Modeling and automation - Exam group 302

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Time 11:05 - 12:55

Permitted aids The single page with your own notes, table of Laplace transforms, calculator

All answers should be readable and well motivated (if nothing else is written). Solutions/motivations should be written on the provided spaces in this exam. Use the last page if more space is needed.

Good luck!

Matricula and name:

Magnetic levitation

Figure 1 shows a one-dimensional magnetic suspension system. The current i in the windings generates a magnetic field which suspends the mass m. Friction is completely negligable, since the mass is not in contact with any other body, and its velocity is typically very small (no air-drag). Hence, there are two forces acting on the mass: gravity and the magnetic force. The magnetic force is proportional to the square of the current i and inverse proportional to the square of the gap distance x. This gives the equation of motion

$$m\ddot{x} = -C\left(\frac{i}{x}\right)^2 + mg. \tag{1}$$

The system is non-linear, so in order to use linear control design, the system must be linearized about an operating point. Introduce the deviation variables y and u

$$\begin{aligned} x(t) &= x_0 + y(t)\\ i(t) &= i_0 - u(t). \end{aligned}$$

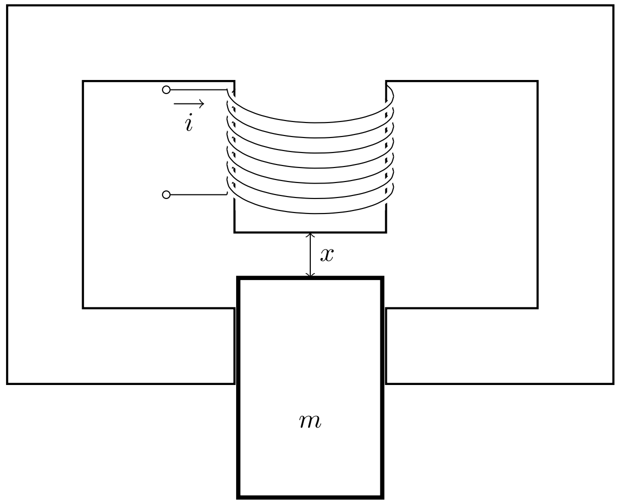
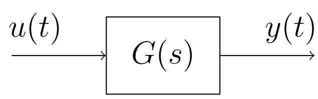


Figure 1: A magnetic suspension system. The force of gravity acts to pull the mass m downwards. The force due to the magnetic field generated by the current i keeps the mass from falling down. The displacement x of the mass is positive downwards.

The input signal to the system is the change, u, in the current in the windings, and the output signal is the change, y, in the gap distance. The negative sign in the definition of the deviation variable u, $i = i_0 - u$ is introduced so that a positive input signal leads to a positive change in the gap distance.



Exercise 1

(a) Show that given an operating point x_0, the corresponding operating point current that gives equilibrium is

$$i_0 = x_0\sqrt{\frac{mg}{C}}.$$

Calculations:

(b) Show that the linearized model of the magnetic levitator system becomes

$$\ddot{y} = \frac{2Ci_0^2}{mx_0^3}(x - x_0) - \frac{2i_0 C}{mx_0^2}(i - i_0) = \frac{2g}{x_0}y + \frac{2\sqrt{Cg}}{x_0\sqrt{m}}u \tag{2}$$

Calculations:

(c) Determine the transfer function of the linear model (2).

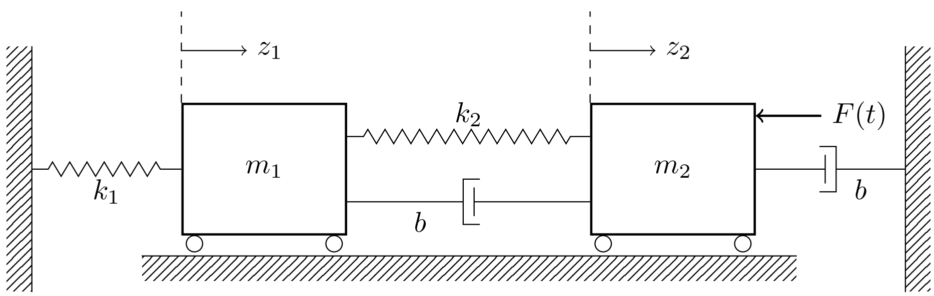
Calculations:

(d) [Bonus exercise worth 5p] Find the poles of the system and determine if the system is stable or not.

Calculations:

A Coupled spring-mass system

The figure below shows a system consisting of two masses connected together and to two rigid walls.



The displacements z_1 and z_2 are deviations from the equilibrium positions of the two masses, respectively, when the force $F(t)$ acting on the right mass is zero. The springs and dampers are considered to be linear.

Problems

Exercise 2

(a) Draw free-body-diagrams and determine the differential equations that describe the system.

Calculations:

(b) Choose a suitable state vector x, and write the system on state-space form

$$\begin{aligned}\dot{x}(t) &= Ax(t) + B\,F(t)\\ y(t) &= Cx(t)\end{aligned}$$

where the output signal consists of the positions of the two masses, $y = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$.

