Discretizing continuous-time controllers

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Context

- Controller $F(s)$ obtained from a design in continuous time.

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- Need discrete approxmation in order to implement on a computer

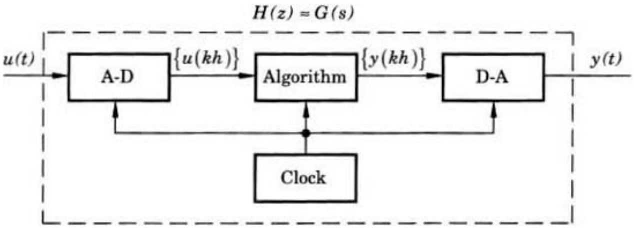
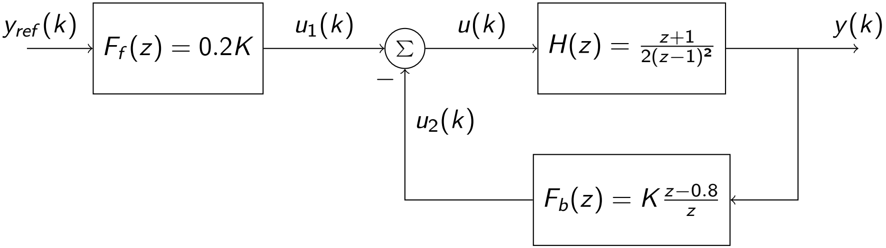


Figure 8.1 Approximating a continuous-time transfer function, $G(s)$, using a computer.

Source: Åström & Wittenmark

Stability for the disk drive arm

Case $\frac{h^2}{J} = 1$. Suggested controller:



Preliminaries

Z-transform of a shifted sequence

$$x(k) \quad \overset{\mathcal{Z}}{\longleftrightarrow} \quad X(z) = \sum_{k=0}^{\infty} x(k) z^{-k}$$

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Proof

$$\begin{aligned}
\mathcal{Z}\left\{x(k+1)\right\} &= \sum_{k=0}^{\infty} x(k+1) z^{-k} = \sum_{n=1}^{\infty} x(n) z^{-(n-1)} \\
&= \sum_{n=1}^{\infty} x(n) z^{-n} z = -zx(0) + z \underbrace{\sum_{n=0}^{\infty} x(n) z^{-n}}_{X(z)} \\
&= zX(z) - zx(0).
\end{aligned}$$

Discrete-time delay

$$x(k-1) \quad \overset{\mathcal{Z}}{\longleftrightarrow} \quad \frac{1}{z}X(z))$$

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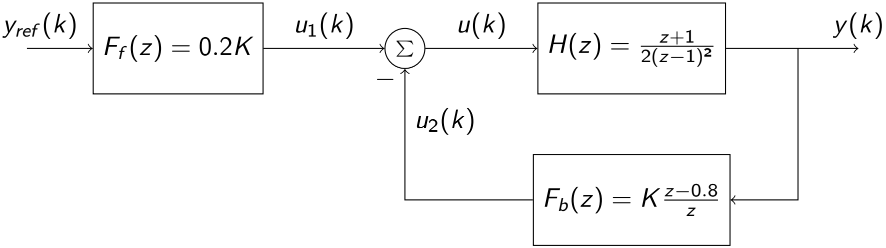
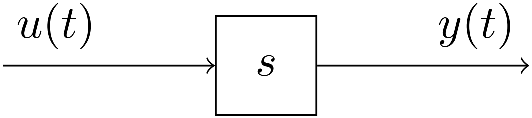
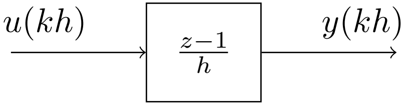
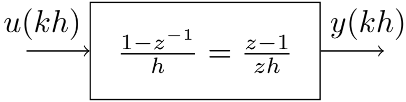


Differentiation



Discrete-time differentiation





Activity Write as difference equation

$$y(kh) =$$

Discretization methods

1. Forward difference. Substitute
$$s = \frac{z-1}{h}$$
in $F(s)$ to get
$$F_d(z) = F(s')|_{s'=\frac{z-1}{h}}.$$

2. Backward difference. Substitute
$$s = \frac{z-1}{zh}$$
in $F(s)$ to get
$$F_d(z) = F(s')|_{s'=\frac{z-1}{zh}}.$$

Discretization methods, contd.

