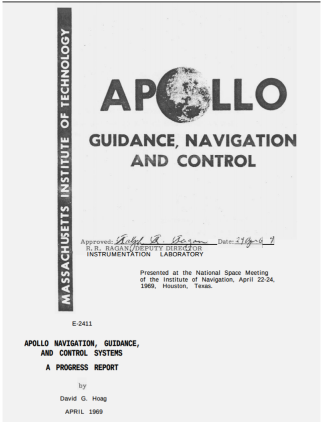
From continuous-time to discrete-time systems

Kjartan Halvorsen

November 4, 2022

The computerized control system of the Apollo LM



MASSACHUSETTS INSTITUTE OF TECHNOLOGY

APOLLO

GUIDANCE, NAVIGATION AND CONTROL

Approved: Date:
R. R. RAGAN, DEPUTY DIRECTOR
INSTRUMENTATION LABORATORY

Presented at the National Space Meeting of the Institute of Navigation, April 22-24, 1969, Houston, Texas.

E-2411

APOLLO NAVIGATION, GUIDANCE, AND CONTROL SYSTEMS

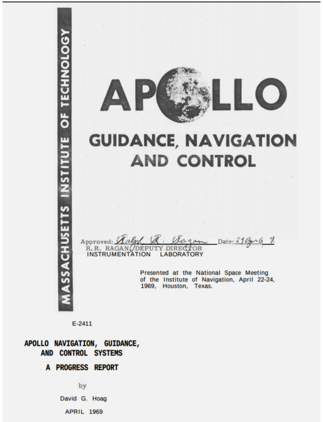
A PROGRESS REPORT

by

David G. Hoag

APRIL 1969

The computerized control system of the Apollo LM


MASSACHUSETTS INSTITUTE OF TECHNOLOGY

APOLLO

GUIDANCE, NAVIGATION AND CONTROL

Approved: Ralph R. Ragan Date: 29 Apr 69
R. R. RAGAN, DEPUTY DIRECTOR
INSTRUMENTATION LABORATORY

Presented at the National Space Meeting of the Institute of Navigation, April 22-24, 1969, Houston, Texas.

E-2411

APOLLO NAVIGATION, GUIDANCE, AND CONTROL SYSTEMS

A PROGRESS REPORT

by

David G. Hoag

APRIL 1969

The original attempt at these autopilot designs was a conventional analog system approach. But in 1964, NASA wisely made the decision to incorporate these autopilots into the CM and LM digital computers. It was easily demonstrated that a direct digital equivalent of the signal processing of the analog autopilot design candidates would not work. Sampling rates would have to be too high in relation to the data processing speed of the computer. Success depended upon design approaches adaptable to the nature of digital processing and capitalizing upon the flexibility and nonlinear computations easily and directly available in a digital computer.

The computerized control system of the Apollo LM

MASSACHUSETTS INSTITUTE OF TECHNOLOGY

APOLLO

GUIDANCE, NAVIGATION AND CONTROL

Approved: Date:
R. R. RAGAN, DEPUTY DIRECTOR
INSTRUMENTATION LABORATORY

Presented at the National Space Meeting of the Institute of Navigation, April 22-24, 1969, Houston, Texas.

E-2411

APOLLO NAVIGATION, GUIDANCE, AND CONTROL SYSTEMS

A PROGRESS REPORT

by

David G. Hoag

APRIL 1969

The original attempt at these autopilot designs was a conventional analog system approach. But in 1964, NASA wisely made the decision to incorporate these autopilots into the CM and LM digital computers. It was easily demonstrated that a direct digital equivalent of the signal processing of the analog autopilot design candidates would not work. Sampling rates would have to be too high in relation to the data processing speed of the computer. Success depended upon design approaches adaptable to the nature of digital processing and capitalizing upon the flexibility and nonlinear computations easily and directly available in a digital computer.

