TTK4130 - Exercise 9

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Problem 1

a

Wish to show that $\mathbf{R}_b^a = \begin{bmatrix} \frac{1}{2}\sqrt{3} & \frac{1}{2} & 0 \\ -\frac{1}{2} & \frac{1}{2}\sqrt{3} & 0 \\ 0 & 0 & 1 \end{bmatrix}$ is a rotation matrix by showing it is in $SO(3) = \{\mathbf{R}|\mathbf{R} \in \mathbb{R}^{3\times 3}, \mathbf{R}^T\mathbf{R} = \mathbf{I}, \det \mathbf{R} = 1\}$. One can see that $\mathbf{R} \in \mathbb{R}^{3\times 3}$ must be true since R_b^a is obviously a 3×3 matrix. The second property must be true as well since

$$
\begin{aligned}
\mathbf{R}^T\mathbf{R} &= \begin{bmatrix} \frac{1}{2}\sqrt{3} & -\frac{1}{2} & 0 \\ \frac{1}{2} & \frac{1}{2}\sqrt{3} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{2}\sqrt{3} & \frac{1}{2} & 0 \\ -\frac{1}{2} & \frac{1}{2}\sqrt{3} & 0 \\ 0 & 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} \frac{3}{4} + \frac{1}{4} + 0 & -\frac{3}{4} + \frac{3}{4} + 0 & 0 + 0 + 0 \\ -\frac{3}{4} + \frac{3}{4} + 0 & \frac{1}{4} + \frac{3}{4} + 0 & 0 + 0 + 0 \\ 0 + 0 + 0 & 0 + 0 + 0 & 0 + 0 + 1 \end{bmatrix} \\
&= \mathbf{I}
\end{aligned}
$$

The last property is also satisfied since

$$\det \mathbf{R} = (\frac{1}{2}\sqrt{3})(\frac{1}{2}\sqrt{3})(1) - (\frac{1}{2})(-\frac{1}{2})(1) = 1$$

And thus the R_b^a is a rotation matrix.

b

The rotation matrix for the z-axis is

$$\mathbf{R}_z(\psi) = \begin{bmatrix} \cos\psi & -\sin\psi & 0 \\ \sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Seeing that $\mathbf{R}_z(30°) = \mathbf{R}^a_b$, meaning that $\mathbf{R}^a_b$ is a 30° rotation about the z-axis

c

$\mathbf{R}^b_a$ is the rotation matrix that rotate $-30°$ about the z-axis. Since $\mathbf{x} = \mathbf{R}^b_a\mathbf{R}^a_b\mathbf{x} = \mathbf{R}^a_b\mathbf{R}^b_a\mathbf{x}$. We also know that $\mathbf{R}^b_a = (\mathbf{R}^a_b)^{-1} = (\mathbf{R}^a_b)^T$.

d

$$\mathbf{u}^a = \mathbf{R}^a_b\mathbf{u}^b = \begin{bmatrix} \frac{1}{2}\sqrt{3} - 1 \\ \frac{1}{2} + \sqrt{3} \\ 3 \end{bmatrix}$$

$$\mathbf{w}^b = \mathbf{R}^b_a\mathbf{w}^a = \begin{bmatrix} \frac{1}{2}\sqrt{3} - \frac{1}{2} \\ -\frac{1}{2} - \frac{1}{2}\sqrt{3} \\ 2 \end{bmatrix}$$

e

i

$(u^a)^T w^a = (u^b)^T w^b$

Proof.

$$\begin{aligned} (u^a)^T w^a &= (\mathbf{R}^a_b u^b)^T \mathbf{R}^a_b w^b \\ &= (u^b)^T (\mathbf{R}^a_b)^T \mathbf{R}^a_b w^b \\ &= (u^b)^T w^b \end{aligned}$$

□

ii

f

Calculating the rotation matrix from a general euler angle can be done like shown below:

$$\begin{aligned}
\mathbf{R}_b^a &= \mathbf{R}_{z,\psi}\mathbf{R}_{y,\theta}\mathbf{R}_{x,\phi} \\
&= \begin{bmatrix} c\psi & -s\psi & 0 \\ s\psi & c\psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c\theta & 0 & s\theta \\ 0 & 1 & 0 \\ -s\theta & 0 & c\theta \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & c\phi & -s\phi \\ 0 & s\phi & c\phi \end{bmatrix} \\
&= \begin{bmatrix} c\psi & -s\psi & 0 \\ s\psi & c\psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c\theta & s\theta s\phi & s\theta c\phi \\ 0 & c\phi & -s\phi \\ -s\theta & c\theta s\phi & c\theta c\phi \end{bmatrix} \\
&= \begin{bmatrix} c\psi c\theta & c\psi s\theta s\phi - s\psi c\phi & c\psi s\theta c\phi + s\psi s\phi \\ c\psi c\theta & s\psi s\theta s\phi + c\psi c\phi & c\psi s\theta c\phi - s\psi s\theta \\ -s\theta & c\theta s\phi & c\theta c\phi \end{bmatrix}
\end{aligned}$$

g

1

It's easy to see that

$$\mathbf{R}_1 = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Since then $\det \mathbf{R}_1 = 1$ and $\mathbf{R}_1\mathbf{R}_1 = \mathbf{I}$. And it's obvious a 3 by 3 matrix.

