TTK4130 - Exercise 4

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11. februar 2016

Problem 1

Considering the butcher array:

$$\begin{array}{c|cc} 0 & & \\ c_2 & a_{21} & \\ \hline & b_1 & b_2 \end{array}$$

a

The first order expansion of k_2 will be:

$$\begin{aligned} k_2 &= f(y_n + ha_{21}k_1, t_n + hc_2) \\ &= f(y_n, t_n) + k_1 h a_{21}\tfrac{\partial f}{\partial y} + hc_2\tfrac{\partial f}{\partial t} + O(h^2 a_{21}^2) + O(h^2 a_{21} c_2) + O(h^2 c_2^2) \\ &= f(y_n, t_n) + k_1 h a_{21}\tfrac{\partial f}{\partial y} + hc_2\tfrac{\partial f}{\partial t} + O(h^2) \end{aligned}$$

Now assuming that $c_2 = a_{21} = K$

$$k_2 = f(y_n, t_n) + hK(k_1\frac{\partial f}{\partial y} + \frac{\partial f}{\partial t}) + O(h^2)$$

Since $k_1 = f(y_n, t_n)$ and $\frac{df}{dt} = f(y_n, t_n)\frac{\partial f}{\partial y} + \frac{\partial f}{\partial t}$

$$k_2 = f(y_n, t_n) + hK\frac{df}{dt} + O(h^2)$$

b

For it to be order p, p must be the smallest integer to satisfy:

$$y_{n+1} = y_n + hf(y_n, t) + \cdots + \frac{h^p}{p!}\frac{d^{p-1}f(y_n, t_n)}{dt^{p-1} + O(h^{p+1})}$$

Meaning that

$$b_1 + b_2 = 1$$

and

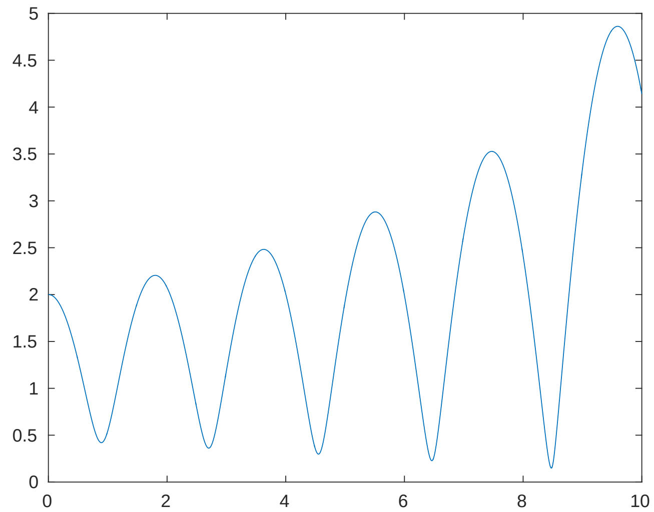
$$b_2 K = \frac{1}{2}$$

is needed for it to be of order 2.

Problem 2

a

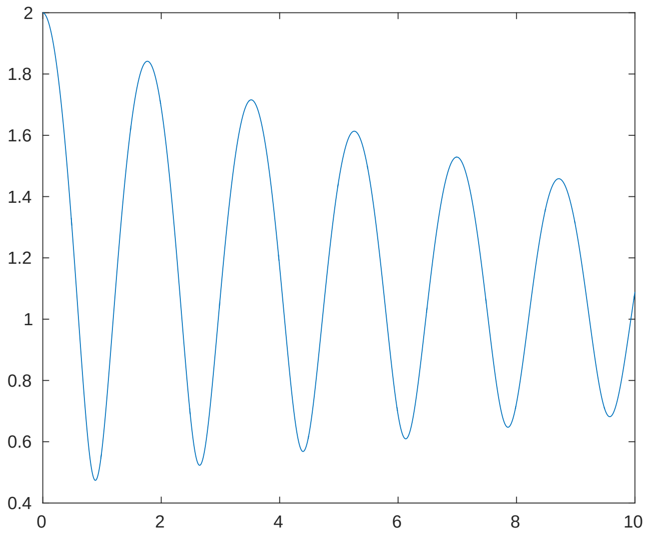
There is no damping here, therefore eulers method wont be stable.



