Modeling and automation - Test exam

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Time When convenient, but before the exam on Thursday June 9

Place

Permitted aids The single page with your own notes, table of Laplace transforms, calculator

All answers should be readable and well motivated (if nothing else is written). Solutions/motivations should be written on the provided spaces in this exam. Use the last page if more space is needed.

Good luck!

Matricula and name:

Velocity control of a pneumatic cylinder

Pneumatic cylinders (figure 1) are much used as actuators in industrial processes, due to relatively low cost and good power to weight ratio. The cylinder can move in a single direction x, and exert a force in this direction by a difference in pressure $\Delta p(t) = p_1(t) - p_2(t)$ between its two chambers. See figure 1. We consider a linearized model about the operating point $\dot{x}_0 = 0$, $x_0 = 0$, $p_1 = p_2 = p$) and with constant supply pressure p_v and constant ambient pressure p_0. The flow through the valve is approximately proportional to the valve position u, which is the input signal to the system. The mass flows $\dot{m}_1$, $\dot{m}_2$ through the valve become

$$\dot{m}_1 = \kappa u, \quad \dot{m}_2 = -\kappa u,$$

and the rate of change in pressures becomes

$$\dot{p}_1(t) = \overbrace{k_u u}^{\text{flow in}} - \overbrace{k_x \dot{x}}^{\text{expansion}} \tag{1}$$

$$\dot{p}_2(t) = -k_u u + k_x \dot{x}. \tag{2}$$

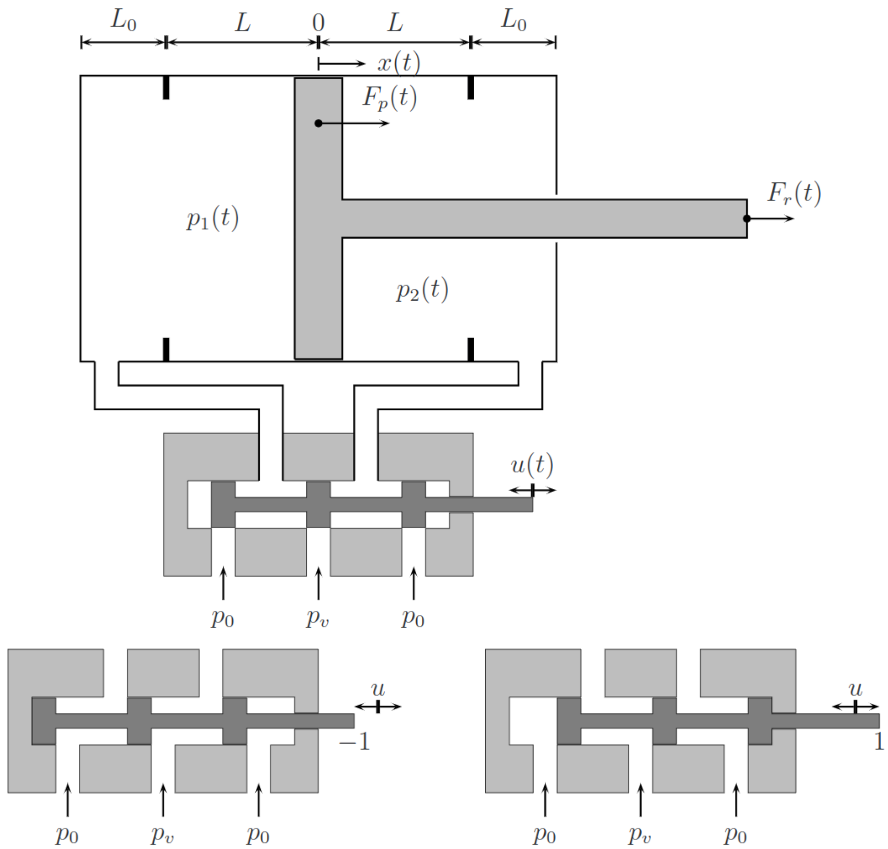


Figure 1: A pneumatic cylinder. The input signal to the system $u(t)$ is the position of the valve. If u is positive the valve core is displaced to the right (bottom right picture), then compressed air flows from the high-pressure inlet p_v to the left chamber, while the air in the right chamber flows out. This causes an increase in the pressure difference $\Delta p(t) = p_1(t) - p_2(t)$. From Ilchmann et al. "Pneumatic cylinders: modelling and feedback force control" Int Journal of Control, 2006

