Computerized control - state feedback

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Goal of today's lecture

- Understand state feedback design

Stability

A system

$$x(k+1) = \Phi x(k), \quad x(0) = x_0$$

is *asymptotically stable* if $\lim_{t\to\infty} x(kh) = 0$ for all $x_0 \in \mathbb{R}^n$.

A system is asymptotically stable if and only if all eigenvalues of Φ are inside the unit circle.

Reachability (controllability)

Reachability is the answer to the question "Can we by choosing a suitable input sequence $u(k),\ k = 0, 1, 2, \ldots, n-1$ reach any point in the state space?"

Consider

$$x(k+1) = \Phi x(k) + \Gamma u(k).$$

With initial state $x(0)$ given. The solution at time n where n is the order of the system is

$$\begin{aligned} x(n) &= \Phi^n x(0) + \Phi^{n-1}\Gamma u(0) + \cdots + \Gamma u(n-1) \\ &= \Phi^n x(0) + W_c U, \end{aligned} \tag{1}$$

where

$$W_c = \begin{bmatrix} \Gamma & \Phi\Gamma & \cdots & \Phi^{n-1}\Gamma \end{bmatrix}$$
$$U = \begin{bmatrix} u(n-1) & u(n-2) & \cdots & u(0) \end{bmatrix}^{\mathrm{T}}$$

Reachability (controllability), contd

To find the input sequence that takes the state to $x(n) = x_d$ we solve the equation

$$x_d = \Phi^n x(0) + W_c U$$

for U.

$$U = W_c^{-1} \left(x_d - \Phi^n x(0) \right)$$

This requires the matrix W_x to be invertible. This gives Theorem 3.7 in Å&W:

THEOREM 3.7 REACHABILITY The state space system above is reachable if and only if the matrix W_c has rank n.

This is equivalent to

$$\det W_c \neq 0.$$

State feedback

Have state space model

$$\begin{aligned} x(k+1) &= \Phi x(k) + \Gamma u(k) \\ y(k) &= Cx(k) \end{aligned} \tag{2}$$

and measurements (or estimates) of the state vector $x(k)$.

Linear state feedback is the control law

$$\begin{aligned} u(k) &= f\big((x(k), u_c(k)\big) = -l_1 x_1(k) - l_2 x_2(k) - \cdots - l_n x_n(k) + m u_c(k) \\ &= -Lx(k) + m u_c(k), \end{aligned}$$

where

$$L = \begin{bmatrix} l_1 & l_2 & \cdots & l_n \end{bmatrix}.$$

Insert the control law into the state space model (3) to get

State feedback

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Linear state feedback is the control law

$$u(k) = -l_1 x_1(k) - l_2 x_2(k) - \cdots - l_n x_n(k) + m u_c(k) = -Lx(k) + m u_c(k),$$

where

$$L = \begin{bmatrix} l_1 & l_2 & \cdots & l_n \end{bmatrix}.$$

Insert the control law into the state space model (3) to get

$$\begin{aligned} x(k+1) &= (\Phi - \Gamma L)\, x(k) + m\Gamma u_c(k) \\ y(k) &= Cx(k) \end{aligned} \tag{4}$$

Pole placement by state feedback

Assume the desired performance of the control system is given as a set of desired closed loop poles $p_1, p_2, \ldots, p_n$, corresponding to the desired characteristic polynomial

$$a_c(z) = (z - p_1)(z - p_2)\cdots(z - p_n) = z^n + \alpha_1 z^{n-1} + \cdots \alpha_n. \tag{5}$$

With state feedback we get the the closed-loop system

$$\begin{aligned} x(k+1) &= (\Phi - \Gamma L)\, x(k) + m\Gamma u_c(k) \\ y(k) &= Cx(k) \end{aligned} \tag{6}$$

with characteristic equation

$$\det\left(zI - (\Phi - \Gamma L)\right) = z^n + \beta_1(l_1, \ldots, l_n) z^{n-1} + \cdots \beta_n(l_1, \ldots, l_n). \tag{7}$$

Equate the coefficients in (5) and (7) to get the system of equations

$$\begin{aligned} \beta_1(l_1, \ldots, l_n) &= \alpha_1 \\ \beta_2(l_1, \ldots, l_n) &= \alpha_2 \\ &\vdots \\ \beta_n(l_1, \ldots, l_n) &= \alpha_n \end{aligned}$$

Pole placement by state feedback, contd.

The system of equations

$$\begin{aligned}\beta_1(l_1, \ldots, l_n) &= \alpha_1 \\ \beta_2(l_1, \ldots, l_n) &= \alpha_2 \\ &\vdots \\ \beta_n(l_1, \ldots, l_n) &= \alpha_n\end{aligned}$$

is always linear in the unknown controller parameters, so it can be written

$$AL^{\mathrm{T}} = \alpha,$$

Where $\alpha^{\mathrm{T}} = \begin{bmatrix}\alpha_1 & \alpha_2 & \cdots & \alpha_n\end{bmatrix}$.

Pole placement and reacability

It can be shown that the controllability matrix W_c is a factor of the matrix A

$$A = \bar{A} W_c.$$

Hence, in general the system of equations

$$\bar{A} W_c L^{\mathrm{T}} = \alpha \tag{8}$$

has a solution only if W_c is invertible, i.e. the system is *reachable*.

Note that equation (8) can still have a solution for unreachable systems if α is in the *column space* of A, i.e. α can be written

$$\alpha = b_1 A_{:,1} + b_2 A_{:,2} + \cdots + b_m A_{:,m}, \ \ m < n$$