State feedback

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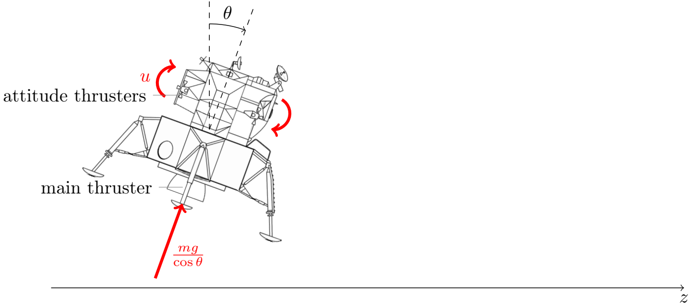
The Apollo lunar module



$$\xrightarrow{u(t)} \boxed{\frac{1}{Js}} \xrightarrow{\dot{\theta}(t)} \boxed{\frac{1}{s}} \xrightarrow{\theta(t)} \boxed{g} \xrightarrow{\ddot{z}(t)} \boxed{\frac{1}{s}} \xrightarrow{\dot{z}(t)}$$

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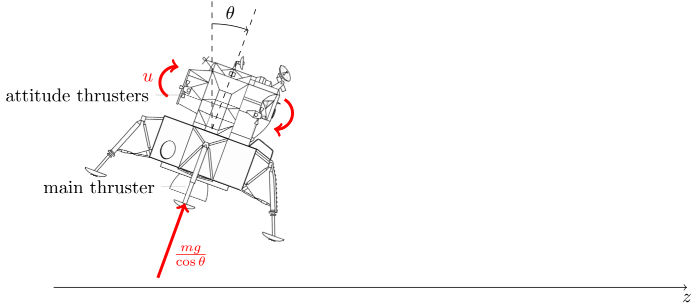
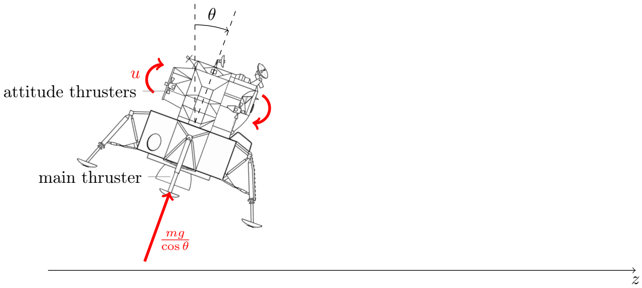


$$u(t) \longrightarrow \boxed{\frac{1}{Js}} \xrightarrow{\dot{\theta}(t)} \boxed{\frac{1}{s}} \xrightarrow{\theta(t)} \boxed{g} \xrightarrow{\ddot{z}(t)} \boxed{\frac{1}{s}} \xrightarrow{\dot{z}(t)}$$

Activity Which is the transfer function of the system?

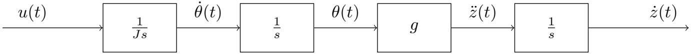
$$1: \quad G(s) = \frac{\frac{g}{J}}{s^2} \qquad 2: \quad G(s) = \frac{\frac{g}{J}}{s(s^2+1)} \qquad 3: \quad G(s) = \frac{\frac{g}{J}}{s^3}$$

State variables



With dynamics

$$\begin{cases} \dot{x}_1 = \ddot{\theta} = \frac{1}{J}u \\ \dot{x}_2 = \dot{\theta} = x_1 \\ \dot{x}_3 = \ddot{z} = g\theta = gx_2 \end{cases}$$



State variables: $x = \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix}^T = \begin{bmatrix} \dot{\theta} & \theta & \dot{z} \end{bmatrix}^T.$

State-space model

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Activity Fill the matrix A and vector B.

$$\dot{x} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \underbrace{\begin{bmatrix} \quad & \quad & \quad \\ & & \\ & & \end{bmatrix}}_{A} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \underbrace{\begin{bmatrix} \quad \\ \\ \end{bmatrix}}_{B} u$$

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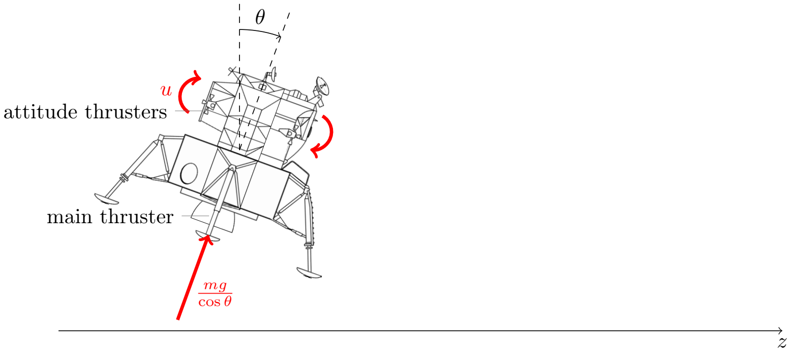
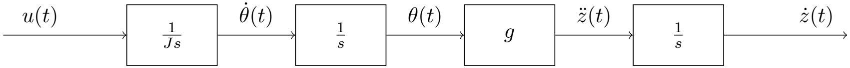
Activity What are the poles of the system?

Sensors





Activity What sensors are needed for state feedback?