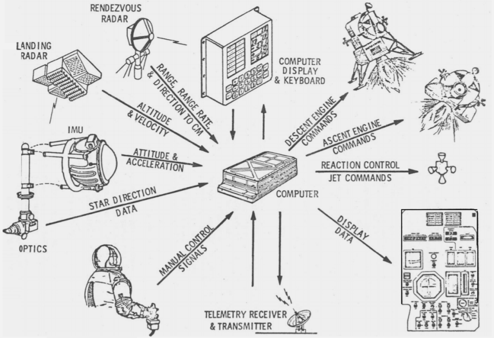


Figure 2 The Lunar Module GN & C System Schematic

Discrete-time systems

The world according to the discrete-time controller

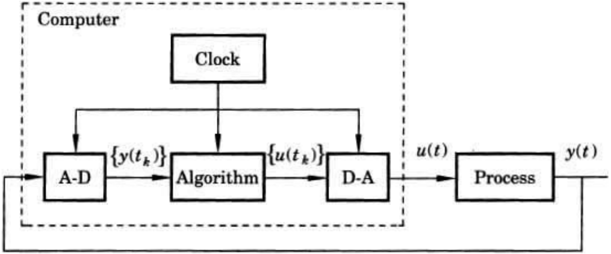


Figure 1.1 Schematic diagram of a computer-controlled system.

Source: Åström and Wittenmark "Computer-controlled systems".

The world according to the discrete-time controller

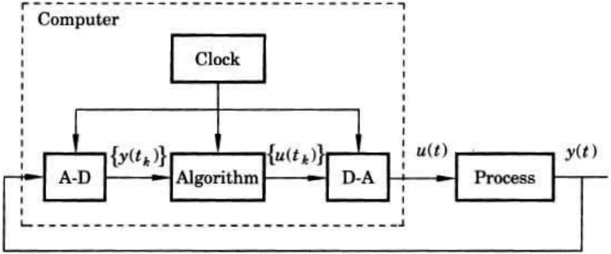
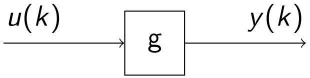


Figure 1.1 Schematic diagram of a computer-controlled system.

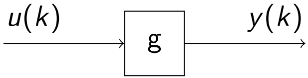
Source: Åström and Wittenmark "Computer-controlled systems".

The sampling leading to the *stroboscopic* model can have some peculiar effects: `https://youtu.be/yIUZ-qKWnXc`

Discrete-time Linear Shift-Invariant systems



Discrete-time Linear Shift-Invariant systems



General case (non-causal)

$$y(k) = g * u = \sum_{n=-\infty}^{\infty} g(n)u(k-n)$$

Discrete-time Linear Shift-Invariant systems

$$\xrightarrow{u(k)} \boxed{g} \xrightarrow{y(k)}$$

General case (non-causal)

$$y(k) = g * u = \sum_{n=-\infty}^{\infty} g(n)u(k-n)$$

Causal case

$$y(k) = g * u = \sum_{n=0}^{\infty} g(n)u(k-n)$$

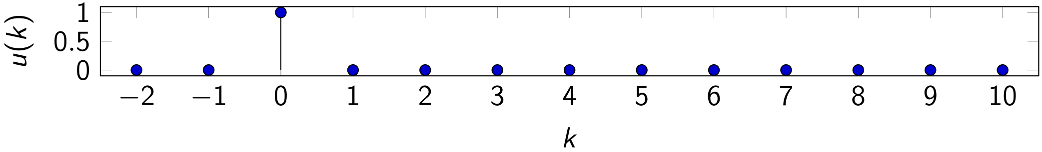
$g(k)$ is called the weighting sequence.

