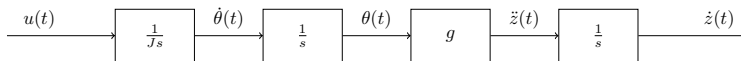
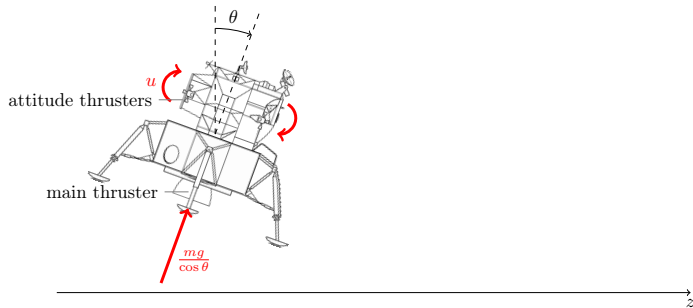


State feedback

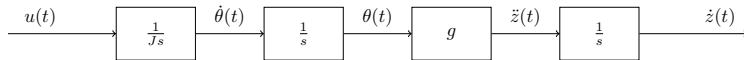
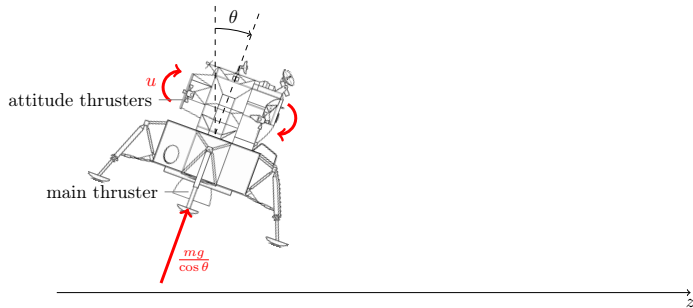
Kjartan Halvorsen

October 13, 2021

The Apollo lunar module



The Apollo lunar module



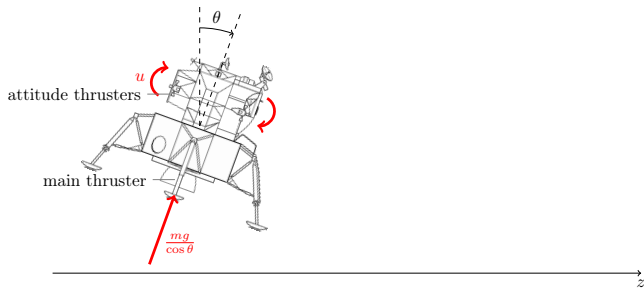
Activity Which is the transfer function of the system?

1 : $G(s) = \frac{g}{s^2}$

2 : $G(s) = \frac{g}{s(s^2 + 1)}$

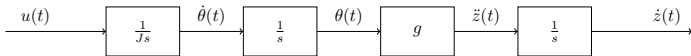
3 : $G(s) = \frac{g}{s^3}$

State variables



With dynamics

$$\begin{cases} \dot{x}_1 = \ddot{\theta} = \frac{1}{J}u \\ \dot{x}_2 = \dot{\theta} = x_1 \\ \dot{x}_3 = \ddot{z} = g\theta = gx_2 \end{cases}$$



State variables: $x = [x_1 \ x_2 \ x_3]^T = [\dot{\theta} \ \theta \ \dot{z}]^T$.

State-space model

State variables: $x = [x_1 \ x_2 \ x_3]^T = [\dot{\theta} \ \theta \ \dot{z}]^T$. With dynamics

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Activity Fill the matrix A and vector B .

$$\dot{x} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \underbrace{\begin{bmatrix} \\ \\ \end{bmatrix}}_A \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \underbrace{\begin{bmatrix} \\ \\ \end{bmatrix}}_B u$$

State-space model

State-space model

State variables: $x = [x_1 \ x_2 \ x_3]^T = [\dot{\theta} \ \theta \ \dot{z}]^T$. With dynamics

$$\begin{cases} \dot{x}_1 = \ddot{\theta} = \frac{1}{J}u \\ \dot{x}_2 = \dot{\theta} = x_1 \\ \dot{x}_3 = \ddot{z} = g\theta = gx_2 \end{cases}$$

$$\dot{x} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & g & 0 \end{bmatrix}}_A \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \underbrace{\begin{bmatrix} \frac{1}{J} \\ 0 \\ 0 \end{bmatrix}}_B u$$

