.

We dig a bit further by plotting the ACF (Autocorrelation function) and the PACF (Partial Autocorrelation Function) and we find no sign of seasonal variations, as presented in figure 6, figure 7 and figure 8.

Figure 6

ACF and PACF graphs of the Inflation Expectations *variable*



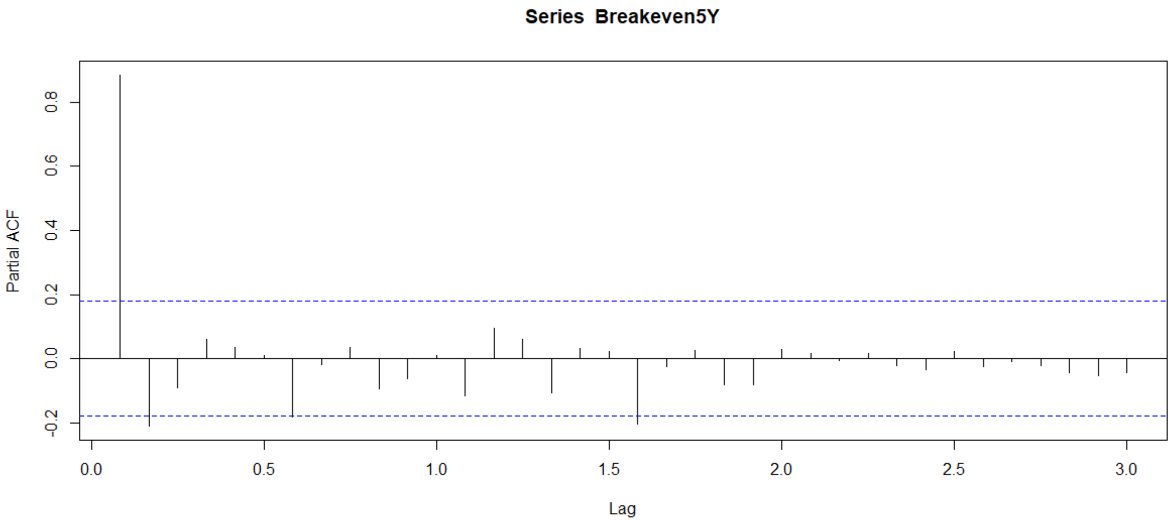
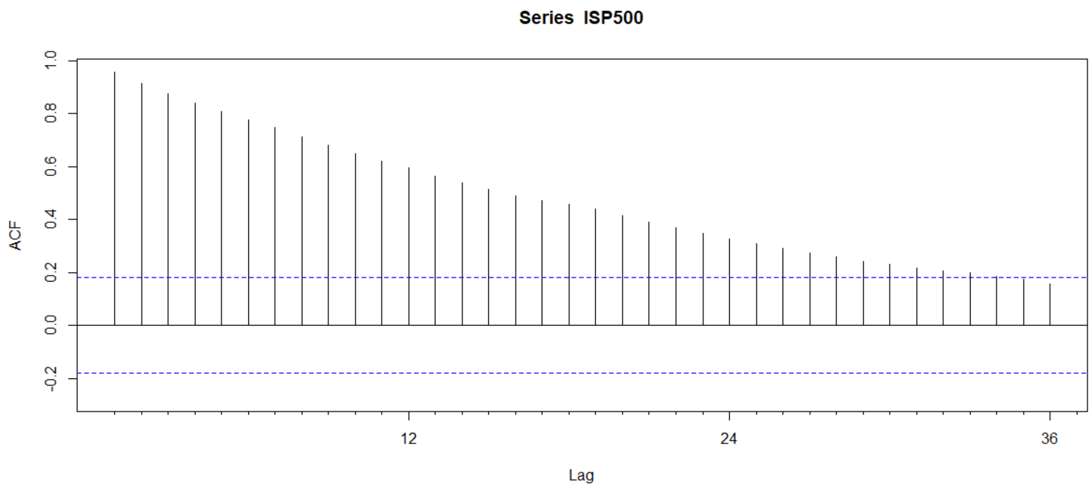


Figure 7

ACF and PACF graphs of the log Stock Index *variable*



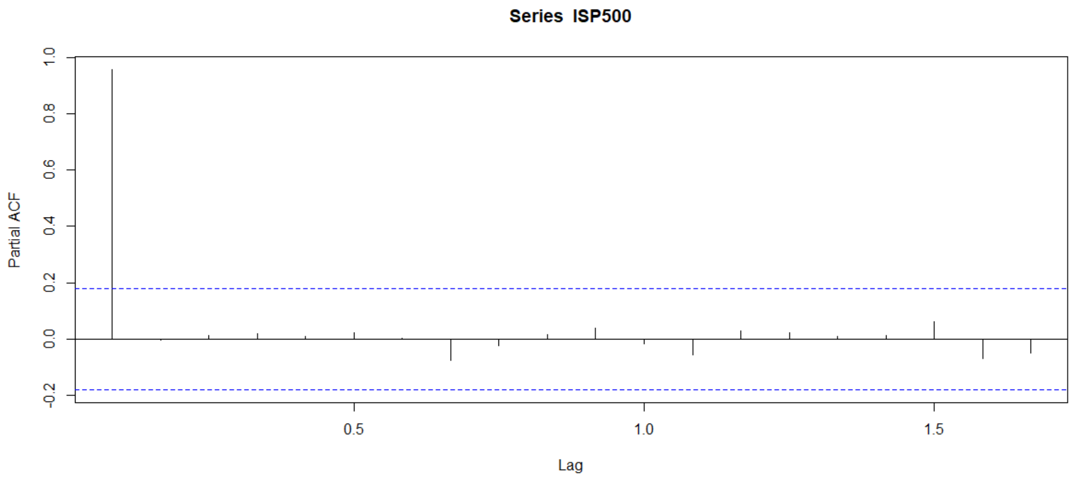
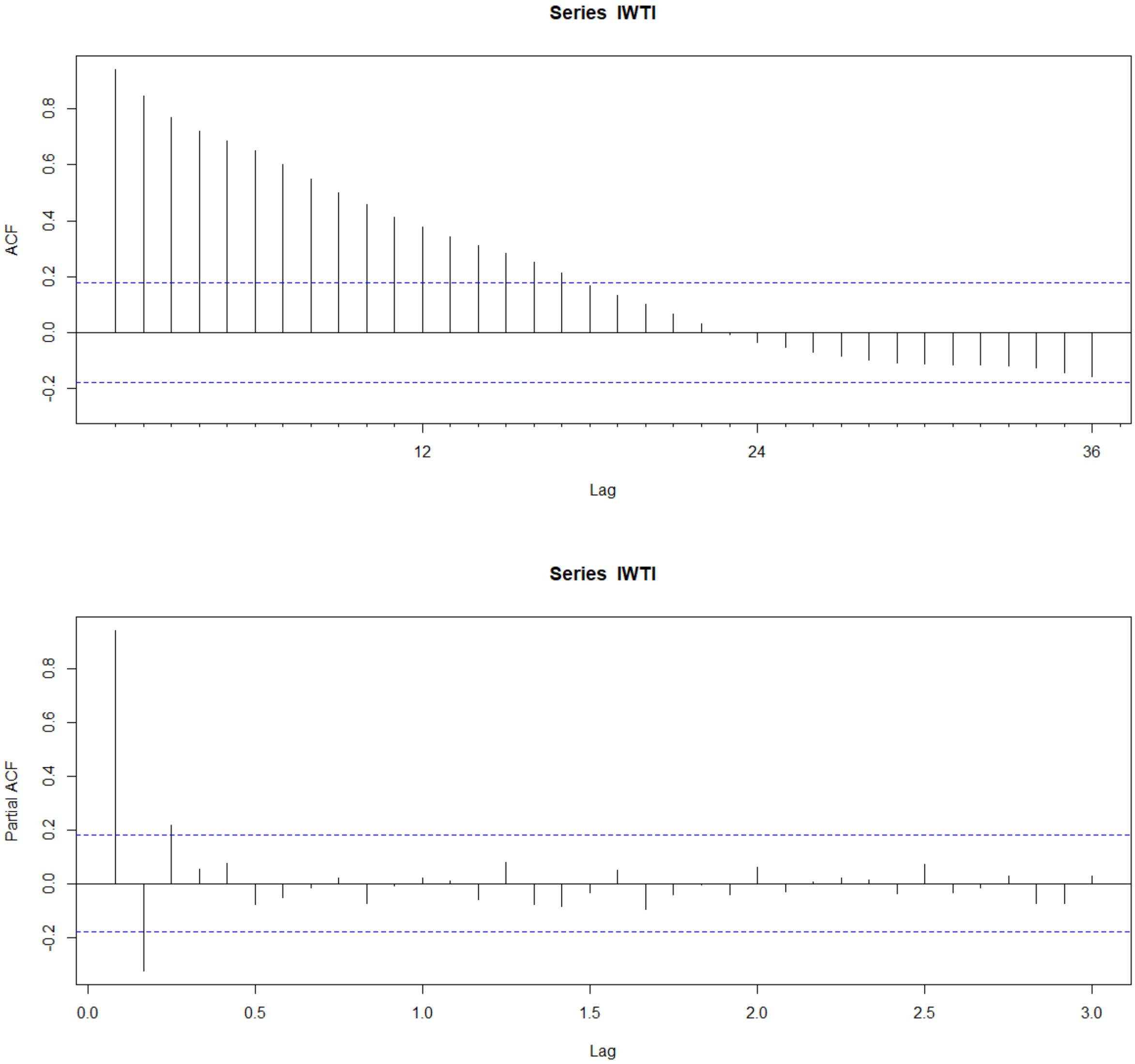


Figure 8

ACF and PACF graphs of the log WTI *variable*



To formalize our observations, we first perform a succinct Webel and Ollech (WO) seasonality test. Then, we test for the significance of the related parameters by removing the deterministic trend in the two variables in which we previously identified a time trend (namely *ln(SP500)* and *ln(WTI)*) and test again for seasonality.

