Frequency response

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Response of LTI systems to sinusoids

$$\xrightarrow{u} \boxed{G(s)} \xrightarrow{y}$$

Let $u(t) = \sin \omega_1 t$. Then, after transients have died out,

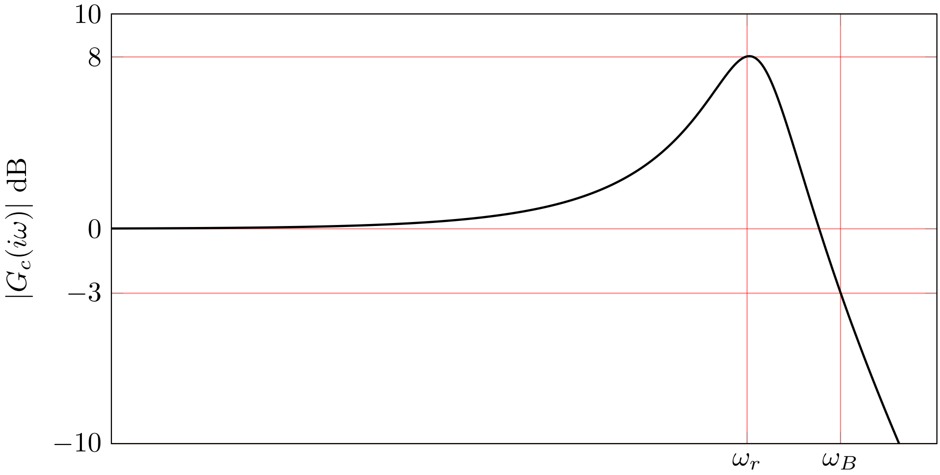
$$y(t) = |G(\omega_1)| \sin \left(\omega_1 t + \arg G(i\omega_1)\right).$$

The Bode diagram

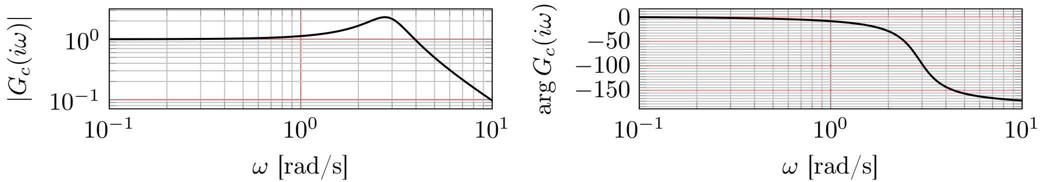
$$y(t) = \underbrace{|G(i\omega_1)|}_{\text{amplification}} \sin\left(\omega_1 t + \underbrace{\arg G(i\omega_1)}_{\text{phase shift}}\right)$$

The Bode diagram shows the magnitude and phase of the transfer function evaluated on the positive imaginary axis. It thus contains all information about the steady-state response of the system to input signals of different frequency.

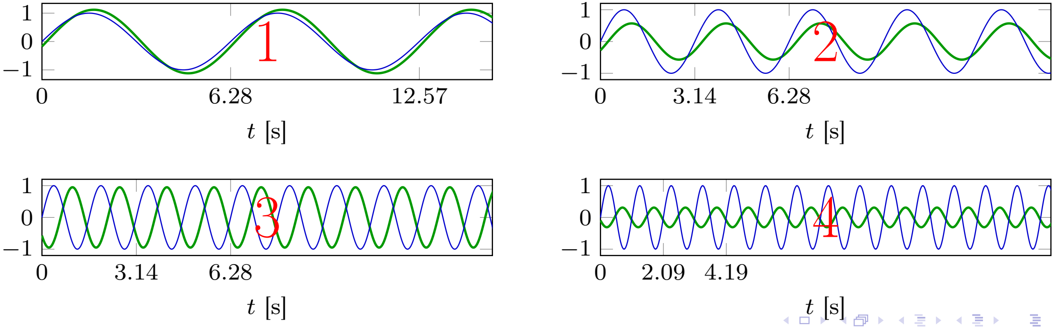
Specifications on the frequency properties of the closed-loop system



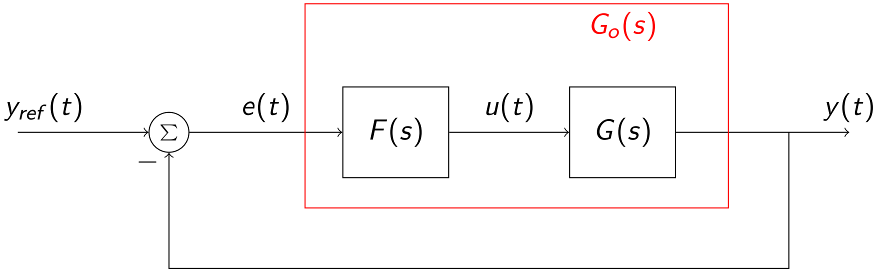
Exercise: Reading the Bode diagram



which of the below responses is not compatible with the Bode diagram?



