Computerized control - homework 4

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Due Friday 2016-04-22

Active suspension

A so-called *quarter model* for the suspension of a car is shown in figure 1. The system consists of two masses, one is the mass of 1/4 of the car, the other is the much smaller suspension mass. The two masses are connected by two passive elements: a spring and a damper, as well as an active element: a linear force actuator. The suspension mass is connected with the ground via a spring, representing the tyre (sometimes there is also a damper included in the model of the tyre).

A slightly more complex model is discussed here. That model also includes a damping in the tyre. You are welcome to study the example. The parameter values are the same as for this homework assignment.

Determine the state space model

1. Draw free-body diagrams for each of the two bodies, and set up the equation of motion of each of the two masses. Note that the variables z_1, z_2 and w are displacements from a static equilibrium, measured with respect to a stationary frame of reference. This means that the force acting on mass m_1 from the spring between the two masses is $F_s = m_1 g - k_1(z_1 - z_2)$, and the force acting on the suspension mass from the spring (tyre) between the suspension mass and ground is $F_t = (m_1 + m_2)g - k_2(z_2 - w)$. Moreover, the force in the damper is proportional with the relative velocity between the car and the suspension mass: $F_d = -b_1(\dot{z}_1 - \dot{z}_2)$.

2. Show that the following state-space model of the system is obtained with the choice

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} z_1 - z_2 \\ \dot{z}_1 - \dot{z}_2 \\ z_2 \\ \dot{z}_2 \end{bmatrix}$$

$$\begin{aligned}\dot{x} &= \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\left(\frac{1}{m_1} + \frac{1}{m_2}\right)k_1 & -\left(\frac{1}{m_1} + \frac{1}{m_2}\right)b_1 & \frac{k_2}{m_2} & 0 \\ 0 & 0 & 0 & 1 \\ \frac{k_1}{m_2} & \frac{b_1}{m_2} & -\frac{k_2}{m_2} & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 \\ -\frac{k_2}{m_2} \\ 0 \\ \frac{k_2}{m_2} \end{bmatrix} w + \begin{bmatrix} 0 \\ \frac{1}{m_1} + \frac{1}{m_2} \\ 0 \\ -\frac{1}{m_2} \end{bmatrix} u \\ y &= \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix} x.\end{aligned}$$

   Note that the model has two inputs: The force from the active suspension u, and the disturbance from the road w.

Simulate step-responses

Implement the model in matlab. Use the following values for the parameters

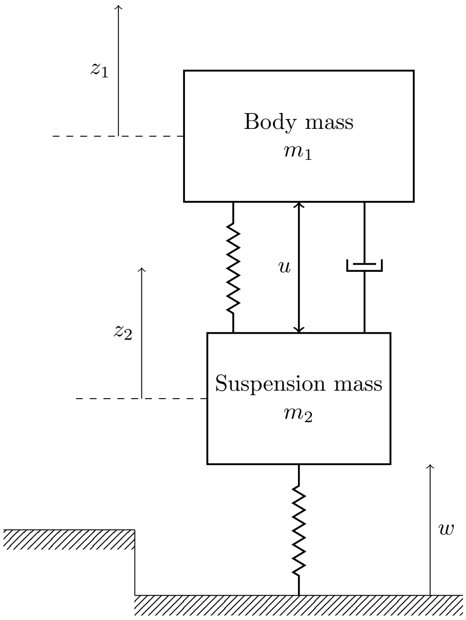


Figure 1: Active suspension model. The distances z_1, z_2 and w are displacements from a static equilibrium. The displacements are measured with respect to a stationary frame of reference.

$$
\begin{aligned}
m1 &= 2500\\
m2 &= 320\\
k1 &= 80000\\
k2 &= 500000\\
b1 &= 350
\end{aligned}
$$

Note that the model has two input signals: The disturbance from the road, w, and the actuator input u. Simulate a step-response from each of the two input signals. Below is some code to help you

```
m1 = 2500;
m2 = 320;
k1 = 80000;
k2 = 500000;
b1 = 350;

A =   % Your code here
Bu = [0
     1/m1+1/m2
     0
     -1/m2 ];
Bw = [ 0
      -k2/m2
      0
      k2/m2];
C=[1 0 0 0];
D=[0];
sys_uy=ss(A,Bu,C,D);
sys_wy=ss(A,Bw,C,D);

% Step responses
figure(1)
clf
step(sys_uy, sys_wy)
```

```
% Simulate step response for first half second only
T = linspace(0, 0.5, 800);
figure(2)
clf
subplot(121)
stepplot(sys_uy, T);
title('Response from u')
xlabel('Time (seconds)')
ylabel('y (meters)')
subplot(122)
stepplot(sys_wy, T);
title('Response from w')
xlabel('Time (seconds)')
ylabel('y (meters)')
```

