Project 3 FYS4150

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Abstract

In this report we have modelled the solar system using Newtons law of gravitation and, for the case of mercury, with a relativistic correction. In order to integrate the dynamics of the solar system both the Euler and the Velocity Verlet methods of integration has been used. Benchmarks and comparisons of energy and angular momentum conservation shows that the Velocity Verlet method is the more stable alternative. The program used in this project can be found at Github.

To do: OBS Unit tests

3c: - Find out which initial velocity that gives a circular motion (plot) - Test stability (energy-stability) as function of dt (both Verlet and Euler) - Plot the earth orbiting the sun - Check (for the circular orbit) that the energy is conserved (plot - both kin and pot separated and together?) - Check that angular moment is conserved

- Discuss differences between Euler and Verlet - number of FLOPS + CPU time

* Plotte ulike dt-er * Plott energi som funk av ulike dt * Referere til convergens - funksjonen. * Vise at angulærmoment bevart

3d: - Find escape velocity (plot) - Compare with numerical results(Result? or Discussion?) - Find exact escape velocity (theory?) - Changing beta (plot) - Comment result + What happens when beta -> 3 ?

Exact løsning escape vel Plots ulike init.hastigheter Bytte gravitasjonskrefter...

3e: - How much does Jupiter alter Earth's orbit? - Position of Jupiter and Earth (plot) - Plot Earth's motion for increased mass of Jupiter (3 masses) - - Discuss stability of velocity verlet (3 body)

* 3 ulike masser * Plotte alle banene * Stabilitet: Energi-plot

3f: - Find center off mass - use as origin - Give sun initial velocity so momentum is zero (origin is fixed) - Compare with 3e) - Extend to all planets (plot) - Discuss difference 3e) and 3f) (3 body) - Discuss result of all planets

3g: - Find perihelion for both relativistic and non-relativisitc (table) - Relativistic - should be a few magnitudes smaller. - Can the observed perihelion precession of Mercury be explained by the general theory of relativity?

FLOPS euler/Verlet Result: - Find out which initial velocity that gives a circular motion (plot) - Test stability (energy-stability) as function of dt (both Verlet and Euler) - Plot the earth orbiting the sun - Check (for the circular orbit) that the energy is conserved (plot - both kin and pot separated and together?) - Check that angular moment is conserved

- Find escape velocity (plot) - Compare exact and numerical results - Find exact escape velocity (theory?) - Changing beta (plot) - Comment result + What happens when beta -> 3 ? (last part in discussion? - How much does Jupiter alter Earth's orbit? - Position of Jupiter and Earth (plot) - Plot Earth's motion for increased mass of Jupiter (3 masses) - Find center off mass - use as origin - Give sun initial velocity so momentum is zero (origin is fixed) - Compare with 3e) - Extend to all planets (plot) - Find perihelion for both relativistic and non-relativisitc (table) - Relativistic - should be a few magnitudes smaller. - Can the observed perihelion precession of Mercury be explained by the general theory of relativity?

Discussion: - Discuss differences between Euler and Verlet - number of FLOPS + CPU time - Discuss stability of velocity verlet (3 body) - Discuss difference 3e) and 3f) (3 body) - Discuss result of all planets

Contents

1 Introduction

There are many physical problems that deals with differential equations. A simple example of this is the ordinary differential equations that governs how objects move in relations to each other, for example the solar system. The aim of this project is to model our solar system using a relatively simple model, relativistic model for the movement of the planets and the sun. In order to do this efficiently to different integration methods will be explored, the Euler method and the Velocity Verlet method.

This report starts by the explaining basic theoretical physics that governs the movement of the objects in the solar system. These equations can be scaled and implemented in a general algorithm for solving 2. order differential equations. The two different integration models will then be discussed alongside how we chose to implement these in our experiment.

The results of benchmark and stability tests of the different methods will be discussed together with more general results of how the planets are moving. This is all wrapped up in a conclusion at the end.

2 Theory

2.1 Classical Solar system

In the classical description of the solar system, there is only a single force working:

$$F_G = -G\frac{M_1 M_2}{r^2} \tag{1}$$

By applying Newtons 2.law on component form, we achieve two more equations, in the case of a two dimensional system.

$$\frac{\mathrm{d}^2\vec{r}}{\mathrm{d}t^2} = \frac{\vec{F}_G}{M_2} \tag{2}$$

Equation 2 is in reality two separate, independent equations, one for the x-direction and the other for the y-direction.

2.2 Units and scaling

A computer has a limited bit-resolution and the distances and time scales are large when computing the solar system. This means that it is important to use appropriate units. The distance between the sun and the earth is defined as 1 Astronomical unit (1 Au) and the time scale used in this project will be in units of 1 year.