State-space model

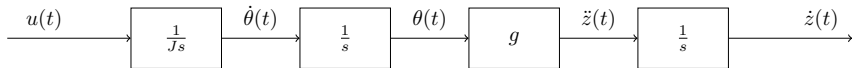
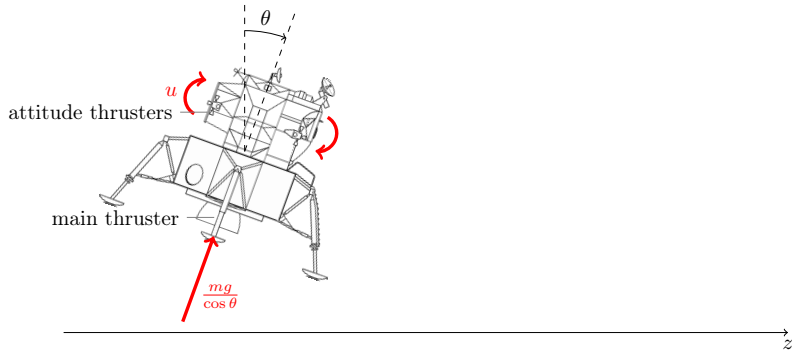
State variables: $x = [x_1 \ x_2 \ x_3]^T = [\dot{\theta} \ \theta \ \dot{z}]^T$. With dynamics

$$\begin{cases} \dot{x}_1 = \ddot{\theta} = \frac{1}{J}u \\ \dot{x}_2 = \dot{\theta} = x_1 \\ \dot{x}_3 = \ddot{z} = g\theta = gx_2 \end{cases}$$

$$\dot{x} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & g & 0 \end{bmatrix}}_A \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \underbrace{\begin{bmatrix} \frac{1}{J} \\ 0 \\ 0 \end{bmatrix}}_B u$$

Activity What are the poles of the system?

Sensors



Activity What sensors are needed for state feedback?

Controllability

$$\dot{x} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & g & 0 \end{bmatrix}}_A \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \underbrace{\begin{bmatrix} \frac{1}{J} \\ 0 \\ 0 \end{bmatrix}}_B u$$

Forming the controllability matrix

$$\mathcal{C} = [B \quad AB \quad A^2B] = \begin{bmatrix} \frac{1}{J} & 0 & 0 \\ 0 & \frac{1}{J} & 0 \\ 0 & 0 & \frac{1}{J}g \end{bmatrix}$$

Controllability

$$\dot{x} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & g & 0 \end{bmatrix}}_A \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \underbrace{\begin{bmatrix} \frac{1}{J} \\ 0 \\ 0 \end{bmatrix}}_B u$$

Forming the controllability matrix

$$\mathcal{C} = [B \quad AB \quad A^2B] = \begin{bmatrix} \frac{1}{J} & 0 & 0 \\ 0 & \frac{1}{J} & 0 \\ 0 & 0 & \frac{1}{J}g \end{bmatrix}$$

Activity Is the system controllable?

Linear state feedback

$$\dot{x} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & g & 0 \end{bmatrix}}_A \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \underbrace{\begin{bmatrix} \frac{1}{J} \\ 0 \\ 0 \end{bmatrix}}_B u$$

Introduce linear state feedback

$$u = -Lx + l_0 r,$$

where r is a reference signal.

Closed-loop system

$$\dot{x} = (A - BL)x + l_0 Br$$

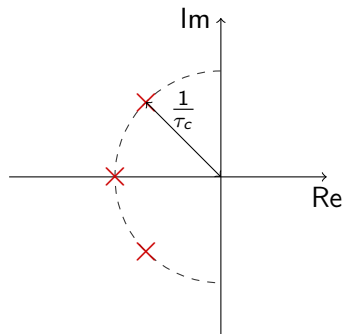
Since the system is **controllable**, we can find a gain vector L that places the eigenvalues of $A - BL$ (the poles of the closed-loop system) at desired locations.

