PID control

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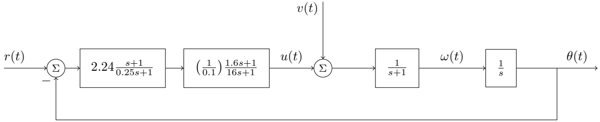
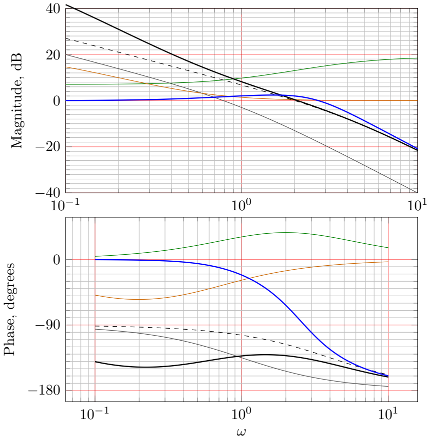
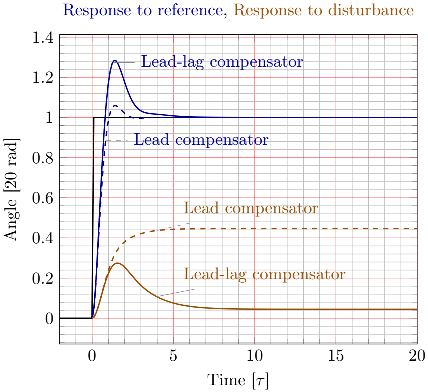
October 5, 2021

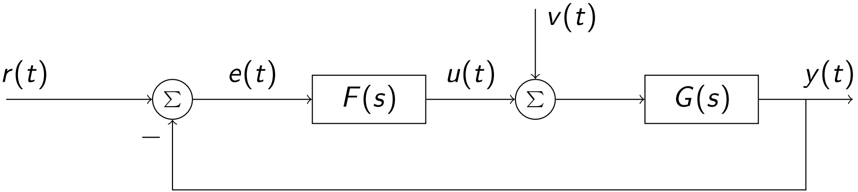
Lead-lag compensator for position control of the DC motor



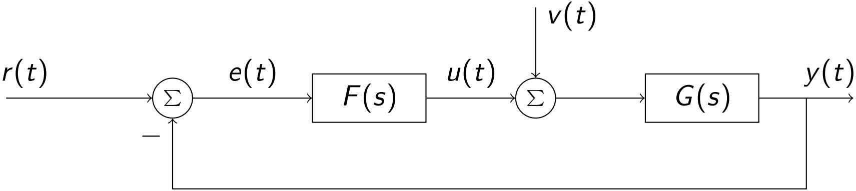




Feedback control



Feedback control



Activity What is the transfer function from the load disturbance $v(t)$ to the control error $e(t)$?

Feedback control - eliminating a constant disturbance

$$\frac{E(s)}{V(s)} = \frac{-G(s)}{1 + G(s)F(s)}$$

The final value theorem

If steady-state exists

$$\lim_{t \to \infty} e(t) = \lim_{s \to 0} sE(s)$$

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Applied to a constant (step input) disturbance

$V(s) = \frac{1}{s}$

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$$= \lim_{s \to 0} \frac{-G(s)}{1 + G(s)F(s)}$$

Feedback control - eliminating a constant disturbance

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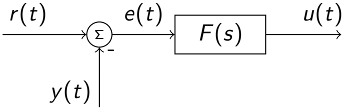
Activity Assume $F(s) = \frac{\bar{F}(s)}{s}$ and $G(0) = b < \infty$. Determine $\lim_{t\to\infty} e(t)$.

The PID controller



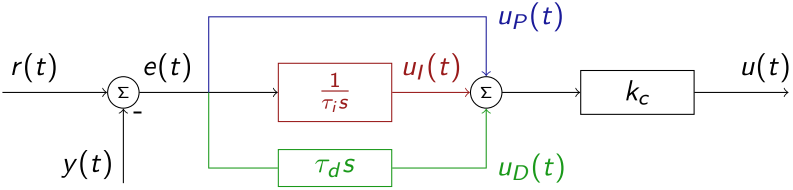
Parallel form (ISA)