The results of the WO tests are given in the following R-Code boxes. For the three variables, the results show no significant seasonality in our time series.

For the *Breakeven5Y* variable:

R-Code 4

WO seasonality test for the Inflation Expectations *variable*

```
> isSeasonal(Breakeven5Y)
[1] FALSE
> summary(wo(Breakeven5Y))
Test used:  WO

Test statistic:  0
P-value:  1 0.6227672 0.01928127

The WO - test does not identify  seasonality
```

For the *ln(SP500)* variable :

R-Code 5
WO seasonality test for the log Stock Index *variable*

```
> isSeasonal(lSP500)
[1] FALSE
> summary(wo(lSP500))
Test used:  WO

Test statistic:  0
P-value:  1 1 0.852026

The WO - test does not identify  seasonality
```

For the *ln(WTI)* variable :

R-Code 6
WO seasonality test for the log WTI *variable*

```
> isSeasonal(lWTI)
[1] FALSE
> summary(wo(lWTI))
Test used:  WO

Test statistic:  0
P-value:  1 1 0.8103998

The WO - test does not identify  seasonality
```

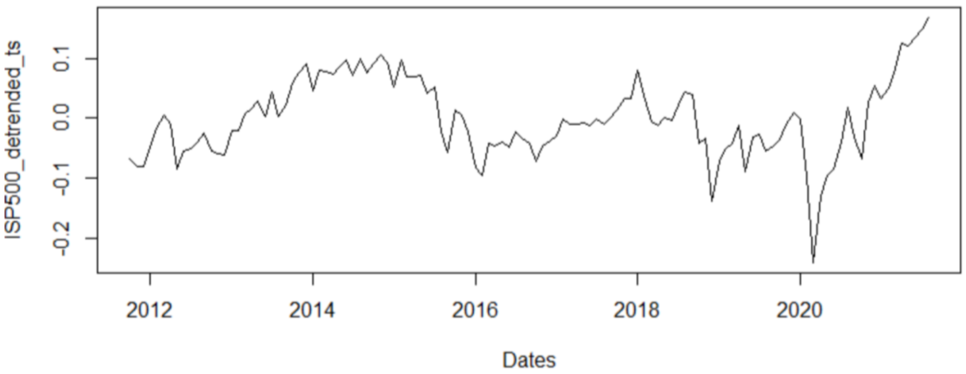
As we want to formally reject the seasonality hypothesis, we perform a 2-steps test with the de-trended variables.

<u>Step 1: De-trending the 2 time-series that exhibit a time trend:</u>

To remove the time trend, we choose to estimate a least-squares fit of a straight line to the data and to subtract the resulting function from the data. The plotted series can be found in figure 9 and figure 10.

For the *ln(SP500)* variable with removed trend:

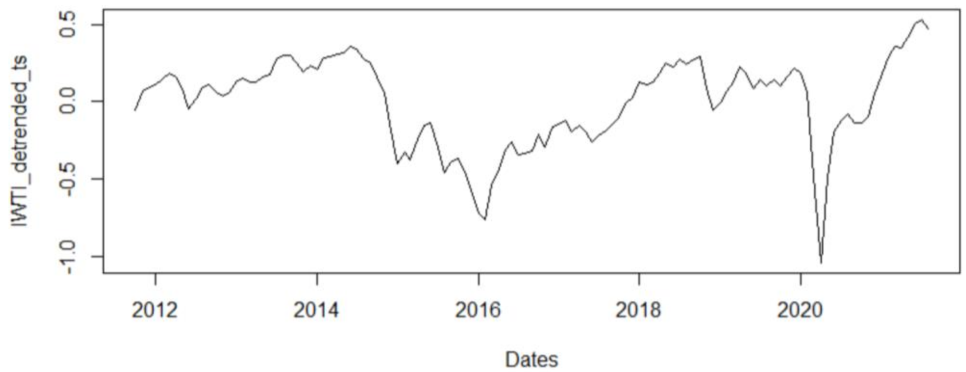
Figure 9
De-trended log Stock Index variable



For the *ln(WTI)* variable with removed trend:

Figure 10

De-trended log WTI variable



Step 2: We use the newly generated de-trended series as well as the *Breakeven5Y* variable to test for the existence of a seasonal pattern in each series by regressing the de-trended variable on seasonal lags:

This test is more powerful than just visual observation of the ACF/PACF graphs and adds a degree of depth to the WO test. The results are displayed in the following R-Code boxes. They confirm the previously obtained results by revealing the absence of seasonality as there are no statistically significant seasonal lags in our time series for each of the three variables.

For the *Breakeven5Y* variable:

R-Code 7

Test for seasonality with lags for the Inflation Expectations *variable*

```
Call:
tslm(formula = Breakeven5Y ~ trend + season)

Residuals:
     Min       1Q   Median       3Q      Max
-1.23992 -0.17472  0.01708  0.19662  0.86167
```

```
Coefficients:
              Estimate Std. Error t value Pr(>|t|)    
(Intercept)  1.8062059  0.1335508  13.524   <2e-16 ***
trend       -0.0003829  0.0010155  -0.377    0.707    
season2      0.0333829  0.1695130   0.197    0.844    
season3     -0.0372343  0.1695222  -0.220    0.827    
season4      0.0621486  0.1695374   0.367    0.715    
season5     -0.0534686  0.1695587  -0.315    0.753    
season6     -0.0500857  0.1695860  -0.295    0.768    
season7     -0.0287028  0.1696195  -0.169    0.866    
season8     -0.0713200  0.1696590  -0.420    0.675    
season9     -0.1732343  0.1741668  -0.995    0.322    
season10    -0.1071486  0.1695374  -0.632    0.529    
season11    -0.1247657  0.1695222  -0.736    0.463    
season12    -0.1053829  0.1695130  -0.622    0.535    
---
Signif. codes:  
0 ‘***’ 0.001 ‘**’ 0.01 ‘*’ 0.05 ‘.’ 0.1 ‘ ’ 1
```

```
Residual standard error: 0.379 on 106 degrees of freedom
Multiple R-squared:  0.03185,   Adjusted R-squared:  -0.0
7775 
F-statistic: 0.2906 on 12 and 106 DF,  p-value: 0.9897
```

For the *detrend_ln(SP500)* variable :

R-Code 8

Test for seasonality with lags for the de-trended log Stock Index variable

```
Call:
tslm(formula = lSP500_detrended_ts ~ season)

Residuals:
     Min       1Q   Median       3Q      Max 
-0.23084 -0.04472 -0.01145  0.04297  0.15051 
```

```
Coefficients:
              Estimate Std. Error t value Pr(>|t|)
(Intercept) -0.004515   0.021497  -0.210    0.834
season2      0.001741   0.030402   0.057    0.954
season3     -0.005397   0.030402  -0.178    0.859
season4      0.010575   0.030402   0.348    0.729
season5      0.003379   0.030402   0.111    0.912
season6      0.008363   0.030402   0.275    0.784
season7      0.023862   0.030402   0.785    0.434
season8      0.021703   0.030402   0.714    0.477
season9     -0.002795   0.031235  -0.089    0.929
season10    -0.011863   0.030402  -0.390    0.697
season11     0.005729   0.030402   0.188    0.851
season12    -0.001845   0.030402  -0.061    0.952

Residual standard error: 0.06798 on 107 degrees of freedo
m
Multiple R-squared:  0.02383,   Adjusted R-squared:  -0.0
7652
F-statistic: 0.2375 on 11 and 107 DF,  p-value: 0.9942
```

For the *detrend_ln(WTI)* variable :

R-Code 9

Test for seasonality with lags for the de-trended log WTI variable

```
Call:
tslm(formula = lWTI_detrended_ts ~ season)

Residuals:
     Min       1Q   Median       3Q      Max
-0.99055 -0.18281  0.07954  0.21543  0.47095
```

```
Coefficients:
              Estimate Std. Error t value Pr(>|t|)
(Intercept) -0.034350   0.093423  -0.368    0.714
season2      0.023964   0.132120   0.181    0.856
season3     -0.003402   0.132120  -0.026    0.980
season4     -0.017791   0.132120  -0.135    0.893
season5      0.060591   0.132120   0.459    0.647
season6      0.080605   0.132120   0.610    0.543
season7      0.095402   0.132120   0.722    0.472
season8      0.077934   0.132120   0.590    0.557
season9      0.045604   0.135740   0.336    0.738
season10     0.033414   0.132120   0.253    0.801
season11     0.010636   0.132120   0.080    0.936
season12     0.006368   0.132120   0.048    0.962

Residual standard error: 0.2954 on 107 degrees of freedom
Multiple R-squared:  0.01617,   Adjusted R-squared:  -0.0
8497
F-statistic: 0.1599 on 11 and 107 DF,  p-value: 0.999
```

As expected, the series do not exhibit a trending behavior but also do not display seasonal significance for any of the three variables tested. We can finally conclude that our time series for the selected variables are not seasonally affected.

