Process Automation Laboratory - Modeling first-order systems

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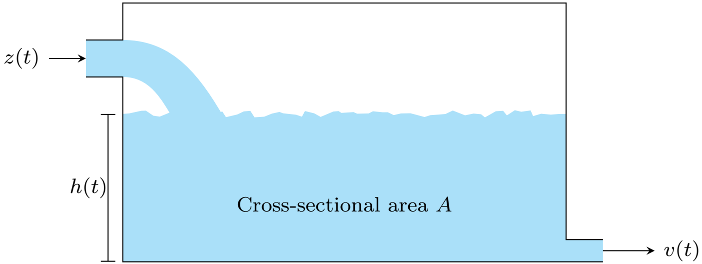
First-order system example: Level control of a tank

What is the input signal and output signal to the system?

First-order system example: Level control of a tank

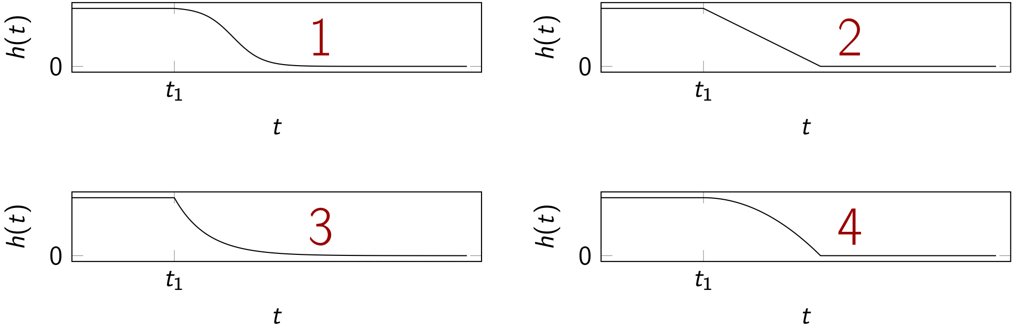


$$\frac{d}{dt}(Ah) = z(t) - x(t) = z(t) - a\sqrt{2gh} \quad \Rightarrow$$

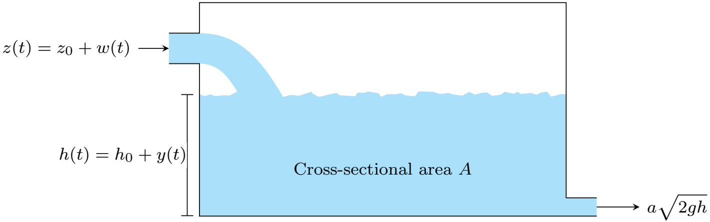
$$\frac{d}{dt}h(t) = -\frac{a\sqrt{2g}}{A}\sqrt{h(t)} + \frac{1}{A}z(t)$$

Intuition

Individual activity A constant inflow has been present since forever, but at time t_1 the flow in is suddenly shut off. Which of the responses of the water level $h(t)$ below is correct?



Deviation variables



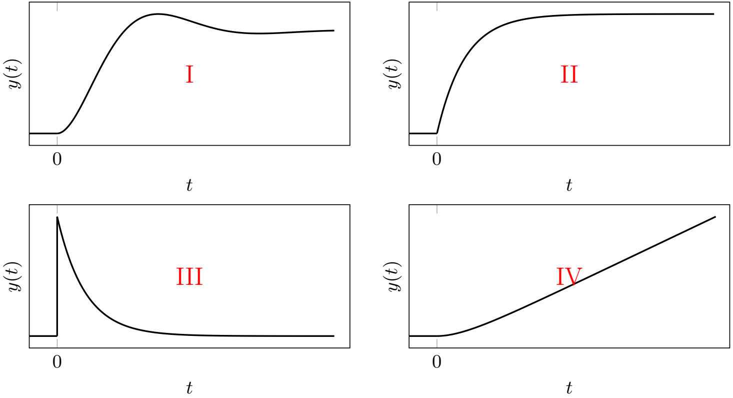
Flow in: $z(t) = z_0 + w(t)$. Level of water: $h(t) = h_0 + y(t)$. The constants h_0 and z_0 define an *operating point*.

$$\frac{d}{dt}h(t) = -\frac{a\sqrt{2g}}{A}\sqrt{h(t)} + \frac{1}{A}z(t)$$

Individual activity Given h_0 determine the operating point for the inflow, z_0, such that the system is in equilibrium at the operating point.

Intuition

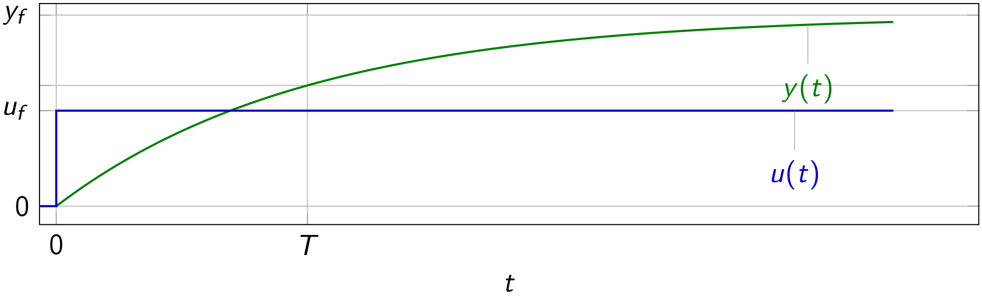
Which change $y(t)$ in the water level corresponds to a step change $w(t)$ in the inflow?



