Process automation laboratory - linearization

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Why linear systems?

Finally, we make some remarks on why linear systems are so important. The answer is simple: because we can solve them!

Richard Feynman

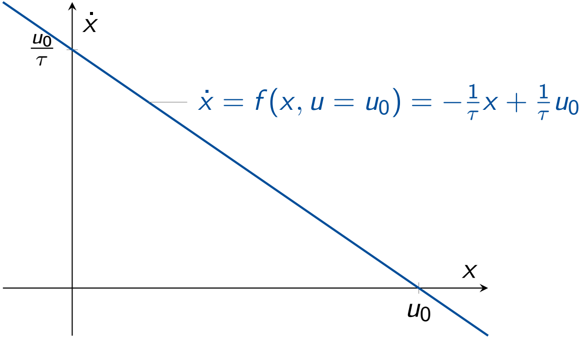
`https://www.feynmanlectures.caltech.edu/I_25.html`

Linear first-order system

$$x + \tau\dot{x} = u \qquad \Leftrightarrow \qquad \dot{x} = \underbrace{-\frac{1}{\tau}x + \frac{1}{\tau}u}_{f(x,u)}$$

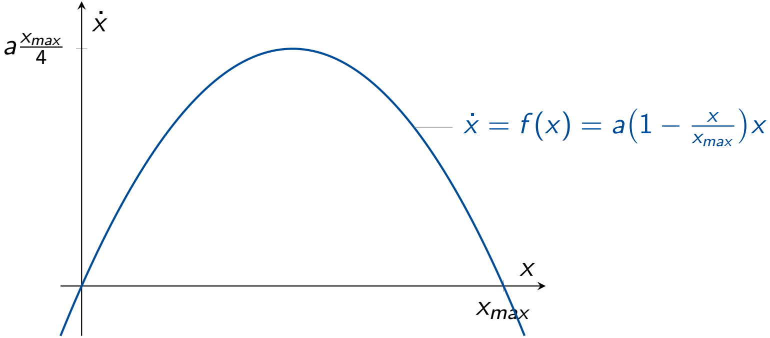
Step response

$$u(t) = \begin{cases} u_0, & t \geq 0, \\ 0, & \text{otherwise} \end{cases}$$



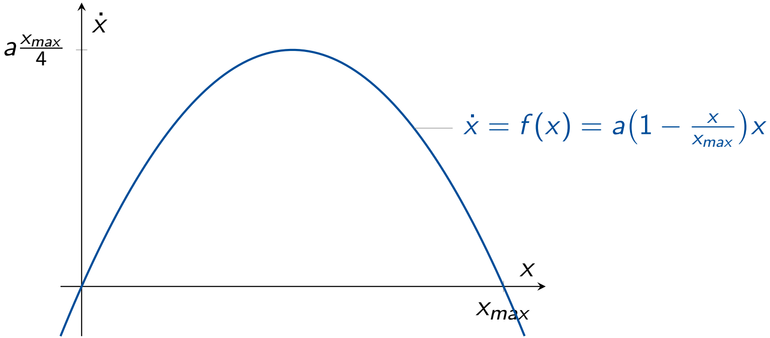
Another relevant first-order system: Logistic growth

$$\dot{x} = \underbrace{a\Big(1 - \frac{x}{x_{max}}\Big)x}_{f(x)}$$



See 3Blue1Brown Exponential growth and epidemics `https://youtu.be/Kas0tIxDvrg`

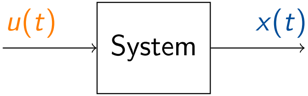
Do in groups



1. In breakout rooms: One of you shares this slide (found on canvas, link in the program for the session)
2. Sketch the solution $x(t)$ using the "bead-on-a-wire" idea for the inital value problem $x(0) = 0.1x_{max}$.

