

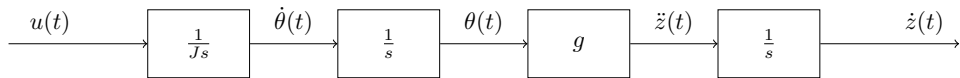
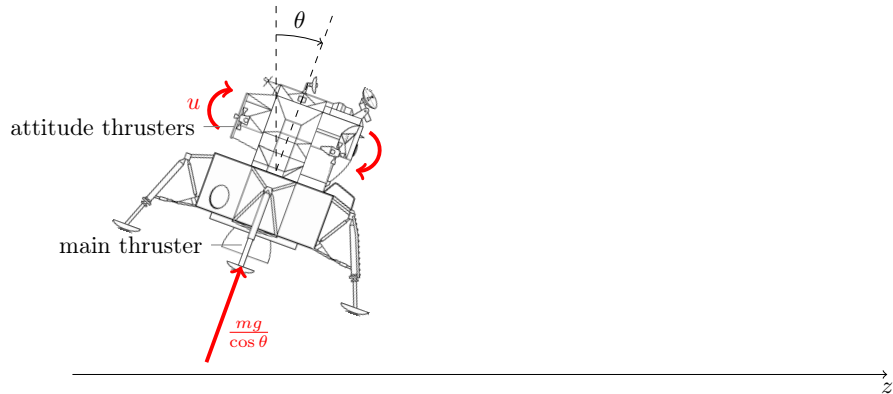
State feedback with observer

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State feedback with reconstructed states

State feedback with reconstructed states



State feedback

Given

$$\begin{aligned}\dot{x} &= Ax + Bu \\ y &= Cx\end{aligned}\tag{1}$$

and measurements (or estimates) of the state vector x .

Linear state feedback is the control law

$$\begin{aligned}u &= f(x, u_c) = -l_1x_1 - l_2x_2 - \cdots - l_nx_n + l_0u_c \\ &= -Lx + l_0u_c,\end{aligned}$$

where

$$L = [l_1 \quad l_2 \quad \cdots \quad l_n].$$

Substituting the control law in the state space model (5) gives

$$\begin{aligned}\dot{x} &= (A - BL)x + l_0Br \\ y(k) &= Cx\end{aligned}\tag{2}$$

Observer design

Given model

$$\dot{x} = Ax + Bu$$

$$y = Cx$$

and measurements of the output signal y .

The observer is given by

$$\dot{\hat{x}} = \underbrace{A\hat{x} + Bu}_{\text{simulation}} + \underbrace{K(y - C\hat{x})}_{\text{correction}} = (A - KC)\hat{x} + Bu + Ky$$

with poles given by the eigenvalues of the matrix $A_o = A - KC$

Rule-of-thumb Choose the poles of the observer (eigenvalues of $A - KC$) at least twice as fast as the poles (eigenvalues) of $A - BL$.

Control by feedback from reconstructed states

The design problem can be separated into two problems

1. Determine the gain vector L and the gain l_0 of the control law

$$u = -L\hat{x} + l_0 r$$

so that the closed-loop system has good reference tracking.

2. Determine the gain vector K of the observer

$$\dot{\hat{x}} = A\hat{x} + Bu + K(y - C\hat{x})$$

to get a good balance between disturbance rejection and noise attenuation.

Computing the observer gain

A matrix M and its transpose M^T have the same eigenvalues. Hence, the problem of determining the gain K to obtain desired eigenvalues of

$$A - KC$$

is equivalent to determining the gain K in

$$(A - KC)^T = A^T - C^T K^T.$$

The last problem has the exact same form as the problem of determining L to obtain desired eigenvalues of

$$A - BL$$

So, the same matlab function can be used for both problems.

Computing the state feedback and observer gains

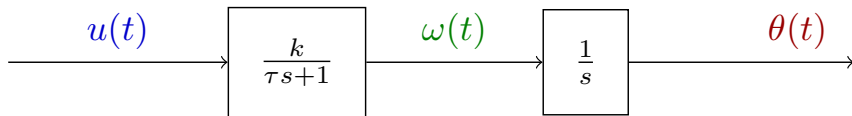
1. Ackerman's method

```
L = acker(A, B, pd)
K = acker(A', C', po)'
```

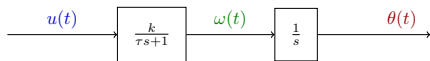
2. More numerically stable method

```
L = place(A, B, pd)
K = place(A', C', po)'
```


Example - Position control of the DC motor



State-space model with physical states



State variables corresponding to physical signals:

