Root locus

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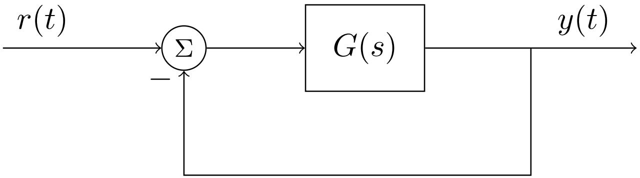
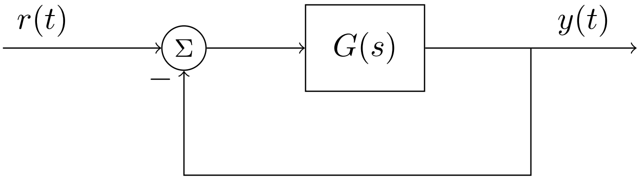
Block diagram algebra



Transfer function from $r(t)$ to $y(t)$:

$$\frac{Y(s)}{R(s)} = \frac{G(s)}{1 + G(s)}$$

Block diagram algebra



Transfer function from $r(t)$ to $y(t)$:

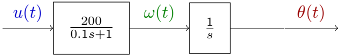
$$\frac{Y(s)}{R(s)} = \frac{G(s)}{1 + G(s)}$$

Mason's gain formula: $G_c(s) = \frac{Y(s)}{R(s)} = \frac{\sum_k P_k \Delta_k}{\Delta}$

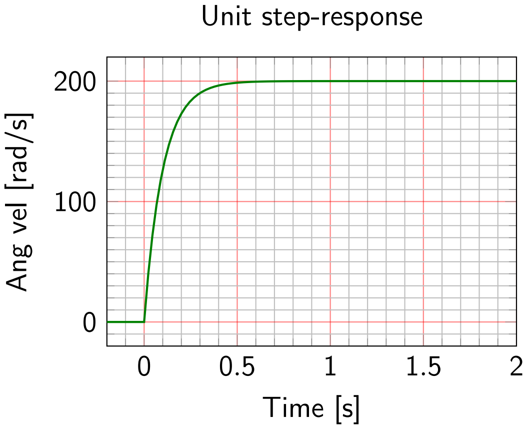
For simple systems with one loop only:

$$G_c(s) = \frac{Y(s)}{R(s)} = \frac{\text{Forward path gain}}{1 + \text{Loop gain}}$$

The DC motor

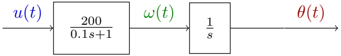


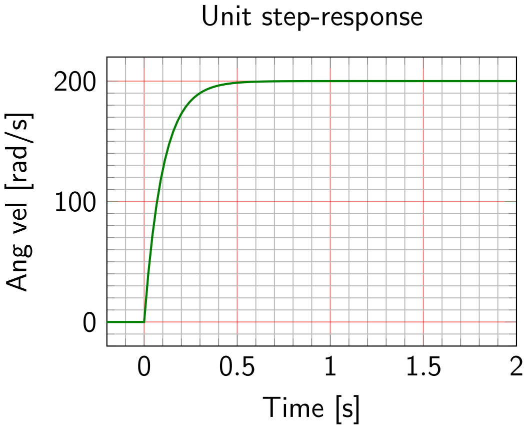
$u(t)$
$\frac{200}{0.1s+1}$
$\omega(t)$
$\frac{1}{s}$
$\theta(t)$



Unit step-response
Ang vel [rad/s]
200
100
0
0
0.5
1
1.5
2
Time [s]

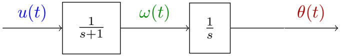
The DC motor

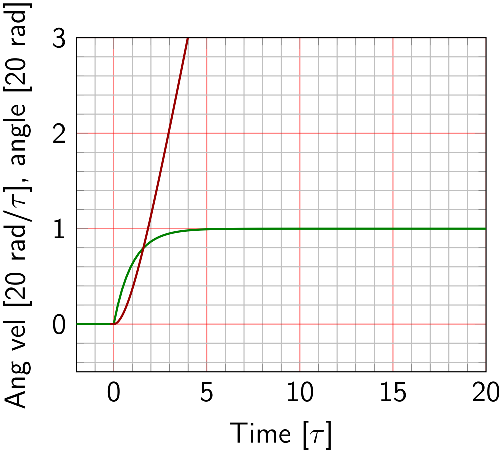




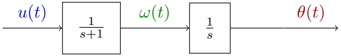
Activity What is the angle (approximately) rotated by the motor after 0.1s starting from still?

