State feedback with observer

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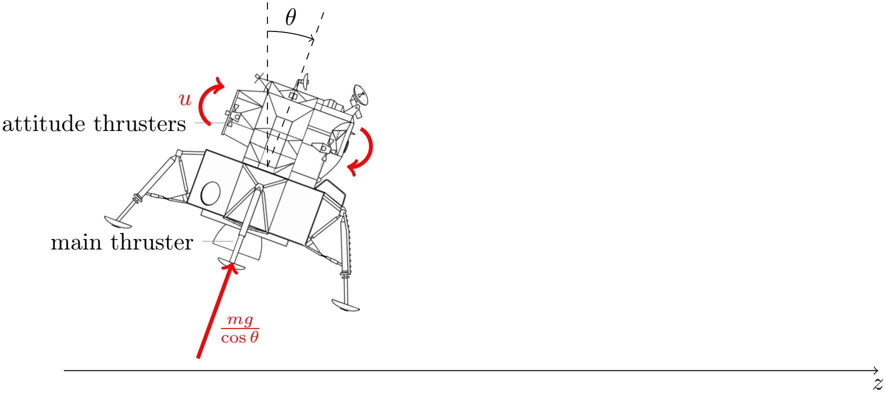
State feedback with reconstructed states

State feedback with reconstructed states



$$u(t) \longrightarrow \boxed{\frac{1}{Js}} \xrightarrow{\dot{\theta}(t)} \boxed{\frac{1}{s}} \xrightarrow{\theta(t)} \boxed{g} \xrightarrow{\ddot{z}(t)} \boxed{\frac{1}{s}} \xrightarrow{\dot{z}(t)}$$

State feedback

Given

$$
\begin{aligned}
\dot{x} &= Ax + Bu \\
y &= Cx
\end{aligned}
\tag{1}
$$

and measurements (or estimates) of the state vector x.

Linear state feedback is the control law

$$
\begin{aligned}
u = f\big((x, u_c\big) &= -l_1 x_1 - l_2 x_2 - \cdots - l_n x_n + l_0 u_c \\
&= -Lx + l_0 u_c,
\end{aligned}
$$

where

$$
L = \begin{bmatrix} l_1 & l_2 & \cdots & l_n \end{bmatrix}.
$$

Substituting the control law in the state space model (5) gives

$$
\begin{aligned}
\dot{x} &= (A - BL)x + l_0 Br \\
y(k) &= Cx
\end{aligned}
\tag{2}
$$

Observer design

Given model

$$\dot{x} = Ax + Bu$$
$$y = Cx$$

and measurements of the output signal y.

The observer is given by

$$\dot{\hat{x}} = \underbrace{A\hat{x} + Bu}_{\text{simulation}} + \underbrace{K(y - C\hat{x})}_{\text{correction}} = (A - KC)\hat{x} + Bu + Ky$$

with poles given by the eigenvalues of the matrix $A_o = A - KC$

Rule-of-thumb Choose the poles of the observer (eigenvalues of $A - KC$) at least twice as fast as the poles (eigenvalues) of $A - BL$.

Control by feedback from reconstructed states

The design problem can be separated into two problems

1. Determine the gain vector L and the gain l_0 of the control law

   $$u = -L\hat{x} + l_0 r$$

   so that the closed-loop system has good reference tracking.

2. Determine the gain vector K of the observer

   $$\dot{\hat{x}} = A\hat{x} + Bu + K\left(y - C\hat{x}\right)$$

   to get a good balance between disturbance rejection and noise attenuation.

Computing the observer gain

A matrix M and its transpose M^{T} have the same eigenvalues. Hence, the problem of determining the gain K to obtain desired eigenvalues of

$$A - KC$$

is equivalent to determining the gain K in

$$(A - KC)^{\mathrm{T}} = A^{\mathrm{T}} - C^{\mathrm{T}} K^{\mathrm{T}}.$$

The last problem has the exact same form as the problem of determining L to obtain desired eigenvalues of

$$A - BL$$

So, the same matlab function can be used for both problems.