The general idea

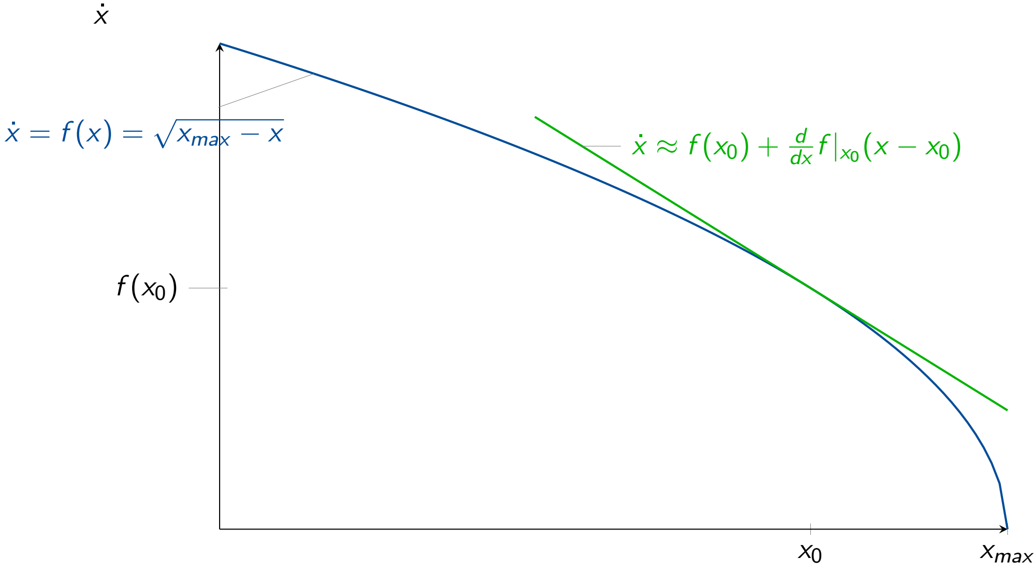
Given a dynamical system described by a nonlinear differential equation



$$\dot{x} = f(x, u)$$

Find a linear approximation to the differential equation about an operating point (x_0, u_0)

The general picture



Linearizing the tank-valve nonlinear model

$$\dot{p} = a_0(u_v - 5)|p_s - p|^{a_1} = f(p, u_v), \quad \text{with } a_0 = 1.1 \text{ and } a_1 = 0.47$$

1. Given operating pressure p_0. Choose operating point u_0 which gives equilibrium $f(p_0, u_0) = 0$.

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1. Given operating pressure p_0. Choose operating point u_0 which gives equilibrium $f(p_0, u_0) = 0$.
2. Introduce deviation variables: $u_v = 5 + u$ and $p = p_0 + y$.

Linearizing the tank-valve nonlinear model

$$\dot{p} = a_0(u_v - 5)|p_s - p|^{a_1} = f(p, u_v), \quad \text{with } a_0 = 1.1 \text{ and } a_1 = 0.47$$

3. Determine partial derivatives

$$\frac{\partial f}{\partial p} = a_0(u_v - 5)a_1|p_s - p|^{a_1 - 1}(-1)$$
$$\frac{\partial f}{\partial u_v} = a_0|p_s - p|^{a_1}$$

4. Evaluate partial derivatives at the operating point $(p_0, u_0$.

$$\frac{\partial f}{\partial p}\Big|_{p_0, u_0} = 0$$
$$\frac{\partial f}{\partial u_v}\Big|_{p_0, u_0} = a_0|p_s - p_0|^{a_1}$$

Linearizing the tank-valve nonlinear model

$$\dot{p} = a_0(u_v - 5)|p_s - p|^{a_1} = f(p, u_v), \quad \text{with } a_0 = 1.1 \text{ and } a_1 = 0.47$$

4. Evaluate partial derivatives at the operating point $(p_0, u_0$.

$$\frac{\partial f}{\partial p}|_{p_0,u_0} = 0$$
$$\frac{\partial f}{\partial u_v}|_{p_0,u_0} = a_0|p_s - p_0|^{a_1}$$

5. Form the linearized model

$$\begin{aligned}\dot{p} = \dot{y} = f(p, u_v) &\approx f(p_0, u_0) + \frac{\partial f}{\partial p}|_{p_0,u_0}(p - p_0) + \frac{\partial f}{\partial u_v}|_{p_0,u_0}(u_v - u_0) \\ &= a_0|p_s - p_0|^{a_1} u.\end{aligned} \tag{1}$$

Linearizing the tank-valve nonlinear model

We arrive at the linear model

$$\dot{y} = a_0|p_s - p_0|^{a_1} u, \qquad \text{which in the Laplace domain is}$$

$$Y(s) = \frac{a_0|p_s - p_0|^{a_1}}{s} U(s)$$

Do in groups

$$\dot{p} = a_0(u_v - 5)|p_s - p|^{a_1} - v = f(p, u_v)$$

1. Given operating pressure p_0. Choose operating point u_0 which gives equilibrium $f(p_0, u_0) = 0$.
2. Introduce deviation variables: $u_v = u_0 + u$ and $p = p_0 + y$.
3. Determine partial derivatives.
4. Evaluate partial derivatives at the operating point.
5. Form the linearized model