$$F(s) = k_c \left(1 + \frac{1}{\tau_i s} + \tau_d s \right)$$

Series form

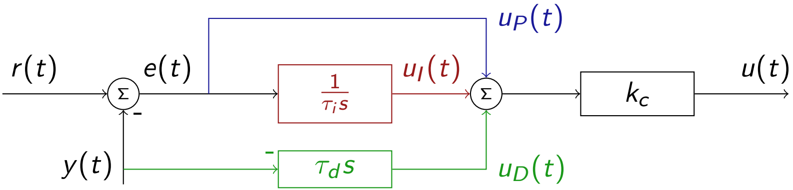
$$F(s) = K_c \left(\frac{\tau_I s + 1}{\tau_I s} \right) (\tau_D s + 1) = \underbrace{\frac{K_c(\tau_I + \tau_D)}{\tau_I}}_{k_c} \left(1 + \frac{1}{\underbrace{(\tau_I + \tau_D)}_{\tau_i} s} + \underbrace{\frac{\tau_I \tau_D}{\tau_I + \tau_D}}_{\tau_d} s \right)$$

The PID - Parallel form



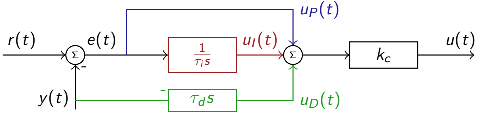
$$u(t) = k_c \left(e(t) + \frac{1}{\tau_i} \int_0^t e(\xi) d\xi + \tau_d \frac{d}{dt} e(t) \right)$$

The PID - Parallel form, modified D-part

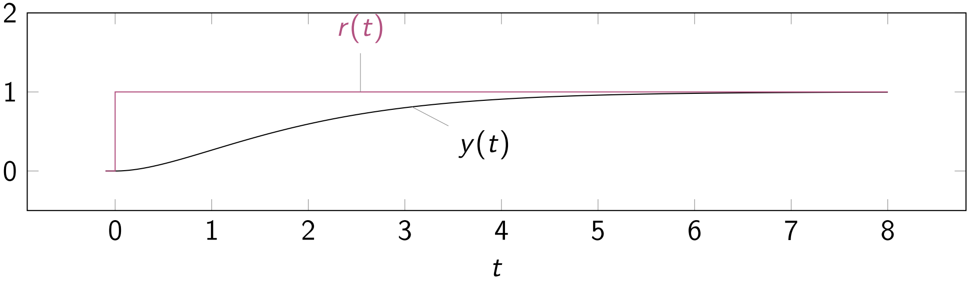


$$u(t) = k_c\left(e(t) + \overbrace{\frac{1}{\tau_i}\int_0^t e(\xi)d\xi}^{u_I(t)} + \underbrace{\tau_d \frac{d}{dt}\big(-y(t)\big)}_{u_D(t)}\right)$$

The PID - Parallel form



$$u(t) = k_c\left(e(t) + \overbrace{\frac{1}{\tau_i}\int_0^t e(\xi)d\xi}^{u_I(t)} + \underbrace{\tau_d \frac{d}{dt}(-y(t))}_{u_D(t)}\right)$$



Activity Sketch the error signal $e(t)$, the derivative signal $u_D(t)$ and the integral signal $u_I(t)$ (use $\tau_i = \tau_d = 1$)

The PID - Parallel form, solution

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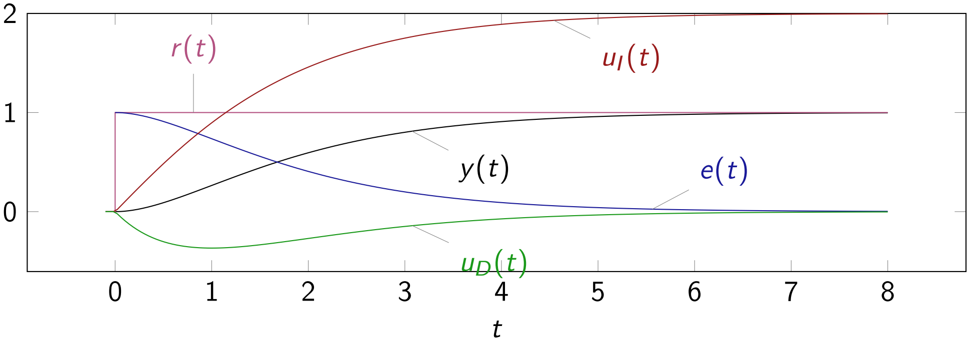
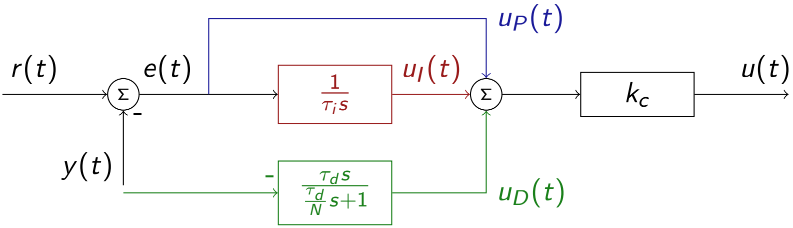
$$u(t) = k_c\Big(e(t) + \overbrace{\frac{1}{\tau_i}\int_0^t e(\xi)d\xi}^{u_I(t)} + \underbrace{\tau_d\frac{d}{dt}\big(-y(t)\big)}_{u_D(t)}\Big)$$