For simplicity we will define a new unite of energy: $J_{ast} = \frac{m_{planet}}{|M_{sun}|}\left(\frac{Au}{year}\right)^2$, where $|M_{sun}|$. In reallity this is simply Joule with a prefactor. This prefactor can be obtained by inserting the values into J_{ast}:

$$\begin{aligned} J_{ast} &= \frac{m_{planet}}{|M_{sun}|}\left(\frac{Au}{year}\right)^2 && (3)\\ &= \frac{m_{planet}}{2\cdot 10^{30}}\left(|v|\frac{1.5\cdot 10^{11}m}{360*24*60*60s}\right)^2 && (4)\\ &\simeq 1.163\cdot 10^{-23}J && (5)\end{aligned}$$

The dimensionality of the variables are as follows: $[v] = m/s$ and $[m_{planet}] = kg$.

We know that the earth needs one year to orbit the sun, meaning that $v = \frac{2\pi r}{1\text{ year}}$. This can be rewritten with $v = \tilde{v}v_0$ and $r = \tilde{r}r_0$. The units of r and v are contained in $v_0 = \frac{1\text{Au}}{1\text{ year}}$ and $r_0 = 1$ Au, giving that $\tilde{v}^2\tilde{r} = 4\pi^2$. In the same way $t = \tilde{t}t_0$, with $t_0 = 1$ year.

For the case of the two body earth - sun system one can assume that the sun is stationary, as $M_{sun} >> M_{earth}$. The force experienced by the earth is thus centrifugal, which means that $a = \frac{v^2}{r}$, with $v = 2\pi r/1year$. Combining this with equations 1 and 2 it is possible to scale the equations in the following manner:

$$\begin{aligned} a_E = \frac{F_E}{M_E} = G\frac{M_{sun}}{r^2} = \frac{v^2}{r} && (6)\\ GM_{sun} = v^2 r = 4\pi^2\frac{(1\text{ Au})^3}{(1\text{ year})^2} && (7)\\ && (8)\end{aligned}$$

This gives that the dimensionless expression can be stated as:

$$\frac{\mathrm{d}\tilde{v}}{\mathrm{d}\tilde{t}} = \frac{4\pi^2}{\tilde{r}^2} \tag{9}$$

In the two dimensional case $r = (x, y) = (r\cos\theta, r\sin\theta)$. Using the notation $\dot{p} = \frac{\mathrm{d}p}{\mathrm{d}t}$, this gives the following coupled differential equations:

$$\dot{v}_x = \frac{4\pi^2 r\cos\theta}{r^3} = \frac{4\pi^2 x}{r^3} \tag{10}$$
$$\dot{v}_y = -\frac{4\pi^2 r\sin\theta}{r^3} = \frac{4\pi^2 y}{r^3} \tag{11}$$
$$\dot{x} = v_x \tag{12}$$
$$\dot{y} = v_y \tag{13}$$
$$\tag{14}$$

2.3 Energy considerations

As with any physical system, the total energy has to be conserved. The potential energy $E_p = \int_{r'}^{\infty} \vec{F}(r') \cdot d\vec{r}' = -\frac{GM_1M_2}{r}$, while the kinetic energy is $E_K = \frac{1}{2}mv^2$. This gives a total energy of:

$$E_{tot} = \frac{1}{2}mv^2 - \frac{GM_1M_2}{r} \tag{15}$$

In addition, the angular momentum ($\vec{L}$) of the system has to be preserved. This is because there are no additional sources of torque ($\vec{\tau}$) once the system has been initialized and $\vec{\tau} = \frac{\mathrm{d}\vec{L}}{\mathrm{d}t} = 0$. As a result all the absolute value of $\vec{L}$ has to be constant.

In order to maintain in the gravitational field of another object, the distance between them needs to be smaller than ∞. However, with a large enough velocity it is possible to escape. By setting equation 15 equal to 0 one obtains the lowest escape velocity v_{esc}:

$$\frac{1}{2}mv_{esc}^2 - \frac{GM_1M_2}{r} = 0 \tag{16}$$
$$v_{esc} = \sqrt{\frac{2GM_1M_2}{mr}} \tag{17}$$

Nevne approksimasjon med barrycentre v sol i sentrum v pos sol @ t=0: Vi har ingen beregninger med sol i sentrum, men bruker sol ved t=0 som posisjon.

Hvorfor bør egentlig være barrycentre til univers?

2.4 Perihelion precession

With a two body problem, only one force working on the planet, Mercury's elliptical orbit should be fixed if there where only non-relativistic Newtonian forces working on it. Mercury's orbit has been observed to rotate even when the forces from the other planets are subtracted away. If we add a relativistic part to the force, the rotation of the elliptical orbit can be explained. The gravitational force with the relativistic part is Equation 18.

$$F_G = \frac{GMm}{r^2}\left[1 + \frac{3l^2}{r^2c^2}\right] \tag{18}$$

Here l is the magnitude of the angular momentum, $l = |\mathbf{r} \times \mathbf{v}|$, of the planet per unit mass and c is the speed of light. To be able to see the rotation, we look at the movement of the perihelion, the position where Mercury is closest to the sun.

