State feedback with observer exercise

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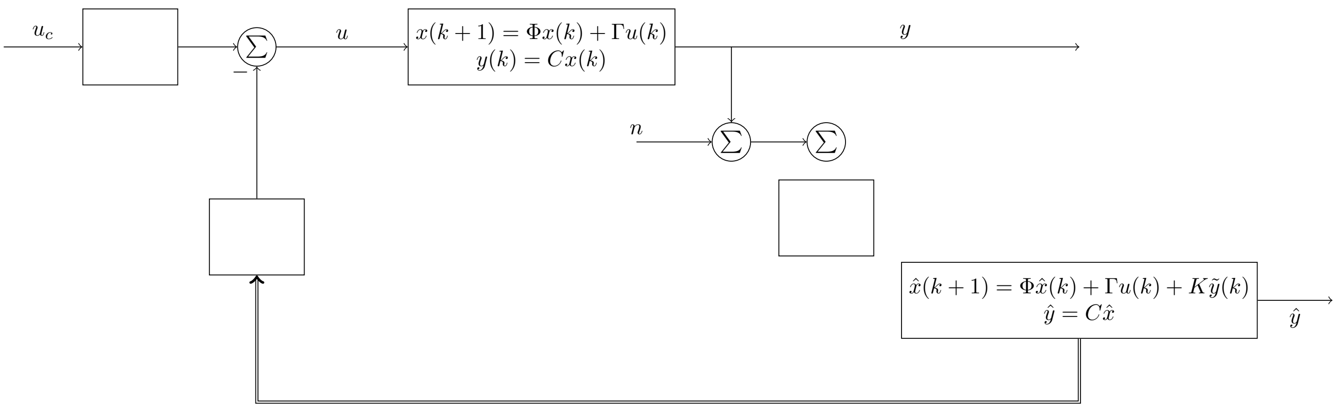
Complete the block diagram



The control law is $u(k) = l_0 u_c(k) - L\hat{x}(k)$. The feedback to the observer is $K\tilde{y}(k) = K\big(y_m(k) - \hat{y}(k)\big) = K\big(y(k) + n(k) - C\hat{x}(k)\big)$

Complete the augmented state-space model

$$\begin{bmatrix} x(k+1) \\ \hat{x}(k+1) \end{bmatrix} = \begin{bmatrix} \quad & \quad \\ \quad & \quad \end{bmatrix} \begin{bmatrix} x(k) \\ \hat{x}(k) \end{bmatrix} + \begin{bmatrix} \quad \\ \quad \end{bmatrix} u_c(k) + \begin{bmatrix} \quad \\ \quad \end{bmatrix} n(k)$$

$$y(k) = \begin{bmatrix} \quad & \quad \end{bmatrix} \begin{bmatrix} x(k) \\ \hat{x}(k) \end{bmatrix}$$

Control of a tank model

Consider the system of two tanks in the figure below. The input signal is the flow of water into the upper tank, and the output signal is the level of the second tank.

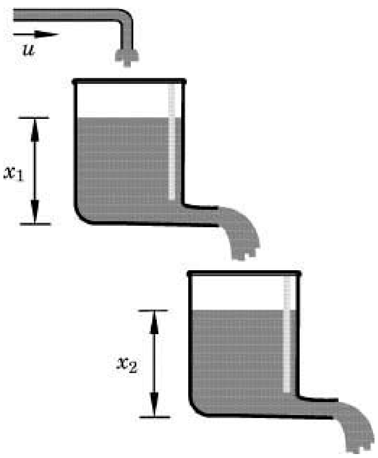


Figure 2.12 The two-tank process.

