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MATH 572

HW #2

02/16/2016

Problem #1

Consider the finite differnce scheme for the 1D steady convection-diffusion equation:

$$\epsilon u^{''} - u^{'} = -1, \quad 0 < x < 1,$$

$$u(0) = 1, \quad u(1) - 3.$$

a)

Verify the exact solution is

$$u(x) = 1 + x + \frac{e^{x/\epsilon} - 1}{e^{1/\epsilon} - 1}$$

Starting from the differential equation rerrange it to be in the form of

$$u^{''} - \frac{1}{\epsilon}u^{'} = -\frac{1}{\epsilon}. \tag{1}$$

From this we can find the general solution from

$$0 = r^2 - \frac{1}{\epsilon}r = r(r - \frac{1}{\epsilon}) \implies r = 0, r = \frac{1}{\epsilon}, \tag{2}$$

therefore our solution is in the form of

$$u(x) = c_1 e^{x/\epsilon} + \psi, \tag{3}$$

where ψ is the particular solution, which we guess the form to be

$$\begin{cases} \psi & = Ax^2 + Bx + C \\ \psi' & = 2Ax + B \\ \psi'' & = 2A \end{cases}. \tag{4}$$

From this we can solve for the unknown coefficients

$$-\frac{1}{\epsilon} = \psi'' \cdot 1 + \psi' \cdot (-\frac{1}{\epsilon}) + \psi \cdot 0 = 2A - \frac{1}{\epsilon}\left(2Ax + B\right), \tag{5}$$

and from this we get a system of equations

$$\begin{cases} -\frac{1}{\epsilon} & = 2A - \frac{1}{\epsilon}B \\ 0 & = -\frac{1}{\epsilon}2Ax \implies A = 0 \\ -\frac{1}{\epsilon} & = -\frac{1}{\epsilon}B \implies B = 1 \end{cases}, \tag{6}$$

therefore our particular solution is

$$\psi = x + C, \tag{7}$$

and our solution has the form of

$$u(x) = c_1 e^{x/\epsilon} + x + C. \tag{8}$$

Now plug in the Boundary Conditions

$$\begin{cases} u(0) = 1 & = c_1 + C \\ u(1) = 3 & = c_1 e^{1/\epsilon} + 1 + C \end{cases}, \tag{9}$$

and we get

$$C = 1 - c_1. \tag{10}$$

Plug this into the second boundary condition equation to get

$$\begin{aligned} 3 &= c_1 e^{1/\epsilon} + 1 + 1 - c_1 \\ 1 &= c_1 \left(e^{1/\epsilon} - 1\right) \implies c_1 = \frac{1}{e^{1/\epsilon} - 1}, \end{aligned} \tag{11}$$

and now we can solve for C to get

$$C = 1 - c_1 = 1 - \frac{1}{e^{1/\epsilon} - 1}. \tag{12}$$

Now plug this constants into Eqn. 8 to get

$$u(x) = \frac{1}{e^{1/\epsilon} - 1} e^{x/\epsilon} + x + 1 - \frac{1}{e^{1/\epsilon} - 1} \implies \boxed{u(x) = 1 + x + \frac{e^{x/\epsilon} - 1}{e^{1/\epsilon} - 1}}, \tag{13}$$

which is the exact solution verified.

b)

To solve the central differnce scheme we set up the following matrix equation with the initial conidtions of $u(0) = 1;\ u(1) = 3,\ AU = f$

$$\frac{1}{h^2}\begin{bmatrix} h^2 & 0 & \cdots & \cdots & \cdots & 0 \\ \left(\epsilon + \frac{h}{2}\right) & -2\epsilon & \left(\epsilon - \frac{h}{2}\right) & 0 & 0 & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 & \vdots \\ \vdots & 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & 0 & 0 & \left(\epsilon + \frac{h}{2}\right) & -2\epsilon & \left(\epsilon - \frac{h}{2}\right) \\ 0 & \cdots & \cdots & \cdots & 0 & h^2 \end{bmatrix} \begin{bmatrix} u_0 \\ u_1 \\ u_2 \\ \vdots \\ u_{m-1} \\ u_m \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ \vdots \\ \vdots \\ -1 \\ 3 \end{bmatrix}. \tag{14}$$

From this we get the following errors for the grid refinement analysis for each of the ϵ cases shown in From this we can see that the order of accuracy is $\mathcal{O}(h^2)$. We also get the different plots for the grid refinement for each ϵ case and the following figures show this.