E. Is there a stochastic trend? Perform a unit root test; if it is the case, examine the first difference of each series and check that there is no more stochastic trend.

After testing each time series of our data for the presence of a trend, a drift and a seasonal component, we perform a set of tests to identify an eventual stochastic trend. Through the Augmented Dickey-Fuller (ADF) and Phillips-Perron (PP) tests, it is possible to identify non-stationarity. For those tests, the null hypothesis is H_0: *Variable is non-stationary*. Rejecting the null hypothesis would then imply that the variable is good for use. Otherwise, differencing is necessary and the first difference undergoes the same test with the same null hypothesis.

The results for the ADF test present non-stationarity issues for all three variables. Variables have been first differenced and the test has been run again. The new results comply with the stationarity condition as all of the variables are stationary at a 1% significance level. The ADF test extended results are displayed in the R-Code 13, R-Code 14 and R-code 15 for the non-differenced variables and in the R-Code 19, R-Code 20 and R-Code 21 for the differenced variables.

<u>Step 1: Perform ADF unit root tests on the non-differenced variables</u>

Before diving into the extended results, we perform a succinct ADF test to identify stationarity issues with our three variables:

R-Code 10

ADF unit root test for the Inflation Expectations variable

```
> adf.test(Breakeven5Y, k=1)

        Augmented Dickey-Fuller Test

data:  Breakeven5Y
Dickey-Fuller = -2.5321, Lag order = 1, p-value = 0.3556
alternative hypothesis: stationary
```

R-Code 11

ADF unit root test for the de-trended log Stock Index variable

```
> adf.test(lSP500_detrended_ts, k=1)

        Augmented Dickey-Fuller Test

data:  lSP500_detrended_ts
Dickey-Fuller = -2.5541, Lag order = 1, p-value = 0.3465
alternative hypothesis: stationary
```

R-Code 12

ADF unit root test for the de-trended log WTI variable

```
> adf.test(lWTI_detrended_ts, k=1)

        Augmented Dickey-Fuller Test

data:  lWTI_detrended_ts
Dickey-Fuller = -3.0646, Lag order = 1, p-value = 0.1345
alternative hypothesis: stationary
```

From the above results, our time series seem to have a unit root as we cannot reject the null hypothesis, meaning that the series do have a unit root and thus a stochastic trend.

We analyze a depth further our data by performing an extended set of ADF tests on our selected variables. The results are displayed in the R-Code boxes below.

R-Code 13

ADF unit root test for the de-trended Inflation Expectations variable

```
###############################################
# Augmented Dickey-Fuller Test Unit Root Test #
###############################################

Test regression trend


Call:
lm(formula = z.diff ~ z.lag.1 + 1 + tt + z.diff.lag)

Residuals:
     Min       1Q   Median       3Q      Max
-0.83695 -0.09280  0.01226  0.09703  0.31783

Coefficients:
              Estimate Std. Error t value Pr(>|t|)
(Intercept)  0.1780353  0.0805973   2.209   0.0292 *
z.lag.1     -0.1070369  0.0422728  -2.532   0.0127 *
tt           0.0001864  0.0004347   0.429   0.6689
z.diff.lag   0.2131685  0.0943732   2.259   0.0258 *
---
Signif. codes:  0 ‘***’ 0.001 ‘**’ 0.01 ‘*’ 0.05 ‘.’ 0.1 ‘ ’ 1

Residual standard error: 0.1575 on 113 degrees of freedom
Multiple R-squared:  0.07869,    Adjusted R-squared:  0.05423
F-statistic: 3.217 on 3 and 113 DF,  p-value: 0.02554


Value of test-statistic is: -2.5321 2.3224 3.418

Critical values for test statistics:
      1pct  5pct 10pct
tau3 -3.99 -3.43 -3.13
phi2  6.22  4.75  4.07
phi3  8.43  6.49  5.47
```

The above R-Code box suggests that we fail to reject the hypothesis of the presence of a unit root. Indeed, in this test, we are interested in the t-stat value of the *z.lag.1* (which corresponds to the unit root presence test) and we benchmark it to the critical values of the “tau3” thresholds. In the case of the expected inflation variable, the t-stat value (-2.5321) is not statistically significant as it does exceeds the 10% critical threshold value set at -3.13. Therefore, the 5-year breakeven inflation series has a unit root, is not stationary and thus exhibits a stochastic trend.

R-Code 14

ADF unit root test for the de-trended log Stock Index variable

```
> SP500_ADF <- ur.df(lSP500_detrended_ts,type="trend",selectlags="AIC")
> summary(SP500_ADF)

###############################################
# Augmented Dickey-Fuller Test Unit Root Test #
###############################################

Test regression trend


Call:
lm(formula = z.diff ~ z.lag.1 + 1 + tt + z.diff.lag)

Residuals:
      Min        1Q    Median        3Q       Max 
-0.161345 -0.013460  0.003486  0.021506  0.079689 

Coefficients:
              Estimate Std. Error t value Pr(>|t|)  
(Intercept)  6.140e-04  6.898e-03   0.089    0.929  
z.lag.1     -1.438e-01  5.629e-02  -2.554    0.012 *
tt           2.363e-05  1.004e-04   0.235    0.814  
z.diff.lag  -1.726e-02  9.635e-02  -0.179    0.858  
---
Signif. codes:  
0 ‘***’ 0.001 ‘**’ 0.01 ‘*’ 0.05 ‘.’ 0.1 ‘ ’ 1

Residual standard error: 0.03656 on 113 degrees of freedom
Multiple R-squared:  0.06473,	Adjusted R-squared:  0.0399 
F-statistic: 2.607 on 3 and 113 DF,  p-value: 0.05516


Value of test-statistic is: -2.5541 2.386 3.3477 

Critical values for test statistics: 
      1pct  5pct 10pct
tau3 -3.99 -3.43 -3.13
phi2  6.22  4.75  4.07
phi3  8.43  6.49  5.47
```

Here again, the above R-Code box suggests that we fail to reject the hypothesis of the presence of a unit root. Indeed, in this test, we are still interested in the t-stat value of the *z.lag.1* and we benchmark it to the critical values of the “tau3” thresholds. In the case of the stock index variable, the t-stat value (-2.5541) is not statistically significant as it does exceeds the 10% critical threshold value set at -3.13. Therefore, the de-trended *ln(SP500)* series has a unit root, is not stationary and thus exhibits a stochastic trend.

R-Code 15

ADF unit root test for the de-trended log WTI variable

```
> WTI_ADF <- ur.df(lWTI_detrended_ts,type="trend",selectlags="AIC")
> summary(WTI_ADF)

###############################################
# Augmented Dickey-Fuller Test Unit Root Test #
###############################################

Test regression trend


Call:
lm(formula = z.diff ~ z.lag.1 + 1 + tt + z.diff.lag)

Residuals:
     Min       1Q   Median       3Q      Max
-0.49933 -0.04805  0.00764  0.05829  0.61332
```

```
Coefficients:
              Estimate Std. Error t value Pr(>|t|)
(Intercept) -0.0069706  0.0215593  -0.323 0.747049
z.lag.1     -0.1190394  0.0388432  -3.065 0.002726 **
tt           0.0001375  0.0003137   0.438 0.661987
z.diff.lag   0.3427638  0.0896120   3.825 0.000215 ***
---
Signif. codes:  0 ‘***’ 0.001 ‘**’ 0.01 ‘*’ 0.05 ‘.’ 0.1 ‘ ’ 1

Residual standard error: 0.1143 on 113 degrees of freedom
Multiple R-squared:  0.1496,    Adjusted R-squared:  0.127
F-statistic: 6.627 on 3 and 113 DF,  p-value: 0.0003647


Value of test-statistic is: -3.0646 3.2532 4.8611

Critical values for test statistics:
      1pct  5pct 10pct
tau3 -3.99 -3.43 -3.13
phi2  6.22  4.75  4.07
phi3  8.43  6.49  5.47
```

Finally, the above R-Code box suggests that we again fail to reject the hypothesis of the presence of a unit root. Indeed, in this test, we are still interested in the t-stat value of the *z.lag.1* and we benchmark it to the critical values of the “tau3” thresholds. In the case of the WTI oil prices variable, the t-stat value (-3.0646) is not statistically significant as it does exceeds the 10% critical threshold value set at -3.13. Therefore, the de-trended *ln(WTI)* series has a unit root, is not stationary and thus exhibits a stochastic trend.

