Process automation laboratory - Root locus, PI control

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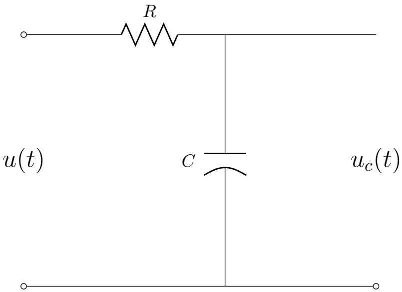
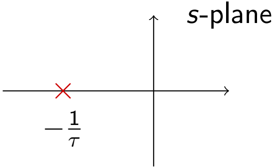
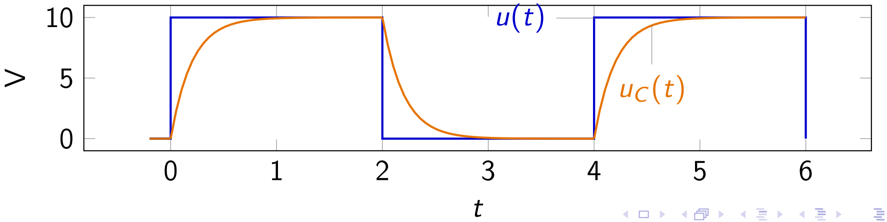
Repetition: The RC-circuit - a first-order system



$$U_C(s) = \frac{1}{s\underbrace{RC}_{\tau}+1}U(s)$$

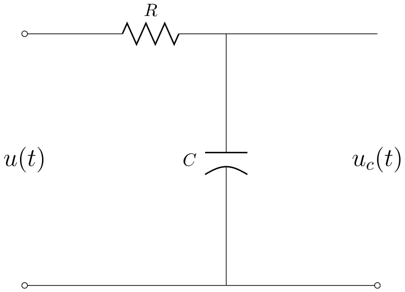


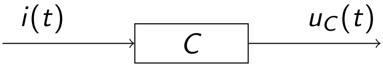
$u_C(t) = 10(1 - \mathrm{e}^{\frac{t}{\tau}})$, for $u(t)$ step of size 10



A concept to keep in mind

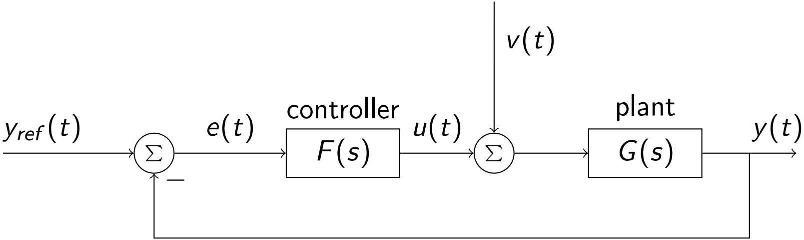
In a system with an integrator steady-state can only exist if the signal to the integrator is zero





$$u_C(t) = u_C(0) + \frac{1}{C}\int i(s)ds$$

Feedback control

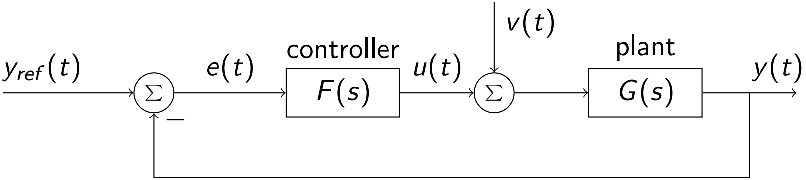


For the closed-loop system we get

$$Y(s) = \frac{G_o(s)}{1 + G_o(s)} Y_{ref}(s) + \frac{G(s)}{1 + G_o(s)} V(s),$$

where $G_o(s) = G(s)F(s)$ is called the *loop gain*.

Feedback control



$$Y(s) = \frac{G_o(s)}{1+G_o(s)} Y_{ref}(s) + \frac{G(s)}{1+G_o(s)} V(s), \quad G_o(s) = G(s)F(s).$$

Let $G(s) = \frac{1}{s}$ and $F(s) = K$.