Figur 1: Eulers method on maginaly stable system

b

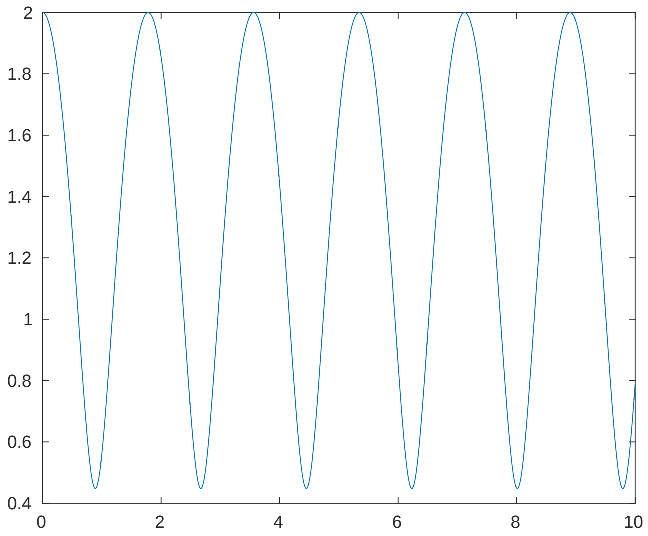
Now the simulation is stable but introduce damping such that the amplitude decreases over time.



Figur 2: Eulers implicit method on maginaly stable system

c

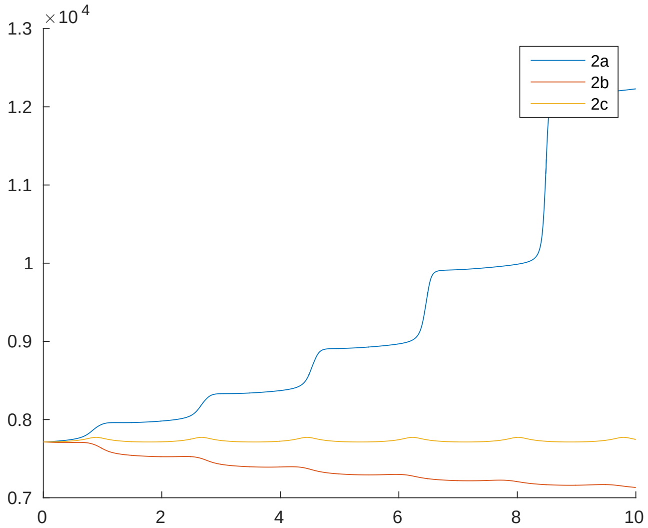
Now everything seems to be physically correct.



Figur 3: Implicit midpoint rule on maginaly stable system

d

Energy plotted for different solutions



Figur 4: Energy

Problem 3

The dc motor system is given by:

$$\begin{aligned} L_a \frac{di_a}{dt} &= -R_a i_a - K_E \omega_m + u_a \\ J_m \frac{d\omega_m}{dt} &= K_t - T_L \end{aligned}$$

a

Let $T_L = u_a = 0$ and $K_E = K_T = K$. Using the lyapunov candidate:

$$V = \frac{1}{2} L_a i_a^2 + \frac{1}{2} J_m \omega_m$$

Which satisfies $V > 0, \forall V \neq 0$ implies that

$$\begin{aligned} \dot{V} &= i_a(-R_a i_a - K\omega_m) + \omega_m (K i_a) \\ &= -R_a i_a^2 + \omega_m i_a (K - K) \\ &= -R_a i_a^2 \end{aligned}$$

Which obviously satisfies $\dot{V} \leq 0$ when $R_a > 0$

b

Given the input and outputs $\mathbf{u} = \begin{bmatrix} u_a \\ -T_L \end{bmatrix}$, $\mathbf{y} = \begin{bmatrix} i_a \\ \omega_m \end{bmatrix}$ and $V = \frac{1}{2}L_a i_a^2 + \frac{1}{2}J_m\omega_m^2$

$$\mathbf{uy} = u_a i_a - T_L\omega_m$$

and

$$\begin{aligned}
\dot{V} \quad &= i_a(-R_a i_a - K\omega_m + u_a + \omega_m(Ki_a - T_L) \\
&= -R_a i_a^2 - K\omega_m i_a + i_a u_a + \omega_m K i_a - \omega_m T_L \\
&= -R_a i_a^2 + i_a u_a - \omega_m T_L \\
&= -R_a i_a^2 + \mathbf{uy}^T \leq \mathbf{uy}^T
\end{aligned}$$

hence passive

c

A feedback connection that asure stability is:

$$\begin{aligned}
u_a \quad &= -R_a i_{ad} - K_E\omega_{md} k K_{P1} i_a - i_{ad} \\
T_L \qquad &= K_T i_{ad} - K_{P2}(\omega_m - \omega md)
\end{aligned}$$