To solve the central-upwind differnce scheme we set up the following matrix equation with the initial conidtions of $u(0) = 1;\ u(1) = 3,\ AU = f$

$$\frac{1}{h^2}\begin{bmatrix} h^2 & 0 & \cdots & \cdots & \cdots & 0 \\ \left(\epsilon + \frac{h}{2}\right) & -(2\epsilon + h) & \epsilon & 0 & 0 & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 & \vdots \\ \vdots & 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & 0 & 0 & \left(\epsilon + \frac{h}{2}\right) & -(2\epsilon + h) & \epsilon \\ 0 & \cdots & \cdots & \cdots & 0 & h^2 \end{bmatrix} \begin{bmatrix} u_0 \\ u_1 \\ u_2 \\ \vdots \\ u_{m-1} \\ u_m \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ \vdots \\ \vdots \\ -1 \\ 3 \end{bmatrix}. \tag{15}$$

From this we get the following errors for the grid refinement analysis for each of the ϵ cases shown in From this we can see that the order of accuracy is $\mathcal{O}(h)$. We also get the different plots for the grid refinement for each ϵ case and the following figures show the numerical soluton against the numerical solutions for each ϵ case.

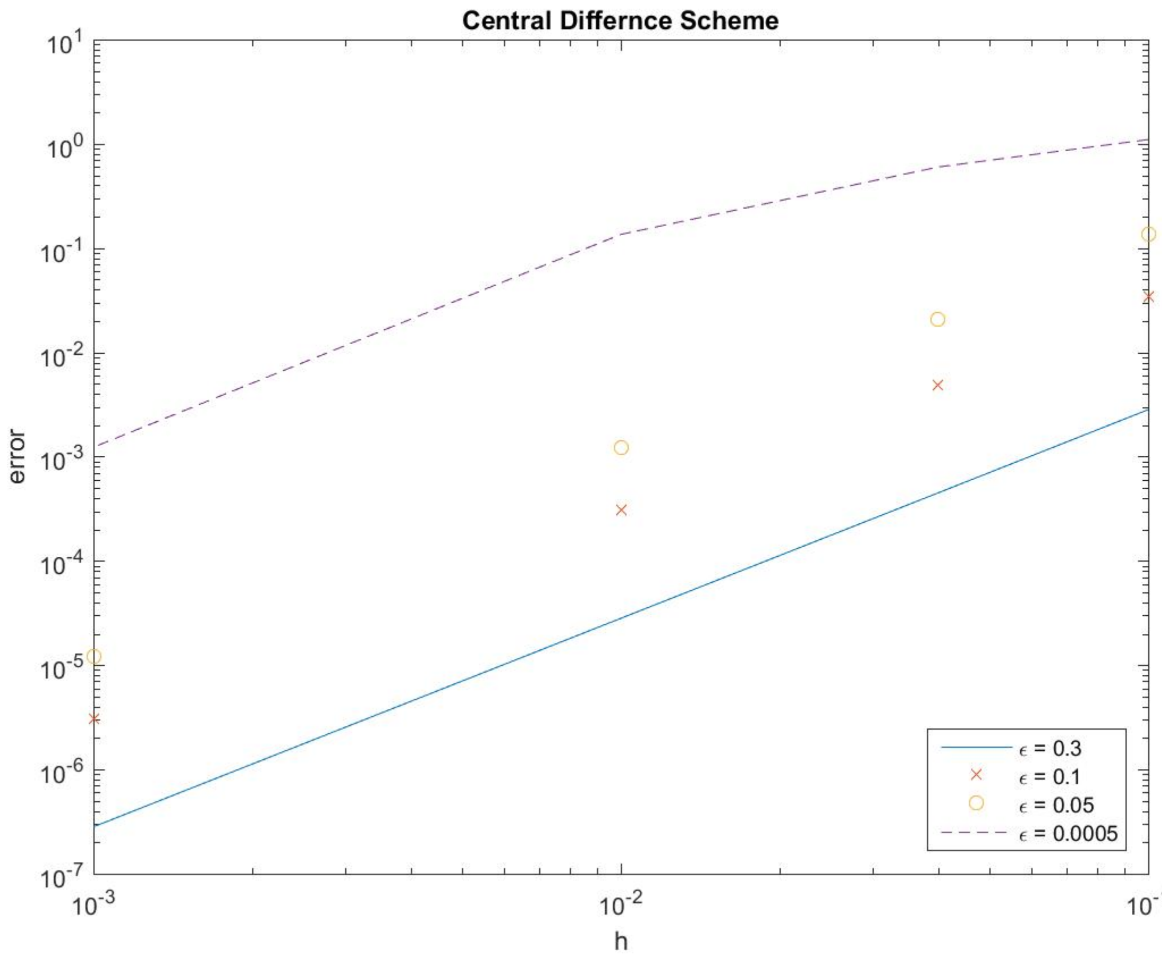

