Process Automation Laboratory - Modeling second-order systems

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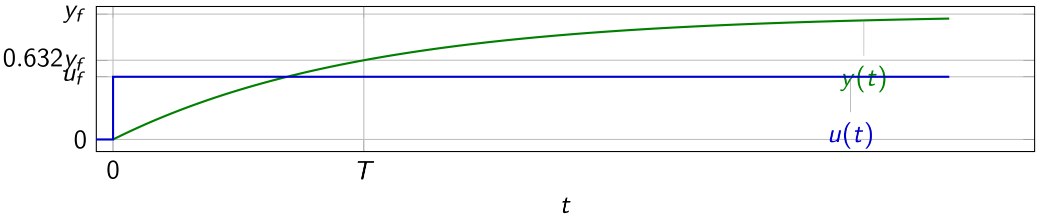
Fitting a first-order model

Assuming a plant model of first-order with time-constant T

$$Y(s) = \frac{K}{sT+1} U(s) \quad \overset{U(s)=\frac{u_f}{s}}{\Longrightarrow} \quad y(t) = u_f K \left(1 - \mathrm{e}^{-\frac{t}{T}}\right) u_H(t)$$

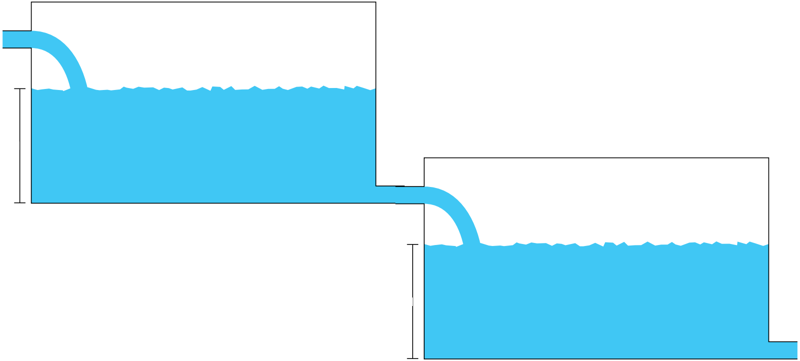


Time-constant: Find the time $t = T$ at which the response has reached 63.2% of its final value

Gain: $y_f = \lim_{t\to\infty} y(t) = K u_f \quad \Rightarrow \quad K = \frac{y_f}{u_f}$

Second-order models

Two first-order models in series



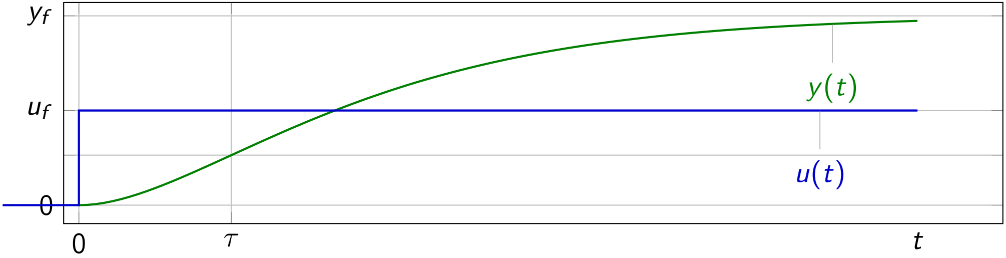
$$u(t) \longrightarrow \boxed{G_1(s)} \longrightarrow \boxed{G_2(s)} \longrightarrow y(t)$$

$$G(s) = G_1(s)G_2(s)$$

Fitting second-order critically-damped model

Model with two identical time-constants. Assuming model

$$Y(s) = \frac{K}{(s\tau+1)^2} U(s) \quad \overset{U(s)=\frac{u_f}{s}}{\Longrightarrow} \quad y(t) = u_f K \left(1 - (1+\frac{t}{\tau})\mathrm{e}^{-\frac{t}{\tau}}\right) u_H(t)$$