Calculations:

Solutions

Magnetic levitation

Exercise 1

(a) At equilibrium the gravitational force and the magnetic force are equal in magnitude but opposite.

$$
\begin{aligned}
0 &= -C\left(\frac{i_0}{x_0}\right)^2 + mg, \quad \text{Solve for } i_0 \\
i_0^2 &= x_0^2\frac{mg}{C} \\
i_0 &= x_0\sqrt{\frac{mg}{C}}.
\end{aligned}
$$

(b) We write the nonlinear ODE as

$$
\ddot{x} = -\frac{Ci^2}{mx^2} + g = f(i, x),
$$

and linearize the nonlinear right-hand side about the operating point x_0, i_0:

$$
\begin{aligned}
f(i, x) &\approx f(i_0, x_0) + \left.\frac{\partial f}{\partial x}\right|_{i_0, x_0}(x - x_0) + \left.\frac{\partial f}{\partial i}\right|_{i_0, x_0}(i - i_0) \\
&= \underbrace{-\frac{Ci_0^2}{mx_0^2} + g}_{=0} + \frac{2Ci_0^2}{mx_0^3}\underbrace{(x - x_0)}_{y} - \frac{2Ci_0}{mx_0^2}\underbrace{(i - i_0)}_{-u} \\
&= \frac{2Cx_0^2mg}{Cmx_0^3}y + \frac{2x_0\sqrt{mg}}{\sqrt{C}mx_0^2}u \\
&= \frac{2g}{x_0}y + \frac{2\sqrt{Cg}}{x_0\sqrt{m}}u.
\end{aligned}
$$

The left-hand side of the ODE becomes

$$
\ddot{x} = \ddot{x}_0 + \ddot{y} = \ddot{y}.
$$

(c) Lumping the parameters (to make life easier), $a = \frac{2g}{x_0}$, $b = \frac{2\sqrt{Cg}}{x_0\sqrt{m}}$, taking the Laplace transform of each term in the model (2), and setting all initial values to zero (we are interested in the transfer function, not the transient response to the initial value) gives

$$
s^2Y(s) = aY(s) + bU(s),
$$

which solved for $Y(s)$ is

$$
Y(s) = \frac{b}{s^2 - a}U(s).
$$

Hence, the transfer function is

$$G(s) = \frac{cs}{s^2 - a}.$$

(d) The poles are found by setting the denominator of the transfer function to zero, i.e. formulating the characteristic equation

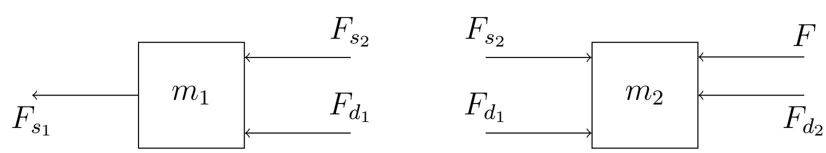
$$s^2 + a = 0 \quad \Rightarrow \quad s = \pm\sqrt{a}.$$

The poles are real, and symmetric about the imaginary axis. The pole $s = \sqrt{a}$ is in the right-half plan, and so the system is unstable.

Spring-mass system

Exercise 2

(a) Free-body-diagrams of the two bodies give



where the spring forces are given by

$$F_{s_1} = k_1 z_1, \qquad F_{s_2} = k_2(z_1 - z_2),$$

and the damper forces

$$F_{d_2} = b\dot{z}_2, \qquad F_{d_1} = b(\dot{z}_1 - \dot{z}_2).$$

Newton's second law for each body gives

$$\begin{aligned} m_1\ddot{z}_1 &= -F_{s_1} - F_{s_2} - F_{d_2} = -k_1 z_1 - k_2(z_1 - z_2) - b(\dot{z}_1 - \dot{z}_2) \\ m_2\ddot{z}_2 &= -F + F_{s_2} + F_{d_2} - F_d = -F + k_2(z_1 - z_2) + b(\dot{z}_1 - \dot{z}_2) - b\dot{z}_2. \end{aligned}$$

(b) One natural choice (among various) of the state vector is

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} z_1 \\ \dot{z}_1 \\ z_1 - z_2 \\ \dot{z}_2 \end{bmatrix},$$

where we focuse on the four energy-storing elements of the system (the deflection of the two springs, and the velocities of the two masses). with this choice, the differential equations can be written

$$\begin{aligned}
\dot{x}_1 &= \dot{z}_1 = x_2\\
\dot{x}_2 &= \ddot{z}_1 = -\frac{k_1}{m_1}z_1 - \frac{k_2}{m_1}(z_1 - z_2) - \frac{b}{m_1}\dot{z}_1 + \frac{b}{m_1}\dot{z}_2 = -\frac{k_1}{m_1}x_1 - \frac{k_2}{m_1}x_3 - \frac{b}{m_1}x_2 + \frac{b}{m_1}x_4\\
\dot{x}_3 &= \dot{z}_1 - \dot{z}_2 = x_2 - x_4\\
\dot{x}_4 &= \ddot{z}_2 = \frac{k}{m_2}(z_1 - z_2) + \frac{b}{m_2}\dot{z}_1 + \frac{b}{m_2}\dot{z}_2 - \frac{b}{m_2}\dot{z}_2 - \frac{1}{m_2}F = \frac{b}{m_2}x_2 + \frac{k_2}{m_2}x_3 - \frac{2b}{m_2}x_4 - \frac{1}{m_2}F.
\end{aligned}$$

The state-space model becomes

$$\begin{aligned}
\dot{z} &= \begin{bmatrix} 0 & 1 & 0 & 0\\ -\frac{k_1}{m_1} & -\frac{b}{m_1} & -\frac{k_2}{m_1} & \frac{b}{m_1}\\ 0 & 1 & 0 & -1\\ 0 & \frac{b}{m_2} & \frac{k_2}{m_2} & -\frac{2b}{m_2} \end{bmatrix} x + \begin{bmatrix} 0\\ 0\\ 0\\ -\frac{1}{M} \end{bmatrix} F\\
y &= \begin{bmatrix} z_1\\ z_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0\\ 1 & 0 & -1 & 0 \end{bmatrix} x.
\end{aligned}$$