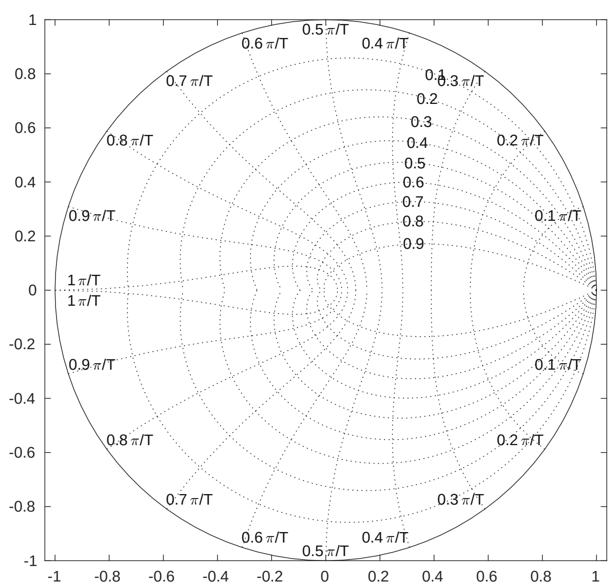
In continuous-time the system is described by the state space system

$$\frac{dx}{dt} = \begin{bmatrix} -0.0197 & 0 \\ 0.0178 & -0.0129 \end{bmatrix} x + \begin{bmatrix} 0.0263 \\ 0 \end{bmatrix} u$$
$$y = \begin{bmatrix} 0 & 1 \end{bmatrix} x.$$

With sampling period $h = 12$ we obtain the discrete-time system

$$x(kh + h) = \begin{bmatrix} 0.790 & 0 \\ 0.176 & 0.857 \end{bmatrix} x(kh) + \begin{bmatrix} 0.281 \\ 0.0296 \end{bmatrix} u(kh)$$
$$y(kh) = \begin{bmatrix} 0 & 1 \end{bmatrix} x(kh).$$

1. Determine an observer with poles that are twice as fast as the fastest mode of the plant.
2. Determine a state feedback gain L such that the closed-loop system has poles in 0.8 ± 0.1.



Implementing the controller

Complete the pseudo-code (matlab) below for the implementation of the controller

```
% The plant model
n = 2;
h = 12;
Phi = [0.79 0; 0.176 0.857];
Gamma = [0.281; 0.0296];
C = [0 1];

% The controller and observer gains

L =

K =

% Initialize variables
u = 0; xhat = [0; 0];
% Run the controller
while run_controller() % The function run_controller will return 0 if the controller should stop
   y = get_output();  % Returns the sampled output signal from the plant
   uc = get_setpoint();  % Returns the desired water leve in the second tank

   % Update the observer

   xhat =

   % Compute the control signal

   u =

   % Send the control signal to the plant
   write_control_signal(u);
   % Wait until next sampling instant
   sleep(h); % Assuming that the computational time is negligable
end
```

Solution

1. The estimation error of the observer satisfies the difference equation

$$\tilde{x}(kh+h) = (\Phi - KC)\,\tilde{x}(kh).$$

We can choose the poles of the matrix $\Phi - KC$ freely if the system is **observable**. Here we have

$$W_o = \begin{bmatrix} C \\ C\Phi \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0.176 & 0.857 \end{bmatrix},$$

so $\det W_o = -0.176 \neq 0$, i.e. the system is **observable**.

The poles of the continuous-time plant are -0.0197 and -0.0129. The fastest poles is thus -0.0197. The discrete-time observer poles should be twice as fast. This is obtained by placing the observer poles in

$$e^{2(-0.0197)h} \approx 0.62.$$

Verify by studying the z-plane grid. This gives the desired characteristic polynomial

$$(z-0.62)^2 = z^2 - 1.24z + 0.3844$$

for the observer.

We have

$$\begin{aligned}
\Phi - KC &= \begin{bmatrix} 0.790 & 0 \\ 0.176 & 0.857 \end{bmatrix} - \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} \begin{bmatrix} 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} 0.790 & 0 \\ 0.176 & 0.857 \end{bmatrix} - \begin{bmatrix} 0 & k_1 \\ 0 & k_2 \end{bmatrix} \\
&= \begin{bmatrix} 0.790 & -k_1 \\ 0.176 & 0.857 - k_2 \end{bmatrix}.
\end{aligned}$$