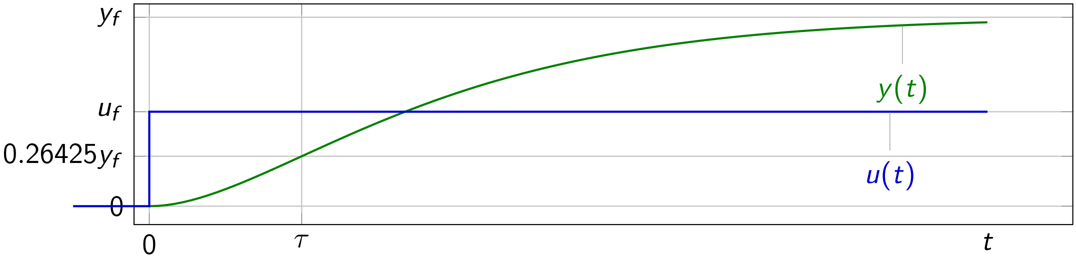


Individual activity Evaluate the response $y(t)$ at the time instants $t = \tau$!

Fitting second-order critically-damped model

Model with two identical time-constants. Assuming model

$$Y(s) = \frac{K}{(s\tau + 1)^2} U(s) \quad \overset{U(s)=\frac{u_f}{s}}{\Longrightarrow} \quad y(t) = u_f K \left(1 - (1 + \frac{t}{\tau})\mathrm{e}^{-\frac{t}{\tau}}\right) u_H(t)$$

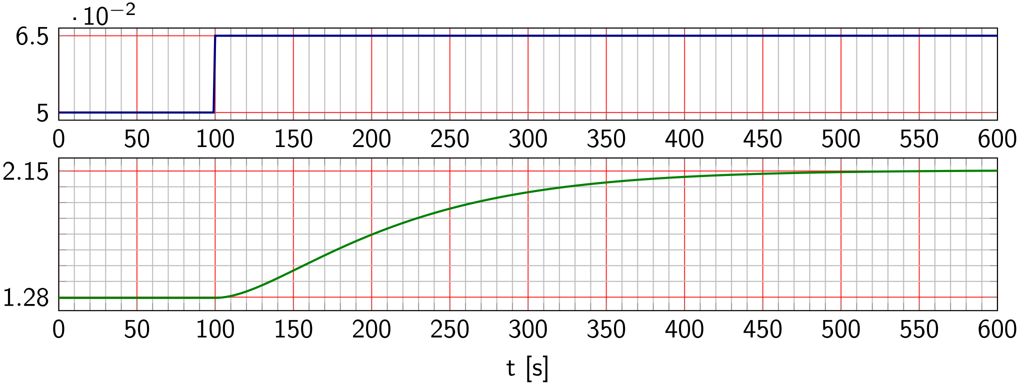


$$y_f = \lim_{t\to\infty} y(t) = u_f K \quad \Rightarrow \quad K = \frac{y_f}{u_f}.$$

Fitting second-order critically-damped model

Assuming model

$$\color{green}{Y(s)} = \frac{K}{(s\tau+1)^2}\color{blue}{U(s)} \quad \overset{U(s)=\frac{u_f}{s}}{\Longrightarrow} \quad \color{green}{y(t)} = u_f K\left(1-(1+\frac{t}{\tau})\mathrm{e}^{-\frac{t}{\tau}}\right)u_H(t)$$



Activity Determine the parameters of the model from the experimental data.

Second-order under-damped models

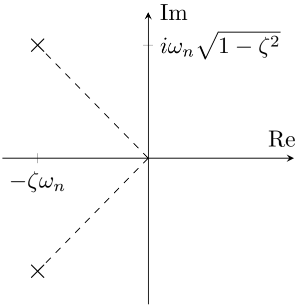
A system with ODE

$$\ddot{y} + 2\zeta\omega_n \dot{y} + \omega_n^2 y = \omega_n^2 u,$$

becomes in the Laplace domain

$$Y(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} U(s).$$

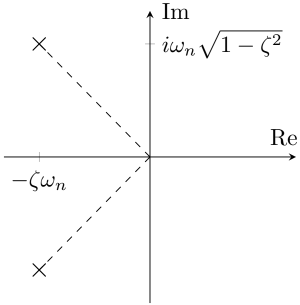
- ζ is called the *damping ratio*.
- ω_n is called the *natural frequency* (of the system).

