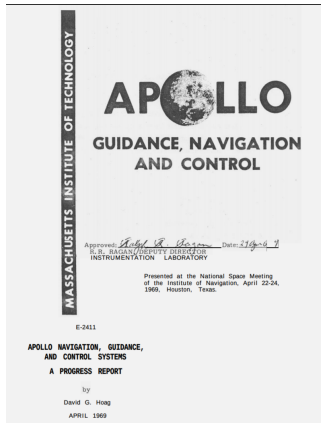


# From continuous-time to discrete-time systems

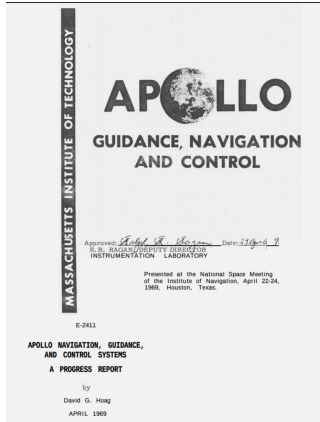
Kjartan Halvorsen

November 5, 2021

# The computerized control system of the Apollo LM

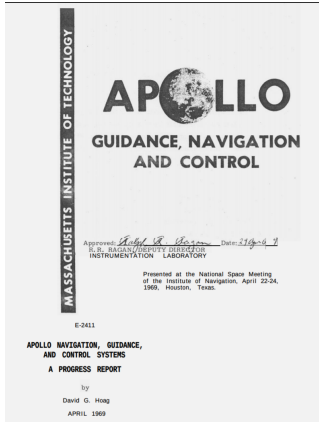


# The computerized control system of the Apollo LM



The original attempt at these autopilot designs was a conventional analog system approach. But in 1964, NASA wisely made the decision to incorporate these autopilots into the CM and LM digital computers. It was easily demonstrated that a direct digital equivalent of the signal processing of the analog autopilot design candidates would not work. Sampling rates would have to be too high in relation to the data processing speed of the computer. Success depended upon design approaches adaptable to the nature of digital processing and capitalizing upon the flexibility and nonlinear computations easily and directly available in a digital computer.

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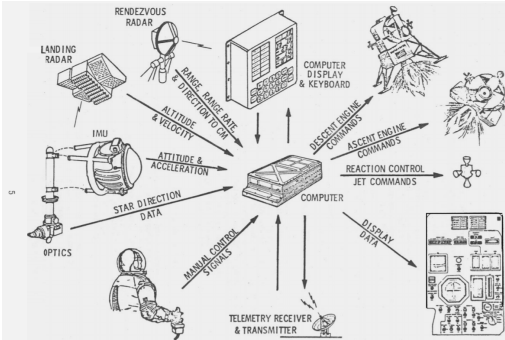
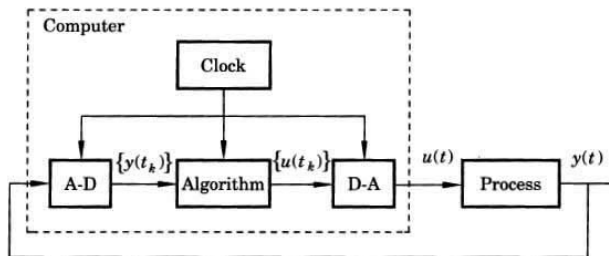


Figure 2 The Lunar Module GN & C System Schematic

# Discrete-time systems

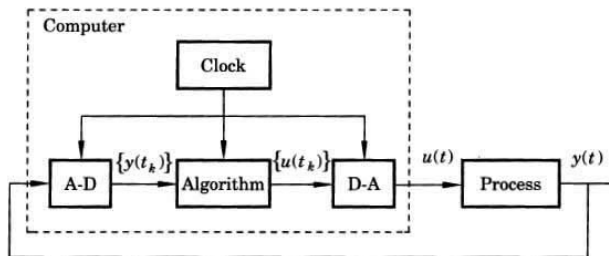
# The world according to the discrete-time controller



**Figure 1.1** Schematic diagram of a computer-controlled system.

Source: Åström and Wittenmark "Computer-controlled systems".