In conclusion of this first step, **each of our three time series exhibit stationarity issues** as we have not rejected the null hypothesis (H_o: *variable is non-stationary*) for each of them. As the stationarity condition is required to obtain an accurate estimation of autoregressive models ‘coefficients, differencing our variables seems necessary.

Step 2 : First-differencing our variables and re-running the ADF tests

In this second step, we first-difference each of our variables and re-run the tests to identify stationarity issues. The newly generated times series are plotted in figure 11, figure 12 and figure 13. As expected, first-differencing solves the stationarity issues. The results can be found in the following R-Code boxes.

As for the non-differenced variables and before diving into the extended results, we run a succinct ADF test for the first-differenced variables.

R-Code 16

ADF unit root test for the first-differenced Inflation Expectations variable

```
> adf.test(Breakeven5Y_DIFF_ts, k=1)

        Augmented Dickey-Fuller Test

data:  Breakeven5Y_DIFF_ts
Dickey-Fuller = -6.7651, Lag order = 1, p-value = 0.01
alternative hypothesis: stationary

Warning message:
In adf.test(Breakeven5Y_DIFF_ts, k = 1) :
  p-value smaller than printed p-value
```

R-Code 17

ADF unit root test for the first-differenced log Stock Index variable

```
> adf.test(lSP500_detrended_DIFF_ts, k=1)

        Augmented Dickey-Fuller Test

data:  lSP500_detrended_DIFF_ts
Dickey-Fuller = -9.3433, Lag order = 1, p-value = 0.01
alternative hypothesis: stationary

Warning message:
In adf.test(lSP500_detrended_DIFF_ts, k = 1) :
  p-value smaller than printed p-value
```

R-Code 18

ADF unit root test for the first-differenced log WTI variable

```
> adf.test(lWTI_detrended_DIFF_ts, k=1)

        Augmented Dickey-Fuller Test

data:  lWTI_detrended_DIFF_ts
Dickey-Fuller = -8.4638, Lag order = 1, p-value = 0.01
alternative hypothesis: stationary

Warning message:
In adf.test(lWTI_detrended_DIFF_ts, k = 1) :
  p-value smaller than printed p-value
```

From the above results, first-differencing seems to have solved the non-stationarity issues as for each of the three variables, the ADF tests allow us to reject the null hypothesis, meaning the variables do not have a stochastic trend and are thus good for use.

Again, we analyze a depth further our data by performing an extended set of ADF tests on our selected variables. The results are displayed in the R-Code boxes below.

R-Code 19

ADF unit root test for the first-differenced Inflation Expectations variable

```
> Breakeven5Y_DIFF_ADF <- ur.df(Breakeven5Y_DIFF_ts,type="trend",selectlags="AIC")
> summary(Breakeven5Y_DIFF_ADF)

###############################################
# Augmented Dickey-Fuller Test Unit Root Test #
###############################################

Test regression trend


Call:
lm(formula = z.diff ~ z.lag.1 + 1 + tt + z.diff.lag)

Residuals:
     Min       1Q   Median       3Q      Max
-0.81574 -0.10064  0.01858  0.09537  0.33771

Coefficients:
              Estimate Std. Error t value Pr(>|t|)
(Intercept) -0.0058917  0.0307536  -0.192    0.848
z.lag.1     -0.8324005  0.1230427  -6.765 6.36e-10 ***
tt           0.0002095  0.0004520   0.463    0.644
z.diff.lag  -0.0312449  0.0942238  -0.332    0.741
---
Signif. codes:  0 ‘***’ 0.001 ‘**’ 0.01 ‘*’ 0.05 ‘.’ 0.1 ‘ ’ 1

Residual standard error: 0.162 on 112 degrees of freedom
Multiple R-squared:  0.4314,    Adjusted R-squared:  0.4162
F-statistic: 28.32 on 3 and 112 DF,  p-value: 1.051e-13


Value of test-statistic is: -6.7651 15.2858 22.9231

Critical values for test statistics:
      1pct  5pct 10pct
tau3 -3.99 -3.43 -3.13
phi2  6.22  4.75  4.07
phi3  8.43  6.49  5.47
```

The above R-Code box suggests that we can now reject the null hypothesis of the presence of a unit root for the *Breakeven5Y* variable. Indeed, the t-stat value of the *z.lag.1* (-6.7651) is statistically significant as it exceeds the 1% critical threshold value of the “tau3” benchmark set at -3.99. Therefore, the first-differenced and de-trended Inflation Expectations series does not have a unit root, meaning it is stationary and thus has not a stochastic trend.

We can even push our analysis of this R-Code box by stating that the test rejects the null hypothesis of time trend (the t-stat of the “phi2” (15.2858) against a critical value set at 6.22

for the 1% significance level) and also rejects the null hypothesis of a drift (the t-stat of the “phi3” (22.9231) against a critical value set at 8.43 for the 1% significance level).

R-Code 20

ADF unit root test for the first-differenced log Stock Index variable

```
> SP500_DIFF_ADF <- ur.df(lSP500_detrended_DIFF_ts,type="trend",selectlags="AIC")
> summary(SP500_DIFF_ADF)

###############################################
# Augmented Dickey-Fuller Test Unit Root Test #
###############################################

Test regression trend


Call:
lm(formula = z.diff ~ z.lag.1 + 1 + tt + z.diff.lag)

Residuals:
      Min        1Q    Median        3Q       Max
-0.160117 -0.010847  0.004896  0.022008  0.075513

Coefficients:
              Estimate Std. Error t value Pr(>|t|)
(Intercept) -1.237e-04  7.041e-03  -0.018   0.9860
z.lag.1     -1.290e+00  1.380e-01  -9.343 1.08e-15 ***
tt           4.705e-05  1.032e-04   0.456   0.6494
z.diff.lag   1.745e-01  9.310e-02   1.874   0.0635 .
---
Signif. codes:  0 ‘***’ 0.001 ‘**’ 0.01 ‘*’ 0.05 ‘.’ 0.1 ‘ ’ 1

Residual standard error: 0.03719 on 112 degrees of freedom
Multiple R-squared:  0.5623,    Adjusted R-squared:  0.5506
F-statistic: 47.97 on 3 and 112 DF,  p-value: < 2.2e-16


Value of test-statistic is: -9.3433 29.1027 43.6506

Critical values for test statistics:
      1pct  5pct 10pct
tau3 -3.99 -3.43 -3.13
phi2  6.22  4.75  4.07
phi3  8.43  6.49  5.47
```

This next R-Code box also suggests that we can now reject the null hypothesis of the presence of a unit root for the ln(*SP500)* variable. Indeed, the t-stat value of the *z.lag.1* (-9.3433) is statistically significant as it exceeds the 1% critical threshold value of the “tau3” benchmark set at -3.99. Therefore, the first-differenced and de-trended log Stock Index series does not have a unit root, meaning it is stationary and thus has not a stochastic trend.

Again, we can push our analysis of this R-Code box by stating that the test rejects the null hypothesis of time trend (the t-stat of the “phi2” (29.1027) against a critical value set at 6.22 for the 1% significance level) and also rejects the null hypothesis of a drift (the t-stat of the “phi3” (43.6206) against a critical value set at 8.43 for the 1% significance level).

R-Code 21

ADF unit root test for the first-differenced log WTI variable

```
> WTI_DIFF_ADF <- ur.df(lWTI_detrended_DIFF_ts,type="trend",selectlags="AIC")
> summary(WTI_DIFF_ADF)

###############################################
# Augmented Dickey-Fuller Test Unit Root Test #
###############################################

Test regression trend


Call:
lm(formula = z.diff ~ z.lag.1 + 1 + tt + z.diff.lag)

Residuals:
     Min       1Q   Median       3Q      Max
-0.52008 -0.05455  0.00915  0.05519  0.58694

Coefficients:
              Estimate Std. Error t value Pr(>|t|)
(Intercept) -0.0095985  0.0217737  -0.441  0.66019
z.lag.1     -0.9256898  0.1093710  -8.464 1.12e-13 ***
tt           0.0002154  0.0003196   0.674  0.50165
z.diff.lag   0.2745748  0.0905532   3.032  0.00302 **
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Residual standard error: 0.1149 on 112 degrees of freedom
Multiple R-squared:  0.411,     Adjusted R-squared:  0.3952
F-statistic: 26.05 on 3 and 112 DF,  p-value: 7.38e-13


Value of test-statistic is: -8.4638 23.881 35.8181

Critical values for test statistics:
      1pct  5pct 10pct
tau3 -3.99 -3.43 -3.13
phi2  6.22  4.75  4.07
phi3  8.43  6.49  5.47
```

Finally, the above R-Code box also suggests that we can now reject the null hypothesis of the presence of a unit root for the ln(*WTI)* variable. Indeed, the t-stat value of the *z.lag.1* (-8.4638) is statistically significant as it exceeds the 1% critical threshold value of the "tau3" benchmark set at -3.99. Therefore, the first-differenced and de-trended log WTI series does not have a unit root, meaning it is stationary and thus has not a stochastic trend.