3. Tustin's method (also known as the bilinear transform). Substitute

$$s = \frac{2}{h}\frac{z-1}{z+1}$$

in $F(s)$ to get

$$F_d(z) = F(s')\big|_{s'=\frac{2}{h}\cdot\frac{z-1}{z+1}}.$$

4. Ramp invariance. This is similar to ZoH, which is step-invariant approximation. Since a unit ramp has z-transform $\frac{zh}{(z-1)^2}$ and Laplace-transform $1/s^2$, the discretization becomes

$$F_d(z) = \frac{(z-1)^2}{zh}\mathcal{Z}\left\{\mathcal{L}^{-1}\left\{\frac{F(s)}{s^2}\right\}\right\}.$$

Frequency warping using Tustin's

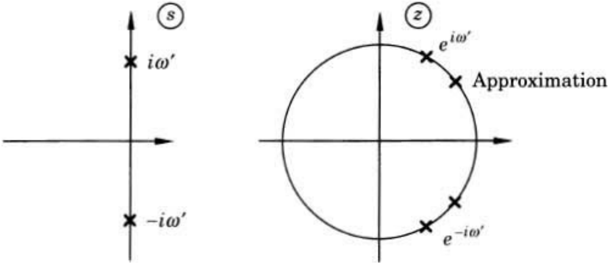
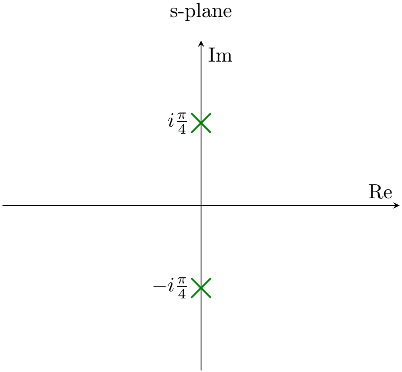


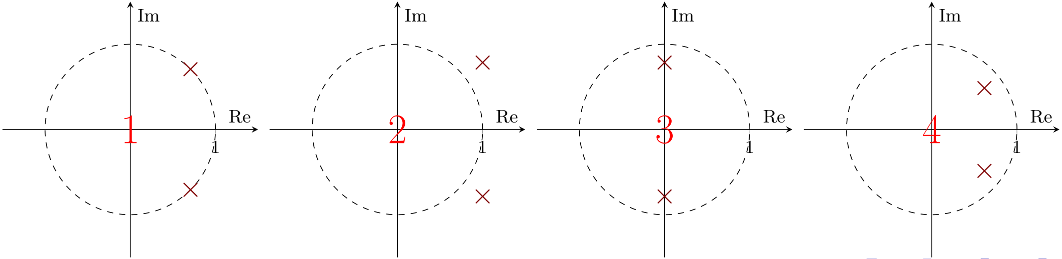
Figure 8.3 Frequency distortion (warping) obtained with approximation.

The infinite positive imaginary axis in the s-plane is mapped to the finite-length upper half of the unit circle in the z-plane.

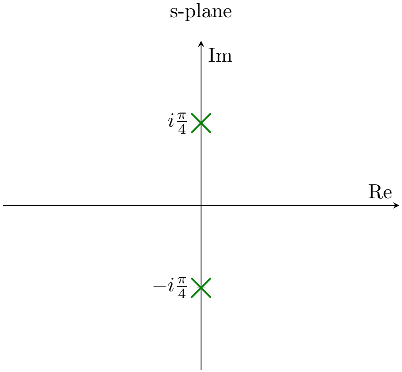
Forward difference exercise



Which of the below figures shows the correct mapping of the continuous-time poles using the **forward difference** $z = 1 + sh$ with $h = 1$?



Backward difference exercise



Which of the below figures shows the correct mapping of the continuous-time poles using the **backward difference** $z = \frac{1}{1-sh}$ with $h = 1$?