# The world according to the discrete-time controller



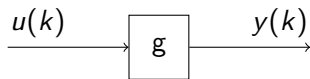
**Figure 1.1** Schematic diagram of a computer-controlled system.

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The sampling leading to the *stroboscopic* model can have some peculiar effects:

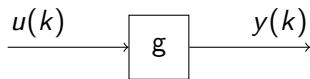
<https://youtu.be/yIUZ-qKWnXc>

## Discrete-time Linear Shift-Invariant systems





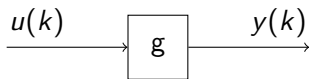
# Discrete-time Linear Shift-Invariant systems



General case (non-causal)

$$y(k) = g * u = \sum_{n=-\infty}^{\infty} g(n)u(k-n)$$

# Discrete-time Linear Shift-Invariant systems



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Causal case

$$y(k) = g * u = \sum_{n=0}^{\infty} g(n)u(k-n)$$

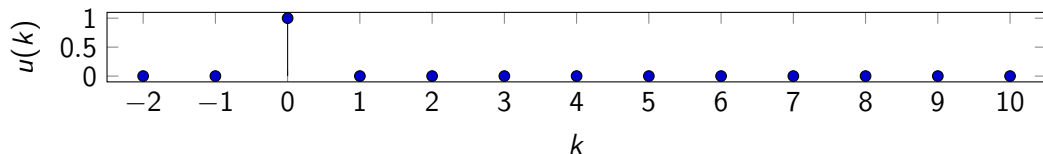
$g(k)$  is called the **weighting sequence**.

# Discrete-time LSI systems

## Impulse response

$$y(k) = g * u = \sum_{n=0}^{\infty} g(n)u(k-n)$$

If the input signal is a unit discrete impulse

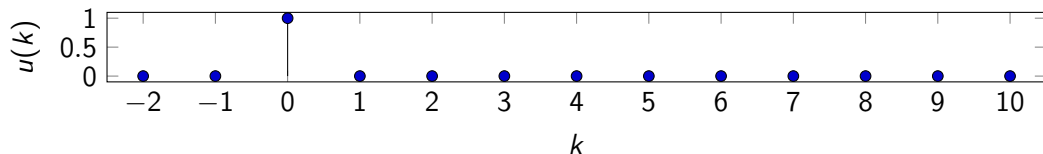


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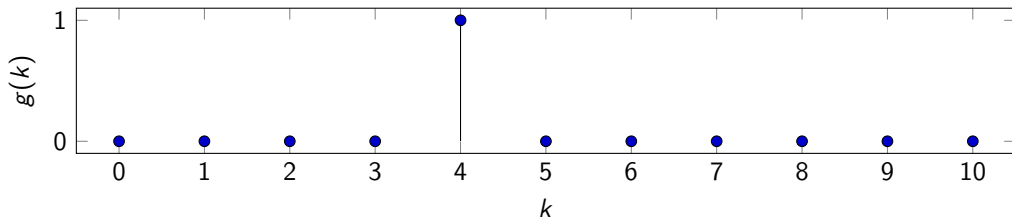
$$y(k) = \sum_{n=0}^{\infty} g(n)\delta(k-n) = g(k)$$

The weighting sequence is equal to the impulse response of the system.

The response of a discrete LSI system is a weighted sum of the current and previous values of the input

$$y(k) = g * u = \sum_{n=0}^{\infty} g(n)u(k-n)$$

**Activity** What is the response of a system to the input signal  $u(k)$  if its impulse response (weighting sequence) is the one below,  $g(k) = \delta(k-4)$  ?



$$y(k) =$$

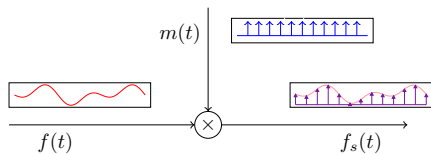
# The z-transform

# The impulse modulation model of sampling

The **impulse train**, a.k.a the **Dirac comb**:  $m(t) = \sum_{k=-\infty}^{\infty} \delta(t - kh)$