Once again, we can push our analysis of this R-Code box by stating that the test rejects the null hypothesis of time trend (the t-stat of the "phi2" (23.881) against a critical value set at 6.22 for the 1% significance level) and also rejects the null hypothesis of a drift (the t-stat of the "phi3" (35.8181) against a critical value set at 8.43 for the 1% significance level).

In order to confirm the ADF results, we also run succinct PP tests. The results are displayed in the R-Code 22, R-Code 23 and R-Code 24. We find that the PP tests 'results are in line with the ones found following the ADF test framework:

- Step 1 : PP tests show that each of the three variables exhibit non-stationarity issues
- Step 2 : first-differencing the variables solves the non-stationarity issues

In the step 1, PP tests show that all variables are non-stationary. In step 2, first differencing the variables solves this issue as the results of the second run display stationarity at a 1% significance level for all three variables.

We are now left with the cyclical (and stationary) component of the 5-year breakeven inflation expectations, the S&P500 stock index and the WTI oil prices time series, as **we have removed the drift, the deterministic trend and the stochastic component by first-differencing our time series**.

F. For the cyclical component, estimate a stationary ARMA model.

Now that we have removed the deterministic and stochastic trends of our time series, we can estimate an ARMA model with their cyclical component term, which has been proven stationary.

In order to estimate the stationary ARMA model for each variables 'component term, we follow a three-step procedure:

- Step 1: we perform an *auto.arima* fit test to let R determine the optimal parameters p, d and q for our ARIMA models
- Step 2: we evaluate and iterate the models by testing various p, and q parameters to be sure the *auto.arima* function has given us the optimal parameters
- Step 3: we finally check the residuals of each model to make sure that there is not autocorrelation

Step 1: Performing an *auto.arima* function and optimal parameters (p, d, q) for our ARIMA model

To determine the optimal parameters of our ARMA model, we could estimate as many models as parameters p, d, and q exist. To alleviate this workload, we use the *auto.arima* function (included in the *forecast* package) which gives us the optimal parameters that fit our data.

The input time series are de-trended and first-differenced. The results are the following:

R-Code 25

Optimal parameters (p, d, q) estimated by the auto.arima function for the Inflation Expectation variable

```
> auto.arima(Breakeven5Y_DIFF_ts, seasonal=FALSE)
Series: Breakeven5Y_DIFF_ts
ARIMA(2,0,2) with zero mean

Coefficients:
         ar1      ar2      ma1     ma2
      0.6208  -0.7443  -0.4328  0.7682
s.e.  0.1841   0.1523   0.1764  0.1335

sigma^2 estimated as 0.02438:  log likelihood=53.54
AIC=-97.08   AICc=-96.55   BIC=-83.23
```

R-Code 26

Optimal parameters (p, d, q) estimated by the auto.arima function for the Stock Index variable

```
> auto.arima(lSP500_detrended_DIFF_ts, seasonal=FALSE)
Series: lSP500_detrended_DIFF_ts
ARIMA(2,0,2) with zero mean

Coefficients:
         ar1      ar2      ma1     ma2
      1.3857  -0.8994  -1.5207  0.9521
s.e.  0.0663   0.0585   0.0592  0.0706

sigma^2 estimated as 0.001261:  log likelihood=227.67
AIC=-445.34   AICc=-444.81   BIC=-431.49
```

R-Code 27

Optimal parameters (p, d, q) estimated by the auto.arima function for the WTI oil price variable

```
> auto.arima(lWTI_detrended_DIFF_ts, seasonal=FALSE)
Series: lWTI_detrended_DIFF_ts
ARIMA(0,0,1) with zero mean

Coefficients:
         ma1
      0.3926
s.e.  0.0895

sigma^2 estimated as 0.01324:  log likelihood=88.12
AIC=-172.24   AICc=-172.14   BIC=-166.7
```

Following the estimations of the auto.arima function, we should fit our ARIMA models such as:

- For the Breakeven5Y variable: ARIMA(2,0,2)
- For the ln(SP500) variable: ARIMA(2,0,2)
- For the ln(WTI) variable: ARIMA(0,0,1)

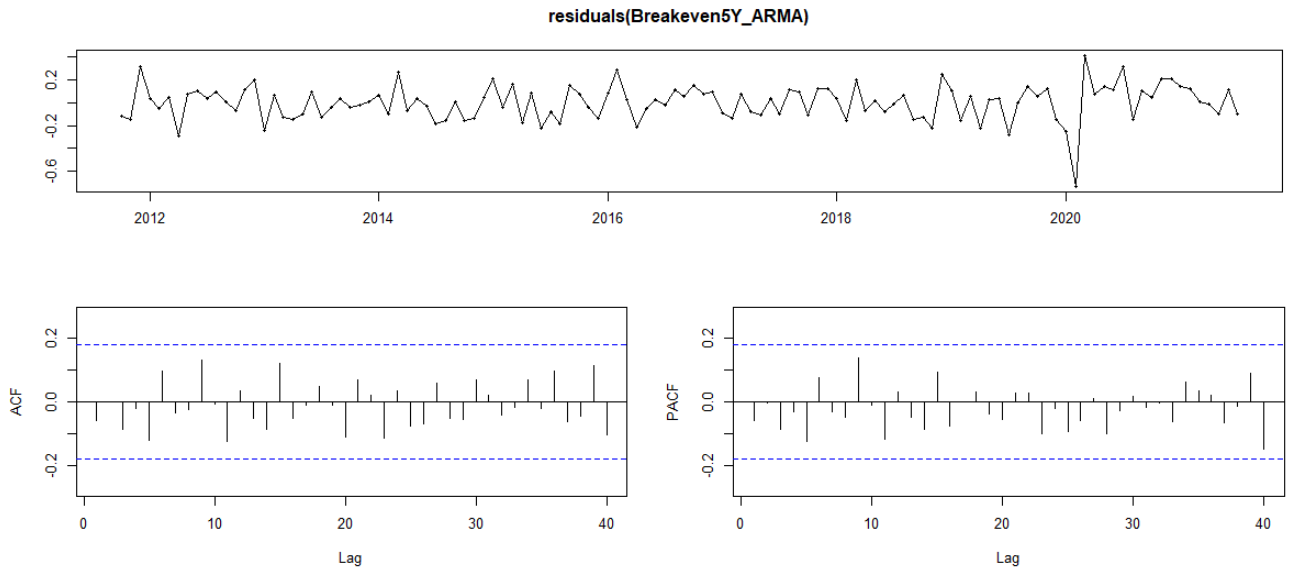
Step 2: Compare the optimal parameters obtained in step 1 to default parameters (1,0,1)

In order to validate the goodness of fit of the parameters obtained in step 1, we benchmark the fitness of the models to a default set of parameters[1] (1,0,1). By plotting the ACF and PACF graphs, we have a visual confirmation of the goodness of fit of the models 'parameters obtained in the first step, the default models displaying some out-of-bounds lags. The comparison between models can be found in figure 14, figure 15 and figure 16.