3 Method

In this project we tested two numerical integration techniques, the Euler method and the velocity Verlet method. This section is largely based on Hjort-Jensen [1]

3.1 The Euler method

When evaluating the function $x(t)$ in the interval between t and $t + h$ it is natural to do a Taylor expansion:

$$x(t+h) = x(t) + \sum_{n=1}^{\infty} \frac{h^n}{n!}\frac{\mathrm{d}^n x(t)}{\mathrm{d}t^n} \tag{19}$$

By choosing a small h, it is sufficient to truncate the sum after n=1, giving $x(t+h) = x(t) + h\frac{\mathrm{d}x}{\mathrm{d}t} + O(h^2)$. The term$O(h^2)$ contains the rest of the infinite sum, often called the truncation error. In order for a computer to use this method it is necessary to discretise the expression, substituting $x(t) \to x(t_i) \to x_i$. The Euler method can thus be expressed as

$$x_{i+1} = x_i + h\dot{x}_i + O(h^2) \tag{20}$$

Combining this with equations 10 and 12 we get an algorithm for doing a 1 dimentional integration for the solar system:

$$a_i = F(t_i) \tag{21}$$
$$v_{i+1} = v_i + ha_i \tag{22}$$
$$x_{i+1} = x_i + hx_i \tag{23}$$

This gives in total 4 floating points operations per iteration, per dimension. For our two dimentional system this results in a total of 8N FLOPS per iteration, with $N = \frac{1}{h}$.

3.2 The Velocity Verlet method

The velocity Verlet can be derived from the same Taylor expansion as the Euler method, equation 19. As with the Euler method, we will truncate the taylor series for position after n=1. However, for the velocity there exist a similar Taylor series: $v_{i+1} = v_i + h\dot{v}_i + \frac{h}{2}\ddot{v}_i + O(h^3)$. Unfortunately, there is no expression for $\ddot{v}_i$, but it can be approximated by $h\ddot{v}_i \simeq \dot{v}_{i+1} - \dot{v}_i$. Adding this to the expression for v_{i+1} and doing some simple algebra one get the final expression for the velocity:

$$v_{i+1} = v_i + \frac{h}{2}\left(\dot{v}_{i+1} + \dot{v}_i\right) + O(h^3) \tag{24}$$

Looking a bit closer at equation 24, there is a immediate problem. In order to calculate v_{i+1} one need to already know v_{i+1}. This problem can be solved by updating the velocity in two steps, which gives the following algorithm:

$$v_{i+\frac{1}{2}} = v_i + \frac{h}{2}a_i \tag{25}$$
$$x_{i+1} = x_i + v_{i+\frac{1}{2}} \tag{26}$$
$$a_{i+1} = \frac{F_{i+1}}{m} \tag{27}$$
$$v_{i+1} = v_{i+\frac{1}{2}}. + \frac{h}{2}a_{i+1} \tag{28}$$
$$\tag{29}$$

The numerical cost of the algorithm can be understood when analysing the number of floating points operations. In the case of the velocity verlet method there are 8 FLOPS per iteration and dimension. As the value $\frac{h}{2}$ a constant, it can be calculated in advance of the loop, thus reducing the FLOPS to 6. This gives a total of 12 FLOPS per iteration for the two dimensional system.

3.3 Choice of origin

As every planet in the solar system is moving, choosing a point of origin is not straight forwards. For the smaller systems, ie. the "sun-earth"-system, we chose to select the suns position at the start as the origin. This allows the sun to move. Another choice of origin is the solar system barycentre, which we utilized when calculating the entire solar system. Using the barrycentre can give a prettier picture of the physics which is (numerically) unfolding, as every object will rotate around this point.

3.4 Object orientation

In order to simplify the calculation of an ensemble of planets, each with velocities, positions, energies and angular momentum, it is useful to generalize the code in an object oriented way. We chose to create one class for the planets, where all the internal dynamics (energies, position, velocity, ...) were stored. The forces experienced by the planets are specific for this project and we kept this in the Planet class, so that the second class, the Solver class, could be more general and easier reused.

This Solver class is were all the technicalities are located, including the different integration methods. For each time step one need the location of every planet in the system and Solver includes therefore a function that for each time step loop over all the planets. In order to update their position it is necessary to again loop over all the different planets in order to find the total gravitational force exerted on the current planet.

When all the calculations are taken care of by the solver class and all the properties of the different planets are stored in each planet object. This means that the main part of the program only needs to initialize the different planetary instances , adding these to the a instance of the

Solver class which is initialized according to what output we want to achieve, see the snippet below for how this looks in 'main'.

```
Planet earth("Initializing inputs");
        Planet sun("Initializing
inputs");
    Solver verlet("Initializing inputs")
;
    verlet.add(earth);
    verlet.add(sun);
    verlet.add(mercury);
    verlet.algorithm("input variables");
```

4 Result

We started with a two body system and did some calculations with both Euler's method and the Velocity Verlet method. Thereafter we added another planet, Jupiter, and saw how this affected the system. At last we added all the planets using initial conditions from NASA [2] [FIX THIS LATER]. Our solar system thus far had been non-relativistic, only Newtonian forces was taken into consideration. To investigate how relativistic forces affect the solar system, we looked at the perihelion precession of Mercury.