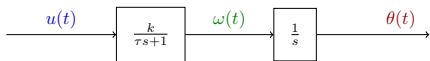
$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \omega \\ \theta \end{bmatrix}$$

With dynamics

$$\tau \dot{\omega} = -\omega + ku$$

$$\dot{\theta} = \omega$$

State-space model with physical states



State variables corresponding to physical signals:

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \omega \\ \theta \end{bmatrix}$$

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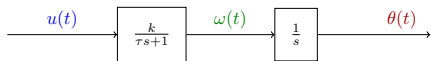
$$\dot{\theta} = \omega$$

Activity Fill the matrix A and vectors B and C .

$$\dot{x} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \underbrace{\begin{bmatrix} \quad \\ \quad \end{bmatrix}}_A \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \underbrace{\begin{bmatrix} \quad \\ \quad \end{bmatrix}}_B u$$

$$y = \theta = \underbrace{\begin{bmatrix} \quad \end{bmatrix}}_C \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

State-space model with physical states



State variables corresponding to physical signals:

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \omega \\ \theta \end{bmatrix}$$

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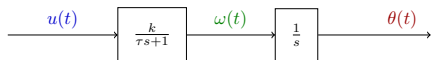
$$\dot{x} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \underbrace{\begin{bmatrix} -\frac{1}{\tau} & 0 \\ 1 & 0 \end{bmatrix}}_A \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \underbrace{\begin{bmatrix} \frac{k}{\tau} \\ 0 \end{bmatrix}}_B u$$

$$y = \theta = \underbrace{\begin{bmatrix} 0 & 1 \end{bmatrix}}_C \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

State-space model on controllable canonical form

The system with transfer function

$$G(s) = \frac{b_1 s^{n-1} + \dots + b_{n-1} s + b_n}{s^n + a_1 s^{n-1} + \dots + a_{n-1} s + a_n}$$



Transfer function

$$G(s) = \frac{\frac{k}{\tau}}{s(s + \frac{1}{\tau})}$$

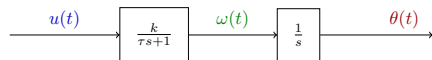
can be represented on state-space form as

$$\dot{x} = \begin{bmatrix} -a_1 & -a_2 & \dots & -a_{n-1} & -a_n \\ 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & 0 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} u$$
$$y = [b_1 \quad b_2 \quad \dots \quad b_n] x$$

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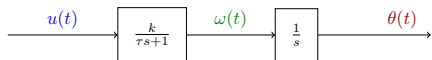
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$$y = [b_1 \quad b_2 \quad \cdots \quad b_n] x$$

Activity Determine the state-space model of the DC motor on controllable canonical form

State-space model on observable canonical form

The system with transfer function

$$G(s) = \frac{b_1 s^{n-1} + \dots + b_{n-1} s + b_n}{s^n + a_1 s^{n-1} + \dots + a_{n-1} s + a_n}$$



Transfer function

$$G(s) = \frac{\frac{k}{\tau}}{s(s + \frac{1}{\tau})}$$

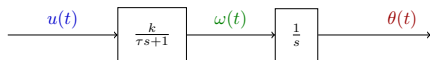
can be represented on state-space form as

$$\dot{x} = \begin{bmatrix} -a_1 & 1 & 0 & \dots & 0 \\ -a_2 & 0 & 1 & \dots & 0 \\ -a_3 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -a_n & 0 & 0 & \dots & 0 \end{bmatrix} x + \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ \vdots \\ b_n \end{bmatrix} u$$
$$y = \begin{bmatrix} 1 & 0 & \dots & 0 \end{bmatrix} x$$

State-space model on observable canonical form

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$$y = [1 \quad 0 \quad \dots \quad 0] x$$

Activity Determine the state-space model of the DC motor on observable canonical form

State-feedback for model with physical states

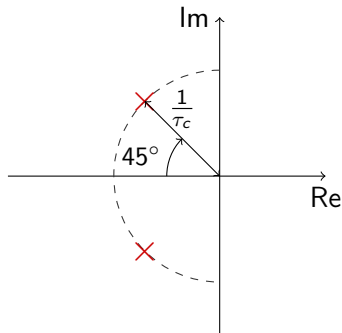
$$\dot{x} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \underbrace{\begin{bmatrix} -\frac{1}{\tau} & 0 \\ 1 & 0 \end{bmatrix}}_A \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \underbrace{\begin{bmatrix} \frac{K}{\tau} \\ 0 \end{bmatrix}}_B u$$

