Controllability and observability

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The concept of state

State The information needed about the history of a dynamical system in order to determine the future behaviour of the system given future input signals.

Formas canonicas

- Forma controlable
- Forma observable

Recurso
https://lpsa.swarthmore.edu/Representations/SysRepTransformations/TF2SS.html

Estabilidad

La solución homógena de $\dot{x} = Ax$ se puede escribir

$$x(t) = \mathrm{e}^{\lambda_1 t}\alpha_1 v_1 + \mathrm{e}^{\lambda_2 t}\alpha_2 v_2 + \cdots + \mathrm{e}^{\lambda_n t}\alpha_n v_n,$$

donde λ_1, λ_2, . . ., λ_n son los eigenvalores de A.

Estabilidad requiere que cada una de las funciones exponenciales va hacia cero.

$$\Rightarrow \qquad \mathrm{Re}\{\lambda_i\} < 0, \ \forall i = 1, 2, 3 \ldots, n$$

Los eigenvalores de A son los polos del sistema.

Los eigenvalores

λ y v es un par de eigenvalor y eigenvector de la matriz A si

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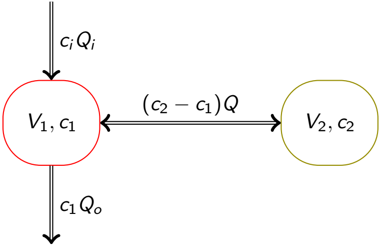
Para que la ecuación tenga soluciones no-triviales

$$\det(\lambda I - A) = 0 \quad \leftarrow \text{Ecuación característica}$$

De espacio de estados a función de transferencia

Modelo compartimental



$$
\begin{aligned}
V_1 \frac{dc_1}{dt} &= Q(c_2 - c_1) - Q_o c_1 + Q_i c_i, && c_1 \geq 0 \\
V_2 \frac{dc_2}{dt} &= Q(c_1 - c_2), && c_2 \geq 0,
\end{aligned}
$$

$$
\begin{aligned}
\dot{x} &= \overbrace{\begin{bmatrix} -\frac{Q+Q_o}{V_1} & \frac{Q}{V_1} \\ \frac{Q}{V_2} & -\frac{Q}{V_2} \end{bmatrix}}^{A} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \overbrace{\begin{bmatrix} \frac{1}{V_1} \\ 0 \end{bmatrix}}^{B} u \\
y &= \underbrace{\begin{bmatrix} 1 & 0 \end{bmatrix}}_{C} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}
\end{aligned}
$$

De espacio de estados a función de transferencia

$$\dot{x} = \overbrace{\begin{bmatrix} -\frac{Q+Q_o}{V_1} & \frac{Q}{V_1} \\ \frac{Q}{V_2} & -\frac{Q}{V_2} \end{bmatrix}}^{A} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \overbrace{\begin{bmatrix} \frac{1}{V_1} \\ 0 \end{bmatrix}}^{B} u = Ax + Bu$$

$$y = \underbrace{\begin{bmatrix} 1 & 0 \end{bmatrix}}_{C} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = Cx$$

Aplicando la transformada de Laplace

$$sX - x(0) = AX + BU$$
$$Y = CX$$

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Aplicando la transformada de Laplace

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Despejando $X(s)$

$$X(s) = (sI - A)^{-1}x(0) + (sI - A)^{-1}BU(s)$$
$$Y(s) = C\left((sI - A)^{-1}x(0) + (sI - A)^{-1}BU(s)\right)$$
$$= \underbrace{C(sI - A)^{-1}x(0)}_{\text{Respuesta transitoria}} + \underbrace{C(sI - A)^{-1}B}_{\text{Función de transf.}}\, U(s)$$

Transformada de Laplace de una función exponencial

$$F(s) = \mathcal{L}\left\{f(t)\right\} = \int_0^\infty f(t)\mathrm{e}^{-st}dt$$

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$$\mathcal{L}\left\{\mathrm{e}^{pt}\right\} = \int_0^\infty \mathrm{e}^{pt}\mathrm{e}^{-st}dt = \int_0^\infty \mathrm{e}^{-(s-p)t}dt = \frac{1}{s-p} = (s-p)^{-1}, \quad \mathrm{Re}\{s\} > \mathrm{Re}\{p\}$$

Solución homógena de sistemas en espacio de estados

$$\begin{aligned} \dot{x} &= Ax, \qquad x(0) = x_0 \\ sX(s) - x(0) &= AX(s) \end{aligned}$$