Discrete-time LSI systems

Impulse response

$$y(k) = g * u = \sum_{n=0}^{\infty} g(n)u(k-n)$$

If the input signal is a unit discrete impulse

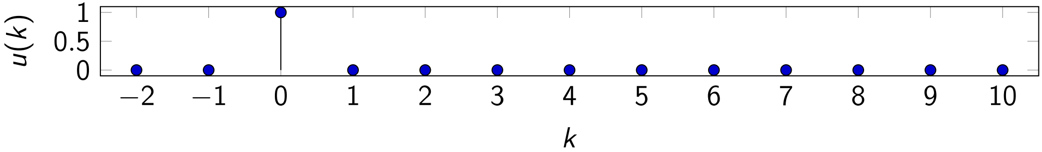


Discrete-time LSI systems

Impulse response

$$y(k) = g * u = \sum_{n=0}^{\infty} g(n)u(k-n)$$

If the input signal is a unit discrete impulse



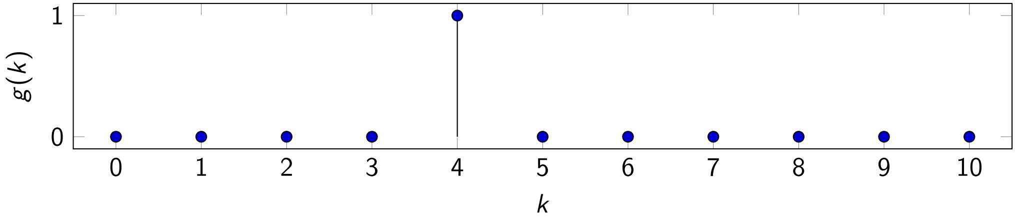
$$y(k) = \sum_{n=0}^{\infty} g(n)\delta(k-n) = g(k)$$

The weighting sequence is equal to the impulse response of the system.

The response of a discrete LSI system is a weighted sum of the current and previous values of the input

$$y(k) = g * u = \sum_{n=0}^{\infty} g(n)u(k-n)$$

Activity What is the response of a system to the input signal $u(k)$ if its impulse response (weighting sequence) is the one below, $g(k) = \delta(k-4)$?



$$y(k) =$$

The z-transform

The Laplace transform

Definition

$$F(s) = \mathcal{L}\left\{f(t)\right\} = \int_0^\infty f(t)\mathrm{e}^{-st} dt$$

Inverse transform

$$f(t) = \mathcal{L}^{-1}\{F(s)\} = \frac{1}{2\pi i}\int_{\gamma - i\infty}^{\gamma + i\infty} F(s)\mathrm{e}^{st}\, ds$$

Note that in control engineering, the one-sided transform is used.

The z-transform

Definition

$$F(z) = \mathcal{Z}\left\{f(kh)\right\} = \sum_{k=0}^{\infty} f(kh) z^{-k}$$

Inverse transform

$$f(kh) = \frac{1}{2\pi i} \oint_r F(z) z^{k-1}\, dz$$

Note that in control engineering, the one-sided transform is used.

One of the most important transform pairs

$$f(kh) = \alpha^{kh}, \quad \alpha \in \mathbb{C}$$

One of the most important transform pairs

$$f(kh) = \alpha^{kh}, \quad \alpha \in \mathbb{C}$$

$$F(z) = \mathcal{Z}\left\{f(kh)\right\} = \sum_{k=0}^{\infty} f(kh) z^{-k} = \sum_{k=0}^{\infty} \alpha^{kh} z^{-k} = \sum_{k=0}^{\infty} \left(\alpha^{h}\right)^{k} z^{-k}$$

$$= \sum_{k=0}^{\infty} \left(\frac{\alpha^h}{z}\right)^{k} = \frac{1}{1 - \frac{\alpha^h}{z}} = \frac{z}{z - \alpha^h}, \quad |\frac{\alpha^h}{z}| < 1$$

One of the most important transform pairs

$$f(kh) = \alpha^{kh}, \quad \alpha \in \mathbb{C}$$

$$F(z) = \mathcal{Z}\left\{f(kh)\right\} = \sum_{k=0}^{\infty} f(kh) z^{-k} = \sum_{k=0}^{\infty} \alpha^{kh} z^{-k} = \sum_{k=0}^{\infty} \left(\alpha^{h}\right)^{k} z^{-k}$$

$$= \sum_{k=0}^{\infty} \left(\frac{\alpha^{h}}{z}\right)^{k} = \frac{1}{1 - \frac{\alpha^{h}}{z}} = \frac{z}{z - \alpha^{h}}, \quad |\frac{\alpha^{h}}{z}| < 1$$

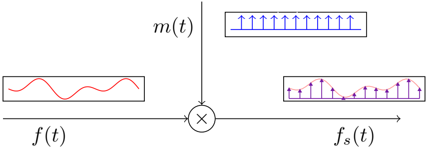
$$\boxed{\alpha^{kh} \quad \overset{\mathcal{Z}}{\longleftrightarrow} \quad \frac{z}{z - \alpha^{h}}}$$

The impulse modulation model of sampling

The impulse train, a.k.a the Dirac comb: $m(t) = \sum_{k=-\infty}^{\infty} \delta(t - kh)$