Sample the system

1. The system is quite oscillative. Choose a sampling period h, such that $\omega_0 h = 0.2$, where ω_0 is the period of the oscillations in radians.
2. Sample the system (numerically, using `c2d` in matlab)

Observability and reachability

Determine observability and reachability of the sampled system. There are matlab commands to do this for you (`obsv` and `ctrb`), but do it also by forming the observability and controllability matrices explicitly. Recall

$$W_C = \begin{bmatrix} \Gamma & \Phi\Gamma & \cdots & \Phi^{n-1}\Gamma \end{bmatrix}$$

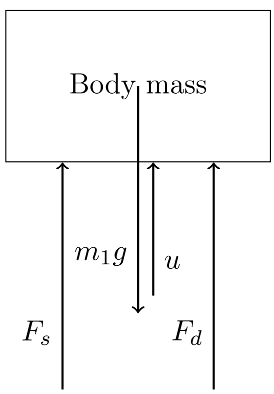
and

$$W_O = \begin{bmatrix} C \\ C\Phi \\ \vdots \\ C\Phi^{n-1} \end{bmatrix}.$$

Solutions

State space model

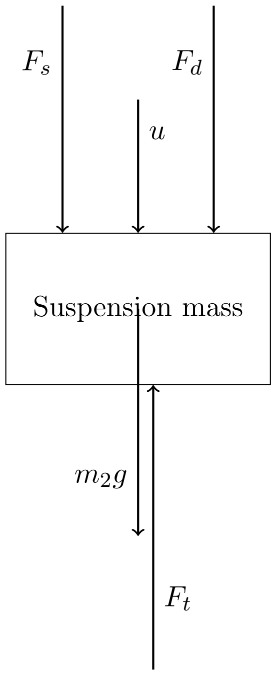
1. There are four forces acting on the car body



The equation of motion in the vertical direction becomes

$$m_1\ddot{z}_1 = \sum_i F_i = -m_1 g + \underbrace{m_1 g - k_1(z_1 - z_2)}_{F_s} + \underbrace{\big(- b_1(\dot{z}_1 - \dot{z}_2)\big)}_{F_d} + u = -k_1(z_1 - z_2) - b_1(\dot{z}_1 - \dot{z}_2) + u. \tag{1}$$

On the suspension mass, there are five forces acting:



The equation of motion in the vertical direction becomes

$$\begin{aligned} m_2\ddot{z}_2 &= -m2_g - \underbrace{\big(m_1 g - k_1(z_1 - z_2)\big)}_{F_s} - \underbrace{\big(- b_1(\dot{z}_1 - \dot{z}_2)\big)}_{F_d} - \underbrace{\big((m_1 + m_2)g - k_2(z_2 - w)\big)}_{F_t} - u \\ &= k_1(z_1 - z_2) + b_1(z_1 - z_2) - k_2(z_2 - w) - u \end{aligned} \tag{2}$$

2. With the state vector

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} z_1 - z_2 \\ \dot{z}_1 - \dot{z}_2 \\ z_2 \\ \dot{z}_2 \end{bmatrix}$$

we can write the four first-order differential equations of the state space model

$$\begin{aligned} \dot{x}_1 &= \dot{z}_1 - \dot{z}_2 = x_2 \\ \dot{x}_2 &= \ddot{z}_1 - \ddot{z}_2 = \frac{1}{m_1}\Big(- k_1(z_1 - z_2) - b_1(\dot{z}_1 - \dot{z}_2) + u\Big) - \frac{1}{m_2}\Big(k_1(z_1 - z_2) + b_1(z_1 - z_2) - k_2(z_2 - w) - u\Big) \\ &= -\Big(\frac{1}{m_1} + \frac{1}{m_2}\Big)k_1 x_1 - \Big(\frac{1}{m_1} + \frac{1}{m_2}\Big)b_1 x_2 + \frac{k_2}{m_2}x_3 - \frac{k_2}{m_2}w + \Big(\frac{1}{m_1} + \frac{1}{m_2}\Big)u \\ \dot{x}_3 &= \dot{z}_2 = x_4 \\ \dot{x}_4 &= \ddot{z}_2 = \frac{1}{m_2}\Big(k_1(z_1 - z_2) + b_1(z_1 - z_2) - k_2(z_2 - w) - u\Big) \\ &= \frac{k_1}{m_2}x_1 + \frac{b_1}{m_2}x_2 - \frac{k_2}{m_2}x_3 + \frac{k_2}{m_2}w - \frac{1}{m_2}u. \end{aligned}$$