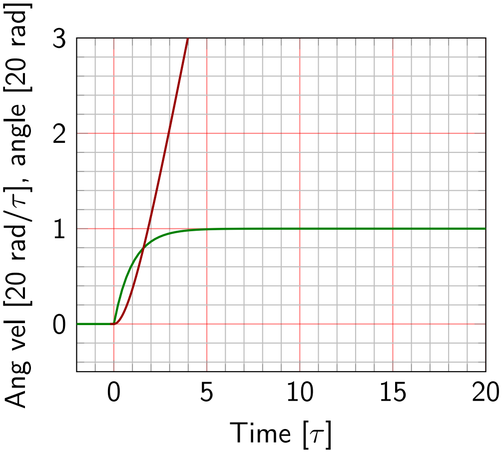
The normalized DC motor





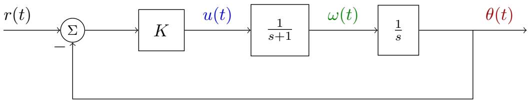
The normalized DC motor



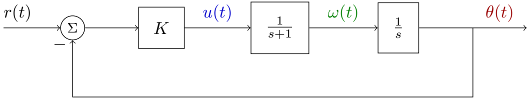


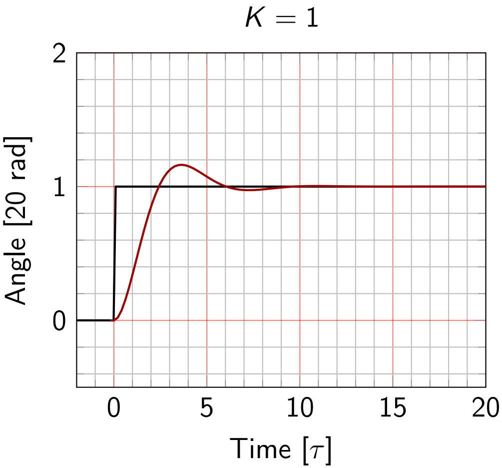
Activity What is the settling time (approximately) for the velocity (in τ and in seconds)?

Proportional control of the normalized DC motor

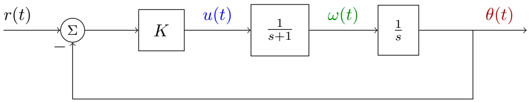


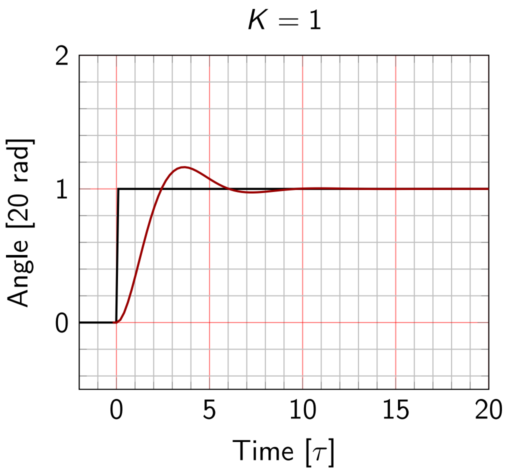
Proportional control of the normalized DC motor





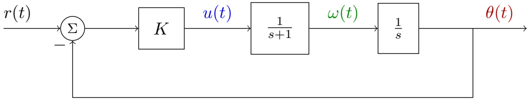
Proportional control of the normalized DC motor





Activity What is the overshoot (in percent) and rise time (in seconds)?