Fitting a first-order model

Assuming a plant model of first-order with time-constant T

$$Y(s) = \frac{K}{sT+1} U(s) \quad \overset{U(s)=\frac{u_f}{s}}{\Longrightarrow} \quad y(t) = u_f K \left(1 - \mathrm{e}^{-\frac{t}{T}}\right) u_H(t)$$

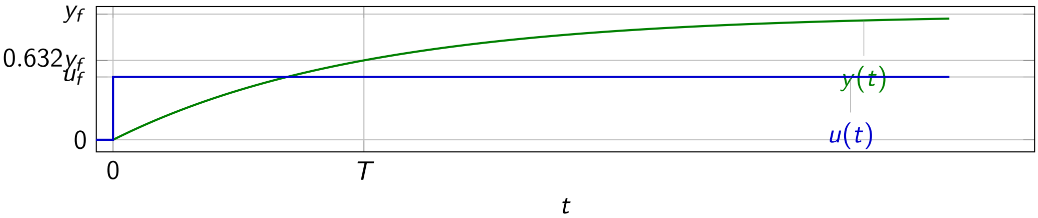


Individual activity Evaluate the response $y(t)$ at the time instant $t = T$ and for $t \to \infty$!

Fitting a first-order model

Assuming a plant model of first-order with time-constant T

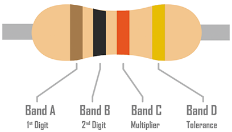
$$Y(s) = \frac{K}{sT+1} U(s) \quad \xRightarrow{U(s)=\frac{u_f}{s}} \quad y(t) = u_f K \left(1 - \mathrm{e}^{-\frac{t}{T}}\right) u_H(t)$$

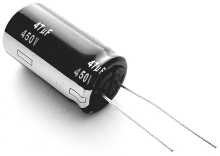


Time-constant: Find the time $t = T$ at which the response has reached 63.2% of its final value

Gain: $y_f = \lim_{t\to\infty} y(t) = K u_f \quad \Rightarrow \quad K = \frac{y_f}{u_f}$

About the tolerance of components and propagation of errors



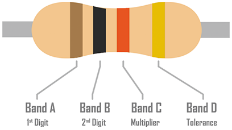


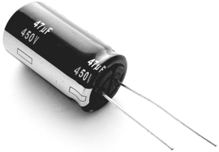
$$R = R_0 \pm \Delta R$$

$$C = C_0 \pm \Delta C$$

$$\tau = RC = \tau_0 + \Delta\tau$$

About the tolerance of components and propagation of errors





$$R = R_0 \pm \Delta R \qquad C = C_0 \pm \Delta C$$

$$\tau = RC = \tau_0 + \Delta\tau$$

Assume the tolerance for the resistor is $\frac{\Delta R}{R_0}$ =5% and for the capacitor $\frac{\Delta C}{C_0}$ =20%. What will the tolerance for the time-constant $\frac{\Delta\tau}{\tau_0}$ be?

1. 5%
2. 20%
3. 25%
4. 100%

About the tolerance of components amd propagation of errors

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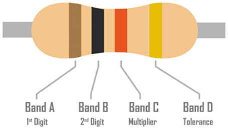

Color	Meaning
Brown	First digit 1
Black	Second digit 0
Orange	Multiply with 10^3
Gold	Tolerance 5%

$$\begin{aligned} R &= R_0 \pm \Delta R \\ &= 10\,\mathrm{k}\Omega \pm 5\% = (10 \pm 0.5)\,\mathrm{k}\Omega \end{aligned}$$

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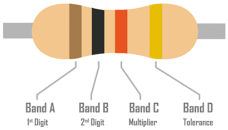

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Two ways to calculate $\Delta\tau$:

1. Direct calculation

$$\begin{aligned} \tau = RC &= (R_0 + \Delta R)(C_0 + \Delta C) \\ &= R_0 C_0 + R_0 \Delta C + C_0 \Delta R + \Delta R \Delta C \\ &\approx \tau_0 + \underbrace{R_0 \Delta C + C_0 \Delta R}_{\Delta\tau} \end{aligned}$$

2. Total derivative

$$\begin{aligned} \Delta\tau &= \left.\frac{\partial\tau}{\partial R}\right|_{R_0, C_0} \Delta R + \left.\frac{\partial\tau}{\partial C}\right|_{R_0, C_0} \Delta C \\ &= C_0 \Delta R + R_0 \Delta C. \end{aligned}$$

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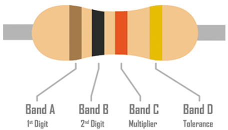

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$$\frac{\Delta\tau}{\tau_0} = \frac{C_0 \Delta R + R_0 \Delta C}{R_0 C_0} = \frac{\Delta R}{R_0} + \frac{\Delta C}{C_0}$$

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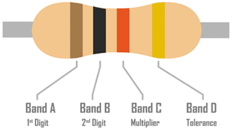

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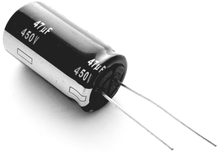

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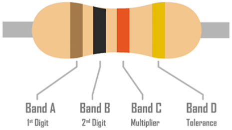

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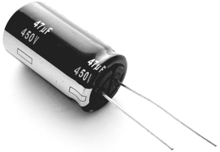

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