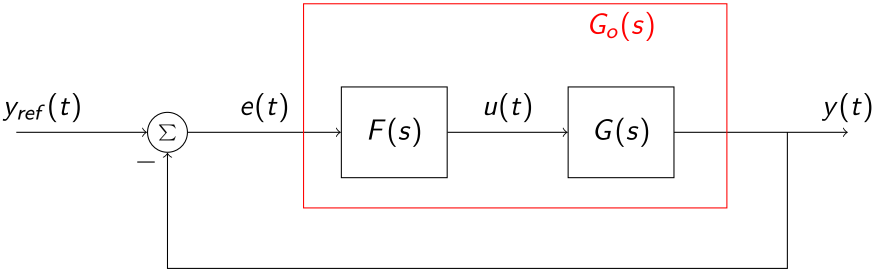
From loop gain to closed-loop gain



$$G_c(i\omega) = \frac{G(i\omega)F(i\omega)}{1 + G(i\omega)F(i\omega)} = \frac{G_o(i\omega)}{1 + G_o(i\omega)}$$

$$|G_c(i\omega)| = \frac{|G_o(i\omega)|}{|1 + G_o(i\omega)|} = \frac{|G_o(i\omega)|}{|G_o(i\omega) - (-1)|}$$

From loop gain to closed-loop gain



$$G_c(i\omega) = \frac{G(i\omega)F(i\omega)}{1 + G(i\omega)F(i\omega)} = \frac{G_o(i\omega)}{1 + G_o(i\omega)}$$

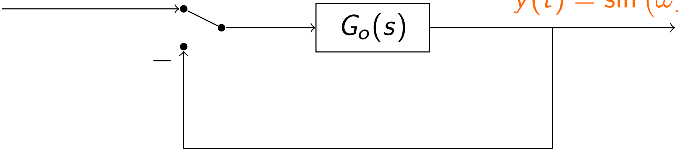
$$|G_c(i\omega)| = \frac{|G_o(i\omega)|}{|1 + G_o(i\omega)|} = \frac{|G_o(i\omega)|}{|G_o(i\omega) - (-1)|}$$

Keep the loop gain $G_o(i\omega)$ away from -1!

If the phase shift is π

$$G_o(i\omega_1) = -1,\ |G_o(i\omega_1)| = 1,\ \arg G_o(i\omega_1) = -\pi$$

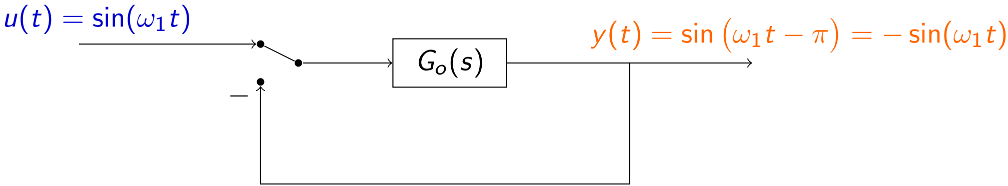
$u(t) = \sin(\omega_1 t)$



$y(t) = \sin\left(\omega_1 t - \pi\right) = -\sin(\omega_1 t)$

If the phase shift is π

$$G_o(i\omega_1) = -1,\ |G_o(i\omega_1)| = 1,\ \arg G_o(i\omega_1) = -\pi$$

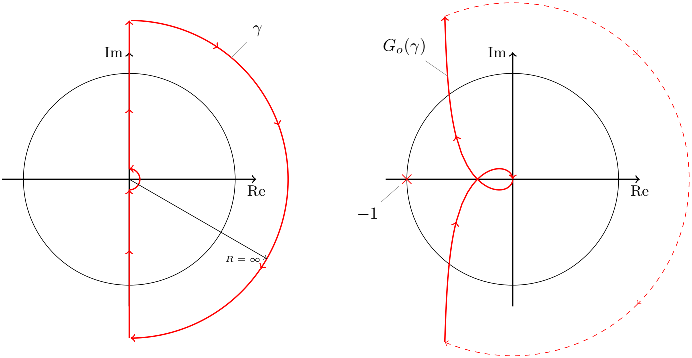


Closed-loop transfer function: $G_c(s) = \frac{G_o(s)}{1+G_o(s)}$ We want

$$1 + G_o(i\omega) \neq 0, \quad \forall\omega$$

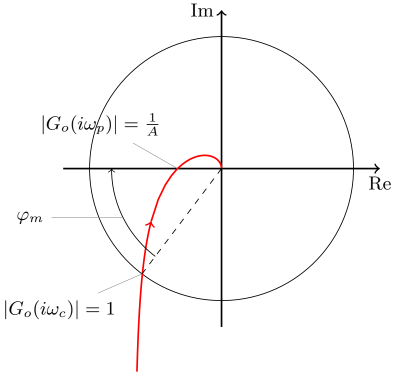
If not, then the closed-loop system will have poles on the imaginary axis (in the s-domain).