Proportional control of the normalized DC motor



Closed-loop transfer function:

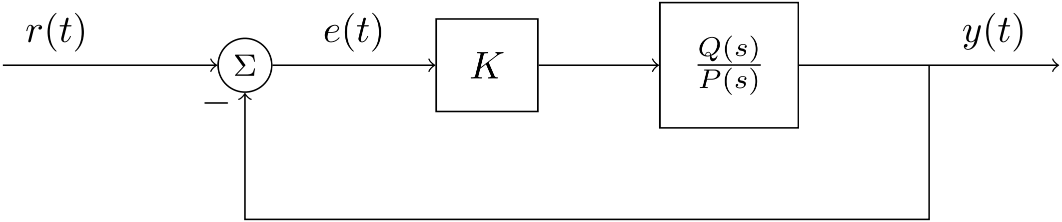
$$G_c(s) = \frac{K}{s(s+1)+K}$$

Characteristic equation:

$$s^2 + s + K = 0$$

Activity Solve the characteristic equation!

Root locus



How do the closed-loop poles depend on K?

Root locus definition

Let

$$\begin{cases} P(s) & = s^n + a_1 s^{n-1} + \cdots + a_n = (s-p_1)(s-p_2)\cdots(s-p_n) \\ Q(s) & = s^m + b_1 s^{m-1} + \cdots + b_m = (s-q_1)(s-q_2)\cdots(s-q_m) \end{cases}, \quad n \geq m$$

The root locus shows how the roots to the equation

$$P(s) + K \cdot Q(s) = 0, \quad 0 \leq K < \infty \tag{1}$$

depend on the parameter K. The root locus consists of the set of all points in the complex plane that are roots to (1) for some non-negative value of K.

Characteristics of the root locus

The polynomial $P(s) + KQ(s) = 0$ above will always have n roots. Each gives a *branch* in the root locus. Since the polynomials $P(s)$ and $Q(s)$ have real-valued coefficients, all roots are either real or complex-conjugated pairs. This means that the root locus is *symmetric about the real axis.* Other characteristics

- Start points - marked by crosses
- End points - marked by circles
- Asymptotes
- Pieces of the real axis

Start- and end points

Start points These are the n roots of $P(s) + KQ(s)$ for $K = 0$, i.e. the roots of $P(s)$. These are the open-loop poles, and are marked with crosses '$\times$'

End points These are the m (finite) roots of $P(s) + KQ(s)$ when $K \to \infty$, and are hence the roots of $Q(s)$. The end points are marked with circles '$\circ$'

The real axis

Those parts of the real axis that have an odd number of real-valued start- or end points to the right (including multiplicity) belong to the root locus.

Asymptotes, directions

The directions of the asymptotes are given by the expression

$$\theta_k = \arg s = \frac{(2k+1)\pi}{n-m}, \; k \in \mathbb{Z}$$

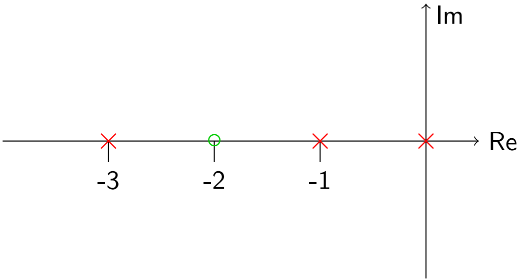
Example: 6 start points and 3 end points gives $n - m = 6 - 3 = 3$ and the directions

$$\theta = \begin{cases} \frac{\pi}{3}, & k = 0 \\ \pi, & k = 1 \\ -\frac{\pi}{3}, & k = -1 \end{cases}.$$

Asymptotes, intersection with the real axis

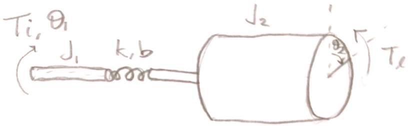
$$ip = \frac{\sum_{i=1}^{n} p_i - \sum_{i=1}^{m} q_i}{n - m},$$

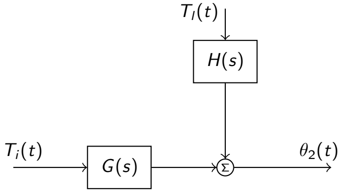
where $\{p_i\}$ are the starting points (open-loop poles) and $\{q_i\}$ are the end points (open-loop zeros).



