

Computerized Control - analysis of discrete-time systems

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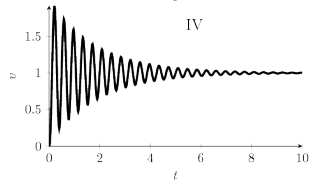
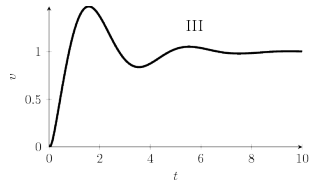
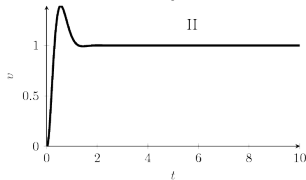
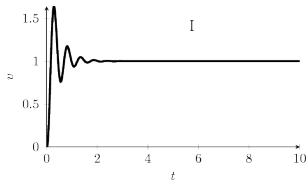
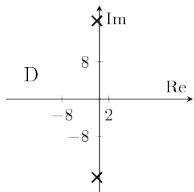
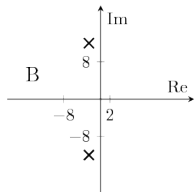
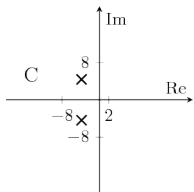
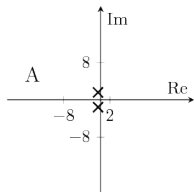
2019-09-05

Result from quizz

- ▶ Very good results!
- ▶ Nyquist plot + criterion
- ▶ Robustness

Pole-placement and time-response

Pair the pole-placement with the correct time-response (from HW1)



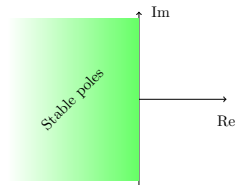
Mapping of poles from continuous time to discrete time

Continuous time

$$Y(s) \triangleq \mathcal{L}\{y(t)\}$$

$$Y(s) = G(s)U(s) = \frac{b}{s+a}U(s)$$

Pole of the system: $s + a = 0 \Rightarrow s = -a$

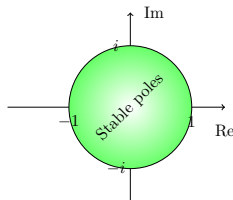


Discrete time

$$Y(z) \triangleq \mathcal{Z}\{y(kh)\}$$

$$Y(z) = H(z)U(z) = \frac{\beta}{z+\alpha}U(z)$$

Pole of the system: $z + \alpha = 0 \Rightarrow z = -\alpha$



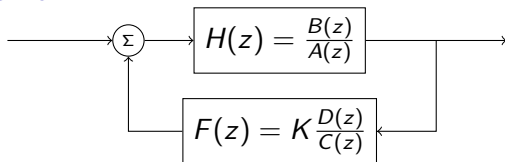
The **s-domain** of continuous-time systems is related to the **z-domain** of discrete-time systems as

$$z = e^{sh}$$

Mapping of poles from continuous time to discrete time

Do exercise on paper!

Root locus: A brief review



- ▶ The loop pulse-transfer function (loop gain) becomes

$$L(z) = H(z)F(z) = K \underbrace{\frac{B(z)D(z)}{A(z)C(z)}}_{P(z)} = K \frac{Q(z)}{P(z)}.$$

- ▶ The roots of $Q(z)$ are called the **open loop zeros**.
- ▶ The roots of $P(z)$ are called the **open loop poles**.
- ▶ The characteristic equation for the closed-loop system is

$$1 + K \frac{Q(z)}{P(z)} = 0 \quad \Leftrightarrow \quad P(z) + KQ(z) = 0$$

Root locus: Definition

Let

$$\begin{cases} P(z) = z^n + a_1 z^{n-1} + \cdots + a_n = (z - p_1)(z - p_2) \cdots (z - p_n) \\ Q(z) = z^m + b_1 z^{m-1} + \cdots + b_m = (z - q_1)(z - q_2) \cdots (z - q_m) \end{cases}, \quad n \geq m$$

The root locus shows how the **solution** to the characteristic equation

$$P(z) + K \cdot Q(z) = 0, \quad 0 \leq K < \infty \quad (1)$$

depend on the parameter K . The root locus consists of the set of all points in the complex plane that are solutions to (1) for some non-negative value of K .