4.1 Two body system

We started out with a two body system consisting of the sun and the earth. The origin of the system was the sun's position at time = 0. We wanted to find out what initial velocity the earth needed to have, for the orbit of the earth to be circular. The result can be found in Figure 4.1.

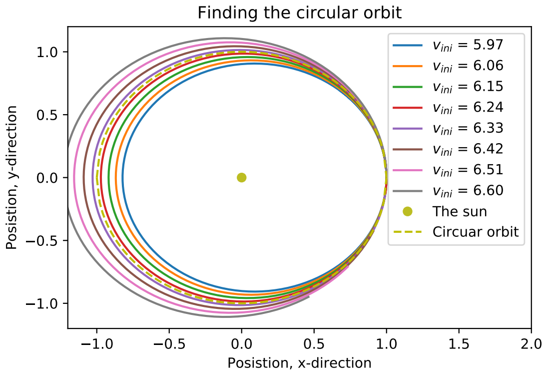

Figure 4.1: This is a plot of the orbit with different initial velocities. The circular orbit has a velocity between 6.24 and 6.33. $2\pi \approx 6.28$ is the initial velocity that gives a circular orbit **??**

As mentioned in the method part of this report, we used two different methods to solve the differential equations, to simulate the motion, Euler's method and the Velocity Verlet method. We started out by plotting the circular motion of the earth with different time steps (see Figure 4.2 and 4.3)to be able to evaluate the two methods.

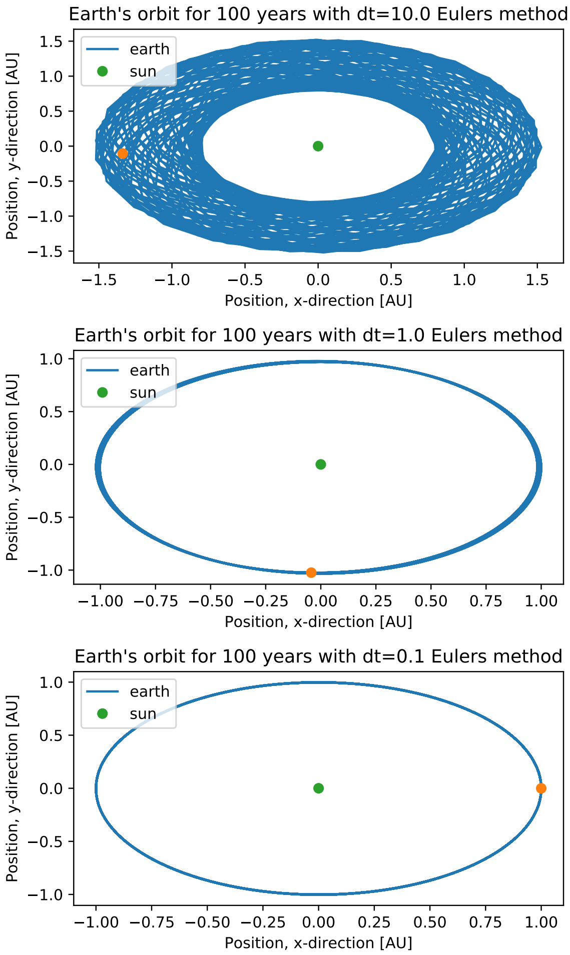

Figure 4.2: This is a plot of the earth's orbit for 100 years using Euler's method, with different timesteps.

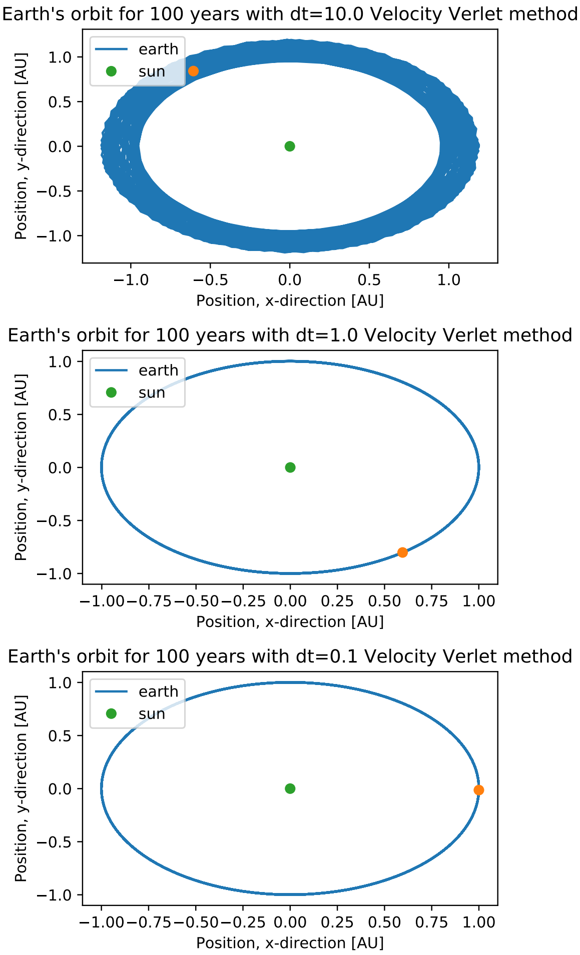

Figure 4.3: This is a plot of the earth's orbit for 100 years using the Velocity Verlet method, with different timesteps.