Figure 1: Central difference scheme grid refinement error analysis

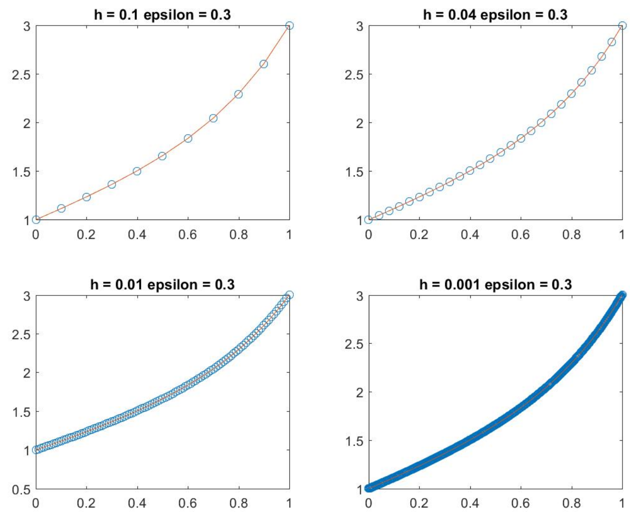

Figure 2: Central difference scheme $\epsilon = 0.3$

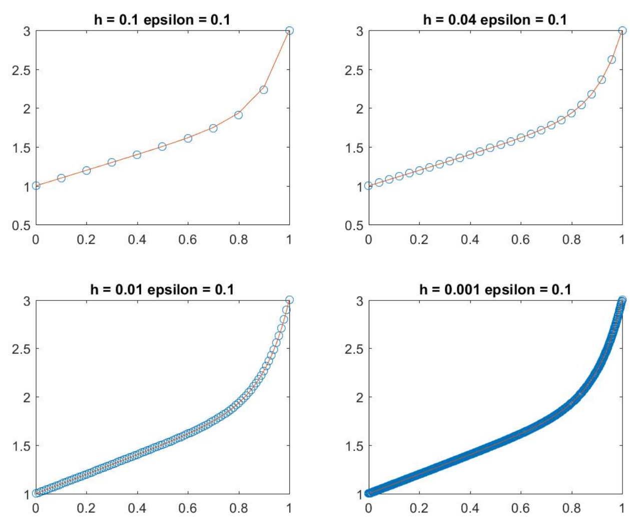

Figure 3: Central difference scheme $\epsilon = 0.1$

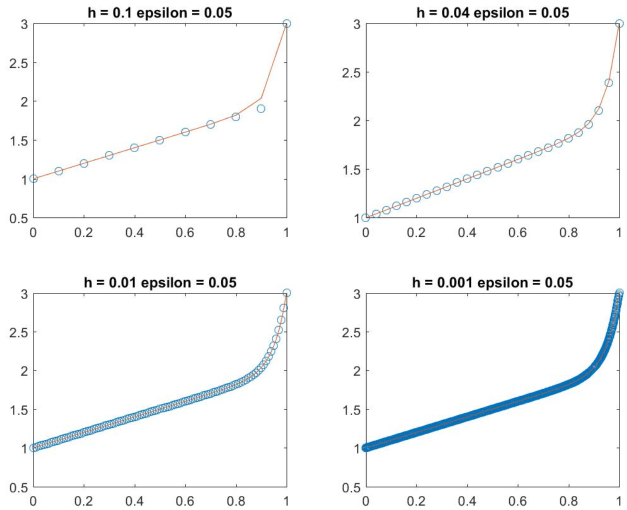

Figure 4: Central difference scheme $\epsilon = 0.05$

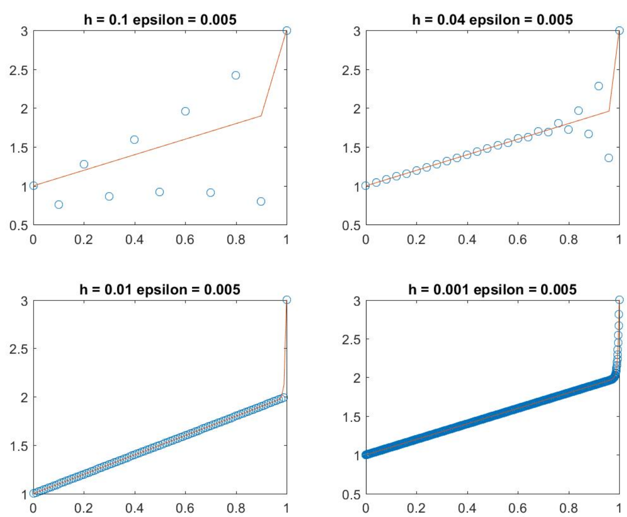

Figure 5: Central difference scheme $\epsilon = 0.005$

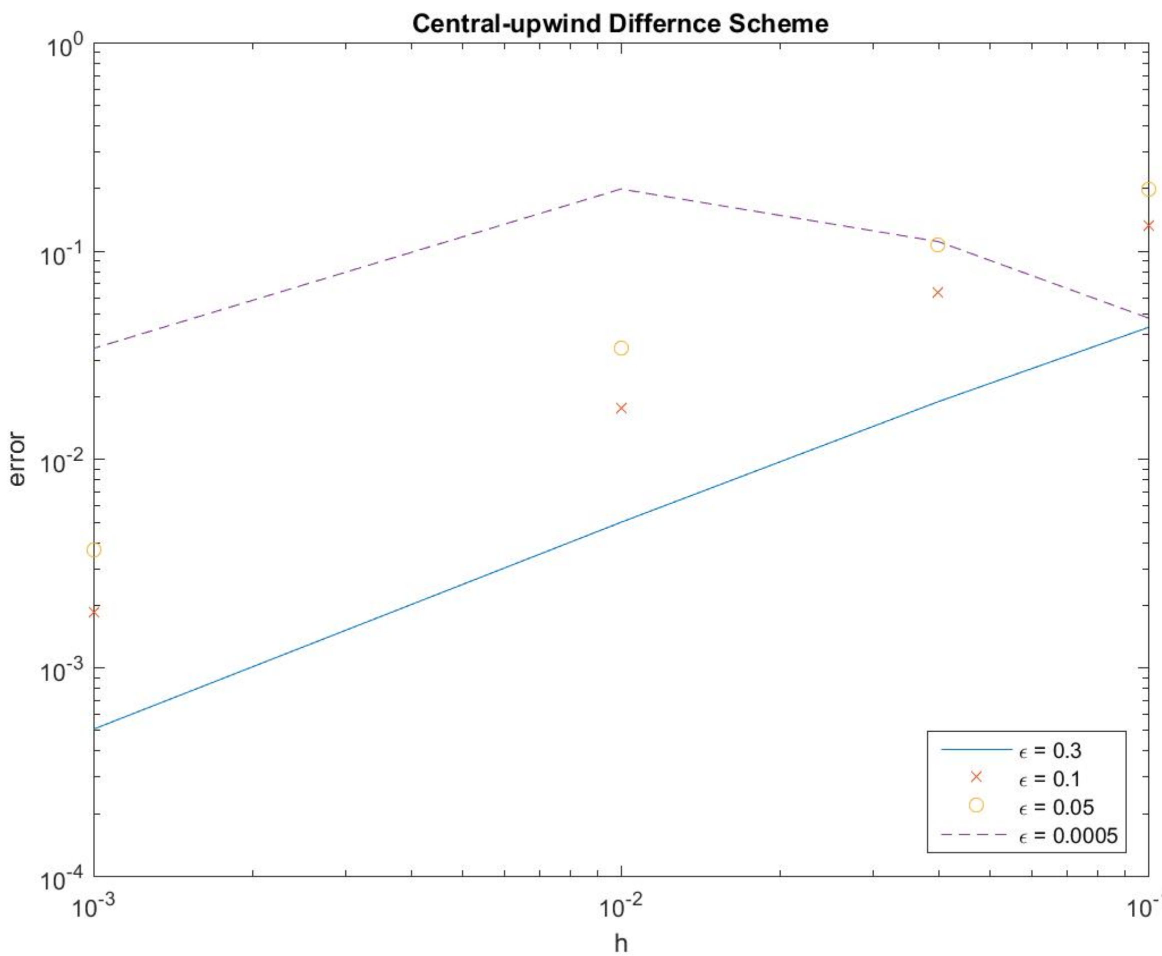

Figure 6: Central-upwind difference scheme grid refinement error analysis