Root locus: Characteristics

Start points The n roots of $P(z)$, marked by crosses

End points The m roots of $Q(z)$, marked by circles

Asymptotes Number equal to the *pole excess* $n - m$

Real axis Some segments of the real axis belong to the root locus

Root locus: Direction of the asymptotes

The characteristic equation $P(z) + KQ(z) = 0$ can be written $\frac{P(z)}{Q(z)} = -K$ and for large z it can be approximated as

$$\frac{z^n}{z^m} = -K \quad \Leftrightarrow \quad z^{n-m} = -K.$$

Taking the argument of both sides of the equation gives $(n - m) \arg z = \pi + k2\pi$, $k \in \mathbb{Z}$. So, the **directions** of the asymptotes are given by the expression

$$\theta_k = \arg z = \frac{(2k + 1)\pi}{n - m}, \quad k \in \mathbb{Z}$$

Root locus: The asymptotes' intersection with the real axis

$$z_{ip} = \frac{\sum_{i=0}^n p_i - \sum_{i=0}^m q_i}{n - m},$$

where $\{p_i\}$ are the starting points (open-loop poles) and $\{q_i\}$ are the end points (open-loop zeros).

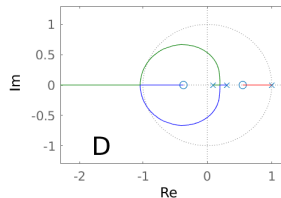
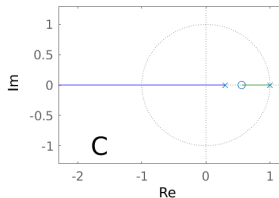
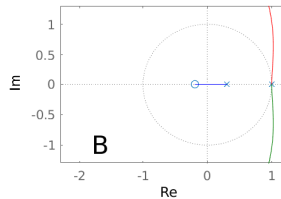
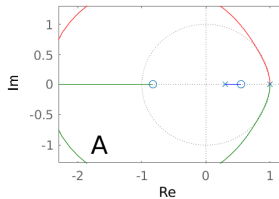
Root locus exercise: Pair the pulse-trf fcn and root locus

$$G_1(z) = K \frac{(z + 2.9)(z + 0.2)}{(z - 1)^2(z - 0.3)}$$

$$G_2(z) = K \frac{(z - 0.5)(z + 0.4)}{(z - 1)(z - 0.3)(z - 0.1)}$$

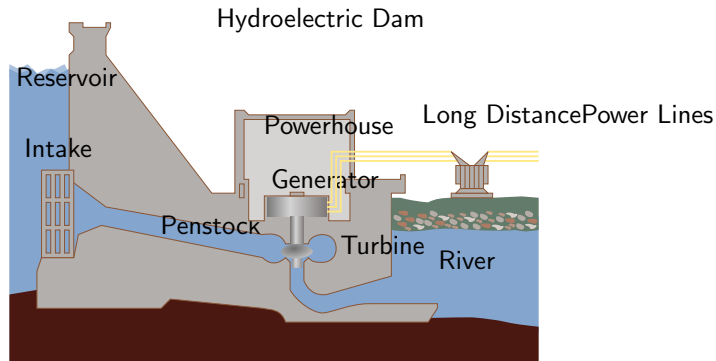
$$G_3(z) = K \frac{(z - 0.5)(z + 0.8)}{(z - 1)^2(z - 0.3)}$$

$$G_4(z) = K \frac{z - 0.6}{(z - 1)(z - 0.3)}$$



Draw a root locus

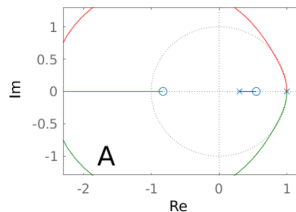
Level control in a hydro power plant dam



Discrete-time model: $y(k+1) - y(k) = \frac{h}{A}u(k) + \frac{h}{A}v(k)$, where $y(k)$ is the deviation in water level from a standard level, $u(k)$ is the (negative) deviation in flow through the dam ports and $v(k)$ is a deviation in other flows (disturbance).