After

Another way to compare two methods are looking at the CPU time. In Table 4.1 the CPU time for the two method are listed. We can see that the difference between them are small, but the velocity Verlet algorithm is slower than the Euler method.

Table 4.1: Comparison of CPU time of Euler's method and the Velocity Verlet method (VV). Calculations were performed with 1000 time steps per year. As the CPU-time also calculate the time spent on updating energies and calling several functions, one will not be able to read out the CPU time of the euler/velocity verlet loop alone.

Years	time Euler (ms)	time VVerlet (ms)	$\Delta time$ (ms)
100	88.728	93.646	4.912
1000	812.486	852.949	40.46
10000	8145.7	8504.06	358.36

4.2 Conservation

First we checked if the angular momentum of the two body system was conserved. Figure 4.4 and 4.5 shows the results. The angular momentum is not totally conserved. The values are oscillating and there is an increase in the angular momentum when we use the Velocity Verlet method and a decrease when we use Euler's method. The scale of the change though is on the level of 10^{-10} for Euler's and 10^{-11} for Velocity Verlet. That is small compared to the whole value that are on the 10^{-5}.

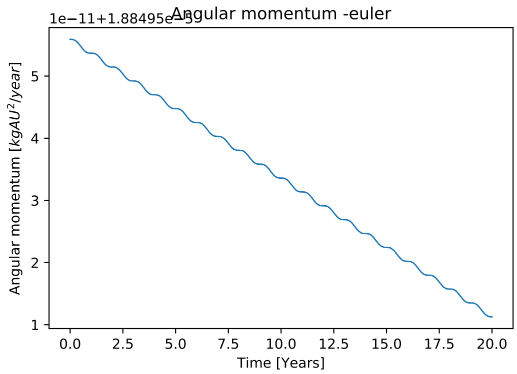

Figure 4.4: This is a plot of the angular momentum of the sun-earth system for 20 years using Euler's method, with 1000 timesteps per year. Notice that the scale of the axis is a sum.

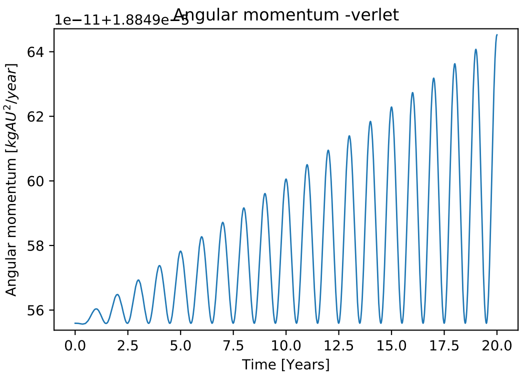

Figure 4.5: This is a plot of the angular momentum of the sun-earth system for 20 years using the velocity Verlet method, with 1000 timesteps per year. Notice that the scale of the axis is a sum.

The other important parameter that needs to be conserved is the energy. Figure 4.6 and 4.7 shows the energy development of the two methods, Euler's and Velocity Verlet respectively. The total energy is oscillating like the angular momentum and slightly changing, but the scales are small here two. Table 4.2 lists the difference in the oscillation amplitudes of the two methods.

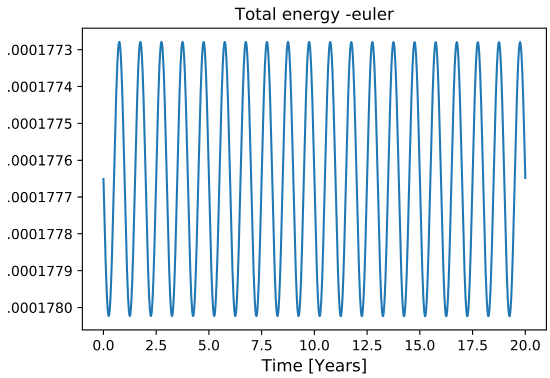

Figure 4.6: This is a plot of the total energy of the sun-earth system for 20 years using Euler's method, with 1000 timesteps per year.