The force $F(t) = F_r(t)$ of the cylinder equals the product of cross-sectional area A of the piston and the difference in pressure between the two sides of the piston head.

$$\frac{d}{dt}F(t) = \frac{d}{dt}A\Delta p(t) = -2Ak_x\dot{x}(t) + 2Ak_u u(t). \tag{3}$$

The pneumatic cylinder is moving a mass m subject to a disturbance force $v(t)$, and we are neglecting any friction in the system. Let the velocity $y = \dot{x}$ be the output signal of the system. Newton's law $m\ddot{x} = m\dot{y} = F + v$, differentiated once, gives

$$m\ddot{y} = \dot{F} + \dot{v} = -2Ak_x y + 2Ak_u u + \frac{d}{dt}v.$$

The final model becomes

$$\frac{d^2}{dt^2}y = -ay + bu + c\frac{d}{dt}v, \tag{4}$$

where $a = \frac{2Ak_x}{m}$, $b = \frac{2Ak_u}{m}$ and $c = \frac{1}{m}$.

Problems

Exercise 1 The model (4) can be written on transfer-function form as

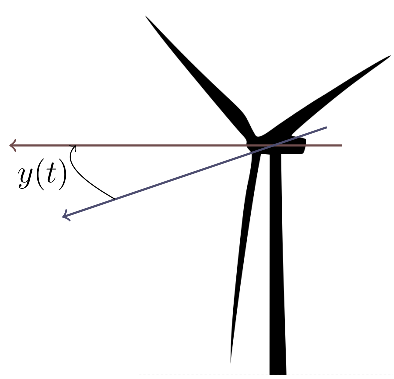
$$Y(s) = G(s)U(s) + G_d(s)V(s), \tag{5}$$

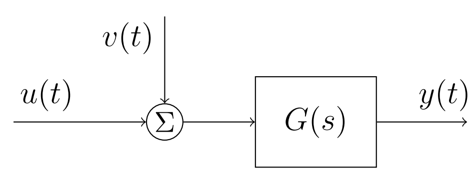
where $G(s) = \frac{b}{(s^2+a)}$. Determine the transfer function $G_d(s)$.

Calculations:

Yaw control of a wind turbine

Yaw is the angle between the direction the wind is blowing from and the direction the wind turbine is facing. Almost all wind turbines in commercial use have active control of the yaw angle in order to optimize energy production.





Assume that the friction force opposing the turning of the turbine is viscuous (proportional to the velocity), and that the wind force is causing a disturbance torque on the head. The dynamics of the system can then be described by the differential equation

$$J\ddot{y}(t) + f\dot{y}(t) = u(t) + v(t), \tag{6}$$

where $y(t)$ is the yaw angle, $u(t)$ is the torque produced by the motor turning the head, $v(t)$ is the disturbance torque, J is the moment of inertia and f is the friction coefficient.

Problems

Exercise 2

(a) Show that the plant model (6) can be expressed in the Laplace-domain as

$$Y(s) = G(s)\big(U(s) + V(s)\big) = \frac{b}{s(s+a)}\big(U(s) + V(s)\big),$$

with $b = \frac{1}{J}$ and $a = \frac{f}{J}$.