Figure 14

ARMA models for the Inflation Expectations variable

Optimal parameters ARMA(2,0,2) model



1 In this case, $d = 0$ since we are already using first-differenced variables

Benchmark parameters ARMA(1,0,1) model

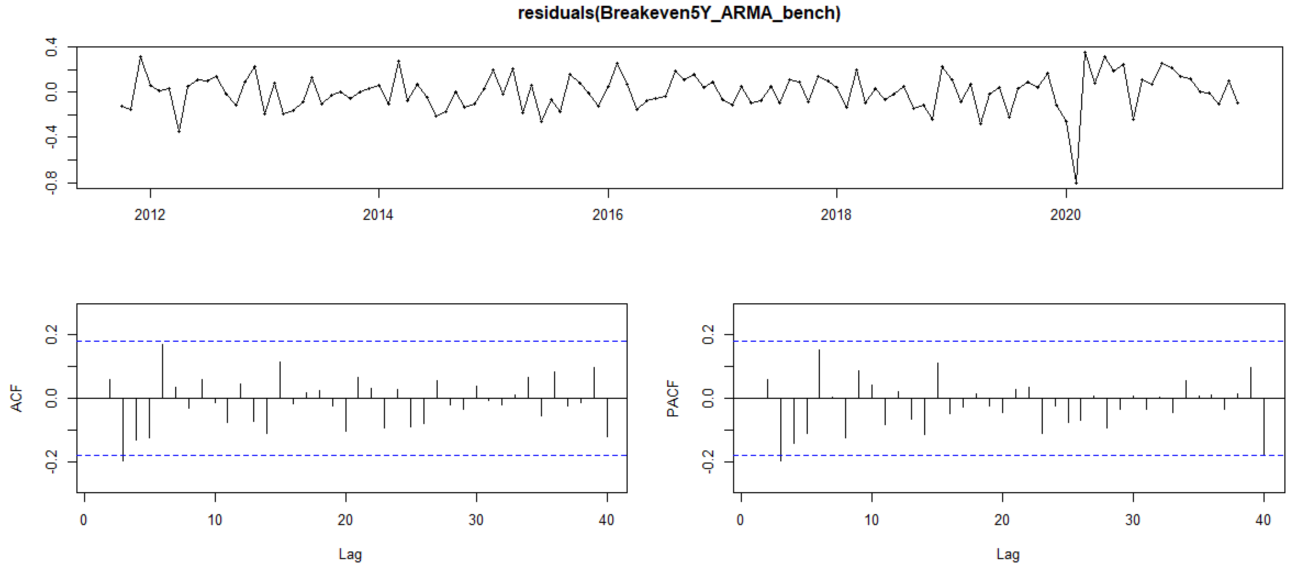
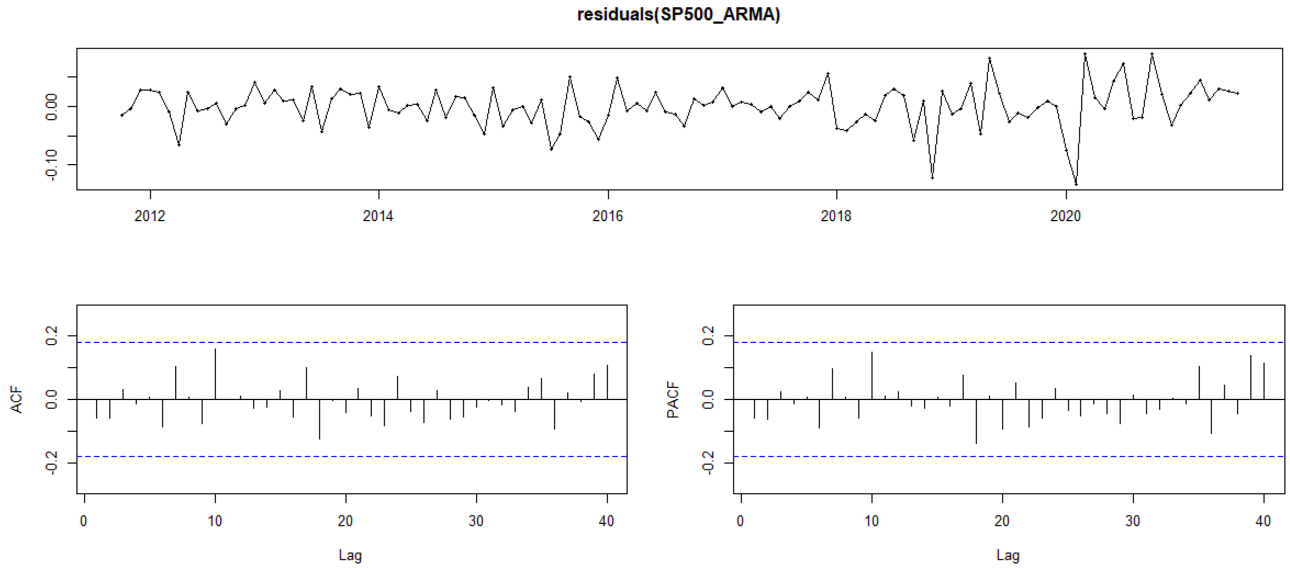


Figure 15

ARMA models for the Stock Index variable

Optimal parameters ARMA(2,0,2) model



Benchmark parameters ARMA(1,0,1) model

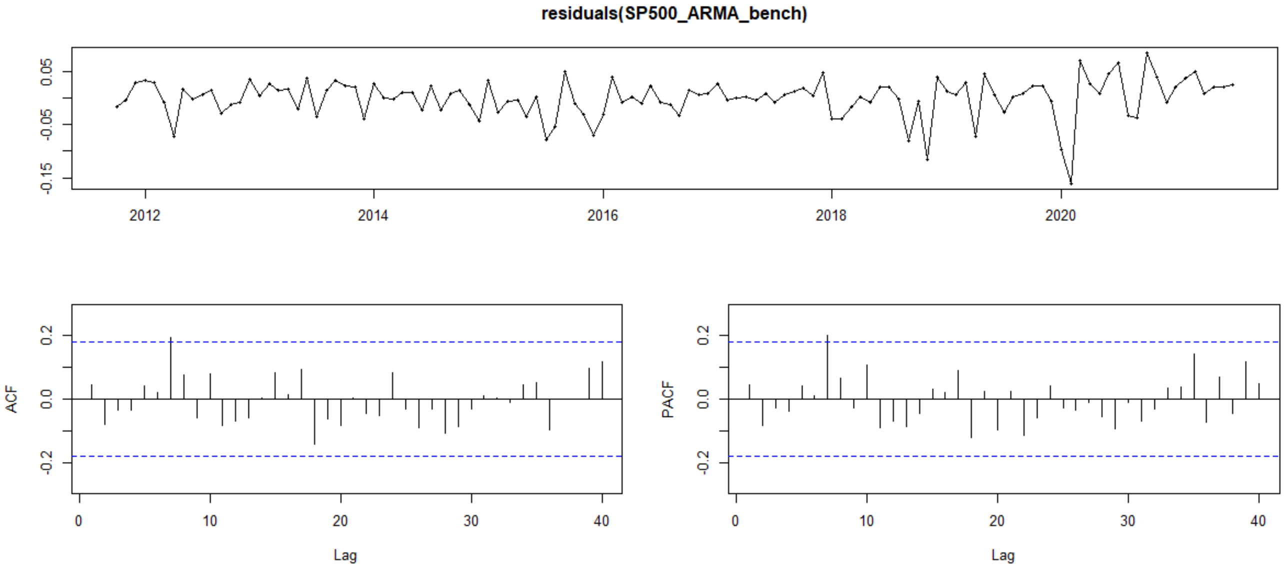
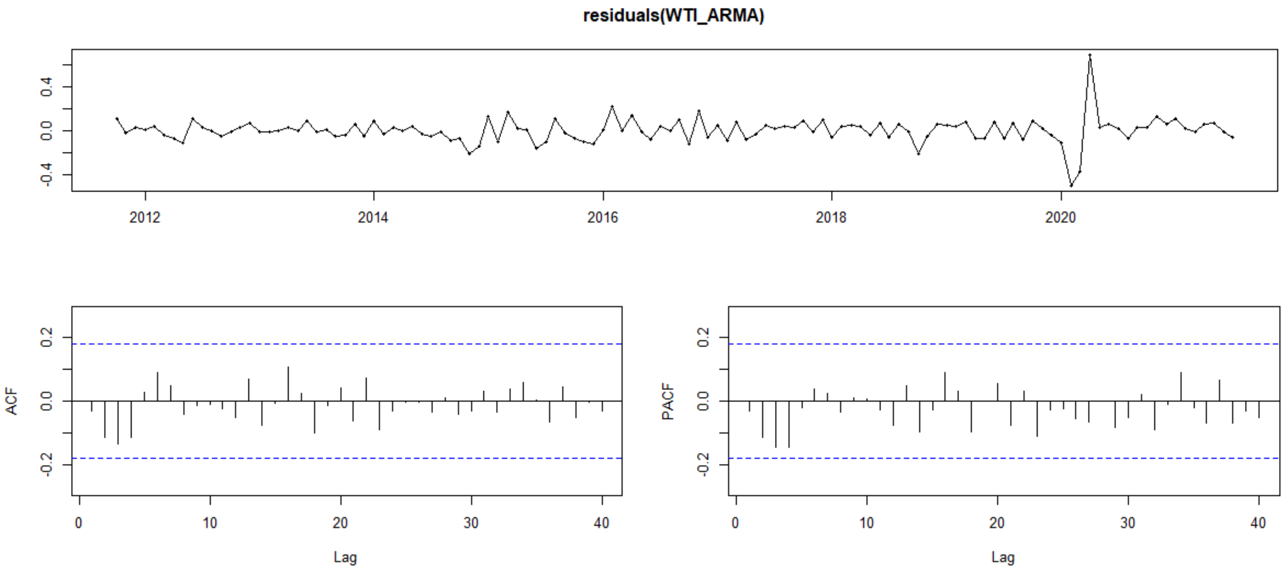