The PID - practical form



The parameter N is chosen to limit the influence of noisy measurements. Typically,

$$3 < N < 20$$

PID tuning

Method by Smith & Corripio using table by Ziegler-Nichols

Given process model (fitted to response of the system)

$$G(s) = K\frac{\mathrm{e}^{-s\theta}}{\tau s + 1}$$

and PID controller

$$F(s) = k_c\left(1 + \frac{1}{\tau_i s} + \tau_d s\right)$$

Choose the PID parameters according to the following table (Ziegler-Nichols, 1943)

Controller	k_c	τ_i	τ_d
P	$\frac{\tau}{\theta K}$		
PI	$\frac{0.9\tau}{\theta K}$	$\frac{\theta}{0.3}$	
PID	$\frac{1.2\tau}{\theta K}$	2θ	$\frac{\theta}{2}$

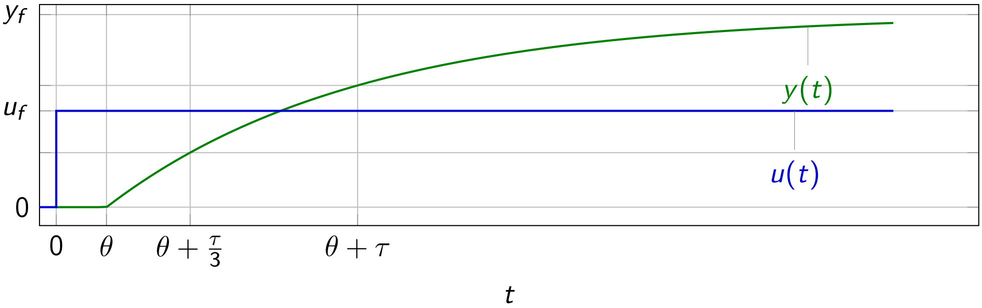
Gives good control for

$$0.1 < \frac{\theta}{\tau} < 0.6.$$

Fitting first-order model with delay

Assuming a plant model of first-order with time constant τ and delay θ

$$\color{green}{Y(s)} = \frac{K\mathrm{e}^{-s\theta}}{s\tau + 1}\color{blue}{U(s)} \quad \overset{U(s)=\frac{u_f}{s}}{\Longrightarrow} \quad \color{green}{y(t)} = u_f K\left(1 - \mathrm{e}^{-\frac{t-\theta}{\tau}}\right)u_H(t-\theta)$$

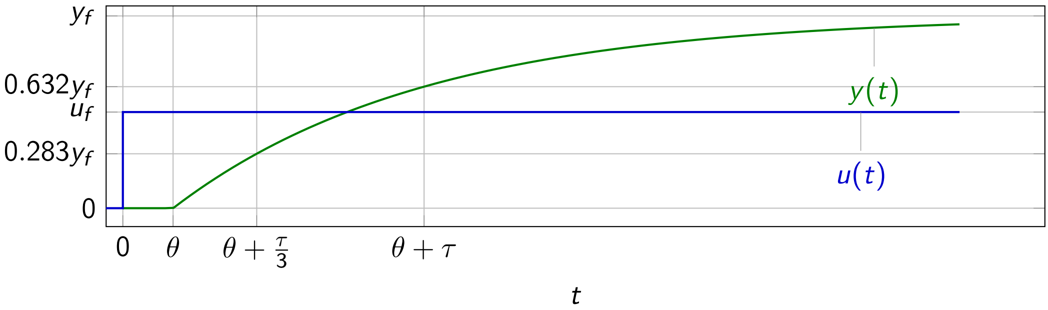


Individual activity Evaluate the response $y(t)$ at the two time instants $t = \theta + \frac{\tau}{3}$ and $t = \theta + \tau$!

