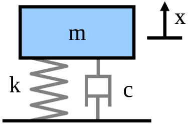
State-space models - canonical forms, transfer function

Kjartan Halvorsen

2022-06-02

Mass-spring-damper



$x = \begin{bmatrix} X & \dot{X} \end{bmatrix}$. We want the acceleration $\ddot{X}$ to be the output signal.

$$\dot{x} = \overbrace{\begin{bmatrix} \quad & \quad \\ \quad & \quad \end{bmatrix}}^{A} x + \overbrace{\begin{bmatrix} \quad \\ \quad \end{bmatrix}}^{B} u$$

$$y = \underbrace{\begin{bmatrix} \quad & \quad \end{bmatrix}}_{C} x + \underbrace{\begin{bmatrix} \quad \end{bmatrix}}_{D} u$$

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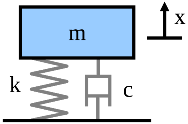

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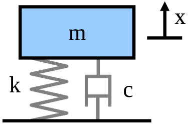

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Canonical forms

- Controllable form (a.k.a. reachable form)
- Observable form

Reference

https://lpsa.swarthmore.edu/Representations/SysRepTransformations/TF2SS.html

Stability

Stability is a key property of the system itself. It does not depend on the input signal.

The homogeneous solution can be written

$$x(t) = \mathrm{e}^{\lambda_1 t}\alpha_1 v_1 + \mathrm{e}^{\lambda_2 t}\alpha_2 v_2 + \cdots + \mathrm{e}^{\lambda_n t}\alpha_n v_n.$$

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Stability requires that each of the exponential functions go to zero.

A sufficient and necessary condition is that *all* the eigenvalues of A has negative real-part.

$$\mathrm{Re}\{\lambda_i\} < 0, \ \forall i = 1, 2, 3 \ldots, n$$

The eigenvalues of A are the poles of the system.

The eigenvalues

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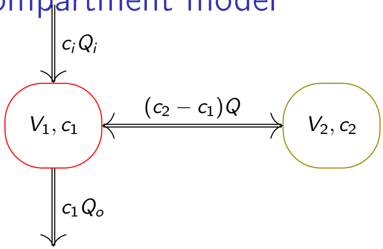
For the equation to have non-trivial solutions:

$$\det(\lambda I - A) = 0 \quad \leftarrow \text{Characteristic equation}$$

From state-space model to transfer function

The compartment model



$$V_1 \frac{dc_1}{dt} = Q(c_2 - c_1) - Q_o c_1 + Q_i c_i, \qquad c_1 \geq 0$$

$$V_2 \frac{dc_2}{dt} = Q(c_1 - c_2), \qquad c_2 \geq 0,$$

$$\dot{x} = \overbrace{\begin{bmatrix} -\frac{Q+Q_o}{V_1} & \frac{Q}{V_1} \\ \frac{Q}{V_2} & -\frac{Q}{V_2} \end{bmatrix}}^{A} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \overbrace{\begin{bmatrix} \frac{1}{V_1} \\ 0 \end{bmatrix}}^{B} u$$

$$y = \underbrace{\begin{bmatrix} 1 & 0 \end{bmatrix}}_{C} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

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$$y = \underbrace{\begin{bmatrix} 1 & 0 \end{bmatrix}}_{C} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = Cx$$

Apply the Laplace transform

$$sX - x(0) = AX + BU$$
$$Y = CX$$

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Solve for $X(s)$

$$X(s) = (sI - A)^{-1}x(0) + (sI - A)^{-1}BU(s)$$
$$Y(s) = C\big((sI - A)^{-1}x(0) + (sI - A)^{-1}BU(s)\big)$$
$$= \underbrace{C(sI - A)^{-1}x(0)}_{\text{Transitory response}} + \underbrace{C(sI - A)^{-1}B}_{\text{Transfer fcn.}}\, U(s)$$

The Laplace transform of the exponential function

$$F(s) = \mathcal{L}\left\{f(t)\right\} = \int_0^\infty f(t)\mathrm{e}^{-st}dt$$

$$\mathcal{L}\left\{\mathrm{e}^{pt}\right\} = \int_0^\infty \mathrm{e}^{pt}\mathrm{e}^{-st}dt = \int_0^\infty \mathrm{e}^{-(s-p)t}dt = \frac{1}{s-p} = (s-p)^{-1}, \quad \mathrm{Re}\{s\} > \mathrm{Re}\{p\}$$

Homogenous solution to linear systems

$$\dot{x} = Ax, \qquad x(0) = x_0$$
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Solution in the Laplace-domain

$$X(s) = (sI - A)^{-1}x(0)$$

Solution in the time-domain

$$x(t) = \Phi(t)x(0) = \mathrm{e}^{At}x(0)$$

Where $\Phi : \mathbb{R} \rightarrow \mathbb{R}^{n \times n}$

$$\Phi(t) = \mathrm{e}^{At} = I + tA + \frac{t^2}{2!}A^2 + \frac{t^3}{3!}A^3 + \cdots$$

The Laplace-transform of the matrix exponential

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$$f(t) = e^{At} \qquad \overset{\mathcal{L}}{\longleftrightarrow} \qquad F(s) = (sI - A)^{-1}$$

$$(sI - A)^{-1} = \frac{1}{\det(sI - A)} \operatorname{adj}(sI - A)$$

$\det(sI - A)$ is a polynomial in s called the characteristic polynomial. Its roots, i.e. the solution to the characteristic equation

$$\det(sI - A) = 0$$

Are the poles of the system and also the eigenvalues of A.