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Solución en dominio de Laplace

$$X(s) = (sI - A)^{-1}x(0)$$

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Solución en dominio de tiempo

$$x(t) = \Phi(t)x(0) = \mathrm{e}^{At}x(0)$$

Donde $\Phi : \mathbb{R} \to \mathbb{R}^{n \times n}$

$$\Phi(t) = \mathrm{e}^{At} = I + tA + \frac{t^2}{2!}A^2 + \frac{t^3}{3!}A^3 + \cdots$$

Solución en dominio de Laplace

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La transformada de Laplace de la exponencial de una matriz

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$$(sI - A)^{-1} = \frac{1}{\det(sI - A)} \operatorname{adj}(sI - A)$$

$\det(sI - A)$ es un polinomio en s, llamado polinomio característico. Sus raíces, es decir las soluciones de la ecuación característica

$$\det(sI - A) = 0$$

son los polos del sistema y los eigenvalores de A.

De espacio de estados a función de transferencia

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$$X(s) = (sI - A)^{-1} x(0) + (sI - A)^{-1} BU(s)$$

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Modelling example

State-space model and transfer function

An nth order system with the state-space representation

$$\begin{cases} \dot{x}(t) & = Ax(t) + Bu(t) \\ y(t) & = Cx(t) + Du(t) \end{cases} \qquad (*)$$

has the transfer function

$$G(s) = C(sI - A)^{-1}B + D = \frac{b(s)}{a(s)}$$

from the input signal u to the output signal y. The denominator polynominal is $a(s) = \det(sI - A)$ and is of order n. If $D \neq 0$ then the numerator polynomial $b(s)$ is of order n, if $D = 0$ then $b(s)$ is of order $\leq n - 1$.

The state-space representation is not unique

Make a change of state variables in the system $(*)$: Let $x = Tz \Leftrightarrow z = T^{-1}x$, where T is an invertible (non-singular) matrix. Then

$$\begin{aligned} \dot{z} &= T^{-1}ATz + T^{-1}Bu \\ y &= CTz + Du \end{aligned}$$

with transfer function

$$G(s) = CT\left(sI - T^{-1}AT\right)^{-1}T^{-1}B + D = C(sI - A)^{-1}B + D$$

This means that a system with transfer function $G(s)$ has infinitely many different state-space representations.

Stability

A system

$$\dot{x} = Ax, \quad x(0) = x_0 \quad \text{(i.e. the system } (*) \text{ with } u \equiv 0)$$

is *asymptotically stable* if $\lim_{t \to \infty} x(t) = 0$ for all $x_0 \in \mathbb{R}^n$.

A system is asymptotically stable if and only if all eigenvalues of the A-matrix have strict negative real parts (are strictly in the LHP).

Different notions of stability

- BIBO stability (Bounded Input - Bounded Output) refers to the effect of the input signal $u(t)$ on the output signal $y(t)$, ignoring the initial state x_0.
- Asymptotic stability refers to the effect of x_0 on the state vector $x(t)$

(it is assumed that $u(t) \equiv 0$, y is ignored).

Connections:

- A system is BIBO stable if it is asymptotically stable.

Controllable/Reachable canonical form

The system with transfer function

$$G(s) = \frac{b_1 s^{n-1} + \cdots + b_{n-1}s + b_n}{s^n + a_1 s^{n-1} + \cdots + a_{n-1}s + a_n}$$

can be represented on state-space form as

$$\dot{x} = \begin{bmatrix} -a_1 & -a_2 & \cdots & -a_{n-1} & -a_n \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} u$$

$$y = \begin{bmatrix} b_1 & b_2 & \cdots & b_n \end{bmatrix} x$$

Controllability/Reachability

A state vector $x^* \in \mathbb{R}^n$ is *reachable* (or *controllable*) if there exists an input signal $u(t)$ such that $x(T) = x^*$, for some $T < \infty$, when $x(0) = 0$. If all $x^* \in \mathbb{R}^n$ are controllable, then the *system (∗)* is controllable.

A system on controllable canonical form is controllable

$$\dot{x}(t) = \begin{bmatrix} -2 & -1 \\ 1 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(t)$$

$$y(t) = \begin{bmatrix} 1 & 1 \end{bmatrix} x(t)$$

$$G(s) = \frac{s+1}{s^2+2s+1} = \frac{s+1}{(s+1)^2} = \frac{1}{s+1}$$

$$\mathcal{C} = \begin{bmatrix} B & AB \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix} \quad \Rightarrow \quad \det \mathcal{C} \neq 0 \quad \Leftrightarrow \quad \text{controllable}$$