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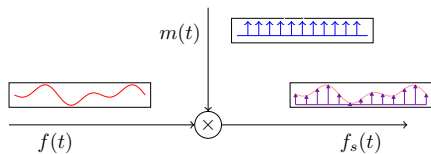
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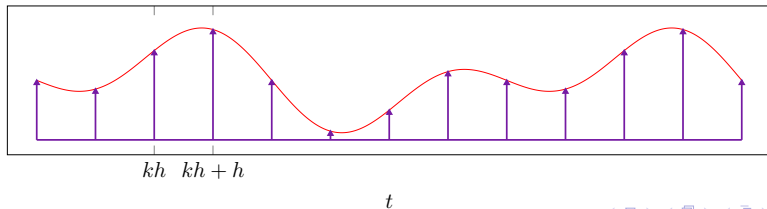


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# The Laplace transform

## Definition

$$F(s) = \mathcal{L}\{f(t)\} = \int_0^{\infty} f(t)e^{-st} dt$$

## Inverse transform

$$f(t) = \mathcal{L}^{-1}\{F(s)\} = \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} F(s)e^{st} ds$$

Note that in control engineering, the one-sided transform is used.

# The z-transform

## Definition

$$F(z) = \mathcal{Z} \{f(kh)\} = \sum_{k=0}^{\infty} f(kh)z^{-k}$$

## Inverse transform

$$f(kh) = \frac{1}{2\pi i} \oint_r F(z)z^{k-1} dz$$

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## The Laplace transform of a sampled signal

Assume right-sided signal  $f(t)$ , meaning it is zero for negative times  $t < 0$ .

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$$\begin{aligned} F_s(s) &= \mathcal{L}\{f_s(t)\} = \int_0^{\infty} \left( \sum_{k=0}^{\infty} f(kh)\delta(t - kh) \right) e^{-st} dt \\ &= \sum_{k=0}^{\infty} \int_0^{\infty} f(kh)\delta(t - kh)e^{-st} dt = \sum_{k=0}^{\infty} f(kh)e^{-skh} \\ &= \sum_{k=0}^{\infty} f(kh) \left( e^{sh} \right)^{-k} \end{aligned}$$

# The Laplace transform of a sampled signal

Note:

$$F_s(s) = \sum_{k=0}^{\infty} f(kh) (e^{sh})^{-k} \quad \text{Laplace transform}$$

$$F(z) = \sum_{k=0}^{\infty} f(kh) z^{-k} \quad \text{z-transform}$$

**Activity** How is the Laplace transform of the sampled signal and the z-transform of the corresponding discrete-time sequence related?

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**Activity** How is the Laplace transform of the sampled signal and the z-transform of the corresponding discrete-time sequence related?

The z-transform of a sampled signal corresponds to its Laplace transform with the following relationship between the s-plane of the Laplace transform and the z-plane of the z-transform.

$$z = e^{sh}$$

## One of the most important transform pairs

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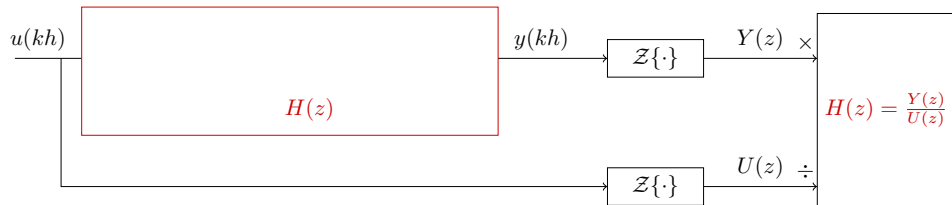
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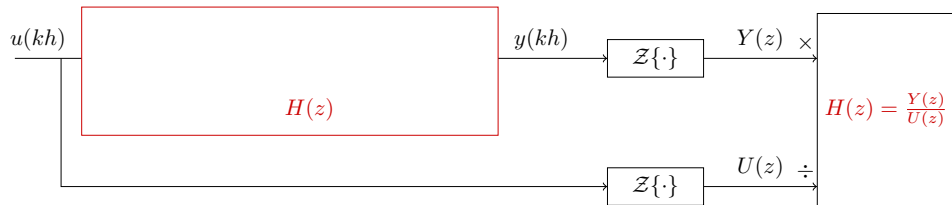
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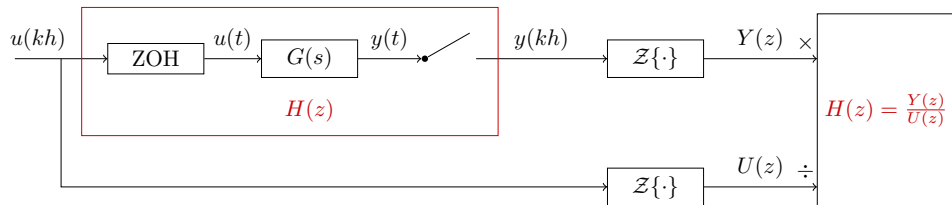
## Step-invariant sampling (a.k.a ZOH sampling)



