System identification of the tank

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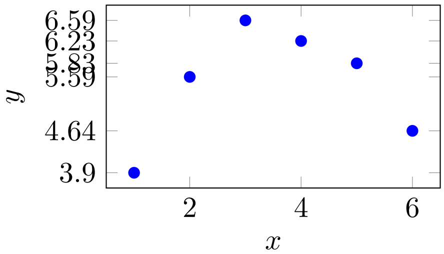
1 Least squares, linear regression

Consider observations from experiments below, where you have set some different values of the dependent variable x, run an experiments and observed the result y.



Can you fit a suitable model to the data?

1.1 Linear regression

Assume the model $y = a_0x^2 + a_1x + a_2 + \epsilon$, where the residual ϵ is the part of the observation y that cannot be explained by the model. We want to find the unknown parameters $\theta = \begin{bmatrix} a_0 & a_1 & a_2 \end{bmatrix}^T$ of the model given a set of experimental data $\mathcal{D} = \{(x_1, y_1),\, (x_2, y_2),\, \ldots,\, (x_N, y_N)\}$.

We can write the model as

$$\begin{aligned}\epsilon &= y - x^2a_0 - xa_1 - a_2 = y - \phi_0(x)a_0 - \phi_1(x)a_1 - \phi_2(x)a_2 = y - \underbrace{\begin{bmatrix}\phi_0(x) & \phi_1(x) & \phi_2(x)\end{bmatrix}}_{\text{regressors}}\begin{bmatrix}a_0\\ a_1\\ a_3\end{bmatrix}\\ &= y - \phi(x)^T\theta.\end{aligned} \tag{1}$$

The model holds for all the observations, which gives

$$\begin{aligned}
\epsilon_1 &= y_1 - \phi(x_1)^T\theta \\
\epsilon_2 &= y_2 - \phi(x_2)^T\theta \\
&\vdots \\
\epsilon_N &= y_n - \phi(x_N)^T\theta
\end{aligned}$$

Linear regression, or least-squares fitting, is defined as the optimization problem

$$\begin{aligned}
\text{minimize} \quad f(\theta) &= \frac{1}{2}\sum_{i=1}^{N}\epsilon_i^2 = \frac{1}{2}\sum_{i=1}^{N}(y_i - \phi(x_i)^T\theta)(y_i - \phi(x_i)^T\theta) \\
&= \frac{1}{2}\sum_{i=1}^{N}\Big(y_i^2 - 2y_i\phi(x_i)^T\theta + (\phi(x_i)^T\theta)^2\Big)
\end{aligned} \tag{2}$$

This optimization problem has a closed-form solution found by setting the derivative of f wrt θ equal to zero

$$\frac{d}{d\theta}f(\theta) = 0$$

$$\frac{1}{2}\sum_{i=1}^{N}\Big(-2y_i\phi(x_i)^T + 2(\phi(x_i)^T\theta)\phi(x_i)^T\Big) = \sum_{i=1}^{N}\Big(-y_i\phi(x_i)^T + (\theta^T\phi(x_i))\phi(x_i)^T\Big) = 0$$

which is equivalent to (by taking the transpose on both sides)

$$\sum_{i=1}^{N}\Big(\phi(x_i)\phi(x_i)^T\theta - \phi(x_i)y_i\Big) = 0.$$

$$\Big(\sum_{i=1}^{N}\phi(x_i)\phi(x_i)^T\Big)\theta - \sum_{i=1}^{N}\phi(x_i)y_i = 0.$$

$$\Big(\sum_{i=1}^{N}\phi(x_i)\phi(x_i)^T\Big)\theta = \sum_{i=1}^{N}\phi(x_i)y_i.$$

Defining the matrices and vectors (with some abuse of notation)

$$
\begin{aligned}
x &= \begin{bmatrix} x_1 & x_2 & \cdots & x_N \end{bmatrix}^T \\
y &= \begin{bmatrix} y_1 & y_2 & \cdots & y_N \end{bmatrix}^T \\
\epsilon &= \begin{bmatrix} \epsilon_1 & \epsilon_2 & \cdots & \epsilon_N \end{bmatrix}^T \\
\Phi(x) &= \begin{bmatrix} \phi(x_1)^T \\ \phi(x_2)^T \\ \vdots \\ \phi(x_N)^T \end{bmatrix}
\end{aligned}
$$

The problem can be written

$$
\text{minimize} \quad f(\theta) = \frac{1}{2}\epsilon^T\epsilon = \frac{1}{2}(y - \Phi(x)\theta)^T(y - \Phi(x)\theta)
$$

with solution

$$
\theta_{LS} = \left(\Phi(x)^T\Phi(x)\right)^{-1}\Phi(x)^T y
$$

1.2 In practice

Given the model $y = a_0x^2 + a_1x + a_2 + \epsilon$ and the data $\mathcal{D} = \{(x_1, y_1), (x_2, y_2), \ldots, (x_N, y_N)\}$, form the vectors and matrices

$$
\begin{aligned}
\Phi(x) &= \begin{bmatrix} x_1^2 & x_1 & 1 \\ x_2^2 & x_2 & 1 \\ \vdots & \vdots & \vdots \\ x_N^2 & x_N & 1 \end{bmatrix} \\
y &= \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_N \end{bmatrix}
\end{aligned}
$$

Find the least-squares solution to

$$
\Phi(x)\theta = y.
$$

In matlab

```
x = [x1, x2, x3, x4, x5, x6]';
y = [y1, y2, y3, y4, y5, y6]';
Phi = [x.^2, x, ones(size(x))];
theta = Phi \ y
```

1.3 What about fitting the model $y = a_0 x^{a_1}$?

Take the logarithm (assuming positive x and y)

$$y = \log a_0 + a_1 \log x$$

2 The tank model

2.1 ODE

$$\frac{d}{dt}p = \alpha A \sqrt{p_s - p} = a(u_v - 5)\sqrt{p_s - p}$$

2.2 Parameter estimation

From experiments filling the tank with different values for u_v one obtains the following table (assuming u_s=1V is p=1bar)

p_s	u_v	p	Δp	Δt	$\dot{p}$
5	5.5	0	0.86	1.51	0.56953642
5	6	0	0.84	0.22	3.8181818
5	9	0	1.16	0.238	4.8739496

The model is

$$\begin{bmatrix} (u_{v_1} - 5)\sqrt{p_s - p_1} \\ (u_{v_2} - 5)\sqrt{p_s - p_2} \\ \vdots \\ (u_{v_N} - 5)\sqrt{p_s - p_N} \end{bmatrix} a = \begin{bmatrix} \dot{p}_1 \\ \dot{p}_2 \\ \vdots \\ \dot{p}_N \end{bmatrix}$$