

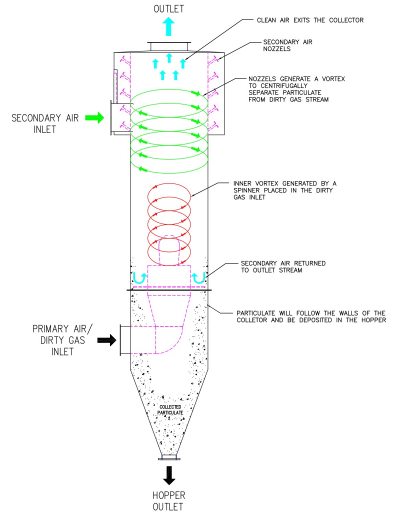
System identification

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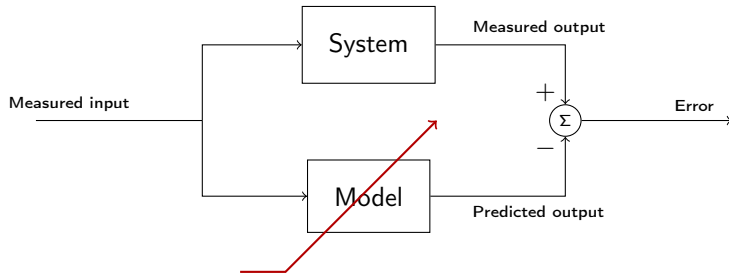
November 14, 2022

A complicated process

From Wikipedia "Cyclonic separation"



System identification



The shift operator q

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Algebra with the shift operator

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Algebra with the shift operator

$$\begin{aligned} \frac{q - q^{-1}}{2h} x(k) &= \frac{1}{2h} (q x(k) - q^{-1} x(k)) = ? \\ &= \frac{x(k+1) - x(k-1)}{2h} \end{aligned}$$

The Auto-Regressive with eXogenous input (ARX) model

$$A(q)y(k) = B(q)u(k) + e(k + n)$$

The error signal $e(k)$ is a zero-mean white noise sequence representing perturbations and modeling errors.

First-order ARX model with one delay

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$$y(k + 1) = -a_1y(k) + b_0u(k) + b_1u(k - 1) + e(k + 1)$$

Using the model to predict the output one step ahead:

$$\begin{aligned}\hat{y}(k + 1) &= -a_1y(k) + b_0u(k) + b_1u(k - 1) = \underbrace{\begin{bmatrix} -y(k) & u(k) & u(k - 1) \end{bmatrix}}_{\varphi_{k+1}^T} \underbrace{\begin{bmatrix} a_1 \\ b_0 \\ b_1 \end{bmatrix}}_{\theta} \\ &= \varphi_{k+1}^T \theta\end{aligned}$$

Parameter estimation - Least squares

Objective Given observations

$$\mathcal{D} = \{(u_1, y_1), (u_2, y_2), \dots, (u_N, y_N)\}$$

and model $\mathcal{M} : y(k+1) = -ay(k) + b_0u(k) + b_1u(k-1) + e(k+1)$, obtain the parameters (a_1, b_0, b_1) which gives the best fit of the model to the data.

Parameter estimation - Least squares

Given observations

$$\mathcal{D} = \{(u_1, y_1), (u_2, y_2), \dots, (u_N, y_N)\}$$

and model \mathcal{M} : $y(k+1) = -ay(k) + b_0u(k) + b_1u(k-1) + e(k+1)$.