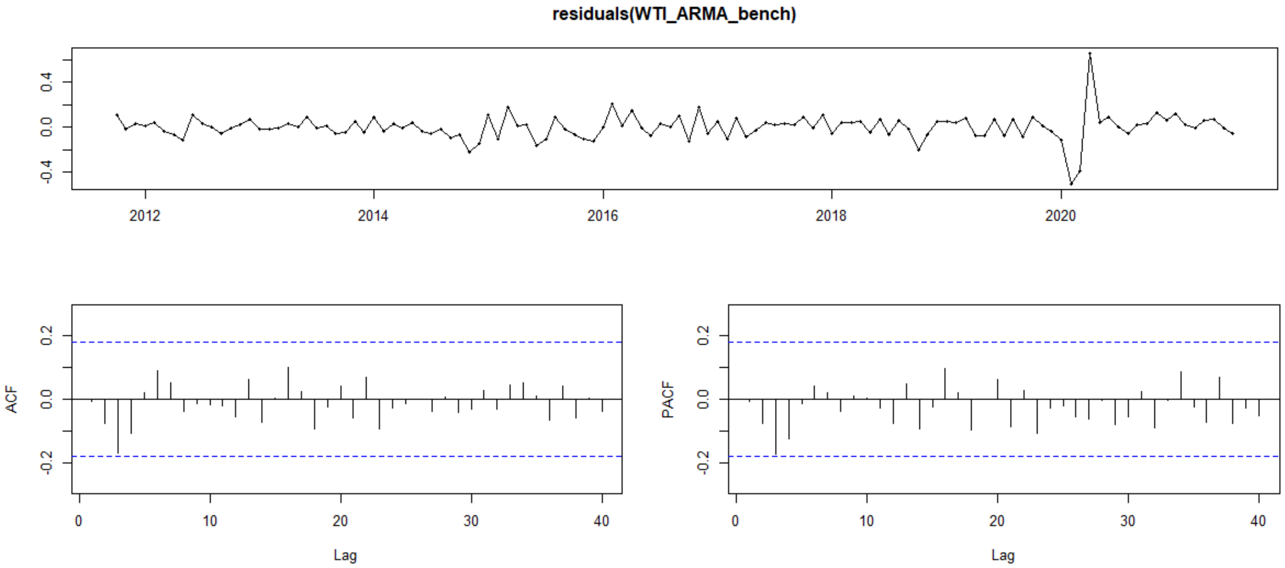
Figure 16

ARMA models for the WTI variable

Optimal parameters ARMA(2,0,2) model



Benchmark parameters ARMA(1,0,1) model



Now that we have a fitted ARMA model, we can check the corresponding coefficients, which are displayed in the following R-Code boxes.

R-Code 28

Estimated ARIMA model with optimal parameters (2,0,2) for the Inflation Expectation variable

```
> summary(Breakeven5Y_ARMA)

Call:
arima(x = Breakeven5Y_DIFF_ts, order = c(2, 0, 2))

Coefficients:
         ar1      ar2      ma1     ma2  intercept
      0.6216  -0.7457  -0.4340  0.7689     0.0049
s.e.  0.1837   0.1516   0.1758  0.1333     0.0168

sigma^2 estimated as 0.02354:  log likelihood = 53.58,  aic = -95.17

Training set error measures:
                       ME      RMSE       MAE MPE MAPE      MASE        ACF1
Training set 0.0001710876 0.1534325 0.1189614 NaN  Inf 0.7664364 -0.05855183
```

R-Code 29

Estimated ARIMA model with optimal parameters (2,0,2) for the Stock Index variable

```
> summary(SP500_ARMA)

Call:
arima(x = lSP500_detrended_DIFF_ts, order = c(1, 0, 1))

Coefficients:
         ar1      ma1  intercept
      0.5042  -0.6915     0.0018
s.e.  0.1974   0.1594     0.0021

sigma^2 estimated as 0.001309:  log likelihood = 224.2,  aic = -440.4

Training set error measures:
                       ME       RMSE        MAE      MPE     MAPE      MASE       ACF1
Training set 2.626439e-05 0.03617855 0.02596813 119.8247 130.0705 0.6640832 0.04432763
```

R-Code 30

Estimated ARIMA model with optimal parameters (0,0,1) for the WTI variable

```
> summary(WTI_ARMA)

Call:
arima(x = lWTI_detrended_DIFF_ts, order = c(0, 0, 1))

Coefficients:
         ma1  intercept
      0.3921     0.0046
s.e.  0.0896     0.0146

sigma^2 estimated as 0.01312:  log likelihood = 88.17,  aic = -170.34

Training set error measures:
                        ME      RMSE        MAE      MPE     MAPE      MASE        ACF1
Training set -0.0003071065 0.1145381 0.07257307 112.9311 140.1764 0.8270899 -0.03053819
```

Step 3 : We control for autocorrelation in the residuals by conducting a Breusch-Godfrey LM test

To confirm the robustness of our models, we control for the possible existence of autocorrelation of the residuals. We check if the residuals are affected by autocorrelation by performing a Breusch-Godfrey LM test. For each tested variable we failed to reject the null hypothesis:

$$H_0 : There\ is\ no\ autocorrelation$$

Those results mean that the residuals of our variables are likely uncorrelated and thus good for use in our ARMA models. The results of the BG-LM tests can be found in the following R-Code boxes.

R-Code 31

Breusch-Godfrey LM test for the Inflation Expectation variable

```
> bgtest(Breakeven5Y_DIFF_ts~lSP500_detrended_DIFF_ts+lWTI_detrended_DIFF_ts, data = data)

	Breusch-Godfrey test for serial correlation of order up to 1

data:  Breakeven5Y_DIFF_ts ~ lSP500_detrended_DIFF_ts + lWTI_detrended_DIFF_ts
LM test = 0.89677, df = 1, p-value = 0.3436
```

R-Code 32

Breusch- Godfrey LM test for the Stock Index variable

```
> bgtest(lSP500_detrended_DIFF_ts~Breakeven5Y_DIFF_ts+lWTI_detrended_DIFF_ts, data = data)

	Breusch-Godfrey test for serial correlation of order up to 1

data:  lSP500_detrended_DIFF_ts ~ Breakeven5Y_DIFF_ts + lWTI_detrended_DIFF_ts
LM test = 3.8682, df = 1, p-value = 0.04921
```

R-Code 33

Breusch- Godfrey LM test for the WTI variable

```
> bgtest(lWTI_detrended_DIFF_ts~lSP500_detrended_DIFF_ts+Breakeven5Y_DIFF_ts, data = data)

	Breusch-Godfrey test for serial correlation of order up to 1

data:  lWTI_detrended_DIFF_ts ~ lSP500_detrended_DIFF_ts + Breakeven5Y_DIFF_ts
LM test = 0.0087964, df = 1, p-value = 0.9253
```