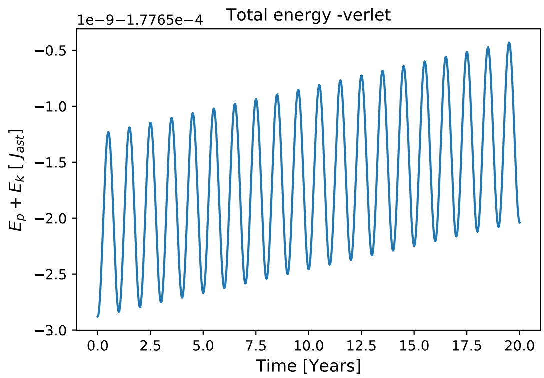

Figure 4.7: This is a plot of the total energy of the sun-earth system for 20 years using the velocity Verlet method, with 1000 timesteps per year.

Table 4.2: This is a table that list how the energy oscillates in the two different method. These are only approximate values that are read form the plots of the energies (Figure 4.6 and 4.7). Plot of kinetic and potential energies can be found in the appendix.

Oscillation of:	Euler's:	Velocity Verlet:
Kinetic energy [J_{ast}]:	$0.8 \cdot 10^{-6}$	$1.5 \cdot 10^{-9}$
Potential energy [J_{ast}]:	$1.5 \cdot 10^{-6}$	$3 \cdot 10^{-9}$
Total energy [J_{ast}]:	$5 \cdot 10^{-6}$	$2 \cdot 10^{-9}$

4.3 Checking gravitational forces

To check the gravitational force working on the earth from the sun in our two body system, we started by testing earth's necessary initial velocity to escape the sun, the escape velocity. Figure 4.8 shows the plot of the exact escape velocity and Figure 4.9 show some of the velocities near the exact escape velocity.

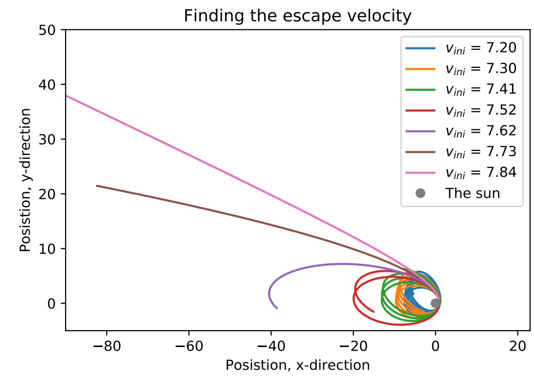

Figure 4.8: This is a plot of the sun-earth system for the exact initial velocity that allows the earth to escape the sun.

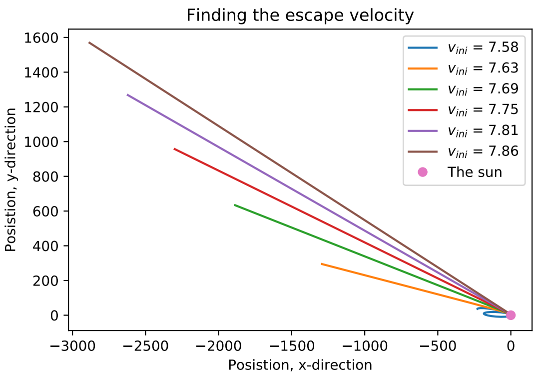

Figure 4.9: This is a plot of the sun-earth system for different initial velocities. The plot shows initial velocities near the exact escapevelocity, $2\pi\sqrt{2} \approx 8.88$.

Next we wanted to see what happened if we changed the gravitational force. The gravitational force is given by $F = \frac{GM_EM_S}{r^\beta}$ where β is 2. In Figure 4.10 we plotted the motion of the earth with different β between 2 and 3.

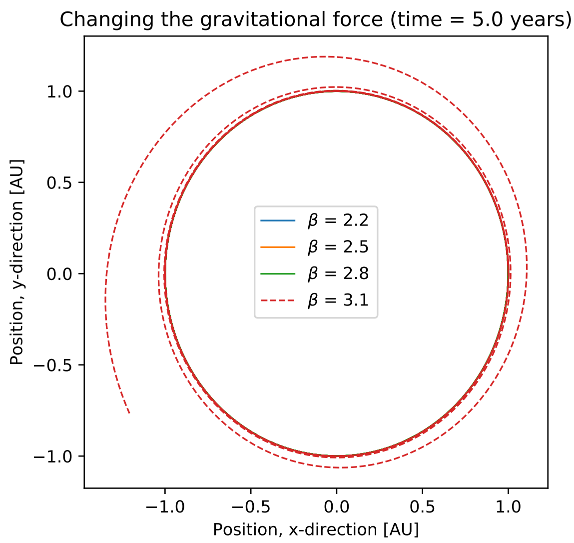

Figure 4.10: This is a plot of the sun-earth system for different gravitational forces. The gravitational force is given by $F = \frac{-GM_EM_S}{r^\beta}$ and in the plot β is changed from it's normal value $\beta = 2$ to $\beta = 3.1$.

4.4 Three body system

Next we added a planet, Jupiter, to our system and got a three body problem. Figure 4.11 shows the motion of the planets.