Step-invariant sampling (zero order hold):  $u(kh) = \begin{cases} 1, & k \geq 0 \\ 0, & k < 0 \end{cases}$

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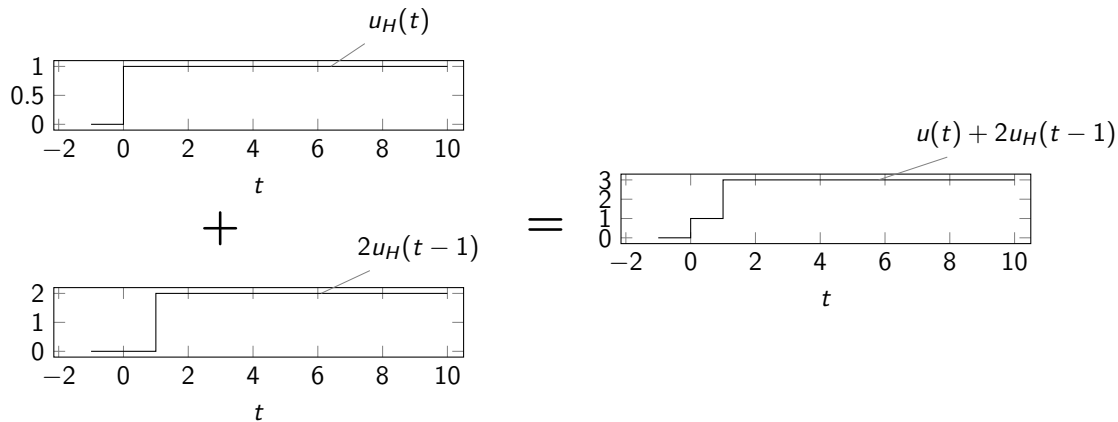
The idea is to sample the continuous-time system's response to a step input, in order to obtain a discrete approximation which is **exact** (at the sampling instants) for such an input signal.



Step-invariant sampling (zero order hold):  $u(kh) = \begin{cases} 1, & k \geq 0 \\ 0, & k < 0 \end{cases}$

## Why is step-invariant sampling a good idea?

A piecewise constant (stair-case shaped) function can be written as a sum of delayed step-responses!



## Step-invariant sampling, or zero-order-hold sampling

Let the input to the continuous-time system be a unit step  $u(t) = u_H(t)$ , which has Laplace transform  $U(s) = \frac{1}{s}$ . In the Laplace-domain we get

$$Y(s) = G(s) \frac{1}{s}$$

1. Obtain the time-response by inverse Laplace:  $y(t) = \mathcal{L}^{-1} \{Y(s)\}$
2. Sample the time-response to obtain the sequence  $y(kh)$  and apply the z-transform to obtain  $Y(z) = \mathcal{Z} \{y(kh)\}$
3. Calculate the pulse-transfer function by dividing with the z-transform of the input signal  $U(z) = \frac{z}{z-1}$ .

$$H(z) = \frac{Y(z)}{U(z)} = \frac{z-1}{z} Y(z)$$



## Example: First-order system

$$G(s) = \frac{1}{\tau s + 1}.$$

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