Computing the state feedback and observer gains

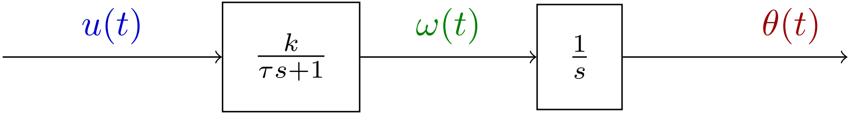
1. Ackerman's method

```
L = acker(A, B, pd)
K = acker(A, C', po)'
```

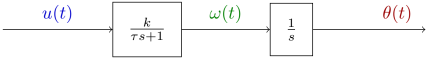
2. More numerically stable method

```
L = place(A, B, pd)
K = place(A, C', po)'
```

Example - Position control of the DC motor



State-space model with physical states



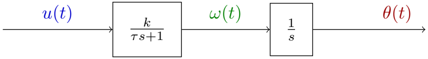
State variables corresponding to physical signals:

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \omega \\ \theta \end{bmatrix}$$

With dynamics

$$\begin{aligned} \tau\dot{\omega} &= -\omega + Ku \\ \dot{\theta} &= \omega \end{aligned}$$

State-space model with physical states



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With dynamics

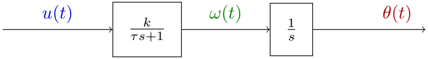
$$\tau\dot{\omega} = -\omega + Ku$$
$$\dot{\theta} = \omega$$

Activity Fill the matrix A and vectors B and C.

$$\dot{x} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \underbrace{\begin{bmatrix} \quad & \quad \\ \quad & \quad \end{bmatrix}}_{A} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \underbrace{\begin{bmatrix} \quad \\ \quad \end{bmatrix}}_{B} u$$

$$y = \theta = \underbrace{\begin{bmatrix} \quad & \quad \end{bmatrix}}_{C} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

State-space model with physical states



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With dynamics

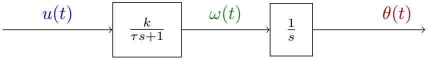
$$\tau\dot{\omega} = -\omega + ku$$
$$\dot{\theta} = \omega$$

Activity Fill the matrix A and vectors B and C.

$$\dot{x} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \underbrace{\begin{bmatrix} -\frac{1}{\tau} & 0 \\ 1 & 0 \end{bmatrix}}_{A} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \underbrace{\begin{bmatrix} \frac{k}{\tau} \\ 0 \end{bmatrix}}_{B} u$$

$$y = \theta = \underbrace{\begin{bmatrix} 0 & 1 \end{bmatrix}}_{C} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

State-space model on controllable canonical form



Transfer function

$$G(s) = \frac{\frac{k}{\tau}}{s(s+\frac{1}{\tau})}$$

The system with transfer function

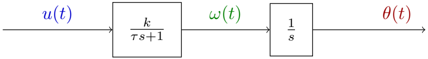
$$G(s) = \frac{b_1 s^{n-1} + \cdots + b_{n-1}s + b_n}{s^n + a_1 s^{n-1} + \cdots + a_{n-1}s + a_n}$$

can be represented on state-space form as

$$\dot{x} = \begin{bmatrix} -a_1 & -a_2 & \cdots & -a_{n-1} & -a_n \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} u$$

$$y = \begin{bmatrix} b_1 & b_2 & \cdots & b_n \end{bmatrix} x$$

State-space model on controllable canonical form



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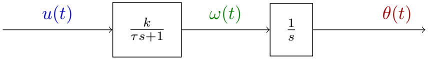
$$y = \begin{bmatrix} b_1 & b_2 & \cdots & b_n \end{bmatrix} x$$

Activity Determine the state-space model of the DC motor on controllable canonical form

State-space model on observable canonical form



Transfer function

$$G(s) = \frac{\frac{k}{\tau}}{s(s+\frac{1}{\tau})}$$

The system with transfer function

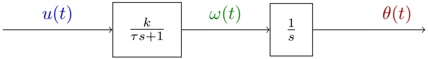
$$G(s) = \frac{b_1 s^{n-1} + \cdots + b_{n-1}s + b_n}{s^n + a_1 s^{n-1} + \cdots + a_{n-1}s + a_n}$$

can be represented on state-space form as

$$\dot{x} = \begin{bmatrix} -a_1 & 1 & 0 & \cdots & 0 \\ -a_2 & 0 & 1 & \cdots & 0 \\ -a_3 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -a_n & 0 & 0 & \cdots & 0 \end{bmatrix} x + \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ \vdots \\ b_n \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix} x$$

