State-space models - stability

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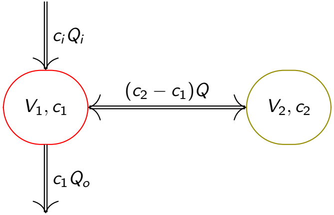
Compartment model



$$V_1 \frac{dc_1}{dt} = Q(c_2 - c_1) - Q_o c_1 + Q_i c_i, \qquad c_1 \ge 0$$

$$V_2 \frac{dc_2}{dt} = Q(c_1 - c_2), \qquad c_2 \ge 0,$$

$$\dot{x} = \overbrace{\begin{bmatrix} \quad & \quad \\ \quad & \quad \end{bmatrix}}^{A} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \overbrace{\begin{bmatrix} \quad \\ \quad \end{bmatrix}}^{B} u$$

$$y = \underbrace{\begin{bmatrix} \quad & \quad \end{bmatrix}}_{C} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

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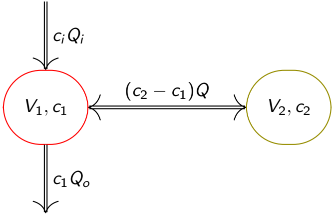

$$
\begin{aligned}
V_1 \frac{dc_1}{dt} &= Q(c_2 - c_1) - Q_o c_1 + Q_i c_i, & c_1 &\geq 0 \\
V_2 \frac{dc_2}{dt} &= Q(c_1 - c_2), & c_2 &\geq 0,
\end{aligned}
$$

$$
\dot{x} = \overbrace{\begin{bmatrix} -\frac{Q+Q_o}{V_1} & \frac{Q}{V_1} \\ \frac{Q}{V_2} & -\frac{Q}{V_2} \end{bmatrix}}^{A} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \overbrace{\begin{bmatrix} \frac{1}{V_1} \\ 0 \end{bmatrix}}^{B} u
$$

$$
y = \underbrace{\begin{bmatrix} 1 & 0 \end{bmatrix}}_{C} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}
$$

Homogeneous solution

$$\dot{x} = Ax, \qquad x(0) = x_0$$

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The matrix exponential

There exists a function $\Phi : \mathbb{R} \mapsto \mathbb{R}^{n \times n}$

$$\Phi(t) = \mathrm{e}^{At} = I + tA + \frac{t^2}{2!}A^2 + \frac{t^3}{3!}A^3 + \cdots$$

that has the property

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The homogeneous solution to the state-space model is

$$x(t) = \Phi(t)x(0)$$

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Eigenvalues and eigenvectors

If the initial value $x(0) = v$ is an eigenvector of the matrix A

$$Av = \lambda v,$$

$$\begin{aligned} x(t) &= \Phi(t)v = \mathrm{e}^{At}v \\ &= \left(I + tA + \frac{t^2}{2!}A^2 + \frac{t^3}{3!}A^3 + \cdots\right)v \\ &= Iv + tAv + \frac{t^2}{2!}A^2v + \frac{t^3}{3!}A^3v + \cdots \\ &= v + t\lambda v + \frac{(t\lambda)^2}{2!}v + \frac{(t\lambda)^3}{3!}v + \cdots \\ &= \mathrm{e}^{\lambda t}v \end{aligned}$$

Stability

Stability is a key property of the system itself. It does not depend on the input signal.

The homogeneous solution can be written

$$x(t) = \mathrm{e}^{\lambda_1 t}\alpha_1 v_1 + \mathrm{e}^{\lambda_2 t}\alpha_2 v_2 + \cdots + \mathrm{e}^{\lambda_n t}\alpha_n v_n.$$

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A sufficient and necessary condition is that *all* the eigenvalues of A has negative real-part.

$$\mathrm{Re}\{\lambda_i\} < 0, \ \forall i = 1, 2, 3 \ldots, n$$

The eigenvalues of A are the poles of the system.