Mathematical foundations

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Concepts

- Differential equations
- Linearization
- The Laplace transform
- Transfer functions

Modeling in the context of automation

hummer-ev.png

The complex exponential function $f(t) = re^{\lambda t}, \quad \lambda \in \mathbb{C}$

$$t \in \mathbb{R}_+, \qquad \lambda = a + i\omega \in \mathbb{C}, \qquad r \in \mathbb{R}$$

$$f: \mathbb{R}_+ \mapsto \mathbb{C}, \qquad f(t) = r\mathrm{e}^{\lambda t} = r\mathrm{e}^{at}\mathrm{e}^{i\omega t} = r\mathrm{e}^{at}\big(\cos(\omega t) + i\sin(\omega t)\big), \qquad 0 < t < \infty$$

Match the value of the parameter λ with the correct time-function (showing the real-part of the function). Let $r = 1$.

A	B	C	D
$\lambda = 2i$	$\lambda = 1 - 2i$	$\lambda = -1 + 2i$	$\lambda = -1$

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[illegible]

The Laplace transform

$$F(s) = \mathcal{L}\{f(t)\} = \int_0^\infty f(t)\mathrm{e}^{-st}dt$$

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$$\mathcal{L}\{\mathrm{e}^{pt}\} = \int_0^\infty \mathrm{e}^{pt}\mathrm{e}^{-st}dt = \int_0^\infty \mathrm{e}^{-(s-p)t}dt = \frac{1}{s-p}, \quad \mathrm{Re}\{s\} > \mathrm{Re}\{p\}$$

The Laplace transform

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$$\mathcal{L}\{\sin(\omega_1 t)\} = \int_0^\infty \frac{1}{2i}\left(\mathrm{e}^{i\omega_1 t} - \mathrm{e}^{-i\omega_1 t}\right)\mathrm{e}^{-st}dt = \frac{\omega}{(s-i\omega_1)(s+i\omega_1)}, \quad \mathrm{Re}\{s\} > 0$$

The Laplace transform of a derivative

$$F(s) = \mathcal{L}\{f(t)\} = \int_0^\infty f(t)\mathrm{e}^{-st}dt$$

$$\mathcal{L}\{\frac{d}{dt}f(t))\} = sF(s) - f(0).$$

Linearized model of the Hummer EV

Parameters: m=5000kg, C_d=0.6, A=4 m^2, C_{rr}=0.01

$$F_d(v) = r + kv^2 = C_{rr}mg + \frac{1}{2}\rho_a C_d A v^2$$
$$= 0.01 \cdot 5000 \cdot 9.8 + \frac{1}{2} 1.2 \cdot 0.6 \cdot 4 v^2 = 490 + 1.44 v^2.$$

Operating point and deviation variables $v_0 = 22$ m/s,

$$v(t) = v_0 + y(t),$$
$$F_m(t) = F_{m_0} + u(t) = F_d(v_0) + u$$

Linearized ODE

$$m\dot{y} = -2kv_0 y + u,$$
$$\dot{y} + \frac{2 \cdot 1.44 \cdot 22}{5000} y = \frac{1}{5000} u,$$
$$\dot{y} + 0.013y = 0.0002u,$$
$$78.9\dot{y} + y = 0.016u.$$

The Laplace transform

$$(78.9s+1)Y(s) = 0.016U(s)$$

Transfer function

$$Y(s) = \underbrace{\frac{\overbrace{0.016}^{K}}{\underbrace{78.9}_{\tau}\, s + 1}}_{G(s)} U(s)$$

Block diagram

$y(t)$ $\frac{K}{\tau s+1}$

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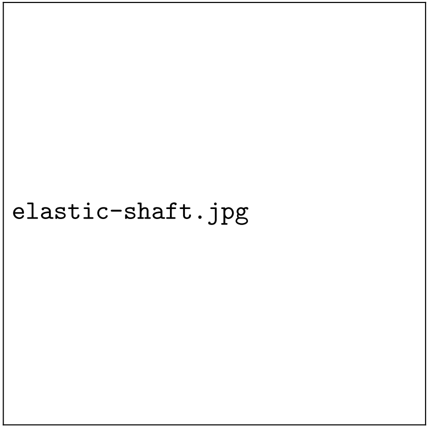
Modeling a mechanical system



A motor drives a load via an elastic shaft. Determine the transfer functions $G(s)$ and $H(s)$.

Modeling a mechanical system

elastic-shaft.jpg

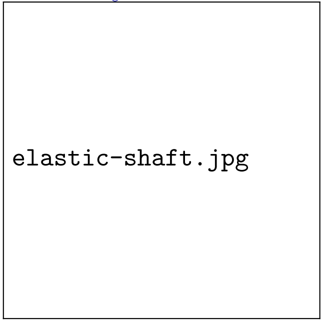
Newton's second law for rotational systems

$$J\ddot{\theta} = \sum_j T_j$$

Free-body diagram

$$\begin{aligned}
\text{Body 1:} &\qquad J_1\ddot{\theta}_1 = T_i - k(\theta_1 - \theta_2) - b(\dot{\theta}_1 - \dot{\theta}_2)\\
\text{Body 2 :} &\qquad J_2\ddot{\theta}_2 = k(\theta_1 - \theta_2) - b(\dot{\theta}_1 - \dot{\theta}_2) - T_l
\end{aligned}$$

Modeling a mechanical system



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Eliminate the variable Θ_1 Substitute $\Theta_1 = \frac{k+bs}{J_1 s^2 + bs + k}\Theta_2 + \frac{1}{J_1 s^2 + bs + k} T_i$ in the equation

$$(J_2 s^2 + bs + k)\Theta_2 = (k + bs)\Theta_1 - T_l$$

$$(J_2 s^2 + bs + k)\Theta_2 = (k+bs)\left(\frac{k+bs}{J_1 s^2 + bs + k}\Theta_2 + \frac{1}{J_1 s^2 + bs + k} T_i\right) - T_l$$

$$(J_2 s^2 + bs + k)\Theta_2 - \frac{(k+bs)^2}{J_1 s^2 + bs + k}\Theta_2 = \frac{k+bs}{J_1 s^2 + bs + k} T_i - T_l$$

$$\frac{(J_2 s^2 + bs + k)(J_1 s^2 + bs + k) - (k+bs)^2}{J_1 s^2 + bs + k}\Theta_2 = \frac{k+bs}{J_1 s^2 + bs + k} T_i - T_l$$