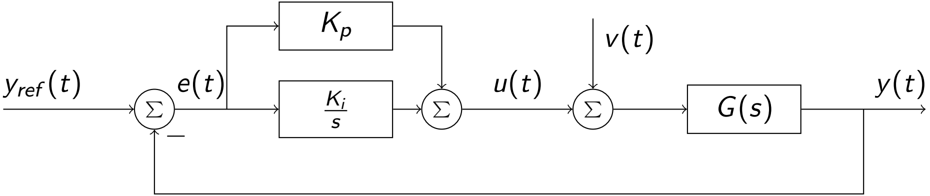
1. Will there be a steady-state control error ($\lim_{t\to\infty} e(t) \neq 0$) if $y_{ref}(t) = 0$ and $v(t)$ is a unit step? Why? Answer on Socrative
2. What is the characteristic equation for the closed-loop system?
3. Sketch the location of the poles in the imaginary plane as the gain K varies from 0 to ∞. Group exercise in breakout room

How to get rid of the steady-state error

Use a proportional-integral controller (PI controller)

$$F(s) = K_p + \frac{K_i}{s}.$$

This gives closed-loop system



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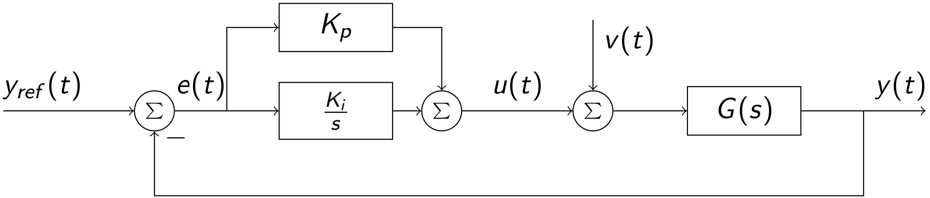


The only way that steady-state can exist is if the input to the integrator of the controller is zero.

Root locus

Given loop gain

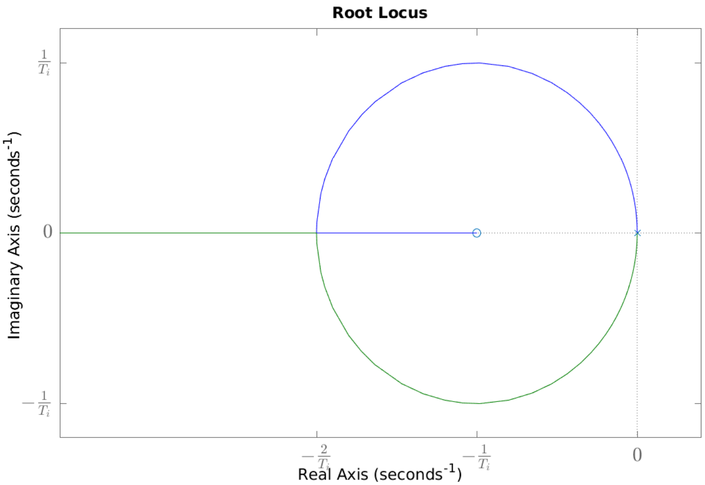
$$G_o(s) = K\frac{Q(s)}{P(s)}$$

how does the solutions to the characteristic equation

$$1 + G_o(s) = 0 \quad \Leftrightarrow \quad P(s) + KQ(s) = 0$$

(i.e. the poles of the closed-loop system) depend on K?

The root locus for PI-control of the integrator



Root locus question

Root locus definition

Let

$$\begin{cases} P(s) & = s^n + a_1 s^{n-1} + \cdots + a_n = (s-p_1)(s-p_2)\cdots(s-p_n) \\ Q(s) & = s^m + b_1 s^{m-1} + \cdots + b_m = (s-q_1)(s-q_2)\cdots(s-q_m) \end{cases}, \quad n \geq m$$

The root locus shows how the roots to the equation

$$P(s) + K \cdot Q(s) = 0, \quad 0 \leq K < \infty \tag{1}$$

depend on the parameter K. The root locus consists of the set of all points in the complex plane that are roots to (1) for some non-negative value of K.