What happens if the poles are **on the** unit circle?

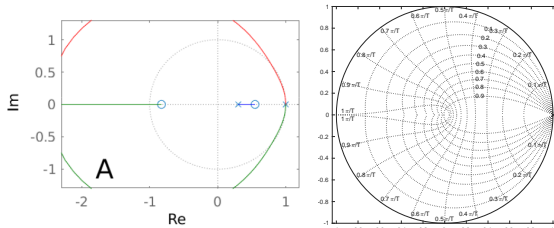
Say, in $z = e^{\pm i\omega_0}$



$$H_c(z) = \frac{kz}{(z - e^{i\omega_0})(z - e^{-i\omega_0})} \underbrace{+ \dots}_{\text{stable term}}$$

What happens if the poles are on the unit circle?

Say, in $z = e^{\pm i\omega_0}$

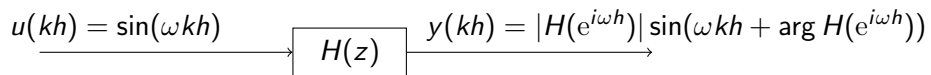


$$\begin{aligned} H_c(z) &= \frac{kz}{(z - e^{i\omega_0})(z - e^{-i\omega_0})} + \dots \\ &= \frac{kz}{z^2 - 2\cos\omega_0 z + 1} + \dots \end{aligned}$$

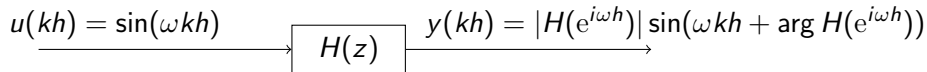
If $\omega_0 = \frac{\pi}{6}$ and the sampling period is 0.4 s, what is the **frequency** (in rad/s and in Hz) of the oscillations in the pulse response?

Bode diagram and Nyquist plots

Sine in — sine out



Sine in — sine out



Prove it! Some hints:

- ▶ Write $\sin(\omega kh) = \text{Im}\{e^{i\omega kh}\}$.
- ▶ Use $H(z)U(z) \stackrel{\mathcal{Z}}{\leftrightarrow} h(k) * u(k)$, and write out the discrete-time convolution $h * u = \sum_{n=-\infty}^{\infty} h(n)u(k-n) = \sum_{n=0}^{\infty} h(n)u(k-n)$
- ▶ Try to rewrite to obtain as a factor $\sum_{n=0}^{\infty} h(n)e^{-i\omega nh} = H(e^{i\omega h})$.

The Nyquist plot

Example of a **Nyquist plot** or **frequency curve**.

$H(z)$ evaluated at $z = e^{i\omega h}$, $0 < \omega < \frac{\pi}{h}$

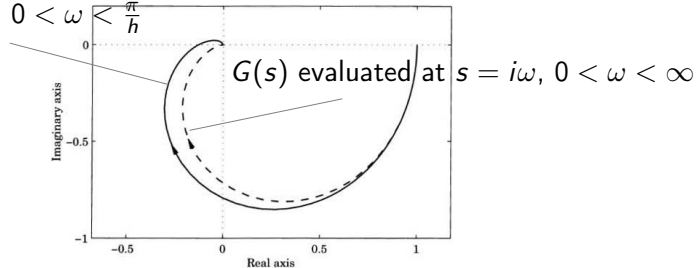


Figure 3.3 The frequency curve of (3.6) (dashed) and for (3.6) sampled with zero-order hold when $h = 0.4$ (solid).

