Computerized Control - discrete-time systems, z-transform

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Result from quizz

- z-transform
- State-space systems

The world according to the discrete-time controller

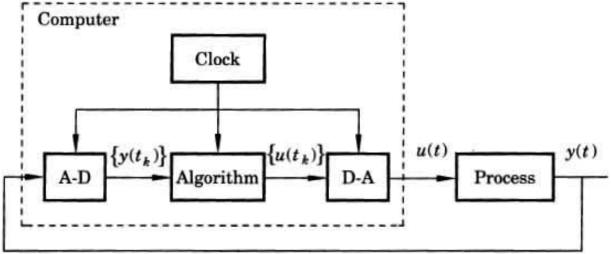
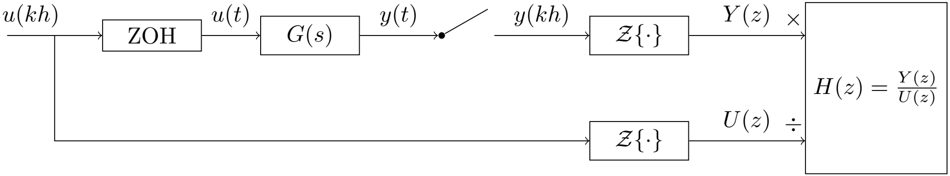


Figure 1.1 Schematic diagram of a computer-controlled system.

Step-invariant sampling (a.k.a ZOH sampling)

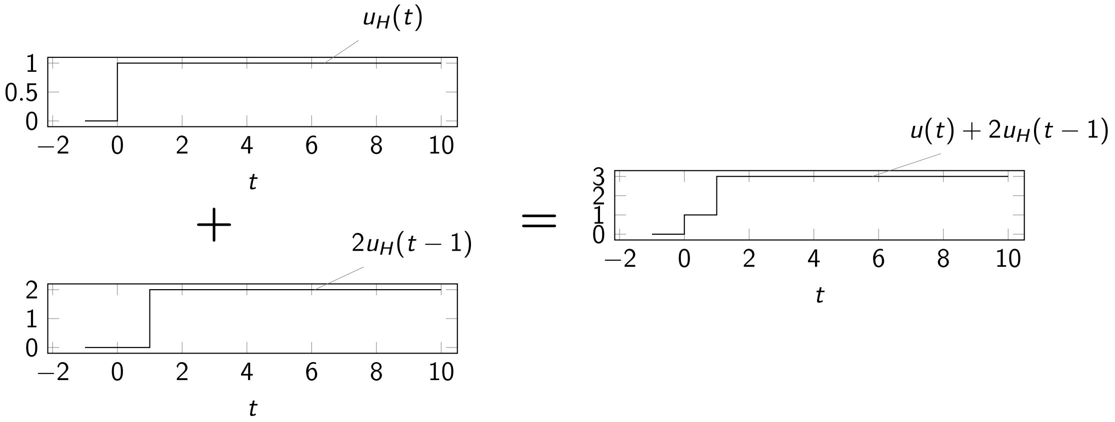
The idea is to sample the continuous-time system's response to a step input, in order to obtain a discrete approximation which is exact (at the sampling instants) for such an input signal.



- Step-invariant sampling (zero order hold): $u(t) = \begin{cases} 1, & t \geq 0 \\ 0, & t < 0 \end{cases}$

Why is step-invariant sampling a good idea?

A piecewise constant (stair-case shaped) function can be written as a sum of delayed step-responses!



Why is step-invariant sampling a good idea? (contd)

Due to the system being LTI (linear time-invariant), the output to a sum of delayed step functions, is the same sum of delayed step-responses.

$$\xrightarrow{u_H(t)} \boxed{\text{LTI}} \xrightarrow{y_H(t)}$$

Hence, $u(t) = \sum_i \alpha_i u_H(t - \tau_i)$ has the response $y(t) =$.

Why is step-invariant sampling a good idea? (contd)

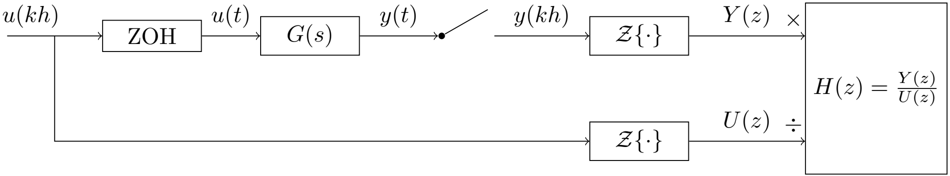
Due to the system being LTI (linear time-invariant), the output to a sum of delayed step functions, is the same sum of delayed step-responses.

$$u_H(t) \longrightarrow \boxed{\text{LTI}} \longrightarrow y_H(t)$$

Hence, $u(t) = \sum_i \alpha_i u_H(t - \tau_i)$ has the response $y(t) = \sum_i \alpha_i y_H(t - \tau_i)$.