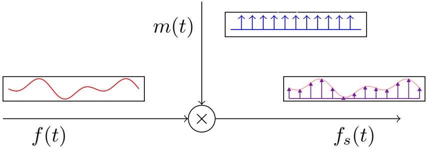
The impulse modulation model of sampling

The impulse train, a.k.a the Dirac comb: $m(t) = \sum_{k=-\infty}^{\infty} \delta(t - kh)$

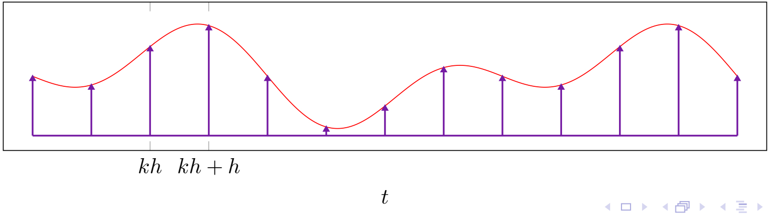


The impulse modulation model of sampling

The impulse train, a.k.a the Dirac comb: $m(t) = \sum_{k=-\infty}^{\infty} \delta(t - kh)$



$$f_s(t) = f(t)m(t) = f(t) \sum_{k=-\infty}^{\infty} \delta(t - kh) = \sum_{k=-\infty}^{\infty} f(t)\delta(t - kh) = \sum_{k=-\infty}^{\infty} f(kh)\delta(t - kh)$$



The Laplace transform of a sampled signal

Assume right-sided signal $f(t)$, meaning it is zero for negative times $t < 0$.

$$f_s(t) = f(t)m(t) = f(t)\sum_{k=0}^{\infty}\delta(t-kh) = \sum_{k=0}^{\infty} f(t)\delta(t-kh) = \sum_{k=0}^{\infty} f(kh)\delta(t-kh)$$

The Laplace transform of a sampled signal

Assume right-sided signal $f(t)$, meaning it is zero for negative times $t < 0$.

$$f_s(t) = f(t)m(t) = f(t)\sum_{k=0}^{\infty}\delta(t-kh) = \sum_{k=0}^{\infty} f(t)\delta(t-kh) = \sum_{k=0}^{\infty} f(kh)\delta(t-kh)$$

$$\begin{aligned} F_s(s) = \mathcal{L}\left\{f_s(t)\right\} &= \int_0^{\infty}\left(\sum_{k=0}^{\infty} f(kh)\delta(t-kh)\right)\mathrm{e}^{-st}\,dt \\ &= \sum_{k=0}^{\infty}\int_0^{\infty} f(kh)\delta(t-kh)\mathrm{e}^{-st}\,dt = \sum_{k=0}^{\infty} f(kh)\mathrm{e}^{-skh} \\ &= \sum_{k=0}^{\infty} f(kh)\left(\mathrm{e}^{sh}\right)^{-k} \end{aligned}$$

The Laplace transform of a sampled signal

Note:

$$F_s(s) = \sum_{k=0}^{\infty} f(kh) \left(\mathrm{e}^{sh}\right)^{-k} \quad \text{Laplace transform}$$

$$F(z) = \sum_{k=0}^{\infty} f(kh) z^{-k} \quad \text{z-transform}$$

Activity How is the Laplace transform of the sampled signal and the z-transform of the corresponding discrete-time sequence related?

The Laplace transform of a sampled signal

Note:

$$F_s(s) = \sum_{k=0}^{\infty} f(kh) \left(\mathrm{e}^{sh}\right)^{-k} \quad \text{Laplace transform}$$

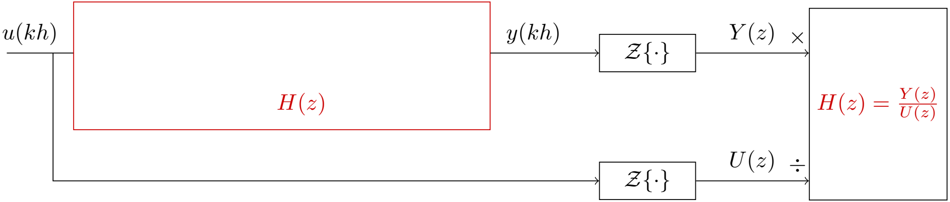
$$F(z) = \sum_{k=0}^{\infty} f(kh) z^{-k} \quad \text{z-transform}$$

Activity How is the Laplace transform of the sampled signal and the z-transform of the corresponding discrete-time sequence related?

