

Process Automation Laboratory - Modeling second-order systems

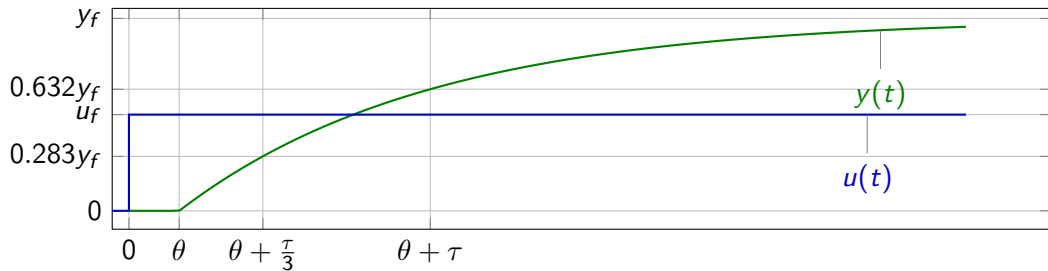
Kjartan Halvorsen

August 26, 2021

Fitting first-order model with delay

Assuming a plant model of first-order with time constant T and delay θ

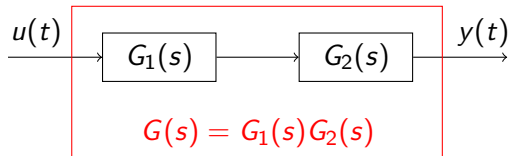
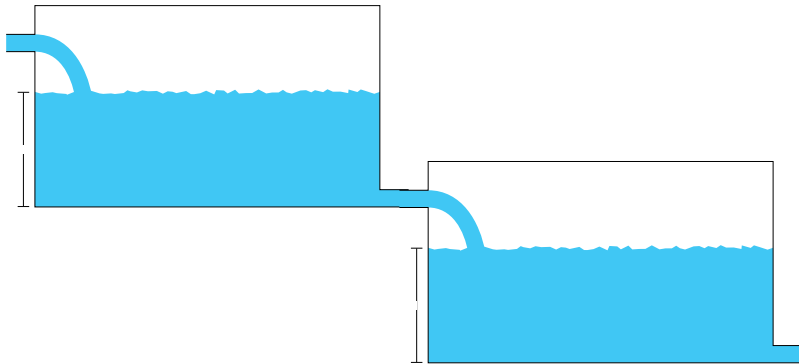
$$Y(s) = \frac{Ke^{-s\theta}}{s\tau + 1} U(s) \quad U(s) = \frac{u_f}{s} \quad \Rightarrow \quad y(t) = u_f K (1 - e^{-\frac{t-\theta}{\tau}}) u_H(t - \theta)$$



$$y_f = \lim_{t \rightarrow \infty} y(t) = u_f K \quad \Rightarrow \quad K = \frac{y_f}{u_f}.$$

Second-order models

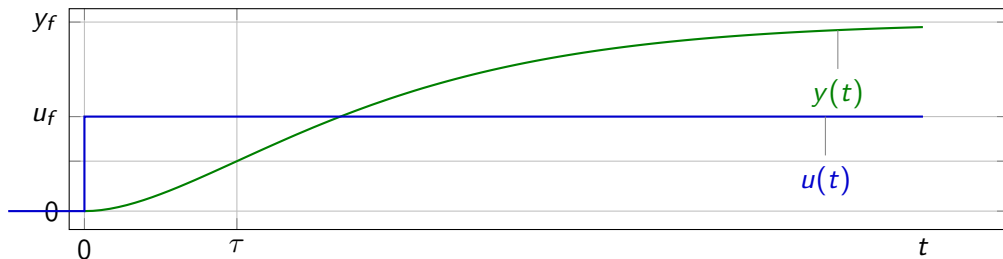
Two first-order models in series



Fitting second-order critically-damped model

Model with two identical time-constants. Assuming model

$$Y(s) = \frac{K}{(s\tau + 1)^2} U(s) \quad \xrightarrow{U(s) = \frac{u_f}{s}} \quad y(t) = u_f K \left(1 - \left(1 + \frac{t}{\tau} \right) e^{-\frac{t}{\tau}} \right) u_H(t)$$

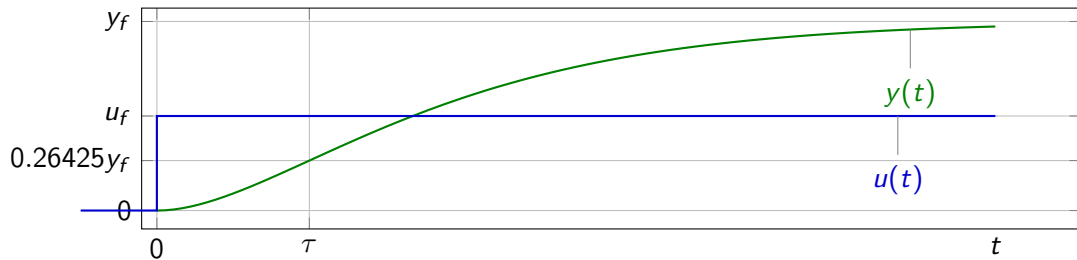


Individual activity Evaluate the response $y(t)$ at the time instants $t = \tau$!