State-space model on observable canonical form



Transfer function

$$G(s) = \frac{\frac{k}{\tau}}{s(s+\frac{1}{\tau})}$$

The system with transfer function

$$G(s) = \frac{b_1 s^{n-1} + \cdots + b_{n-1}s + b_n}{s^n + a_1 s^{n-1} + \cdots + a_{n-1}s + a_n}$$

can be represented on state-space form as

$$\dot{x} = \begin{bmatrix} -a_1 & 1 & 0 & \cdots & 0 \\ -a_2 & 0 & 1 & \cdots & 0 \\ -a_3 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -a_n & 0 & 0 & \cdots & 0 \end{bmatrix} x + \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ \vdots \\ b_n \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix} x$$

Activity Determine the state-space model of the DC motor on observable canonical form

State-feedback for model with physical states

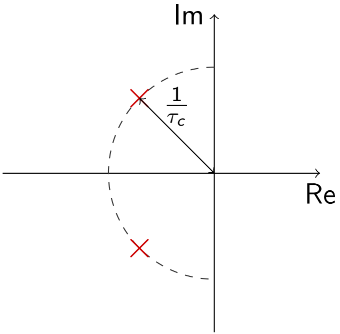
$$\dot{x} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \underbrace{\begin{bmatrix} -\frac{1}{\tau} & 0 \\ 1 & 0 \end{bmatrix}}_{A} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \underbrace{\begin{bmatrix} \frac{K}{\tau} \\ 0 \end{bmatrix}}_{B} u$$

Feedback $u = -Lx + l_0 r$ gives closed-loop system

$$\begin{aligned} \dot{x} &= (A - BL)x + l_0 Br \\ &= \begin{bmatrix} -\frac{1}{\tau} - l_1 \frac{k}{\tau} & -l_2 \frac{k}{\tau} \\ 1 & 0 \end{bmatrix} x + l_0 Br \end{aligned}$$

with characteristic polynomial

$$s^2 + (\frac{1}{\tau} + l_1 \frac{k}{\tau})s + l_2 \frac{k}{\tau}$$



Desired closed-loop characteristic polynomial

$$(s - p_1)(s - p_2) = s^2 + \frac{\sqrt{2}}{\tau_c} s + \frac{1}{\tau_c^2}$$

State-feedback for model with physical states

Characteristic polynomial obtained with state feedback

$$s^2 + (\frac{1}{\tau} + l_1\frac{k}{\tau})s + l_2\frac{k}{\tau}$$

Desired characteristic polynomial

$$s^2 + \frac{\sqrt{2}}{\tau_c}s + \frac{1}{\tau_c^2}$$

State-feedback for model with physical states

Characteristic polynomial obtained with state feedback

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Activity Determine the feedback gains!

State-feedback for model with physical states

Characteristic polynomial obtained with state feedback

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Desired characteristic polynomial

$$s^2 + \frac{\sqrt{2}}{\tau_c} s + \frac{1}{\tau_c^2}$$

Activity Determine the feedback gains!

Solution

State-feedback for model with physical states

Characteristic polynomial obtained with state feedback

$$s^2 + (\frac{1}{\tau} + l_1 \frac{k}{\tau})s + l_2 \frac{k}{\tau}$$

Desired characteristic polynomial

$$s^2 + \frac{\sqrt{2}}{\tau_c} s + \frac{1}{\tau_c^2}$$

Activity Determine the feedback gains!