Linear state feedback

The poles of $\dot{x} = (A - BL)x + l_0 Br$ are given by the solutions to the characteristic equation

$$\det(sI - (A - BL)) = 0$$
$$\det \left(\begin{bmatrix} s & 0 & 0 \\ 0 & s & 0 \\ 0 & 0 & s \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & g & 0 \end{bmatrix} + \begin{bmatrix} \frac{1}{J}l_1 & \frac{1}{J}l_2 & \frac{1}{J}l_3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \right) = 0$$
$$\det \begin{bmatrix} s + \frac{1}{J}l_1 & \frac{1}{J}l_2 & \frac{1}{J}l_3 \\ -1 & s & 0 \\ 0 & -g & s \end{bmatrix} = 0$$
$$(s + \frac{1}{J}l_1)s^2 + \frac{1}{J}l_2s + \frac{1}{J}gl_3 = 0$$
$$s^3 + \frac{1}{J}l_1s^2 + \frac{1}{J}l_2s + \frac{1}{J}gl_3 = 0$$

Where to place the closed-loop poles



Desired closed-loop characteristic polynomial

$$\begin{aligned}(s - p_1)(s - p_2)(s - p_3) &= \left(s + \frac{1}{\tau_c}\right)\left(s^2 + \frac{\sqrt{2}}{\tau_c}s + \frac{1}{\tau_c^2}\right) \\ &= s^3 + \frac{1 + \sqrt{2}}{\tau_c}s^2 + \frac{1 + \sqrt{2}}{\tau_c^2}s + \frac{1}{\tau_c^3}\end{aligned}$$

Determining the state feedback gain

By linear state feedback we have characteristic polynomial

$$\det(sI - (A - BL)) = s^3 + \frac{1}{J}l_1s^2 + \frac{1}{J}l_2s + \frac{1}{J}gl_3.$$

And we want to achieve the characteristic polynomial

$$s^3 + \frac{1 + \sqrt{2}}{\tau_c}s^2 + \frac{1 + \sqrt{2}}{\tau_c^2}s + \frac{1}{\tau_c^3}.$$

Activity What do we do next?

Determining the state feedback gain

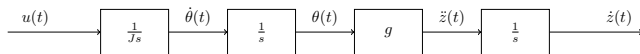
Set the characteristic polynomial obtained from $\det(sI - (A - BL))$ equal to the desired characteristic polynomial

$$s^3 + \frac{1}{J}l_1s^2 + \frac{1}{J}l_2s + \frac{1}{J}gl_3 = s^3 + \frac{1 + \sqrt{2}}{\tau_c}s^2 + \frac{1 + \sqrt{2}}{\tau_c^2}s + \frac{1}{\tau_c^3}$$

Solve for the gains by setting corresponding coefficients equal.

$$\left. \begin{array}{l} s^2 : \quad \frac{1}{J}l_1 = \frac{1 + \sqrt{2}}{\tau_c} \\ s^1 : \quad \frac{1}{J}l_2 = \frac{1 + \sqrt{2}}{\tau_c^2} \\ s^0 : \quad \frac{1}{J}gl_3 = \frac{1}{\tau_c^3} \end{array} \right\} \Rightarrow \left. \begin{array}{l} l_1 = \frac{J(1 + \sqrt{2})}{\tau_c} \\ l_2 = \frac{J(1 + \sqrt{2})}{\tau_c^2} \\ l_3 = \frac{J}{g\tau_c^3} \end{array} \right\}$$

The gain l_0



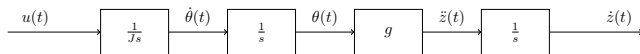
$$G(s) = \frac{g}{s^3}$$

It can be shown that state feedback does not change the numerator of the transfer function, only the denominator, so

$$G_c(s) = l_0 \frac{\frac{g}{J}}{s^3 + \frac{1+\sqrt{2}}{\tau_c} s^2 + \frac{1+\sqrt{2}}{\tau_c^2} s + \frac{1}{\tau_c^3}}$$

Want unit static gain, $G_c(0) = 1$

The gain l_0



$$G(s) = \frac{g}{s^3}$$

It can be shown that state feedback does not change the numerator of the transfer function, only the denominator, so

$$G_c(s) = l_0 \frac{\frac{g}{J}}{s^3 + \frac{1+\sqrt{2}}{\tau_c} s^2 + \frac{1+\sqrt{2}}{\tau_c^2} s + \frac{1}{\tau_c^3}}$$

Want unit static gain, $G_c(0) = 1$

Activity Determine the gain l_0