Fitting second-order critically-damped model

Model with two identical time-constants. Assuming model

$$Y(s) = \frac{K}{(s\tau + 1)^2} U(s) \quad \xrightarrow{U(s) = \frac{u_f}{s}} \quad y(t) = u_f K \left(1 - \left(1 + \frac{t}{\tau} \right) e^{-\frac{t}{\tau}} \right) u_H(t)$$



$$y_f = \lim_{t \rightarrow \infty} y(t) = u_f K \quad \Rightarrow \quad K = \frac{y_f}{u_f}.$$

Second-order under-damped models

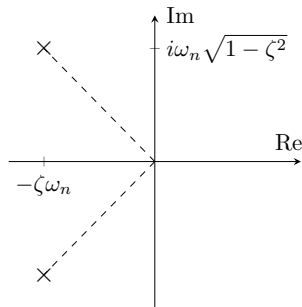
A system with ODE

$$\ddot{y} + 2\zeta\omega_n\dot{y} + \omega_n^2 y = \omega_n^2 u,$$

becomes in the Laplace domain

$$Y(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} U(s).$$

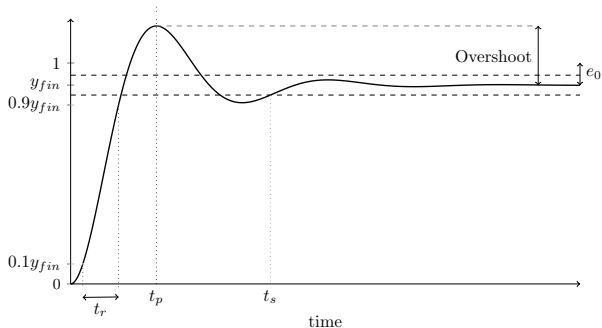
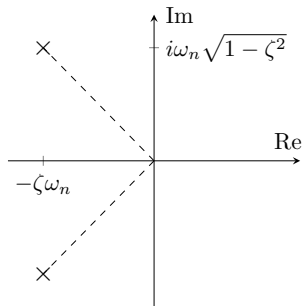
- ▶ ζ is called the *damping ratio*.
- ▶ ω_n is called the *natural frequency* (of the system).



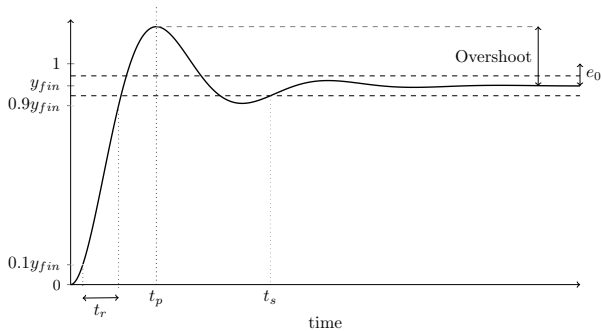
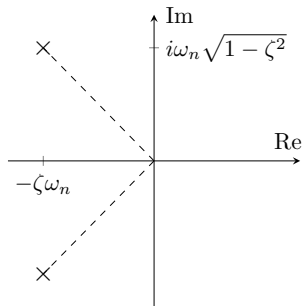
Second-order under-damped models

$$Y(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} U(s), \quad U(s) = \frac{u_f}{s}$$

$$y(t) = 1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}} \sin(\sqrt{1-\zeta^2}\omega_n t + \phi)$$



Second-order under-damped models



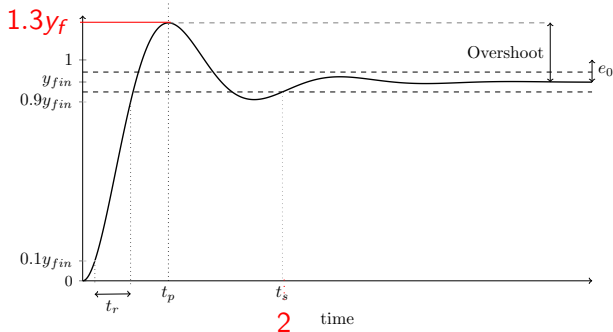
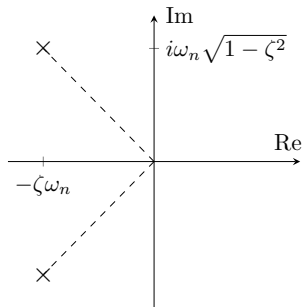
$$t_r \approx \frac{\pi}{2\omega_n}, \quad t_s \approx \frac{4}{\zeta\omega_n},$$

$$t_p \approx \frac{\pi}{\sqrt{1-\zeta^2}\omega_n},$$

$$\zeta \approx \sqrt{\frac{(\ln \frac{PO}{100})^2}{\pi^2 + (\ln \frac{PO}{100})^2}}$$

Second-order under-damped models

Activity in pairs Determine the poles of the system!



$$t_s \approx \frac{4}{\zeta\omega_n},$$

$$\zeta \approx \sqrt{\frac{(\ln \frac{PO}{100})^2}{\pi^2 + (\ln \frac{PO}{100})^2}}$$