Solution

$$l_1 = \frac{\tau}{k}(\frac{\sqrt{2}}{\tau_c} - \frac{1}{\tau}) = \frac{\sqrt{2}\tau - \tau_c}{k\tau_c}$$
$$l_2 = \frac{\tau}{k\tau_c^2}$$

Observer for the model with physical states

The observer is given by

$$\begin{aligned}
\dot{\hat{x}} &= \underbrace{A\hat{x} + Bu}_{\text{simulation}} + \underbrace{K\big(y - C\hat{x}\big)}_{\text{correction}} \\
&= (A - KC)\,\hat{x} + Bu + Ky \\
&= \begin{bmatrix} -\frac{1}{\tau} & -k_1 \\ 1 & -k_2 \end{bmatrix} \hat{x} + Bu + Ky
\end{aligned}$$

with poles given by the eigenvalues of the matrix $A_o = A - KC$, which has characteristic polynomial

$$\begin{aligned}
&\det\left(\begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} -\frac{1}{\tau} & -k_1 \\ 1 & -k_2 \end{bmatrix} \right) \\
&= s^2 + \left(\frac{1}{\tau} + k_2\right)s + \left(k_1 + \frac{k_2}{\tau}\right)
\end{aligned}$$

Observer for the model with physical states

The observer is given by

$$\dot{\hat{x}} = \underbrace{A\hat{x} + Bu}_{\text{simulation}} + \underbrace{K(y - C\hat{x})}_{\text{correction}}$$

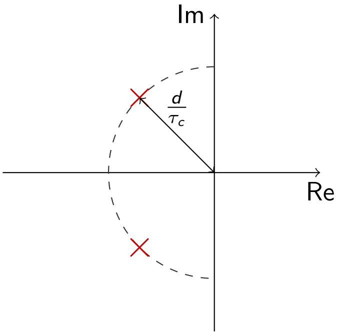
$$= (A - KC)\,\hat{x} + Bu + Ky$$

$$= \begin{bmatrix} -\frac{1}{\tau} & -k_1 \\ 1 & -k_2 \end{bmatrix} \hat{x} + Bu + Ky$$

with poles given by the eigenvalues of the matrix $A_o = A - KC$, which has characteristic polynomial

$$\det\left(\begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} -\frac{1}{\tau} & -k_1 \\ 1 & -k_2 \end{bmatrix}\right)$$

$$= s^2 + (\frac{1}{\tau} + k_2)s + (k_1 + \frac{k_2}{\tau})$$



Choose d between 2 and 10 and obtain desired closed-loop characteristic polynomial

$$(s - p_{o,1})(s - p_{o,2}) = s^2 + \frac{d\sqrt{2}}{\tau_c}s + \frac{d^2}{\tau_c^2}$$

Observer for the model with physical states

Characteristic polynomial of the observer

$$s^2 + (\frac{1}{\tau} + k_2)s + (k_1 + \frac{k_2}{\tau})$$

Desired characteristic polynomial

$$s^2 + \frac{\sqrt{2}d}{\tau_c}s + \frac{d^2}{\tau_c^2}$$

Observer for the model with physical states

Characteristic polynomial of the observer

$$s^2 + \left(\frac{1}{\tau} + k_2\right)s + \left(k_1 + \frac{k_2}{\tau}\right)$$

Desired characteristic polynomial

$$s^2 + \frac{\sqrt{2}d}{\tau_c}s + \frac{d^2}{\tau_c^2}$$

Activity Determine the observer gains!

Observer for the model with physical states

Characteristic polynomial of the observer

$$s^2 + (\frac{1}{\tau} + k_2)s + (k_1 + \frac{k_2}{\tau})$$

Desired characteristic polynomial

$$s^2 + \frac{\sqrt{2}d}{\tau_c}s + \frac{d^2}{\tau_c^2}$$

Activity Determine the observer gains!

Solution

Observer for the model with physical states

Characteristic polynomial of the observer

$$s^2 + (\frac{1}{\tau} + k_2)s + (k_1 + \frac{k_2}{\tau})$$

Desired characteristic polynomial

$$s^2 + \frac{\sqrt{2}d}{\tau_c}s + \frac{d^2}{\tau_c^2}$$

Activity Determine the observer gains!

Solution

$$k_1 = \frac{d^2}{\tau_c^2} - \frac{\sqrt{2}d}{\tau_c \tau} + \frac{1}{\tau^2}$$

$$k_2 = \frac{\sqrt{2}d}{\tau_c} - \frac{1}{\tau}$$