Fitting first-order model with delay

Assuming a plant model of first-order with time constant τ and delay θ

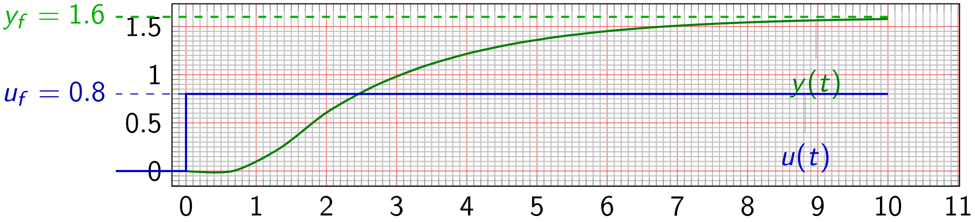
$$Y(s) = \frac{K\mathrm{e}^{-s\theta}}{s\tau + 1} U(s) \quad \overset{U(s)=\frac{u_f}{s}}{\Longrightarrow} \quad y(t) = u_f K \left(1 - \mathrm{e}^{-\frac{t-\theta}{\tau}}\right) u_H(t-\theta)$$



$$y_f = \lim_{t\to\infty} y(t) = u_f K \quad \Rightarrow \quad K = \frac{y_f}{u_f}.$$

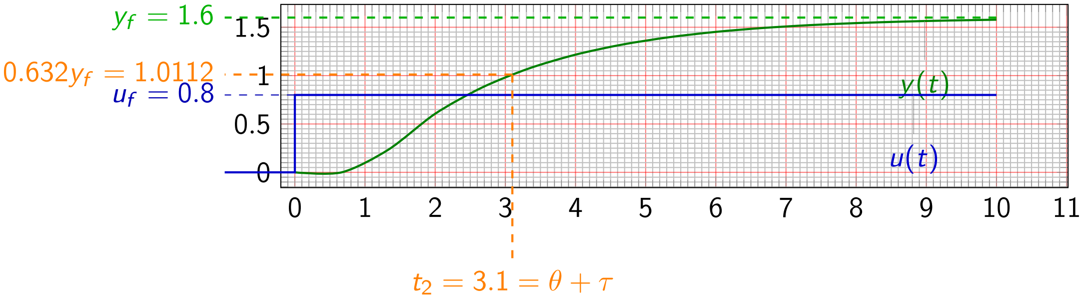
First-order model with delay - example

$$Y(s) = \frac{K\mathrm{e}^{-s\theta}}{s\tau + 1} U(s) \quad \overset{U(s)=\frac{u_f}{s}}{\Longrightarrow} \quad y(t) = u_f K \left(1 - \mathrm{e}^{-\frac{t-\theta}{\tau}}\right) u_s(t - \theta)$$



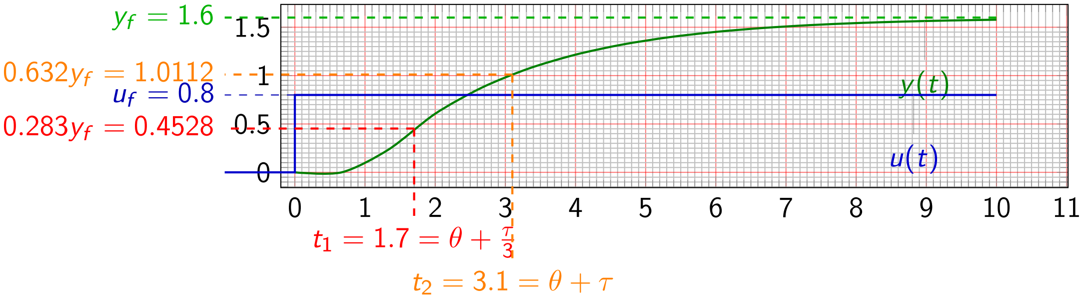
First-order model with delay - example

$$Y(s) = \frac{K\mathrm{e}^{-s\theta}}{s\tau + 1} U(s) \quad \overset{U(s)=\frac{u_f}{s}}{\Longrightarrow} \quad y(t) = u_f K \left(1 - \mathrm{e}^{-\frac{t-\theta}{\tau}}\right) u_s(t-\theta)$$



