System identification

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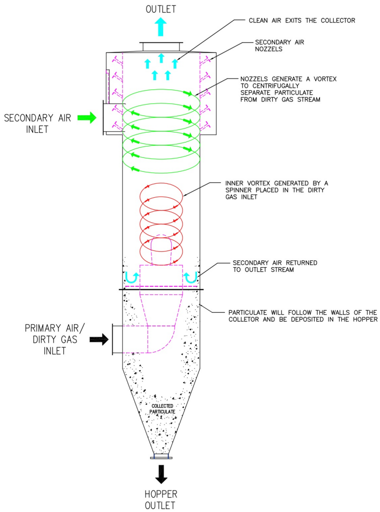
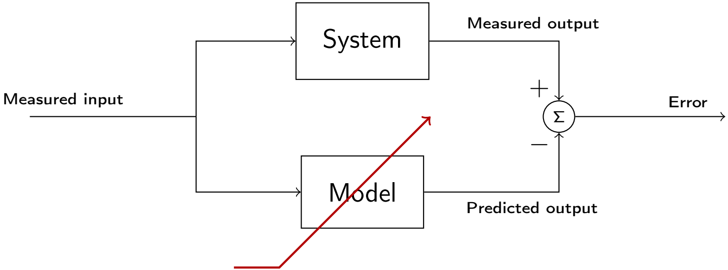
A complicated process

From Wikipedia "Cyclonic separation"



System identification



The shift operator q

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Algebra with the shift operator

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$$= \frac{x(k+1) - x(k-1)}{2h}$$

The Auto-Regressive with eXogenous input (ARX) model

$$A(\mathrm{q})y(k) = B(\mathrm{q})u(k) + e(k+n)$$

The error signal $e(k)$ is a zero-mean white noise sequence representing perturbations and modeling errors.

First-order ARX model with one delay

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First-order ARX model with one delay

$$(q + a_1)y(k) = (b_0 q + b_1) q^{-1} u(k) + e(k+1)$$

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$$y(k+1) =$$

$$y(k+1) = -a_1 y(k) + b_0 u(k) + b_1 u(k-1) e(k+1)$$

Using the model to predict the output one step ahead:

$$\hat{y}(k+1) = -a_1 y(k) + b_0 u(k) + b_1 u(k-1) = \underbrace{\begin{bmatrix} -y(k) & u(k) & u(k-1) \end{bmatrix}}_{\varphi_{k+1}^T} \underbrace{\begin{bmatrix} a_1 \\ b_0 \\ b_1 \end{bmatrix}}_{\theta}$$

$$= \varphi_{k+1}^T \theta$$

Parameter estimation - Least squares

Objective Given observations

$$\mathcal{D} = \{(u_1, y_1), (u_2, y_2), \ldots, (u_N, y_N)\}$$

and model $\mathcal{M}: \; y(k+1) = -ay(k) + b_0 u(k) + b_1 u(k-1) + e(k+1)$, obtain the parameters $(a_1, \; b_0, \; b_1)$ which gives the best fit of the model to the data.

Parameter estimation - Least squares

Given observations

$$\mathcal{D} = \{(u_1, y_1), (u_2, y_2), \ldots, (u_N, y_N)\}$$

and model $\mathcal{M}: \; y(k+1) = -ay(k) + b_0 u(k) + b_1 u(k-1) + e(k+1).$

1. Form the one-step ahead prediction

$$\hat{y}_{k+1} = -a_1 y_k + b_0 u_k + b_1 u_{k-1} = \underbrace{\begin{bmatrix} -y_k & u_k & u_{k-1} \end{bmatrix}}_{\varphi_{k+1}^T} \underbrace{\begin{bmatrix} a_1 \\ b_0 \\ b_1 \end{bmatrix}}_{\theta}$$