Feedback $u = -Lx + l_0 r$ gives closed-loop system

$$\begin{aligned} \dot{x} &= (A - BL)x + l_0 Br \\ &= \begin{bmatrix} -\frac{1}{\tau} - l_1 \frac{k}{\tau} & -l_2 \frac{k}{\tau} \\ 1 & 0 \end{bmatrix} x + l_0 Br \end{aligned}$$

with characteristic polynomial

$$s^2 + \left(\frac{1}{\tau} + l_1 \frac{k}{\tau}\right)s + l_2 \frac{k}{\tau}$$



Desired closed-loop characteristic polynomial

$$(s - p_1)(s - p_2) = s^2 + \frac{\sqrt{2}}{\tau_c} s + \frac{1}{\tau_c^2}$$

State-feedback for model with physical states

Characteristic polynomial obtained with state feedback

$$s^2 + \left(\frac{1}{\tau} + l_1 \frac{k}{\tau}\right)s + l_2 \frac{k}{\tau}$$

Desired characteristic polynomial

$$s^2 + \frac{\sqrt{2}}{\tau_c}s + \frac{1}{\tau_c^2}$$

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Activity Determine the feedback gains!

State-feedback for model with physical states

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Activity Determine the feedback gains!

Solution

State-feedback for model with physical states

Characteristic polynomial obtained with state feedback

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Desired characteristic polynomial

$$s^2 + \frac{\sqrt{2}}{\tau_c}s + \frac{1}{\tau_c^2}$$

Activity Determine the feedback gains!

Solution

$$l_1 = \frac{\tau}{k} \left(\frac{\sqrt{2}}{\tau_c} - \frac{1}{\tau} \right) = \frac{\sqrt{2}\tau - \tau_c}{k\tau_c}$$
$$l_2 = \frac{\tau}{k\tau_c^2}$$

Observer for the model with physical states

The observer is given by

$$\begin{aligned}\dot{\hat{x}} &= \underbrace{A\hat{x} + Bu}_{\text{simulation}} + \underbrace{K(y - C\hat{x})}_{\text{correction}} \\ &= (A - KC)\hat{x} + Bu + Ky \\ &= \begin{bmatrix} -\frac{1}{\tau} & -k_1 \\ 1 & -k_2 \end{bmatrix} \hat{x} + Bu + Ky\end{aligned}$$

with poles given by the eigenvalues of the matrix $A_o = A - KC$, which has characteristic polynomial

$$\begin{aligned}\det \left(\begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} -\frac{1}{\tau} & -k_1 \\ 1 & -k_2 \end{bmatrix} \right) \\ = s^2 + \left(\frac{1}{\tau} + k_2 \right) s + \left(k_1 + \frac{k_2}{\tau} \right)\end{aligned}$$

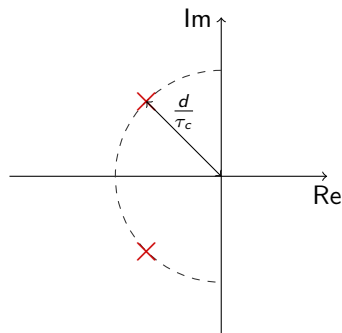
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Choose d between 2 and 10 and obtain desired closed-loop characteristic polynomial

$$(s - p_{o,1})(s - p_{o,2}) = s^2 + \frac{d\sqrt{2}}{\tau_c}s + \frac{d^2}{\tau_c^2}$$

Observer for the model with physical states

Characteristic polynomial of the observer

$$s^2 + \left(\frac{1}{\tau} + k_2\right)s + \left(k_1 + \frac{k_2}{\tau}\right)$$

Desired characteristic polynomial

$$s^2 + \frac{\sqrt{2}d}{\tau_c}s + \frac{d^2}{\tau_c^2}$$

Observer for the model with physical states

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Activity Determine the observer gains!

Observer for the model with physical states

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Activity Determine the observer gains!

Solution

Observer for the model with physical states

Characteristic polynomial of the observer

$$s^2 + \left(\frac{1}{\tau} + k_2\right)s + \left(k_1 + \frac{k_2}{\tau}\right)$$

Desired characteristic polynomial

$$s^2 + \frac{\sqrt{2}d}{\tau_c}s + \frac{d^2}{\tau_c^2}$$

Activity Determine the observer gains!

Solution

$$k_1 = \frac{d^2}{\tau_c^2} - \frac{\sqrt{2}d}{\tau_c\tau} + \frac{1}{\tau^2}$$

$$k_2 = \frac{\sqrt{2}d}{\tau_c} - \frac{1}{\tau}$$