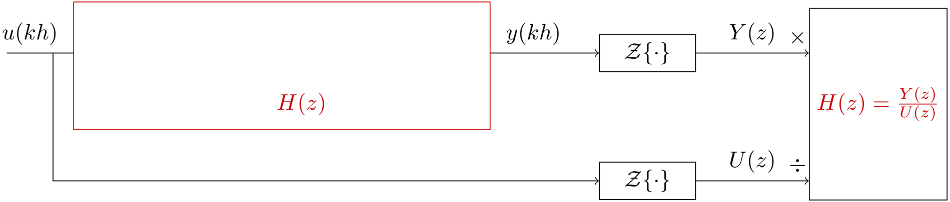
> The z-transform of a sampled signal corresponds to its Laplace transform with the following relationship between the s-plane of the Laplace transform and the z-plane of the z-plane of the z-transform.
>
> $$z = \mathrm{e}^{sh}$$

Step-invariant sampling (a.k.a ZOH sampling)

Step-invariant sampling (a.k.a ZOH sampling)



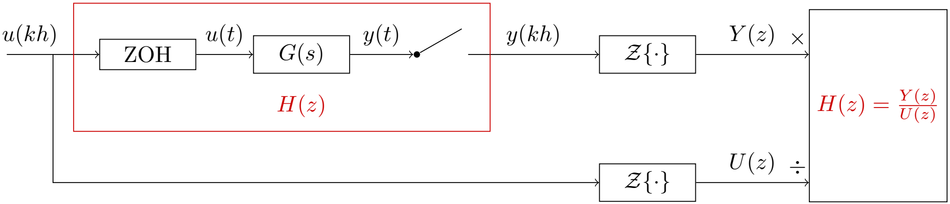
Step-invariant sampling (a.k.a ZOH sampling)



Step-invariant sampling (zero order hold): $u(kh) = \begin{cases} 1, & k \geq 0 \\ 0, & k < 0 \end{cases}$

Step-invariant sampling (a.k.a ZOH sampling)

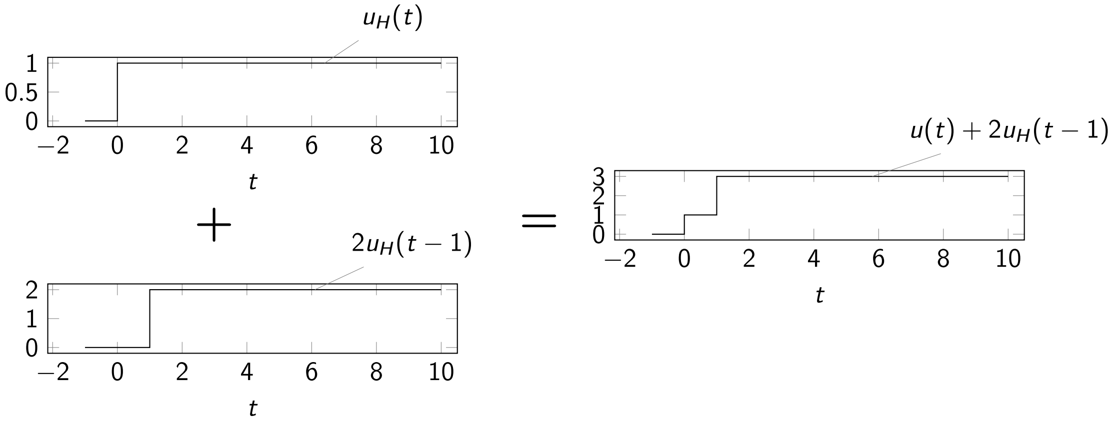
The idea is to sample the continuous-time system's response to a step input, in order to obtain a discrete approximation which is exact (at the sampling instants) for such an input signal.



Step-invariant sampling (zero order hold): $u(kh) = \begin{cases} 1, & k \geq 0 \\ 0, & k < 0 \end{cases}$

Why is step-invariant sampling a good idea?