and the prediction error

$$\epsilon_{k+1} = y_{k+1} - \hat{y}_{k+1} = y_{k+1} - \varphi_{k+1}^T \theta.$$

Parameter estimation - Least squares

2. Combine all the observations y_k and predictions $\hat{y}_k$ on vector form

$$\epsilon = \begin{bmatrix} \epsilon_3 \\ \epsilon_4 \\ \vdots \\ \epsilon_N \end{bmatrix} = \begin{bmatrix} y_3 \\ y_4 \\ \vdots \\ y_N \end{bmatrix} - \begin{bmatrix} \hat{y}_3 \\ \hat{y}_4 \\ \vdots \\ \hat{y}_N \end{bmatrix} = \begin{bmatrix} y_3 \\ y_4 \\ \vdots \\ y_N \end{bmatrix} - \begin{bmatrix} \varphi_3^T \theta \\ \varphi_4^T \theta \\ \vdots \\ \varphi_N^T \theta \end{bmatrix}$$

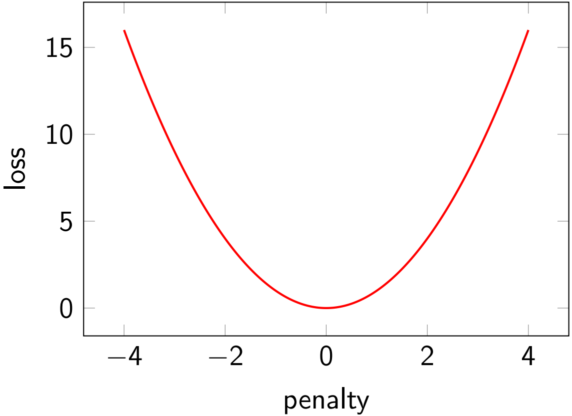
$$= y - \underbrace{\begin{bmatrix} \varphi_3^T \\ \varphi_4^T \\ \vdots \\ \varphi_N^T \end{bmatrix}}_{\Phi} \theta = y - \Phi\theta$$

3. Solve $\arg\min \; J(\theta) = \frac{1}{2}\epsilon^T\epsilon = \frac{1}{2}\sum_{i=3}^{N} \epsilon_i(\theta)^2 \quad \Rightarrow \hat{\theta}_{LS} = (\Phi^T\Phi)^{-1}\Phi^T y$

The problem with least squares

$$\text{minimize} \sum_k g(\epsilon_k)$$

where $g(u) = u^2$

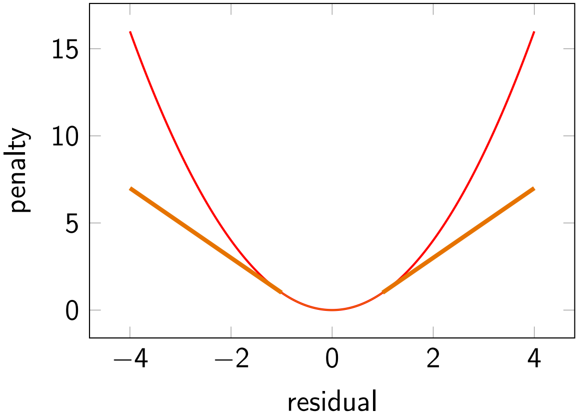


More robust: The Huber loss function

Also known as robust regression

$$\text{minimize} \sum_k g_{hub}(\epsilon_k)$$

$$\text{where } g_{hub}(u) = \begin{cases} u^2 & |u| \le M \\ M(2|u| - M) & |u| > M \end{cases}$$



First-order ARX model without delay

$$(\mathrm{q} + a_1)y(k) = (b_0\,\mathrm{q} + b_1)u(k) + e(k+1)$$

Activity

1. Determine the one-step ahead predictor $\hat{y}_{k+1}$ and the prediction error ϵ_{k+1}.
2. Form the system of equations $\Phi\theta = y$

The ARX model

$$A(\mathrm{q})y(k) = B(\mathrm{q})u(k) + e(k+n)$$

Activity Fill the empty blocks.