2

It's easy to see that

$$\mathbf{R}_2 = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Since then $\det \mathbf{R}_2 = 1$ and $\mathbf{R}_2\mathbf{R}_2 = \mathbf{I}$. And it's obvious a 3 by 3 matrix.

3

To find $\mathbf{R}_3$ one need to solve the set of equations:

$$\det \mathbf{R}_3 = 1$$
$$\mathbf{R}_3\mathbf{R}_3^T = \mathbf{I}$$

Since it's obviously a 3 by 3 matrix.

Problem 2

a

The general expression for A_i is:

$$\begin{aligned}
\mathbf{A}_i &= Rot_{z,\theta_i} Trans_{x,d_i} Trans_{x,a_i} Rot_{x,\alpha_i} \\
&= \begin{bmatrix} \mathbf{R}_z(\theta_i) & \mathbf{0} \\ \mathbf{0}^T & 1 \end{bmatrix} \begin{bmatrix} \mathbf{I} & \begin{bmatrix} 0 \\ 0 \\ d_i \end{bmatrix} \\ \mathbf{0} & 1 \end{bmatrix} \begin{bmatrix} \mathbf{I} & \begin{bmatrix} a_i \\ 0 \\ 0 \end{bmatrix} \\ \mathbf{0} & 1 \end{bmatrix} \begin{bmatrix} \mathbf{R}_z(\theta_i) & \mathbf{0} \\ \mathbf{0}^T & 1 \end{bmatrix} \\
&= \begin{bmatrix} \mathbf{R}_z(\theta_i) & \begin{bmatrix} 0 \\ 0 \\ d_i \end{bmatrix} \\ \mathbf{0}^T & 1 \end{bmatrix} \begin{bmatrix} \mathbf{R}_x(\alpha_i) & \begin{bmatrix} a_1 \\ 0 \\ 0 \end{bmatrix} \\ \mathbf{0}^T & 1 \end{bmatrix} \\
&= \begin{bmatrix} \mathbf{R}_z(\theta_i)\mathbf{R}_x(\alpha_i) & \begin{bmatrix} a_i \cos\theta_i \\ a_i \sin\theta_i \\ d_i \end{bmatrix} \\ \mathbf{0}^T & 1 \end{bmatrix}
\end{aligned}$$

b

Manipilator A

$$A_1 = \begin{bmatrix} \mathbf{R}_z(q_1) & \begin{bmatrix} l_1 \cos q_1 \\ l_1 \sin q_1 \\ 0 \end{bmatrix} \\ \mathbf{0}^T & 1 \end{bmatrix} = \begin{bmatrix} cq_1 & -sq_1 & 0 & l_1 cq_1 \\ sq_1 & cq_1 & 0 & l_1 sq_1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$A_2 = \begin{bmatrix} \mathbf{I} & \begin{bmatrix} q_2 \\ 0 \\ 0 \end{bmatrix} \\ \mathbf{0}^T & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & q_2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Manipulator B

$$A_1 = \begin{bmatrix} \mathbf{R}_z(q_1) & \begin{bmatrix} l_1 \cos q_1 \\ l_1 \sin q_1 \\ 0 \end{bmatrix} \\ \mathbf{0}^T & 1 \end{bmatrix} = \begin{bmatrix} cq_1 & -sq_1 & 0 & l_1 cq_1 \\ sq_1 & cq_1 & 0 & l_1 sq_1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$A_2 = \begin{bmatrix} \mathbf{R}_z(q_2) & \begin{bmatrix} l_2 \cos q_2 \\ l_2 \sin q_2 \\ 0 \end{bmatrix} \\ \mathbf{0}^T & 1 \end{bmatrix} = \begin{bmatrix} cq_2 & -sq_2 & 0 & l_2 cq_2 \\ sq_2 & cq_2 & 0 & l_2 sq_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

c

Manipulator A

$$\begin{aligned} T_2^0 &= A_1 A_2 = \begin{bmatrix} cq_1 & -sq_1 & 0 & l_1 cq_1 \\ sq_1 & cq_1 & 0 & l_1 sq_1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & q_2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} cq_1 & -sq_1 & 0 & (q_2 + l_1) cq_1 \\ sq_1 & cq_1 & 0 & (q_2 + l_1) sq_1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{aligned}$$