Characteristics of the root locus

The polynomial $P(s) + KQ(s) = 0$ above will always have n roots. Each gives a *branch* in the root locus. Since the polynomials $P(s)$ and $Q(s)$ have real-valued coefficients, all roots are either real or complex-conjugated pairs. This means that the root locus is *symmetric about the real axis.* Other characteristics

- Start points - marked by crosses
- End points - marked by circles
- Asymptotes
- Pieces of the real axis

Start- and end points

Start points These are the n roots of $P(s) + KQ(s)$ for $K = 0$, i.e. the roots of $P(s)$. These are the open-loop poles, and are marked with crosses '$\times$'

End points These are the m (finite) roots of $P(s) + KQ(s)$ when $K \to \infty$, and are hence the roots of $Q(s)$. The end points are marked with circles '$\circ$'

The real axis

Those parts of the real axis that have an odd number of real-valued start- or end points to the right (including multiplicity) belong to the root locus.

Asymptotes

With n starting points and m end points, then m of the branches will go to end points. The rest will go out towards infinity along $n - m$ asymptotes. The asymptotes go out symmetrically from a point on the real axis.

Asymptotes, directions

The directions of the asymptotes are given by the expression

$$\theta_k = \arg s = \frac{(2k+1)\pi}{n-m}, \; k \in \mathbb{Z}$$

Example: 6 start points and 3 end points gives $n - m = 6 - 3 = 3$ and the directions

$$\theta = \begin{cases} \frac{\pi}{3}, & k = 0 \\ \pi, & k = 1 \\ -\frac{\pi}{3}, & k = -1 \end{cases}.$$

Asymptotes, intersection with the real axis

$$i.p. = \frac{\sum_{i=0}^{n} p_i - \sum_{i=0}^{m} q_i}{n-m},$$

where $\{p_i\}$ are the starting points (open-loop poles) and $\{q_i\}$ are the end points (open-loop zeros).

PI-Control of the integrator

Write the controller

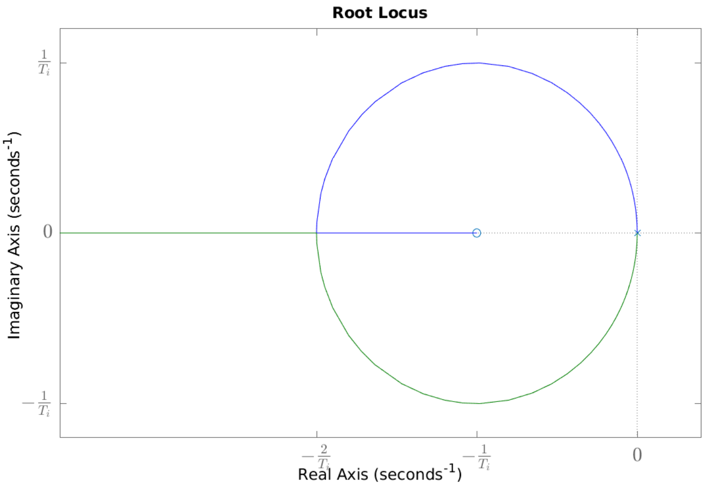
$$F(s) = K_p + \frac{K_i}{s} = K\big((1 + \frac{1}{sT_i}\big) = K/T_i \frac{sT_i + 1}{s},$$

and let $T_i = 2$. The characteristic equation can be written

$$s^2 + \frac{K}{2}(2s + 1) = 0$$

- Start points: $n = 2$, in $s = 0$
- End points: $m = 1$, $s = -\frac{1}{2}$
- Asymptotes: $m - n = 1$, with directions $\theta = \pi$
- The real line: The real-line left of the end-point is part of the root locus.

PI-Control of the integrator



Do on your own: First-order system

Instead of the plant being an integrator

$$F(s) = \frac{1}{s}$$

consider a stable first order system

$$F(s) = \frac{1}{s + a}$$

How does the root locus change?