The simplified Nyquist criterion in the s-plane



If the open-loop system (the loop gain) is not unstable, i.e. $G_o(s)$ has no poles in the right-half plane, then the closed-loop system will be stable if the Nyquist curve **do not encircle the point** $s = -1$. The point $s = -1$ should stay on the left side of the Nyquist curve when we go along the curve from low to high frequencies.

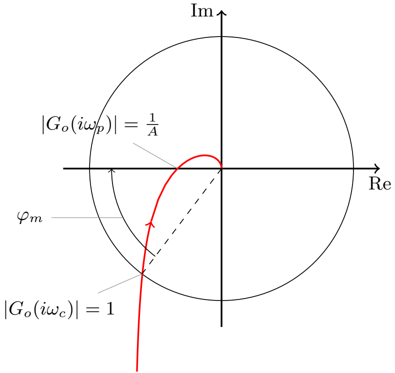
Stability margins



- Cross-over frequency: The frequency ω_c for which $|G_o(i\omega)| = 1$.
- Phase margin: The angle φ_m to the negative real axis for the point where the Nyquist curve intersects the unit circle.

$$\varphi_m = \arg G_o(i\omega_c) - (-180^\circ) = \arg G_o(i\omega_c) + 180^\circ$$

Stability margins



- phase-cross-over frequency: The frequency ω_p for which $\arg G_o(i\omega) = -180°$.
- Gain margin: The gain $K = A$ that would make the Nyquist curve of $KG_o(i\omega h)$ go through the point $-1 + i0$. This means that

$$|G_o(i\omega_p h| = \frac{1}{A}.$$

How to achieve the frequency-domain specifications

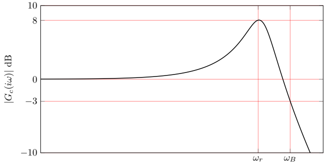
$$G_c(i\omega) = \frac{G_o(i\omega)}{1 + G_o(i\omega)}$$

Activity

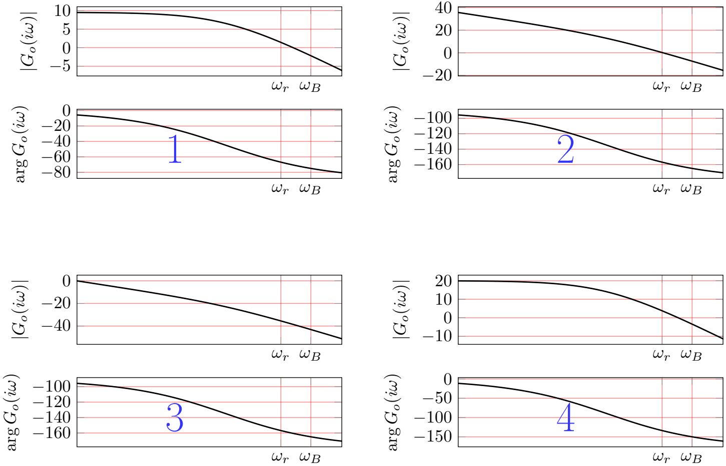
1. If $Go(i\omega_1) = -0.5$ what is $|G_c(i\omega_1)|$?
2. If $Go(i\omega_1) = -i$ what is $|G_c(i\omega_1)|$?

How to achieve the frequency-domain specifications

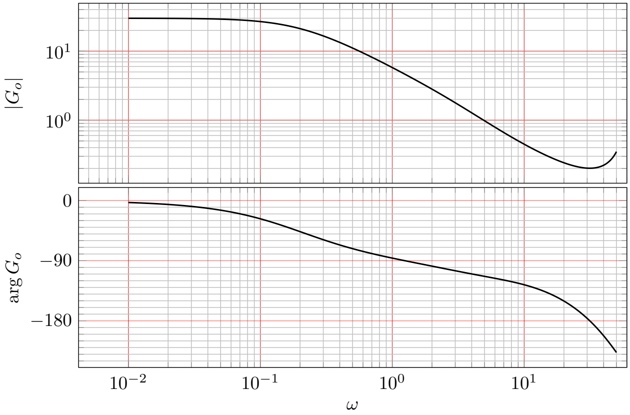
$$G_c(i\omega) = \frac{G_o(i\omega)}{1 + G_o(i\omega)}$$



Which of the Bode plots to the right shows the correct loop gain $G_o(i\omega)$?



Stability margins excercise



Activity Determine the cross-over frequency ω_c, the phase cross-over frequency ω_p, the phase margin and the amplitude margin.