First-order model with delay - example

$$Y(s) = \frac{K\mathrm{e}^{-s\theta}}{s\tau + 1}U(s) \quad \xRightarrow{U(s)=\frac{u_f}{s}} \quad y(t) = u_f K\left(1 - \mathrm{e}^{-\frac{t-\theta}{\tau}}\right)u_s(t-\theta)$$



First-order model with delay - example

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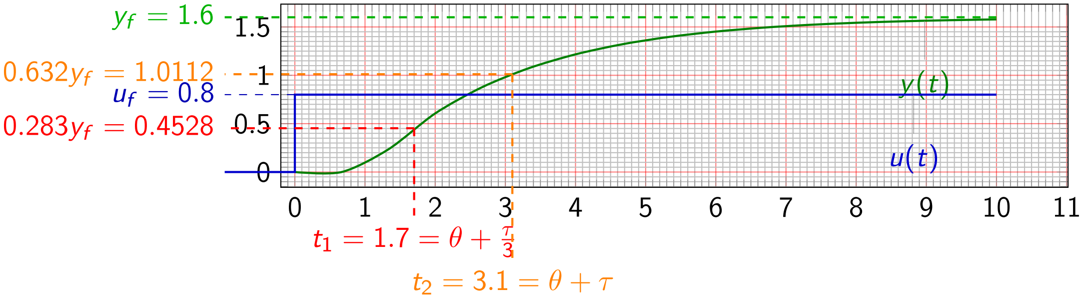

$$\begin{cases} 1.7 = \theta + \frac{\tau}{3} \\ 3.1 = \theta + \tau \end{cases} \Rightarrow \begin{cases} \theta = 1 \\ \tau = 2.1 \end{cases}, \qquad K = \frac{y_f}{u_f} = \frac{1.6}{0.8} = 2$$

Analytical PID design

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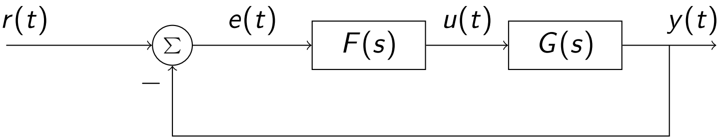


Activity Solve for $F(s)$ in the closed-loop transfer function

$$G_c(s) = \frac{G(s)F(s)}{1 + G(s)F(s)}$$

Analytical PID design - Solution

Analytical PID design - Solution

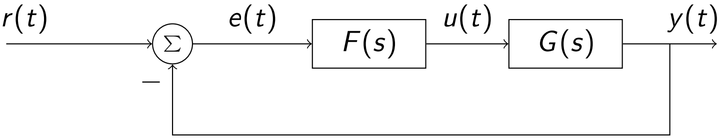
Solving for $F(s)$ in the closed-loop transfer function $G_c(s) = \frac{G(s)F(s)}{1+G(s)F(s)}$

$$\left(1 + G(s)F(s)\right) G_c(s) = G(s)F(s)$$

$$G_c(s) = \left(1 - G_c(s)\right) G(s)F(s)$$

$$F(s) = \frac{\frac{G_c(s)}{G(s)}}{1 - G_c(s)}$$

Analytic PID tuning - first-order with delay



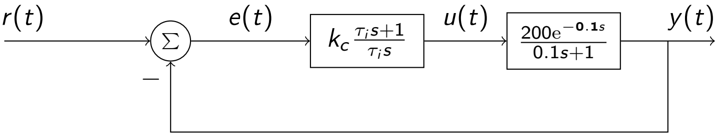
Given model $G(s) = K\frac{\mathrm{e}^{-s\theta}}{\tau s+1}$ of the process and desired closed-loop transfer function $G_c(s) = \frac{\mathrm{e}^{-s\theta}}{\tau_c s+1}$

Activity Show that the controller becomes

$$F(s) = \frac{1}{K}\left(\frac{\tau s+1}{\tau_c s + 1 - \mathrm{e}^{-s\theta}}\right) \approx \frac{1}{K}\left(\frac{\tau s+1}{(\tau_c+\theta)s}\right) = \underbrace{\frac{\tau}{K(\tau_c+\theta)}}_{k_c}\left(1 + \frac{1}{\underbrace{\tau}_{\tau_i} s}\right).$$

Which is a PI-controller with $k_c = \frac{\tau}{K(\tau_c+\theta)}$ and $\tau_i = \tau$.

Example



$k_c = \frac{\tau}{K(\tau_c+\theta)}$ and $\tau_i = \tau$.

Activity Determine the controller for the choice $\tau_c = \tau$