Calculations:

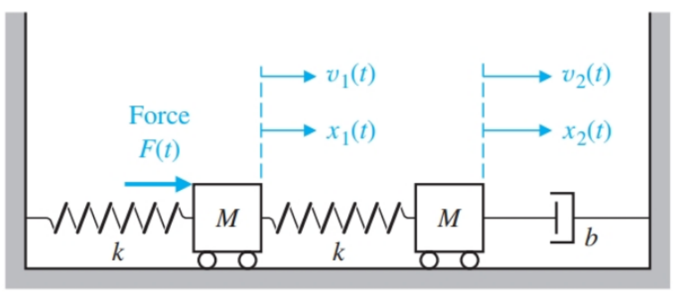
(b) Choose a state vector x, and write the system (6) on state-space form.

$$\begin{aligned} \dot{x} &= Ax + B \begin{bmatrix} u \\ v \end{bmatrix} \\ y &= Cx \end{aligned}$$

Calculations:

A Coupled spring-mass system

The figure below shows a system consisting of two masses connected together and to two rigid walls.



The displacements x_1 and x_2 are deviations from the equilibrium positions when the force acting on the left mass is zero.

Problems

Exercise 3

(a) Determine the differential equations that describe the system.

Calculations:

(b) Choose a suitable state vector z, and write the system on state-space form

$$\begin{aligned}\dot{z}(t) &= Az(t) + B\,F(t)\\ y(t) &= Cz(t)\end{aligned}$$

where the interesting output is the position of the second mass, $y = x_2$.

Calculations:

(c) The two springs are actually progressive, and the spring deflection (the relationship between spring force and length-change) can be described as

$$F_s(x) = kx^3. \tag{7}$$

Consider a situation in which the force acting on the left mass can be written as a deviation around a typical, positive value F_0

$$F(t) = F_0 + u(t).$$

In equilibrium (steady-state) with constant $F(t) = F_0$, what is the position $x_1(t) = x_0$ in terms of the spring constant k and the force F_0?

Calculations:

(d) Determine the constant a in the linearized model

$$v = aw, \quad \text{where } F_s = F_0 + v,\ x_1 = x_0 + w \tag{8}$$

for the left-most spring with operating point (F_0, x_0)

Calculations:

Solutions

Pneumatic cylinder

Exercise 1 Taking the Laplace transform of each term in the model (4), and setting all initial values to zero (we are interested in the transfer function, not the transient response to the initial value) gives

$$s^2Y(s) = -aY(s) + bU(s) + csV(s),$$

which solved for $Y(s)$ is

$$Y(s) = \frac{b}{s^2+a}U(s) + \frac{cs}{s^2+a}V(s).$$

Hence

$$G_d(s) = \frac{cs}{s^2+a}.$$

Wind turbine

Exercise 2

(a) Since we are interested in the transfer function only, not the response to initial values, we set all initial values to zero when applying the Laplace transform to (6). This gives

$$Js^2Y(s) + fsY(s) = U(s) + V(s)$$

solving for Y(s) gives

$$Y(s) = \frac{\frac{1}{J}}{s(s+\frac{f}{J})}U(s) + \frac{\frac{1}{J}}{s(s+\frac{f}{J})}V(s) = \frac{\frac{1}{J}}{s(s+\frac{f}{J})}(U(s)+V(s))$$

(b) One suitable choice of state vector is

$$x = \begin{bmatrix} y \\ \dot{y} \end{bmatrix},$$

which gives the model

$$\begin{aligned} \dot{x}_1 &= \dot{y} = x_2 \\ \dot{x}_2 &= -\frac{f}{J}\dot{y} + \frac{1}{J}u + \frac{1}{J}v = -\frac{f}{J}x_2 + \frac{1}{J}u + \frac{1}{J}v \end{aligned}$$