The system $G(s) = \frac{1}{s^2 + 1.4s + 1}$ is sampled with ZOH-sampling ($h = 0.4$ s) to get
 $H(z) = \frac{0.066z + 0.055}{z^2 - 1.45z + 0.571}$

Simplified Nyquist criterion

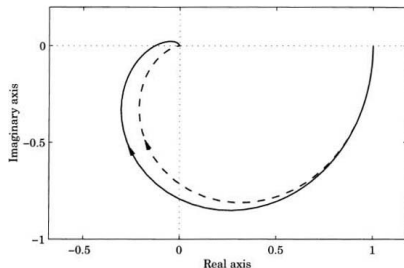
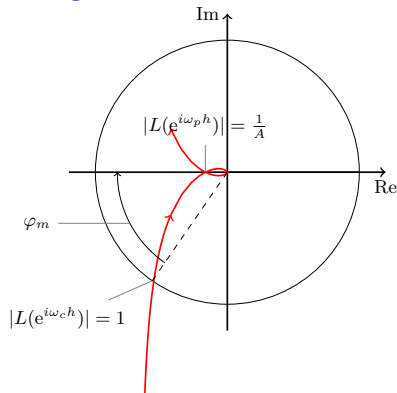


Figure 3.3 The frequency curve of (3.6) (dashed) and for (3.6) sampled with zero-order hold when $h = 0.4$ (solid).

Consider the loop pulse-transfer function $L(z)$ of a closed-loop system. If $L(z)$ is stable (no poles outside the unit circle), then the closed-loop system with characteristic equation $1 + L(z) = 0$ will be stable iff $L(z)$ evaluated on the unit circle (i.e, the Nyquist plot of L) has the point **-1 to the left**.

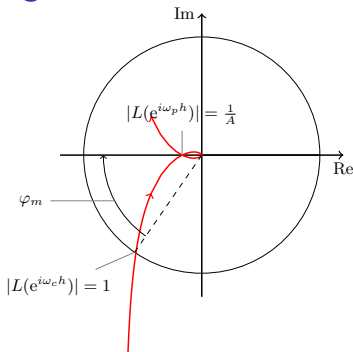
Stability margins - phase margin



- ▶ Cross-over frequency: The frequency ω_c for which $|L(e^{i\omega h})| = 1$.
- ▶ Phase margin: The angle φ_m to the negative real axis for the point where the Nyquist curve intersects the unit circle.

$$\varphi_m = \arg L(e^{i\omega_c h}) - (-180^\circ) = \arg L(e^{i\omega_c h}) + 180^\circ$$

Stability margins - gain margin



- ▶ phase-cross-over frequency: The frequency ω_p for which $\arg L(e^{i\omega h}) = -180^\circ$.
- ▶ Gain margin: The gain $K = A$ that would make the Nyquist curve of $KL(e^{i\omega h})$ go through the point $-1 + i0$. This means that

$$|L(e^{i\omega_p h})| = \frac{1}{A}.$$

The relationship between Bode diagrams and frequency curves (Nyquist plots)

They are both showing the value of a pulse-transfer function $H(z)$ evaluated for $z = e^{i\omega h}$, $0 < \omega \leq \frac{\pi}{h}$.

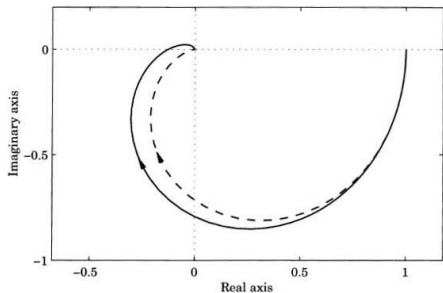


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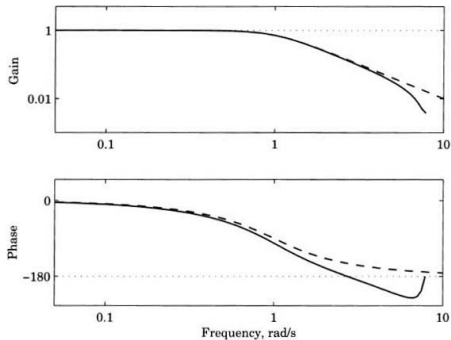


Figure 3.4 The Bode diagram of (3.6) (dashed) and of (3.6) sampled with zero-order hold when $h = 0.4$ (solid).