Computerized Control - LTIs, impulse response, difference equations

Kjartan Halvorsen

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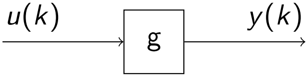
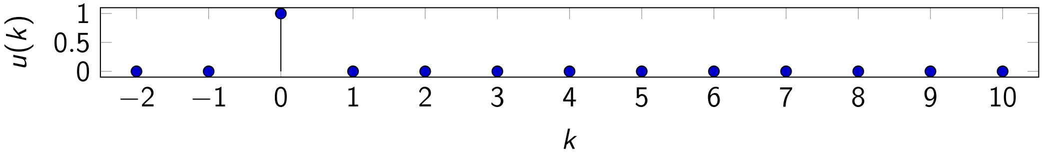
The discrete causal linear time-invariant system



$$y(k) = g * u = \sum_{n=0}^{\infty} g(n)u(k-n)$$

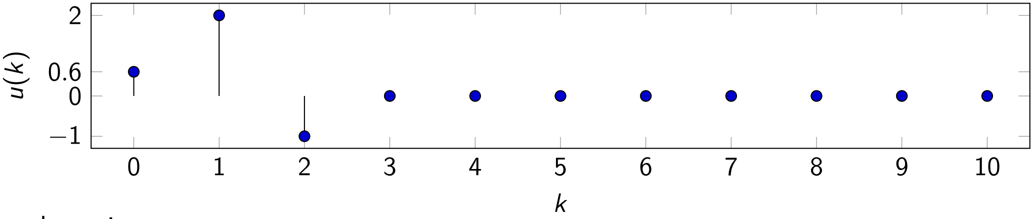
If input signal is a pulse (delta-function)



$$y(k) = \sum_{n=0}^{\infty} g(n)\delta(k-n) = g(k)$$

Linearity, time invariance and the pulse response

The input signal



Can be written

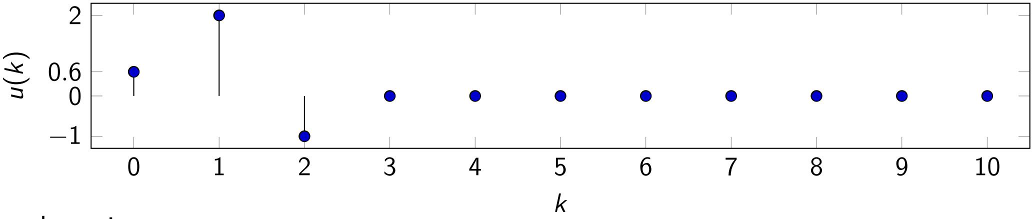
$$u(k) = 0.6\delta(k) + 2\delta(k-1) - \delta(k-2)$$

Since the system's response to a pulse is given by $g(k)$, the output signal is

$$y(k) = ?$$

Linearity, time invariance and the pulse response

The input signal



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$$u(k) = 0.6\delta(k) + 2\delta(k-1) - \delta(k-2)$$

Since the system's response to a pulse is given by $g(k)$, the output signal is

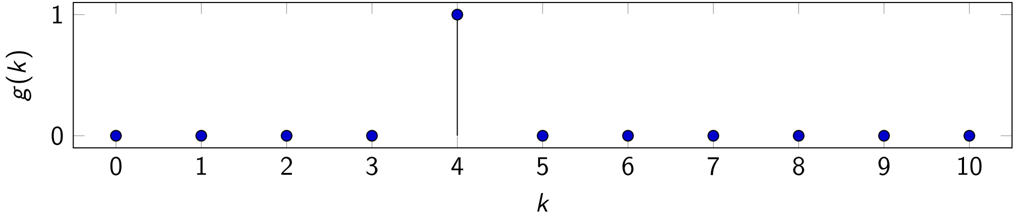
$$y(k) = 0.6g(k) + 2g(k-1) - g(k-2)$$

The output of a causal, linear discrete-time system is a weighted sum of previous input

$$y(k) = g * u = \sum_{n=0}^{\infty} g(n)u(k-n)$$

The weighting function $g(k)$ is the pulse response of the system.

What if the weighting function looks like this



$$y(k) =$$

Exercise: Impulse response

The shift operator

- For difference equations the shift operator q is very useful.
- The shift operator is defined for double-infinite sequences x_k, i.e. the sequence x_k must be infinitely long both for negative and positive k.
- The operator shifts the sequence ahead one step:

$$\mathrm{q}\, x_k = x_{k+1}$$

The difference equation is a representation of a discrete-time dynamical systems

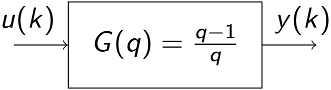
$$u(k) \longrightarrow \boxed{G(\mathrm{q})} \longrightarrow y(k)$$

$$y_{k+n} + a_1 y_{k+n-1} + \cdots + a_n y_k = b_0 u_{k+m} + b_1 u_{k+m-1} + \cdots + b_m u_k$$

$$\left(\mathrm{q}^n + a_1 \mathrm{q}^{n-1} + \cdots + a_n\right) y(k) = \left(b_0 \mathrm{q}^m + b_1 \mathrm{q}^{m-1} + \cdots + b_m\right) u(k)$$

$$y(k) = \frac{b_0 \mathrm{q}^m + b_1 \mathrm{q}^{m-1} + \cdots + b_m}{\mathrm{q}^n + a_1 \mathrm{q}^{n-1} + \cdots + a_n} u(k) = \frac{B(\mathrm{q})}{A(\mathrm{q})} u(k) = G(\mathrm{q}) u(k)$$

First order systems

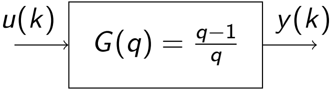


The system with pulse-transfer operator $G(q) = \frac{q-1}{q}$ corresponds to the difference equation

$$y(k) = G(q)u(k) \Leftrightarrow y(k) = \frac{q-1}{q}u(k)$$

$$y(k+1) = ?$$

First order systems



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$$y(k) = G(q)u(k) \Leftrightarrow y(k) = \frac{q-1}{q}u(k)$$

$$y(k+1) = u(k+1) - u(k), \quad \text{i.e. a discrete-time differentiator}$$

First order systems

$$
\begin{tikzcd}
{} \arrow[r, "u(k)"] & \boxed{G(q) = \frac{q}{q-a}} \arrow[r, "y(k)"] & {}
\end{tikzcd}
$$

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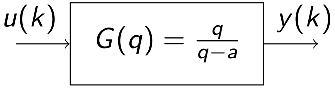

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$$y(k) = G(q)u(k) \Leftrightarrow y(k) = \frac{q}{q-a}u(k)$$

$y(k+1) = ay(k) + u(k+1)$. If $a = 1$, the system is a discrete-time integrator

Pulse-response of a first order system

$$y(k+1) = ay(k) + u(k+1)$$

Pulse response of a first order system

$$y(k+1) = ay(k) + u(k+1)$$

Pair the pulse response to each of the values of a

I) $a = 1$ II) $a = 2$ III) $a = 0.5$ IV) $a = -0.9$