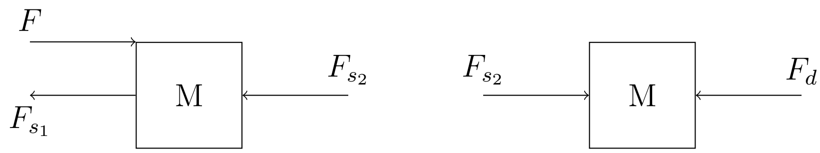
which can be written

$$\begin{aligned} \dot{x} &= \begin{bmatrix} 0 & 1 \\ 0 & -\frac{f}{J} \end{bmatrix} x + \begin{bmatrix} 0 & 0 \\ \frac{1}{J} & \frac{1}{J} \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix}. \\ y &= \begin{bmatrix} 1 & 0 \end{bmatrix} x. \end{aligned}$$

Spring-mass system

Exercise 3

(a) Free-body-diagrams of the two bodies gives



where the spring forces are given by

$$F_{s_1} = kx_1, \qquad F_{s_2} = k(x_1 - x_2),$$

and the damper force

$$F_d = b\dot{x}_2$$

Newton's second law for each body gives

$$\begin{aligned} M\ddot{x}_1 &= F - F_{s_1} - F_{s_2} = F - kx_1 - k(x_1 - x_2) \\ M\ddot{x}_2 &= F_{s_2} - F_d = k(x_1 - x_2) - b\dot{x}_2. \end{aligned}$$

(b) One choice of state vector is

$$z = \begin{bmatrix} x_1 \\ \dot{x}_1 \\ x_1 - x_2 \\ dotx_2 \end{bmatrix},$$

with which the differential equations can be written

$$\begin{aligned} \dot{z}_1 &= \dot{x}_1 = z_2 \\ \dot{z}_2 &= \ddot{x}_1 = -\frac{k}{M}x_1 - \frac{k}{M}(x_1 - x_2) + \frac{1}{M}F = -\frac{k}{M}z_1 - \frac{k}{M}z_3 + \frac{1}{M}F \\ \dot{z}_3 &= \dot{x}_1 - \dot{x}_2 = z_2 - z_4 \\ \dot{z}_4 &= \ddot{x}_2 = \frac{k}{M}(x_1 - x_2) - \frac{b}{M}\dot{x}_2 = \frac{k}{M}z_3 - \frac{b}{M}z_4. \end{aligned}$$

The state-space model becomes

$$\begin{aligned} \dot{z} &= \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\frac{k}{M} & 0 & -\frac{k}{M} & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & \frac{k}{M} & -\frac{b}{M} \end{bmatrix} z + \begin{bmatrix} 0 \\ \frac{1}{M} \\ 0 \\ 0 \end{bmatrix} \\ y &= x_2 = \begin{bmatrix} 1 & 0 & -1 & 0 \end{bmatrix} z. \end{aligned}$$

(c) In equilibrium with a constant force $F(t) = F_0$, the masses will be shifted to the right a distance $x_1 = x_2 = x_0$. For the left-most mass, we get that, since $\ddot{x}_1 = 0$ and the forces are in balance,

$$0 = F_0 - F_s = F_0 - kx_0^3.$$

Solving for x_0 gives

$$x_0 = \left(\frac{F_0}{k}\right)^{\frac{1}{3}}.$$

(d) The Taylor-expansion of $F_s(x_1) = kx_1^3$ about the point x_0 is given by

$$F_s(x_1) = F_s(x_0) + 3kx_0^2(x_1 - x_0) + \frac{6kx_0}{2!}(x_1 - x_0)^2 + \frac{6k}{3!}(x_1 - x_0)^3,$$

so the linear model becomes

$$F_s(x_1) \approx F_s(x_0) + 3kx_0^2(x_1 - x_0) = F_0 + 3kx_0^2 w$$

$$F_s - F_0 = 3kx_0^2 w$$

$$v = 3kx_0^2 w.$$

The parameter we are looking for is

$$a = 3kx_0^2 = 3k\left(\left(\frac{F_0}{k}\right)^{\frac{1}{3}}\right)^2 = 3k\left(\frac{F_0}{k}\right)^{\frac{2}{3}} = 3k^{\frac{1}{3}}F_0^{\frac{2}{3}}.$$