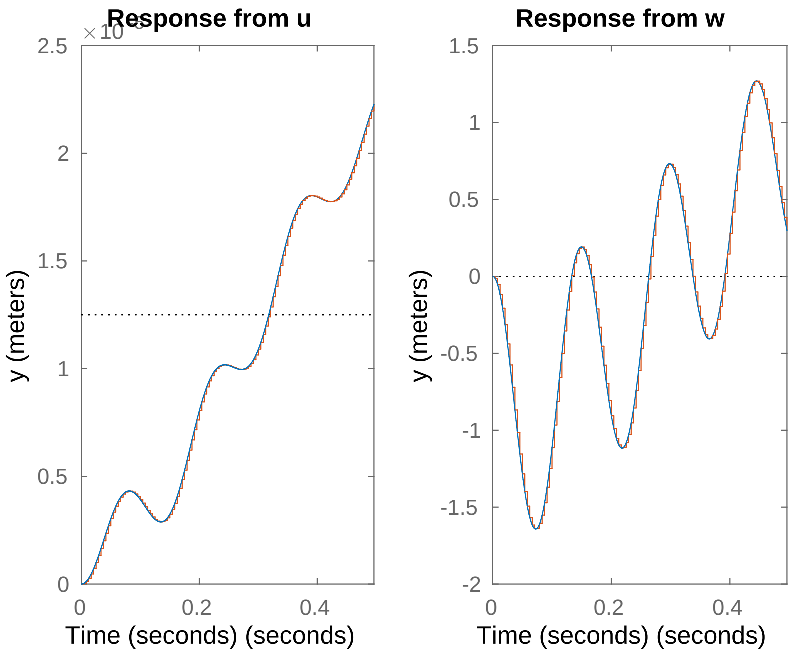
In state-space form, this can be written

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\left(\frac{1}{m_1} + \frac{1}{m_2}\right)k_1 & \left(\frac{1}{m_1} + \frac{1}{m_2}\right)b_1 & \frac{k_2}{m_2} & 0 \\ 0 & 0 & 0 & 1 \\ \frac{k_1}{m_2} & \frac{b_1}{m_2} & -\frac{k_2}{m_2} & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 \\ -\frac{k_2}{m_2} \\ 0 \\ \frac{k_2}{m_2} \end{bmatrix} w + \begin{bmatrix} 0 \\ \frac{1}{m_1} + \frac{1}{m_2} \\ 0 \\ -\frac{1}{m_2} \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix} x.$$

step-response

The figure below shows the step-responses of both the continuous- and the sampled systems.



Sampling the system

1. The fast oscillations in the beginning of the step-response have period 0.13 s, so the frequency is $\omega_0 = \frac{2\pi}{0.13} \approx 48$ /s. This gives
$$h = \frac{0.2}{\omega_0} \approx 0.0042 \text{ s}.$$

2. Sampling the system is straightforward

```
sys_uy_d = c2d(sys_uy, h); % The system from input u to y
sys_wy_d = c2d(sys_wy, h); % From input w to y
```

Observability and reachability

Obtain the Φ, Γ, C and D matrices from the discretized system. Then form W_o and W_c and calculate the determinant

```
% Check observability and reachability
[Ad, Bd, Cd, Dd] = ssdata(sys_uy_d);
Wo = [Cd;Cd*Ad;Cd*Ad*Ad; Cd*Ad*Ad*Ad]
det(Wo)
Wc = [Bd  Ad*Bd  Ad*Ad*Bd Ad*Ad*Ad*Bd]
det(Wc)
```

The result is

$$\det W_o = 1.26 \cdot 10^{-8}$$

and

$$\det W_c = 5.92 \cdot 10^{-30}.$$

So, the system is observable, but not reachable.