ANNEXES – EMPIRICAL APPLICATION 1

Figure 1

Time plot of untransformed variables

Figure 2

Time plot of transformed variables

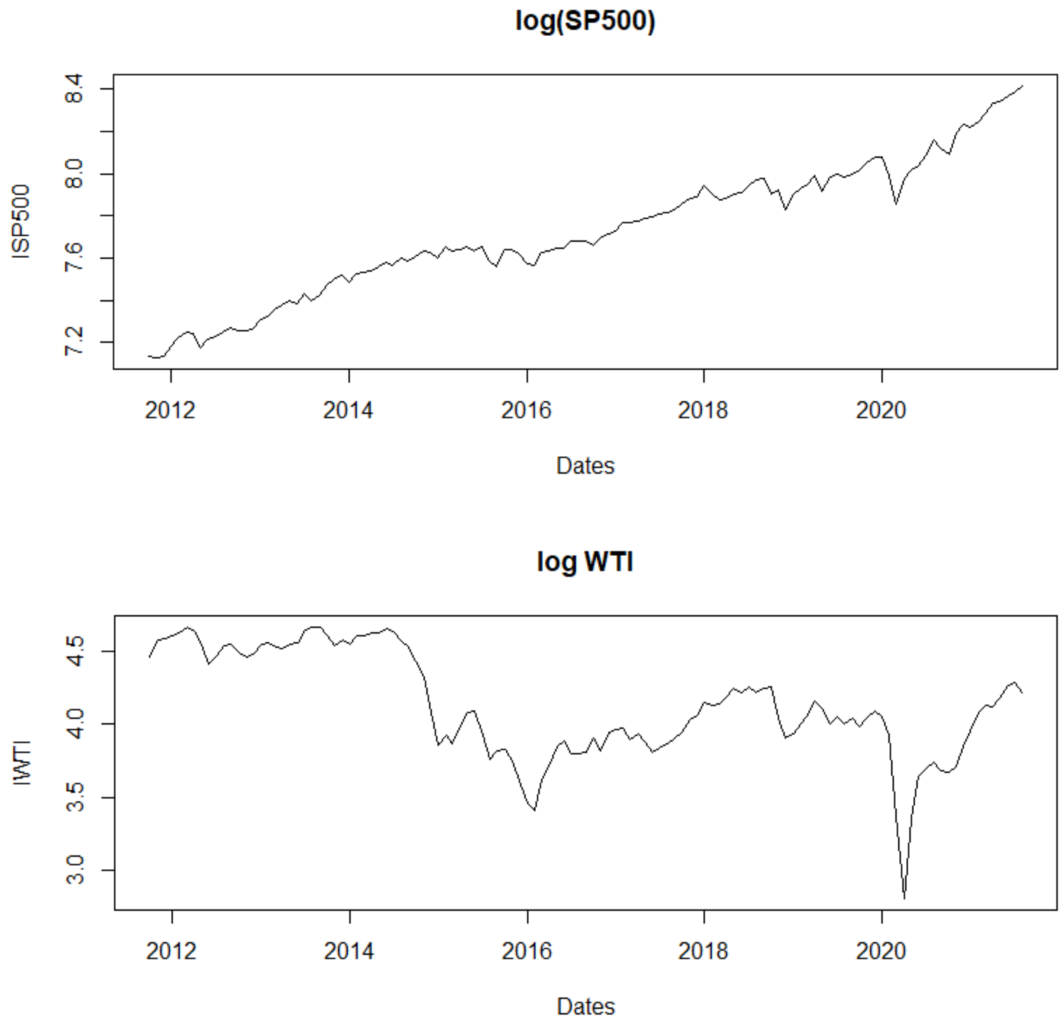


Figure 11

Time plot of de-trended and first-differenced Inflation Expectation variable

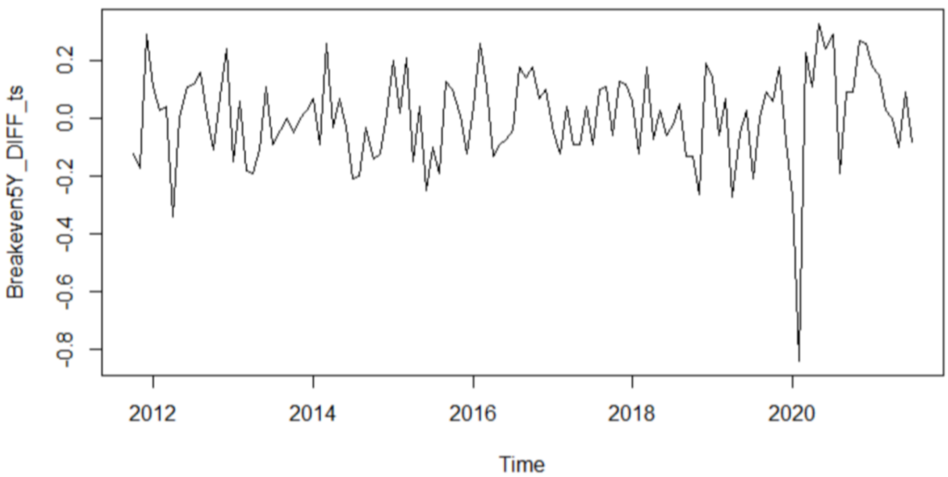


Figure 12

Time plot of de-trended and first-differenced log Stock Index variable

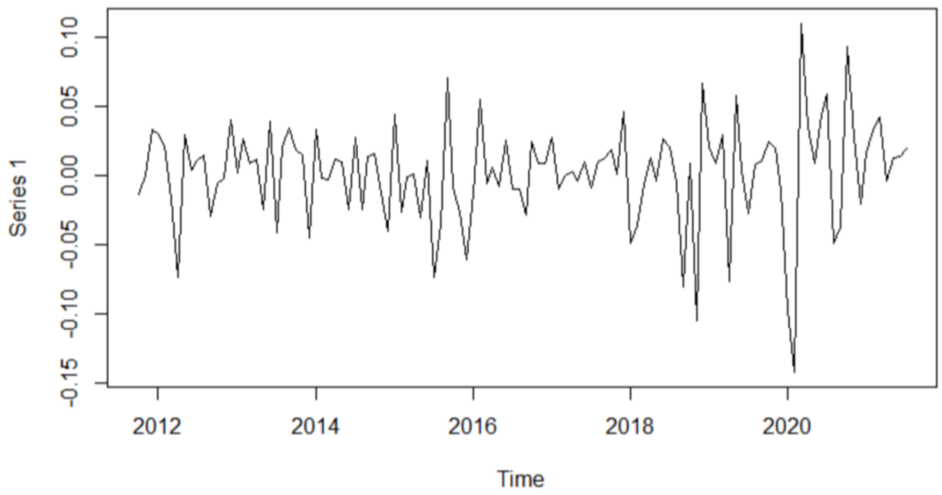


Figure 13

Time plot of de-trended and first-differenced log WTI variable

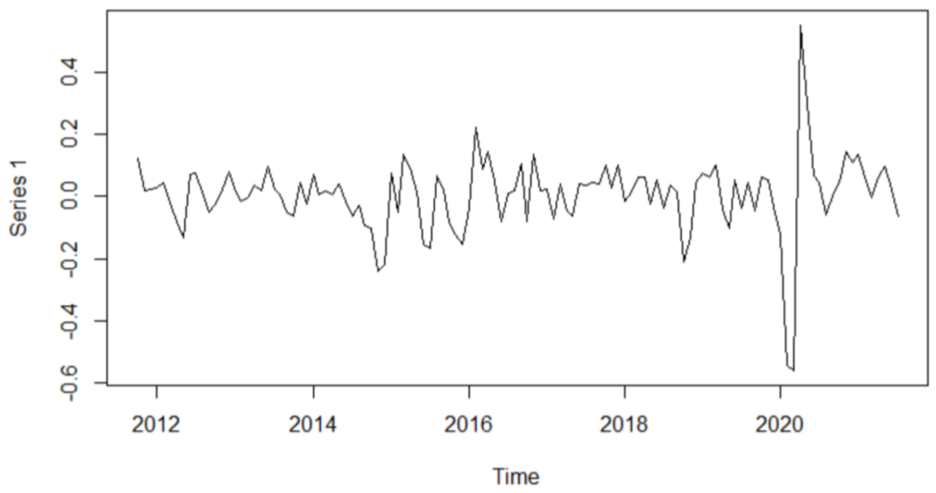


Table 1

Summary of variables

Variable	Description	Units	Data source	Period
Breakeven5Y	The breakeven inflation rate represents a measure of expected inflation derived from 5-Year Treasury Constant Maturity Securities and 5-Year Treasury Inflation-Indexed Constant Maturity Securities. The latest value implies what market participants expect inflation to be in the next 5 years, on average	Monthly, Percent, Not Seasonally Adjusted	FRED, Federal Reserve Bank of St. Louis	November 2011 – August 2021
SP500	The S&P 500 is regarded as a gauge of the large cap U.S. equities market. The index includes 500 leading companies in leading industries of the U.S. economy, which are publicly held on either the NYSE or NASDAQ	Monthly, End of period Index, Not Seasonally Adjusted	S&P Dow Jones Indices LLC, S&P 500	November 2011 – August 2021
WTI	A crude stream produced in Texas and southern Oklahoma which serves as a reference for pricing a number of other crude streams and which is traded in the domestic spot market at Cushing, Oklahoma.	Monthly, Average, Dollars per Barrel, Not Seasonally Adjusted	Dow Jones & Company, Spot Oil Price: WTI (Discountinued in july 2013) U.S. Energy Information Administration, Crude Oil Prices: WTI (starting in August 2013)	November 2011 – August 2021

Notes: All variables are gapless over the November 2011 – August 2021 period. Variables Breakeven5Y, SP500 and WTI are retrieved from the Federal Reserve Bank of St. Louis' Economic Database. Frequency of the time series is monthly.

R-Code 22

PP unit root test for the Expected Inflation variable

Non-differenced – Not significant: Non-stationary

```
> pp.test(Breakeven5Y)

        Phillips-Perron Unit Root Test

data:  Breakeven5Y
Dickey-Fuller Z(alpha) = -12.075, Truncation lag parameter = 4, p-value = 0.4175
alternative hypothesis: stationary
```

First-differenced – Significant at 1% confidence level: Stationary

```
> pp.test(Breakeven5Y_DIFF_ts)

        Phillips-Perron Unit Root Test

data:  Breakeven5Y_DIFF_ts
Dickey-Fuller Z(alpha) = -91.838, Truncation lag parameter = 4, p-value = 0.01
alternative hypothesis: stationary

Warning message:
In pp.test(Breakeven5Y_DIFF_ts) : p-value smaller than printed p-value
```

R-Code 23

PP unit root test for the log Stock Index variable

Non-differenced – Not significant: Non-stationary

```
> pp.test(lSP500_detrended_ts)

	Phillips-Perron Unit Root Test

data:  lSP500_detrended_ts
Dickey-Fuller Z(alpha) = -14.209, Truncation lag parameter = 4, p-value = 0.294
alternative hypothesis: stationary
```

First-differenced – Significant at 1% confidence level: Stationary

```
> pp.test(lSP500_detrended_DIFF_ts)

	Phillips-Perron Unit Root Test

data:  lSP500_detrended_DIFF_ts
Dickey-Fuller Z(alpha) = -108.59, Truncation lag parameter = 4, p-value = 0.01
alternative hypothesis: stationary

Warning message:
In pp.test(lSP500_detrended_DIFF_ts) : p-value smaller than printed p-value
```

R-Code 24

PP unit root test for the log WTI variable

Non-differenced – Not significant: Non-stationary

```
> pp.test(lWTI_detrended_ts)

	Phillips-Perron Unit Root Test

data:  lWTI_detrended_ts
Dickey-Fuller Z(alpha) = -11.2, Truncation lag parameter = 4, p-value = 0.4682
alternative hypothesis: stationary
```

First-differenced – Significant at 1% confidence level: Stationary

```
> pp.test(lWTI_detrended_DIFF_ts)

	Phillips-Perron Unit Root Test

data:  lWTI_detrended_DIFF_ts
Dickey-Fuller Z(alpha) = -67.209, Truncation lag parameter = 4, p-value = 0.01
alternative hypothesis: stationary

Warning message:
In pp.test(lWTI_detrended_DIFF_ts) : p-value smaller than printed p-value
```