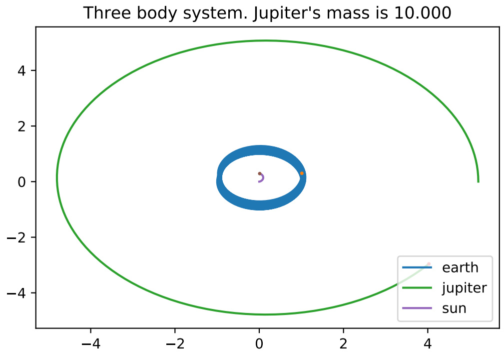

Figure 4.11: This is a plot of the three body system, Jupiter, Earth and Sun, with the origin in sun at time = 0.

Ok - if initial values are correct.

- Plot Earth's motion for increased mass of Jupiter (3 masses)

same.

- Find center off mass - use as origin

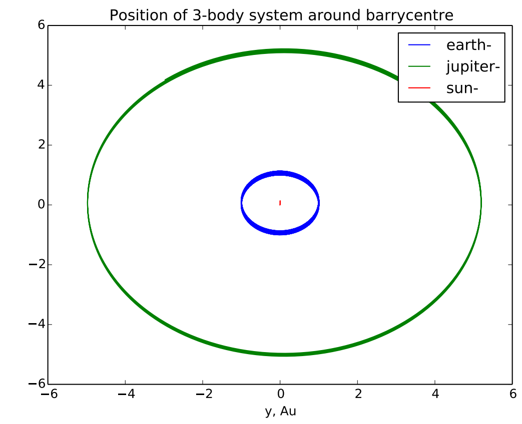

Figure 4.12

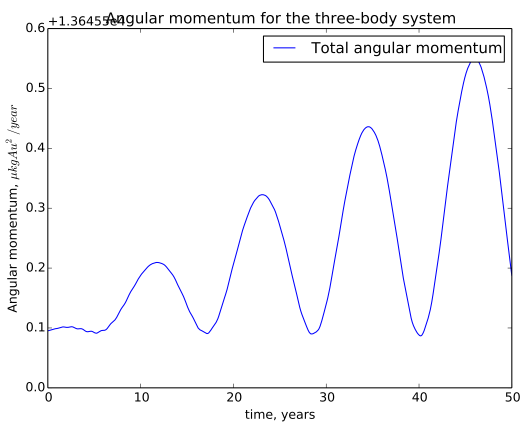

Figure 4.13

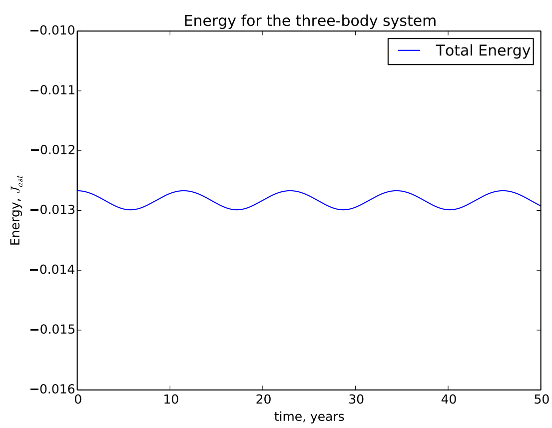

Figure 4.14

Found and incorporated.

- Give sun initial velocity so momentum is zero (origin is fixed)

How? What momentum? velocity?

- Compare with 3e) - Extend to all planets (plot)

4.5 All planets

At last we added all the planets with initial conditions form NASA [2]. Figure 4.15 shows the orbits of all the planets and Figure 4.16 shows the inner planet's orbits, because they are difficult to distinguish in the other plot.

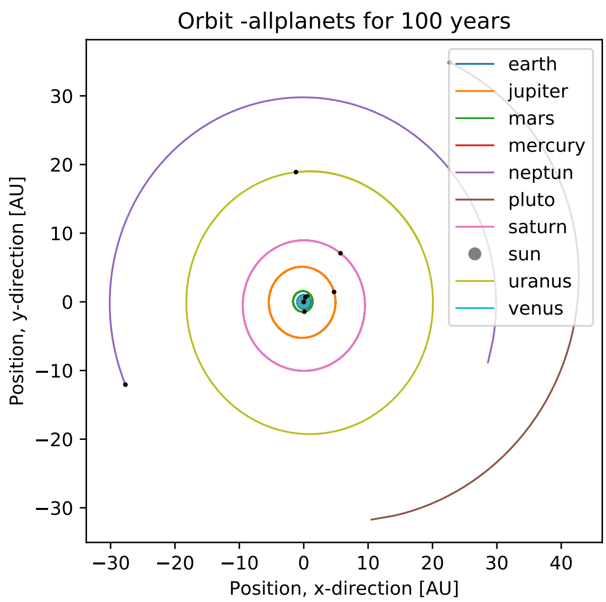

Figure 4.15: This is a plot of the Solar System after 100 years of motion.