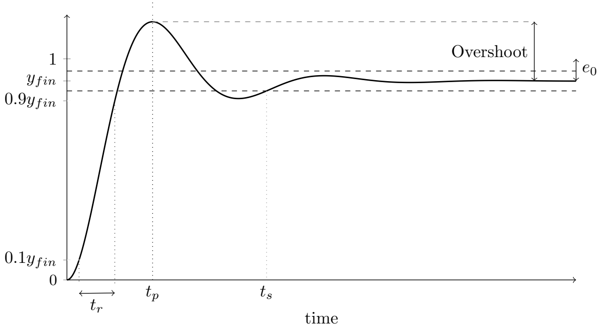


Second-order under-damped models

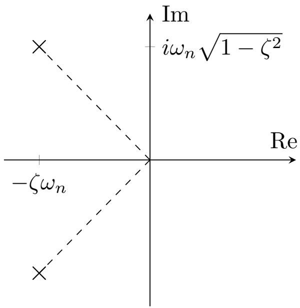
$$Y(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} U(s), \qquad \overset{U(s)=\frac{u_f}{s}}{\Longrightarrow}$$

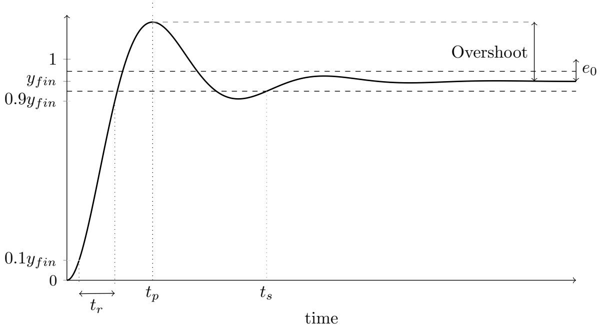
$$y(t) = 1 - \frac{\mathrm{e}^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}} \sin\left(\sqrt{1-\zeta^2}\omega_n t + \phi\right)$$





Second-order under-damped models

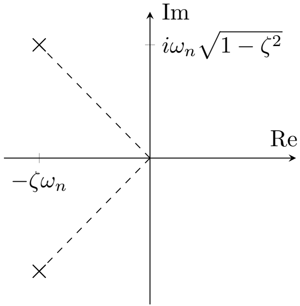


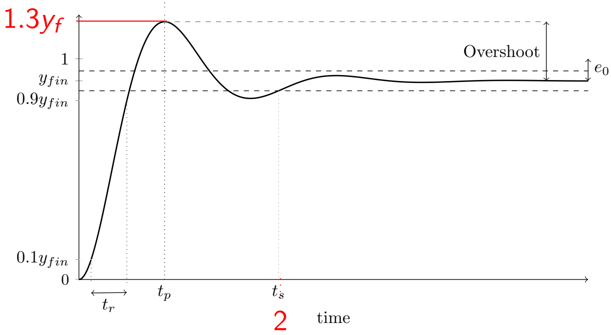


$$t_r \approx \frac{\pi}{2\omega_n}, \qquad t_s \approx \frac{4}{\zeta\omega_n}, \qquad t_p \approx \frac{\pi}{\sqrt{1-\zeta^2}\omega_n}, \qquad \zeta \approx \sqrt{\frac{(\ln\frac{PO}{100})^2}{\pi^2 + (\ln\frac{PO}{100})^2}}$$

Second-order under-damped models

Activity in pairs Determine the poles of the system!





$$t_s \approx \frac{4}{\zeta\omega_n},$$

$$\zeta \approx \sqrt{\frac{(\ln \frac{PO}{100})^2}{\pi^2 + (\ln \frac{PO}{100})^2}}$$