Which gives the characteristic polynomial

$$\begin{aligned}
\det\left(zI - (\Phi - KC)\right) &= \det \begin{bmatrix} z-0.790 & k_1 \\ -0.176 & z - 0.857 + k_2 \end{bmatrix} \\
&= (z-0.790)(z-0.857+k_2) + 0.176k_1 = z^2 + (-0.790 - 0.857 + k_2)z - 0.790(-0.857 + \\
&= z^2 + (k_2 - 1.647)z + 0.176k_1 - 0.790k_2 + 0.677
\end{aligned}$$

Setting the coefficients of the two characteristic polynomials equal gives the system of equations

$$\begin{aligned}
k_2 - 1.647 = -1.24 \quad &\Rightarrow \quad k_2 = 0.407 \\
0.176k_1 - 0.790k_2 + 0.677 = 0.3844 \quad &\Rightarrow \quad k_1 \approx 0.164.
\end{aligned}$$

2. Now design the feedback gain L. The system is **reachable** since

$$W_c = \begin{bmatrix} \Gamma & \Phi\Gamma \end{bmatrix} = \begin{bmatrix} 0.281 & 0.222 \\ 0.0296 & 0.0748 \end{bmatrix}$$

$$\det W_c = 0.0145 \neq 0$$

.

The desired characteristic polynomial is

$$(z - 0.8 - 0.1i)(z - 0.8 + 0.1i) = z^2 - 1.6z + 0.65.$$

The closed-loop system characteristic polynomial is obtained by

$$\det\left(zI - (\Phi - \Gamma L)\right).$$

with

$$\Gamma L = \begin{bmatrix} 0.281 \\ 0.0296 \end{bmatrix} \begin{bmatrix} l_1 & l_2 \end{bmatrix} = \begin{bmatrix} 0.281l_1 & 0.281l_2 \\ 0.0296l_1 & 0.0296l_2 \end{bmatrix}$$

we get

$$
\begin{aligned}
\det\left(zI - (\Phi - \Gamma L)\right) &= \det \begin{bmatrix} z - 0.790 + 0.281 l_1 & 0.281 l_2 \\ -0.176 + 0.0296 l_1 & z - 0.857 + 0.0296 l_2 \end{bmatrix} \\
&= (z - 0.790 + 0.281 l_1)(z - 0.857 + 0.0296 l_2) - (-0.176 + 0.0296 l_1)(0.281 l_2) \\
&= z^2 + (-0.790 + 0.281 l_1 - 0.857 + 0.0296 l_2) z + (-0.790 + 0.281 l_1)(-0.857 + 0.0296 l_2) + \\
&= z^2 + (0.281 l_1 + 0.0296 l_2 - 1.647) z + 0.677 - 0.023 l_2 - 0.241 l_1 + 0.0296 \cdot 0.281 l_1 l_2 + 0.04 \\
&= z^2 + (0.281 l_1 + 0.0296 l_2 - 1.647) z + 0.677 - 0.241 l_1 + 0.026 l_2.
\end{aligned}
$$

Setting the coefficients of the two characteristic polynomials equal gives the system of equations

$$
\begin{aligned}
0.281 l_1 + 0.0296 l_2 &= -1.6 + 1.647 \\
-0.241 l_1 + 0.026 l_2 &= 0.65 - 0.677
\end{aligned}
$$

or

$$
\begin{bmatrix} 0.281 & 0.0296 \\ -0.241 & 0.026 \end{bmatrix} \begin{bmatrix} l_1 \\ l_2 \end{bmatrix} = \begin{bmatrix} 0.047 \\ -0.027 \end{bmatrix}
$$

with solution

$$
\begin{bmatrix} l_1 \\ l_2 \end{bmatrix} = \frac{1}{0.281 \cdot 0.026 + 0.241 \cdot 0.0296} \begin{bmatrix} 0.026 & -0.0296 \\ 0.241 & 0.281 \end{bmatrix} \begin{bmatrix} 0.047 \\ -0.027 \end{bmatrix} = \begin{bmatrix} 0.14 \\ 0.26 \end{bmatrix}.
$$