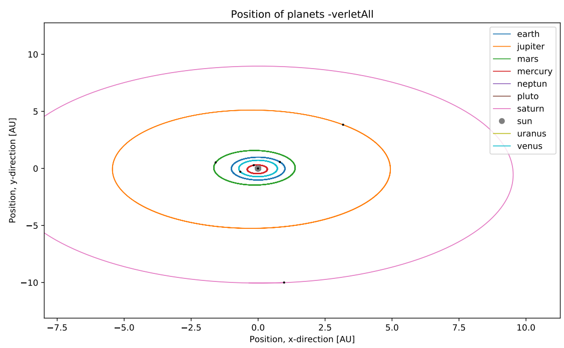

Figure 4.16: This is a plot of planets closest to the sun in the Solar System after 30 years of motion.

4.6 Considering relativistic force

As a last point we wanted to see how relativistic forces affect the orbits. We used the perihelion precession of Mercury to explore this. Table 4.3 shows the perihelion position of Mercury after 100 years of orbiting the sun alone with no other planets affecting the movement. The table lists the position with non-relativistic forces and relativistic forces.

Table 4.3: This is a table with the Perihelion information about Mercury after one century, 100 years. The perihelion precession is observed to be 43 arc seconds and, as we can see, this fits well with our result.

	Position [AU]	Angle [arc
Initial values:	$(0.3075, 0)$	0
Newtonian	$(0.3075, -1.519 \cdot 10^{-7})$	-0.102
Relativistic	$(0.3075, 6.447 \cdot 10^{-5})$	43.247

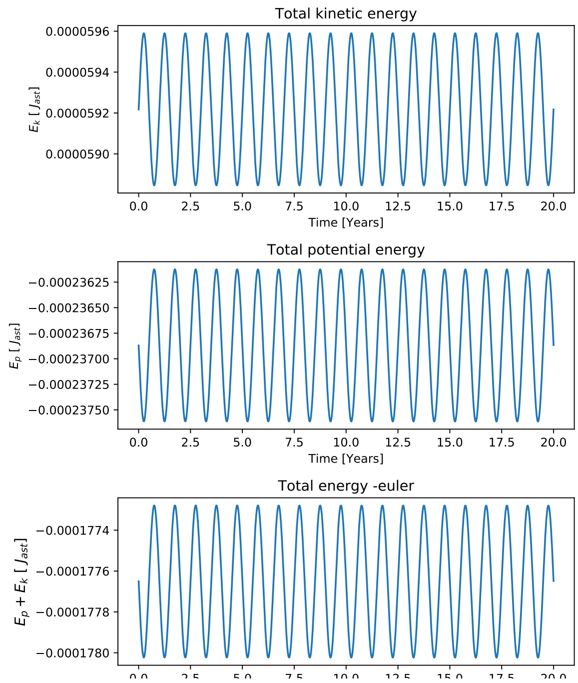

Figure 6.1: This is a plot of the total energies of the sun-earth system for 20 years using Euler's method, with 1000 timesteps per year.

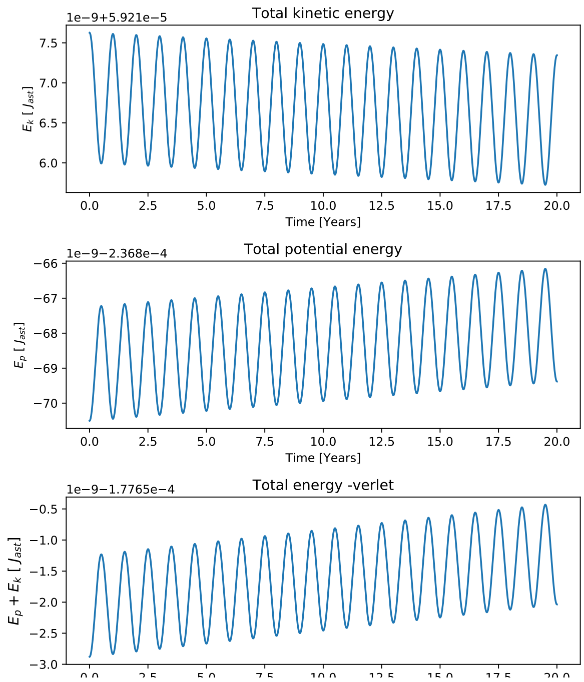

Figure 6.2: This is a plot of the total energies of the sun-earth system for 20 years using the velocity Verlet method, with 1000 timesteps per year.

5 Discussion

Can the observed perihelion precession of Mercury be explained by the general theory of relativity?

6 Conclusion

References

[1] Morten Hjorth-Jensen. Computational physics: Lecture notes fall 2015. Department of Physics, University of Oslo, 8 2015. Chapter 2 and 6.

[2] Solar system dynamics. https://ssd.jpl.nasa.gov/horizons.cgi#top. Accessed: 2017-10-28.

Appendix

appendix