Examples

Motor driving an elastic shaft

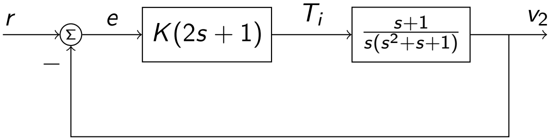


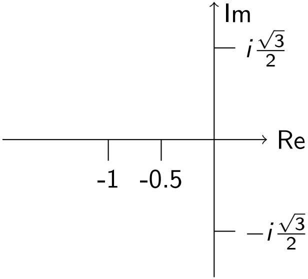


$$\Theta_2(s) = \underbrace{\frac{k + bs}{s^2(J_1 J_2 s^2 + bs + k)}}_{G(s)} T_i(s) - \underbrace{\frac{J_1 s^2 + bs + k}{s^2(J_1 J_2 s^2 + bs + k)}}_{H(s)} T_l(s)$$

Motor driving an elastic shaft

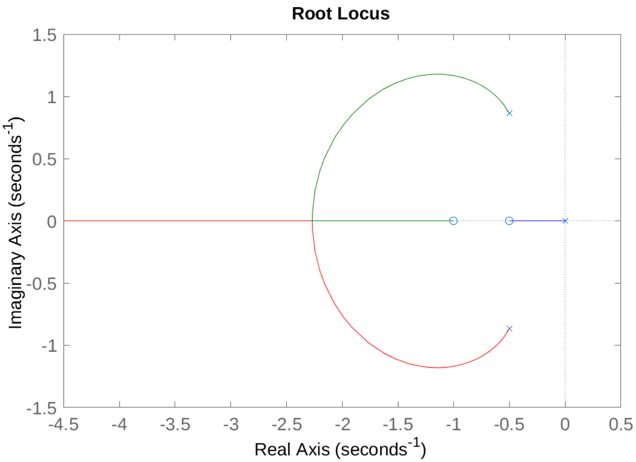
PD-control





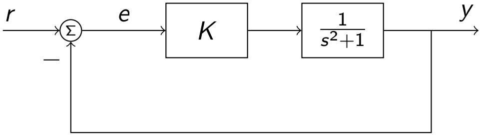
Activity Indicate the start- and end points.

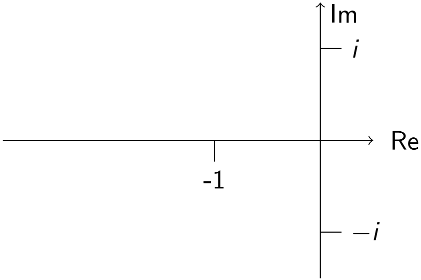
Motor driving an elastic shaft



Harmonic oscillator

P-control

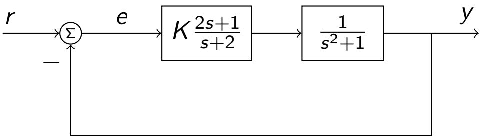


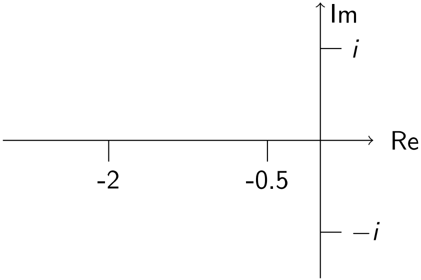


Activity Indicate the start- and end points, and the asymptotes.

Harmonic oscillator

Lead-compensator





Activity Indicate the start- and end points, and the asymptotes.

Pair the root locus plots with the correct transfer function

$$G_1(s) = K\frac{s+2}{s(s+4)}$$

$$G_2(s) = K\frac{s+2}{s(s+4)(s+8)}$$

$$G_3(s) = K\frac{s+2}{s^2(s+4)}$$

$$G_4(s) = K\frac{1}{s^2(s+4)}.$$