Controllability

$$\dot{x} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & g & 0 \end{bmatrix}}_{A} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \underbrace{\begin{bmatrix} \frac{1}{J} \\ 0 \\ 0 \end{bmatrix}}_{B} u$$

Forming the controllability matrix. Note that

$$A^2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ g & 0 & 0 \end{bmatrix}$$

$$\mathcal{C} = \begin{bmatrix} B & AB & A^2B \end{bmatrix} = \begin{bmatrix} \frac{1}{J} & 0 & \\ 0 & \frac{1}{J} & \\ 0 & 0 & \end{bmatrix}$$

Controllability

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Activity Is the system controllable?

Linear state feedback

$$\dot{x} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & g & 0 \end{bmatrix}}_{A} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \underbrace{\begin{bmatrix} \frac{1}{J} \\ 0 \\ 0 \end{bmatrix}}_{B} u$$

Introduce linear state feedback

$$u = -Lx + l_0 r,$$

where r is a reference signal.

Closed-loop system

$$\dot{x} = (A - BL)x + l_0 Br$$

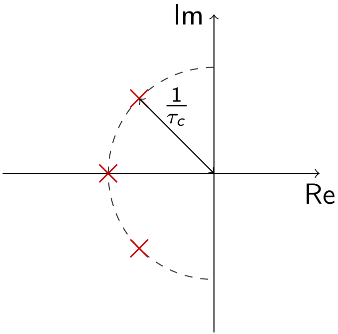
Since the system is controllable, we can find a gain vector L that places the eigenvalues of $A - BL$ (the poles of the closed-loop system) at desired locations.

Linear state feedback

The poles of $\dot{x} = (A - BL)x + l_0 Br$ are given by the solutions to the characteristic equation

$$
\begin{aligned}
\det\Big(sI - (A - BL)\Big) &= 0 \\
\det\left(\begin{bmatrix} s & 0 & 0 \\ 0 & s & 0 \\ 0 & 0 & s \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & g & 0 \end{bmatrix} + \begin{bmatrix} \frac{1}{J} l_1 & \frac{1}{J} l_2 & \frac{1}{J} l_3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\right) &= 0 \\
\det\begin{bmatrix} s + \frac{1}{J} l_1 & \frac{1}{J} l_2 & \frac{1}{J} l_3 \\ -1 & s & 0 \\ 0 & -g & s \end{bmatrix} &= 0 \\
(s + \frac{1}{J} l_1)s^2 + \frac{1}{J} l_2 s + \frac{1}{J} g l_3 &= 0 \\
s^3 + \frac{1}{J} l_1 s^2 + \frac{1}{J} l_2 s + \frac{1}{J} g l_3 &= 0
\end{aligned}
$$

Where to place the closed-loop poles



Desired closed-loop characteristic polynomial

$$
\begin{aligned}
(s-p_1)(s-p_2)(s-p_3) &= (s+\frac{1}{\tau_c})(s^2+\frac{\sqrt{2}}{\tau_c}s+\frac{1}{\tau_c^2}) \\
&= s^3+\frac{1+\sqrt{2}}{\tau_c}s^2+\frac{1+\sqrt{2}}{\tau_c^2}s+\frac{1}{\tau_c^3}
\end{aligned}
$$

Determining the state feedback gain

By linear state feedback we have characteristic polynomial

$$\det\Big(sI - (A - BL)\Big) = s^3 + \frac{1}{J}l_1 s^2 + \frac{1}{J}l_2 s + \frac{1}{J}g\,l_3.$$

And we want to achieve the characteristic polynomial

$$s^3 + \frac{1+\sqrt{2}}{\tau_c}s^2 + \frac{1+\sqrt{2}}{\tau_c^2}s + \frac{1}{\tau_c^3}.$$

Activity What do we do next?

Determining the state feedback gain

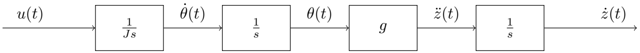
Set the characteristic polynomial obtained from det $\left(sI - (A-BL)\right)$ equal to the desired characteristic polynomial

$$s^3 + \frac{1}{J} l_1 s^2 + \frac{1}{J} l_2 s + \frac{1}{J} g\, l_3 = s^3 + \frac{1+\sqrt{2}}{\tau_c} s^2 + \frac{1+\sqrt{2}}{\tau_c^2} s + \frac{1}{\tau_c^3}$$

Solve for the gains by setting corresponding coefficients equal.

$$\left.\begin{array}{ll} s^2: & \frac{1}{J} l_1 = \frac{1+\sqrt{2}}{\tau_c} \\ s^1: & \frac{1}{J} l_2 = \frac{1+\sqrt{2}}{\tau_c^2} \\ s^0: & \frac{1}{J} g\, l_3 = \frac{1}{\tau_c^3} \end{array}\right\} \Rightarrow \left.\begin{array}{ll} l_1 & = \frac{J(1+\sqrt{2})}{\tau_c} \\ l_2 & = \frac{J(1+\sqrt{2})}{\tau_c^2} \\ l_3 & = \frac{J}{g\tau_c^3} \end{array}\right\}$$

The gain l_0



$$G(s) = \frac{\frac{g}{J}}{s^3}$$

It can be shown that state feedback does not change the numerator of the transfer function, only the denominator, so

$$G_c(s) = l_0 \frac{\frac{g}{J}}{s^3 + \frac{1+\sqrt{2}}{\tau_c} s^2 + \frac{1+\sqrt{2}}{\tau_c^2} s + \frac{1}{\tau_c^3}}$$

We want unit static gain, $G_c(0) = 1$

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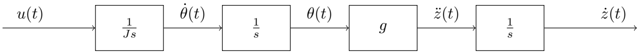

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Activity Determine the gain l_0