Controllability and observability

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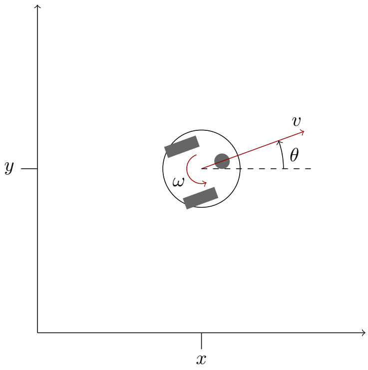
October 11, 2022

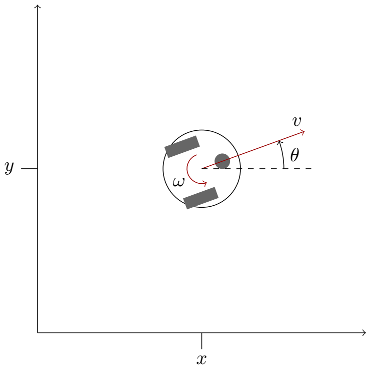
The concept of state

State The information needed about the history of a dynamical system in order to determine the future behaviour of the system given future input signals.

Example: Mobile robot



Example: Mobile robot

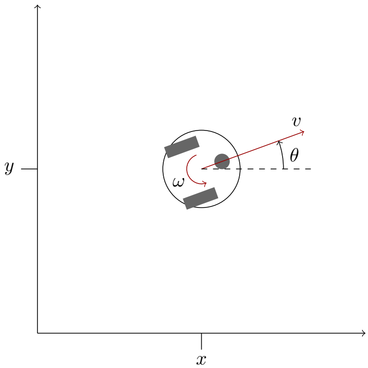


$$\xi = \begin{bmatrix} \theta \\ x \\ y \end{bmatrix}, \quad u = \begin{bmatrix} \omega \\ v \end{bmatrix}$$

$$\frac{d}{dt}\xi = \begin{bmatrix} \dot{\theta} \\ \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} \omega \\ v\cos\theta \\ v\sin\theta \end{bmatrix}$$

Called uni-cycle model.

Example: Mobile robot



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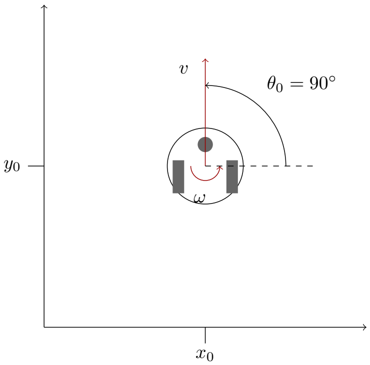
Called uni-cycle model.
Activity Can we reach any point in state space $\begin{bmatrix} x & y & \theta \end{bmatrix}^T$ by a suitably designed input signal sequence $u(t)$?

Example: Mobile robot



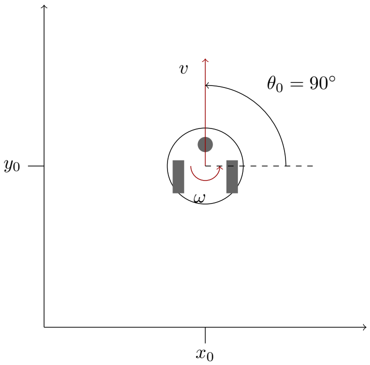
$$\frac{d}{dt}\xi = \begin{bmatrix} \dot\theta \\ \dot x \\ \dot y \end{bmatrix} = \begin{bmatrix} \omega \\ v\cos\theta \\ v\sin\theta \end{bmatrix}$$

Linearized model using deviation variables

$$\xi(t) = \xi_0 + z(t), \quad \frac{d}{dt}\xi = \frac{d}{dt}z, \quad \theta_0 = 90°, v_0 = 0$$

$$\frac{d}{dt}z = \begin{bmatrix} \omega \\ v\cos\theta_0 \\ v\sin\theta_0 \end{bmatrix} = \underbrace{0}_{A} z + \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}}_{B} \begin{bmatrix} \omega \\ v \end{bmatrix}$$

Example: Mobile robot



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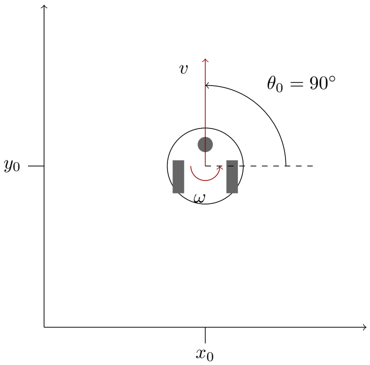
$$\begin{aligned} \mathcal{C} &= \begin{bmatrix} B & AB & A^2B \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix} \end{aligned}$$

Example: Mobile robot



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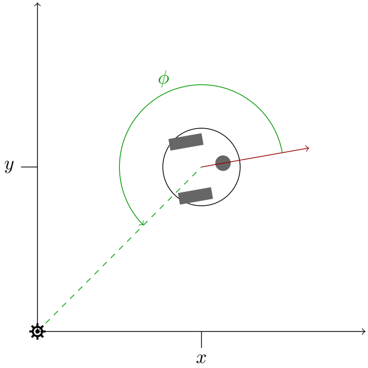
$$= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Activity Is the linearized model controllable? (Hint: What is rank $\mathcal{C}$?)

Example: Mobile robot

Single bearing measurement.



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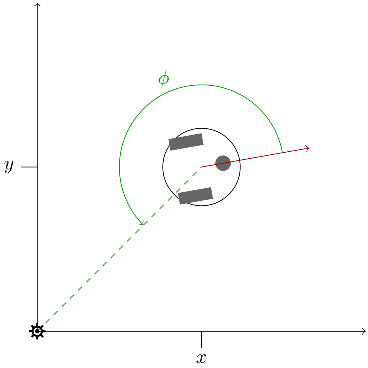
Example: Mobile robot

Single bearing measurement.

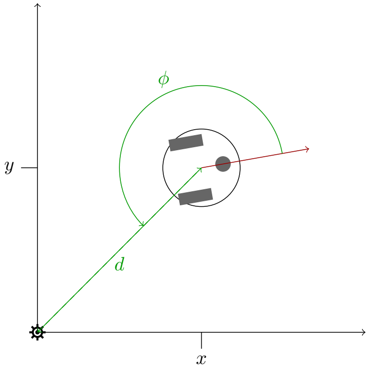


$$\xi = \begin{bmatrix} \theta \\ x \\ y \end{bmatrix}$$

Activity Is the system observable with one bearing-only measurement?

Example: Mobile robot

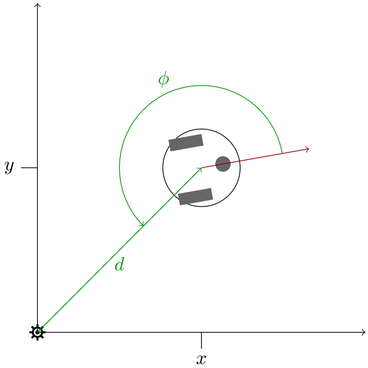
Bearing and distance measurement.



$$\xi = \begin{bmatrix} \theta \\ x \\ y \end{bmatrix}$$

Example: Mobile robot

Bearing and distance measurement.



$$\xi = \begin{bmatrix} \theta \\ x \\ y \end{bmatrix}$$

Activity Is the system observable with one bearing and one distance measurement?