If the sampling method is exact for step input signals, it will also be exact for piecwise-constant step input signals, and this is exactly what the ZOH-block produces!

Impulse- step- and ramp-invariant sampling



- Impulse-invariant sampling: $u(t) = \delta(t)$
- Step-invariant sampling (zero order hold): $u(t) = \begin{cases} 1, & t \geq 0 \\ 0, & t < 0 \end{cases}$
- Ramp-invariant sampling: $u(t) = \begin{cases} t, & t \geq 0 \\ 0, & t < 0 \end{cases}$

Step-invariant sampling, or zero-order-hold sampling

Let the input to the continuous-time system be a step $u(t) = \begin{cases} 0, & t < 0 \\ 1, & t \geq 0 \end{cases}$, which has Laplace transform $U(s) = \frac{1}{s}$. In the Laplace-domain we get

$$Y(s) = G(s)\frac{1}{s}$$

1. Obtain the time-response by inverse Laplace: $y(t) = \mathcal{L}^{-1}\{Y(s)\}$
2. Sample the time-response to obtain the sequence $y(kh)$ and apply the z-transform to obtain $Y(z) = \mathcal{Z}\{y(kh)\}$
3. Calculate the pulse-transfer function by dividing with the z-transform of the input signal $U(z) = \frac{z}{z-1}$.

$$H(z) = \frac{Y(z)}{U(z)} = \frac{z-1}{z}Y(z)$$

Example: First-order system

Let's apply step-invariant sampling to the system

$$G(s) = \frac{1}{s+a}.$$

Do on your own: The double integrator

$$G(s) = \frac{1}{s^2}$$

Another important property of the z-transform

The z-transform and the solution to difference equations

Taking the z-transform of a difference equation

$$(\mathrm{q}^2 + a_1\,\mathrm{q} + a_2) y_k = \left(b_0\,\mathrm{q}^2 + b_1\,\mathrm{q} + b_2\right) u_k$$

gives

$$\begin{aligned} z^2 Y - z^2 y(0) - zy(1) + a_1 zY - a_1 zy(0) + a_2 Y = \\ b_0 z^2 U - b_0 z^2 u(0) - b_0 zu(1) + b_1 zU - b_1 zu(0) + b_2 U \end{aligned}$$

$$Y(z) = \underbrace{\frac{\big(y(0) - b_0 u(0)\big) z^2 + \big(y(1) + a_1 y(0) - b_0 u(1) - b_1 u(0)\big) z}{z^2 + a_1 z + a_2}}_{\text{transient response}}$$

$$+ \underbrace{\underbrace{\frac{b_0 z^2 + b_1 z + b_2}{z^2 + a_1 z + a_2}}_{\text{pulse-transfer function}} U(z)}_{\text{response to input}}$$

The z-transform and the solution to difference equations

In general, the output of the discrete-time LTI

$$\left(\mathrm{q}^n + a_1\,\mathrm{q}^{n-1} + \cdots + a_n\right) y(k) = \left(b_0\,\mathrm{q}^m + b_1\,\mathrm{q}^{m-1} + \cdots + b_m\right) u(k)$$

is

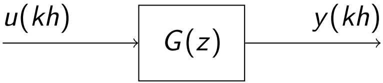
$$Y(z) = \frac{\beta(z)}{A(z)} + \frac{B(z)}{A(z)} U(z)$$

For systems that are intially at rest

$$Y(z) = \frac{B(z)}{A(z)} U(z) = G(z)U(z)$$

Convolution in the time-domain is multiplication in the z-domain

$$\mathcal{Z}\left\{g * u)\right\} = \mathcal{Z}\left\{g(kh)\right\}\mathcal{Z}\left\{u(kh)\right\} = \left(\sum_{k=0}^{\infty} g(kh)z^{-k}\right)\left(\sum_{k=0}^{\infty} u(kh)z^{-k}\right)$$



$$y(kh) = g(kh) * u(kh)$$

$$\mathcal{Z}\left\{y(kh)\right\} = \mathcal{Z}\left\{g(kh) * u(kh)\right\}$$

$$Y(z) = G(z)U(z).$$

The z-transform plays the same role for discrete-time control systems as the Laplace transform for continuous-time ontrol systems!