Manipulator B

$$\begin{aligned} T_2^0 &= A_1 A_2 = \begin{bmatrix} cq_1 & -sq_1 & 0 & l_1 cq_1 \\ sq_1 & cq_1 & 0 & l_1 sq_1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} cq_2 & -sq_2 & 0 & l_2 cq_2 \\ sq_2 & cq_2 & 0 & l_2 sq_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} c(q_1 + q_2) & -s(q_1 + q_2) & 0 & l_2 c(q_1 + q_2) + l_1 cq_1 \\ s(q_1 + q_2) & c(q_1 + q_2) & 0 & l_2 s(q_1 + q_2) + l_1 sq_1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{aligned}$$

d

Manipulator A

$$
\begin{aligned}
g_0 &= T_2^0 g^2 = \begin{bmatrix} cq_1 & -sq_1 & 0 & (q_2+l_1)cq_1 \\ sq_1 & cq_1 & 0 & (q_2+l_1)sq_1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \\
&= \begin{bmatrix} (q_2+l_1+1)cq_1 - sq_1 \\ (q_2+l_1+1)sq_1 + sq_1 \\ 1 \\ 1 \end{bmatrix}
\end{aligned}
$$

Manipulator B

$$
\begin{aligned}
g_0 &= T_2^0 g^2 = \begin{bmatrix} c(q_1+q_2) & -s(q_1+q_2) & 0 & l_2c(q_1+q_2)+l_1cq_1 \\ s(q_1+q_2) & c(q_1+q_2) & 0 & l_2s(q_1+q_2)+l_1sq_1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \\
&= \begin{bmatrix} (l_2+1)c(q_1+q_2) - s(q_1+q_2) + l_1cq_1 \\ (l_2+1)s(q_1+q_2) + c(q_1+q_2) + l_1sq_1 \\ 1 \\ 1 \end{bmatrix}
\end{aligned}
$$

3

a

From (5) in the assignment text we know that:

$$\mathbf{R} = \mathbf{e}^\times + \cos\theta\mathbf{I} + \mathbf{k}\mathbf{k}^T(1-\cos\theta)$$

where

$$\mathbf{e}^\times = \sin\theta \begin{bmatrix} 0 & -k_3 & k_2 \\ k_3 & 0 & -k_1 \\ -k_2 & k_1 & 0 \end{bmatrix}$$

Meaning that:

$$
\begin{aligned}
r_{11} &= (1-k_1^2)\cos\theta + k_1^2 \\
r_{22} &= (1-k_2^2)\cos\theta + k_2^2 \\
r_{33} &= (1-k_3^2)\cos\theta + k_3^2
\end{aligned}
$$

Since we know that $k_1^2 + k_2^2 + k_3^2 = 1$

$$r_{11} + r_{22} + r_{33} = (3 - (k_1^2 + k_2^2 + k_3^2)) \cos\theta + (k_1^2 + k_2^2 + k_3^2)$$

Which means that

$$\cos\theta = \frac{r_{11} + r_{22} + r_{33} - 1}{2}$$

b

By calculating terms in $\mathbf{R}$ we find

$$\begin{aligned}
r_{32} &= k_1 \sin\theta + k_3 k_2 (1 - \cos\theta) \\
r_{23} &= -k_1 \sin\theta + k_3 k_2 (1 - \cos\theta) \\
r_{13} &= k_2 \sin\theta + k_1 k_3 (1 - \cos\theta) \\
r_{31} &= -k_2 \sin\theta + k_1 k_3 (1 - \cos\theta) \\
r_{21} &= k_3 \sin\theta + k_2 k_1 (1 - \cos\theta) \\
r_{12} &= -k_3 \sin\theta + k_2 k_1 (1 - \cos\theta)
\end{aligned}$$

Which means that

$$\mathbf{e} = \sin\theta \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} r_{32} - r_{23} \\ r_{13} - r_{31} \\ r_{21} - r_{12} \end{bmatrix}$$

c

Both the direction of the axis and the angle will be reversed such that the rotation will be the same.

d

The code

```
R = [0.2133, -0.2915, 0.9325; 0.9209, -0.2588,
    -0.2915; 0.3263, 0.9209, 0.2133];

theta = acos((R(1,1) + R(2,2) + R(3,3) - 1)/2);
e = 1/2 * [R(3,2) - R(2,3);R(1,3) - R(3,1);R(2,1) - R
    (1,2)];
k = e ./ sin(theta);

theta
k
```

Produce the result

```
theta =

    1.9999


k =

    0.6667
    0.3333
    0.6667
```