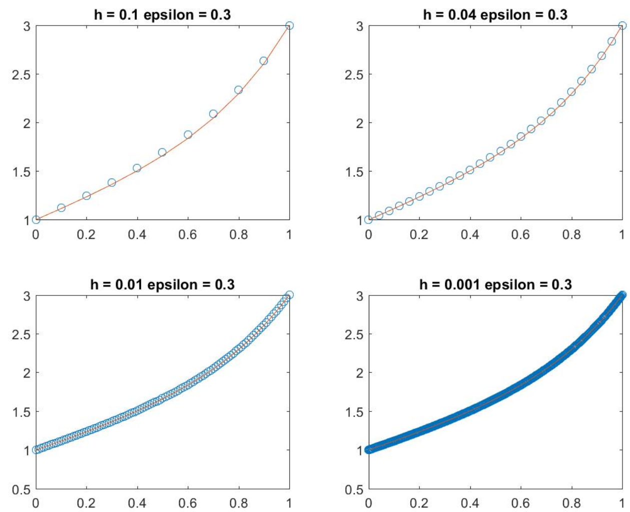

Figure 7: Central-upwind difference scheme $\epsilon = 0.3$

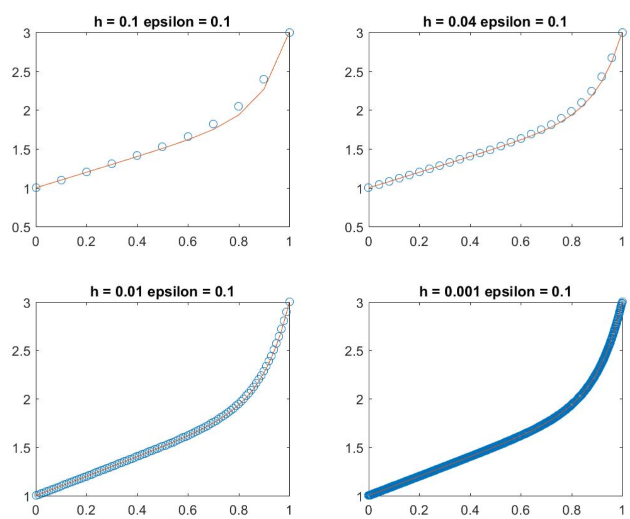

Figure 8: Central-upwind difference scheme $\epsilon = 0.1$

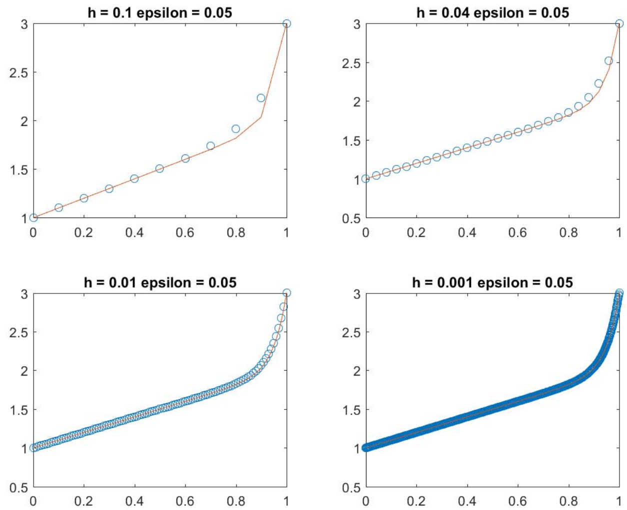

Figure 9: Central-upwind difference scheme $\epsilon = 0.05$

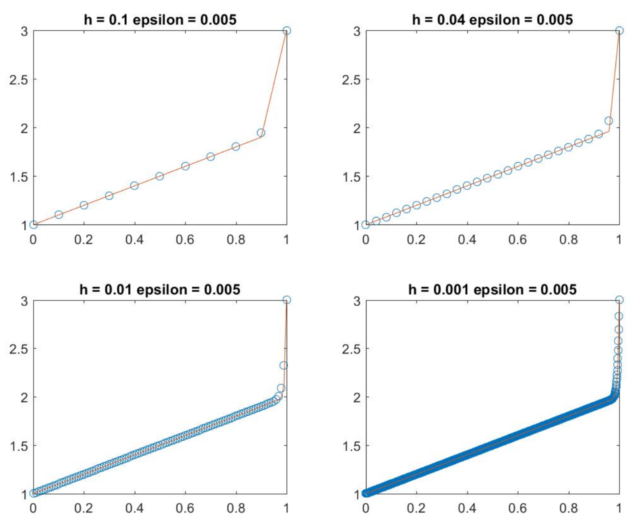

Figure 10: Central-upwind difference scheme $\epsilon = 0.005$

c)

From these results the order of accuracy tells us that the Central difference scheme is best, but for cases where ϵ is small the Central-upwind difference scheme is best when our grid is small. This is seen in both the error plots of Figures 1 and 6, as well as the plots of the analytic solution against the numerical solutions in Figures 5 and 10.

Problem #2

a)

Write a program to solve the boundary value problem for the nonlinear pendulum as discussed in the text. See if you an find yet another solution for the boundary conditions illustrated in Figures 2.4 and 2.5.

Using the method given in the book the following plots were recreated for Figures 2.4 and 2.5.