1. Form the one-step ahead prediction

$$\hat{y}_{k+1} = -a_1y_k + b_0u_k + b_1u_{k-1} = \underbrace{\begin{bmatrix} -y_k & u_k & u_{k-1} \end{bmatrix}}_{\varphi_{k+1}^T} \underbrace{\begin{bmatrix} a_1 \\ b_0 \\ b_1 \end{bmatrix}}_{\theta}$$

and the prediction error

$$\epsilon_{k+1} = y_{k+1} - \hat{y}_{k+1} = y_{k+1} - \varphi_{k+1}^T \theta.$$

Parameter estimation - Least squares

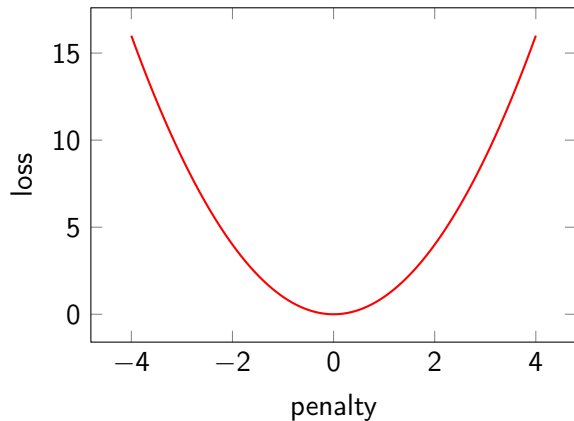
2. Combine all the observations y_k and predictions \hat{y}_k on vector form

$$\begin{aligned}\epsilon &= \begin{bmatrix} \epsilon_3 \\ \epsilon_4 \\ \vdots \\ \epsilon_N \end{bmatrix} = \begin{bmatrix} y_3 \\ y_4 \\ \vdots \\ y_N \end{bmatrix} - \begin{bmatrix} \hat{y}_3 \\ \hat{y}_4 \\ \vdots \\ \hat{y}_N \end{bmatrix} = \begin{bmatrix} y_3 \\ y_4 \\ \vdots \\ y_N \end{bmatrix} - \begin{bmatrix} \varphi_3^T \theta \\ \varphi_4^T \theta \\ \vdots \\ \varphi_N^T \theta \end{bmatrix} \\ &= y - \underbrace{\begin{bmatrix} \varphi_3^T \\ \varphi_4^T \\ \vdots \\ \varphi_N^T \end{bmatrix}}_{\Phi} \theta = y - \Phi \theta\end{aligned}$$

3. Solve $\arg \min J(\theta) = \frac{1}{2} \epsilon^T \epsilon = \frac{1}{2} \sum_{i=3}^N \epsilon_i(\theta)^2 \Rightarrow \hat{\theta}_{LS} = (\Phi^T \Phi)^{-1} \Phi^T y$

The problem with least squares

$$\begin{aligned} &\text{minimize } \sum_k g(\epsilon_k) \\ &\text{where } g(u) = u^2 \end{aligned}$$

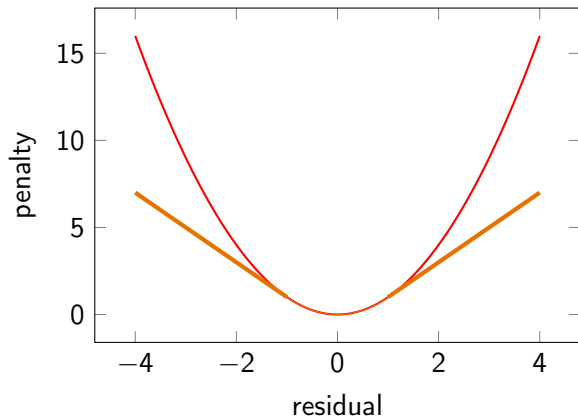


More robust: The Huber loss function

Also known as **robust regression**

$$\text{minimize } \sum_k g_{hub}(\epsilon_k)$$

$$\text{where } g_{hub}(u) = \begin{cases} u^2 & |u| \leq M \\ M(2|u| - M) & |u| > M \end{cases}$$



First-order ARX model without delay

$$(q + a_1)y(k) = (b_0 q + b_1)u(k) + e(k + 1)$$

Activity

1. Determine the one-step ahead predictor \hat{y}_{k+1} and the prediction error ϵ_{k+1} .
2. Form the system of equations $\Phi\theta = y$

The ARX model

$$A(q)y(k) = B(q)u(k) + e(k + n)$$

Activity Fill the empty blocks.