A piecewise constant (stair-case shaped) function can be written as a sum of delayed step-responses!



Step-invariant sampling, or zero-order-hold sampling

Let the input to the continuous-time system be a unit step $u(t) = u_H(t)$, which has Laplace transform $U(s) = \frac{1}{s}$. In the Laplace-domain we get

$$Y(s) = G(s)\frac{1}{s}$$

1. Obtain the time-response by inverse Laplace: $y(t) = \mathcal{L}^{-1}\{Y(s)\}$
2. Sample the time-response to obtain the sequence $y(kh)$ and apply the z-transform to obtain $Y(z) = \mathcal{Z}\{y(kh)\}$
3. Calculate the pulse-transfer function by dividing with the z-transform of the input signal $U(z) = \frac{z}{z-1}$.

$$H(z) = \frac{Y(z)}{U(z)} = \frac{z-1}{z}Y(z)$$

Example: First-order system

$$G(s) = \frac{1}{\tau s + 1}.$$

Example: First-order system

$$G(s) = \frac{1}{\tau s + 1}.$$

1. Step response: $y(t) = \left(1 - \mathrm{e}^{-\frac{t}{\tau}}\right) u_H(t)$

Example: First-order system

$$G(s) = \frac{1}{\tau s + 1}.$$

1. Step response: $y(t) = \big(1 - \mathrm{e}^{-\frac{t}{\tau}}\big) u_H(t)$
2. Sampling and applying the z-transform:

$$y(kh) = \big(1 - \mathrm{e}^{-\frac{kh}{\tau}}\big) u_H(kh) = u_H(kh) - \big(\mathrm{e}^{-\frac{h}{\tau}}\big)^k u_H(kh)$$

Example: First-order system

$$G(s) = \frac{1}{\tau s + 1}.$$

1. Step response: $y(t) = \big(1 - \mathrm{e}^{-\frac{t}{\tau}}\big) u_H(t)$

2. Sampling and applying the z-transform:

$$y(kh) = \big(1 - \mathrm{e}^{-\frac{kh}{\tau}}\big) u_H(kh) = u_H(kh) - \big(\mathrm{e}^{-\frac{h}{\tau}}\big)^k u_H(kh)$$

$$Y(z) = \frac{z}{z-1} - \frac{z}{z - \mathrm{e}^{-\frac{h}{\tau}}} = \frac{z\big(z - \mathrm{e}^{-\frac{h}{\tau}} - (z-1)\big)}{(z-1)(z - \mathrm{e}^{-\frac{h}{\tau}})} = \frac{z(1 - \mathrm{e}^{-\frac{h}{\tau}})}{(z-1)(z - \mathrm{e}^{-\frac{h}{\tau}})}$$

Example: First-order system

$$G(s) = \frac{1}{\tau s + 1}.$$

1. Step response: $y(t) = \big(1 - \mathrm{e}^{-\frac{t}{\tau}}\big)u_H(t)$
2. Sampling and applying the z-transform:

$$y(kh) = \big(1 - \mathrm{e}^{-\frac{kh}{\tau}}\big)u_H(kh) = u_H(kh) - \big(\mathrm{e}^{-\frac{h}{\tau}}\big)^k u_H(kh)$$

$$Y(z) = \frac{z}{z-1} - \frac{z}{z - \mathrm{e}^{-\frac{h}{\tau}}} = \frac{z\big(z - \mathrm{e}^{-\frac{h}{\tau}} - (z-1)\big)}{(z-1)(z - \mathrm{e}^{-\frac{h}{\tau}})} = \frac{z(1 - \mathrm{e}^{-\frac{h}{\tau}})}{(z-1)(z - \mathrm{e}^{-\frac{h}{\tau}})}$$

3. Calculate the pulse-transfer function

$$H(z) = \frac{Y(z)}{U(z)} = \frac{z-1}{z} \cdot \frac{z(1 - \mathrm{e}^{-\frac{h}{\tau}})}{(z-1)(z - \mathrm{e}^{-\frac{h}{\tau}})} = \frac{1 - \mathrm{e}^{-\frac{h}{\tau}}}{z - \mathrm{e}^{-\frac{h}{\tau}}}$$