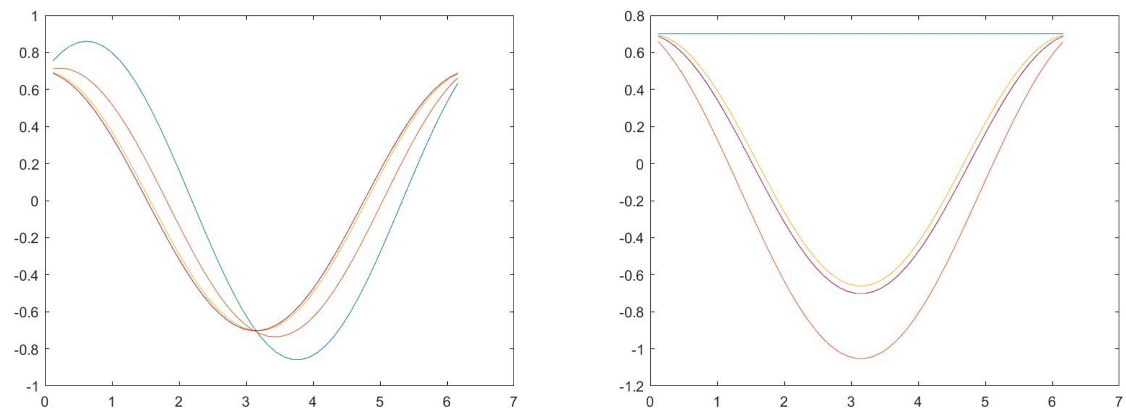

Figure 11: Figures 2.4 recreated

Both of these converged with 7 itterations for $m = 100$ in the example given in the book. Another solution can be found by starting with

$$\theta(t_i) = 0.7 \cdot \cos(t_i) + 0.6 \cdot \sin(t_i), \tag{16}$$

and

$$\theta(t_i) = 0.7 \cdot \cos(t_i). \tag{17}$$

The solution from Equation 16 converges in 8 steps, and the solution from Equation 17 converges in 5 steps. The plots from these solutions are

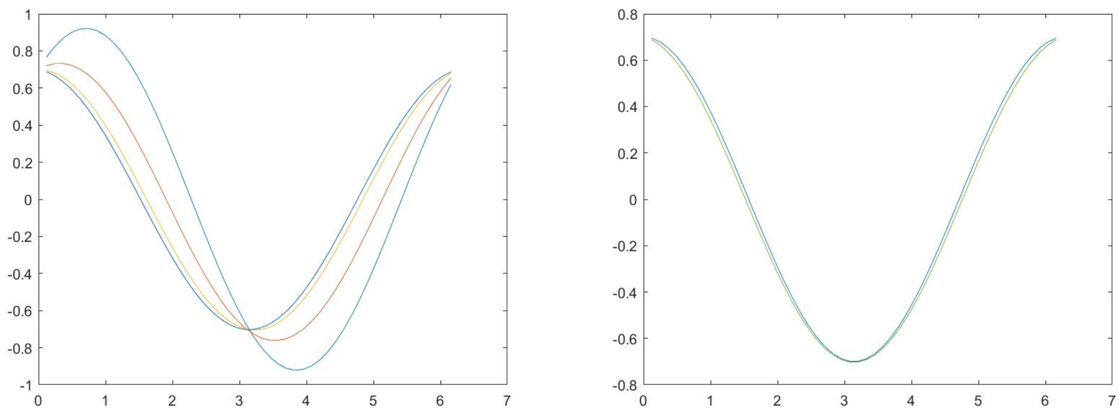

Figure 12: Alternate solutions (right Eqn. 16, left Eqn. 17)

b)

Find numerical solution to this BVP with the same general behavior as seen in Figure 2.5 for the case of a longer time interval, say $T = 20$, again with $\alpha = \beta = 0.7$. Try larger values of T. What does $\max_i \theta_i$ approach as T is increased? Note that for large T this solution exhibits "boundary layers".

The original solution was recreated with a $\max \theta_i = 2.897$ for $T = 2\pi$. The following is the recreated figure

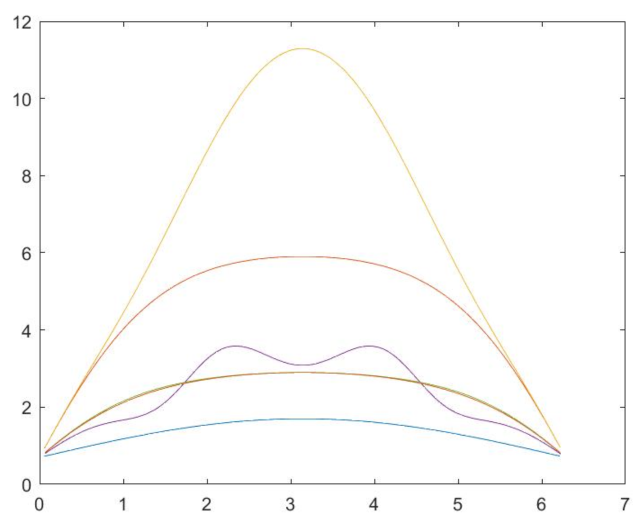

Figure 13: Problem 2b, Figure 2.5

As T is increased, the $\max \theta_i$ approaches infinity. The following plots show the solution for $T = 20, 50, 100, 200$.

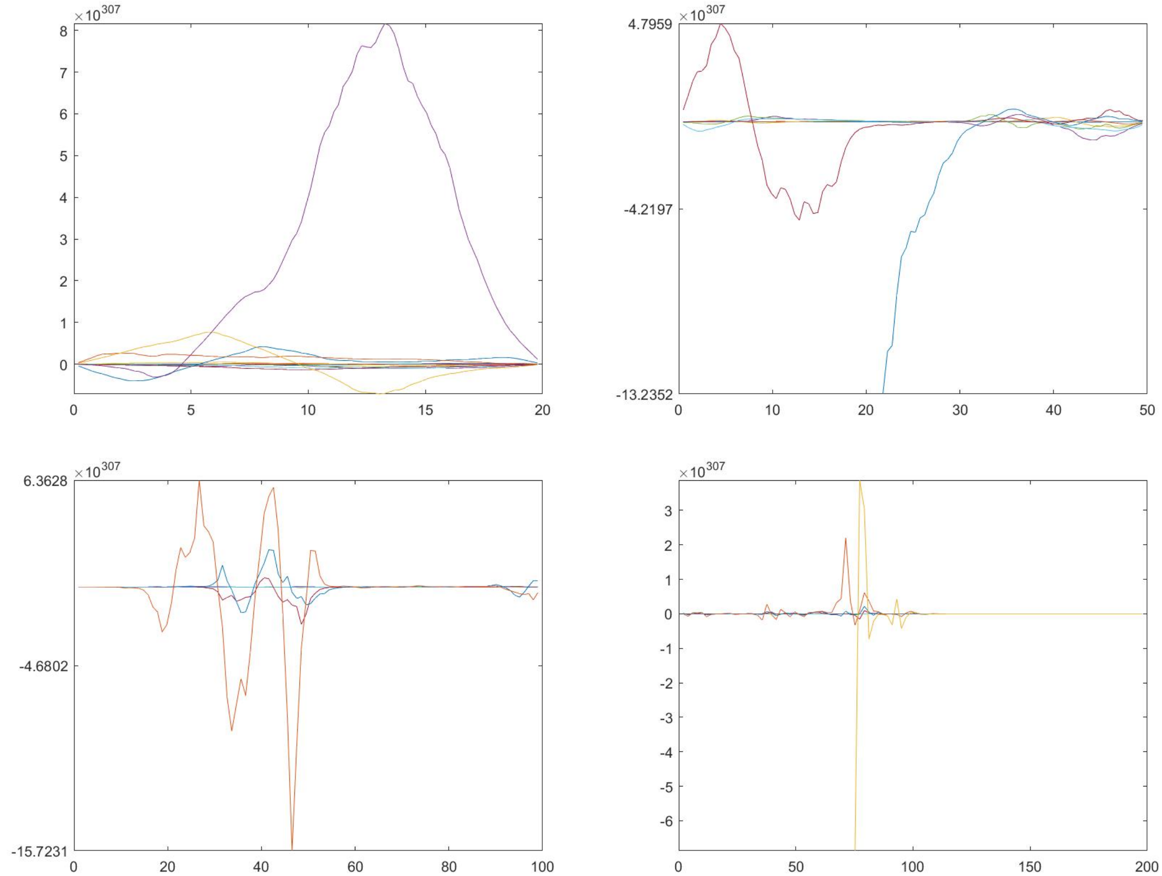

Figure 14: Problem 2b, increasing T (Top left T=20, Top right T=50, Bottom left T=100, Bottom right T=200)

Problem #3

a)

The function appears to converge when $N = 30$, which gives us the plot of

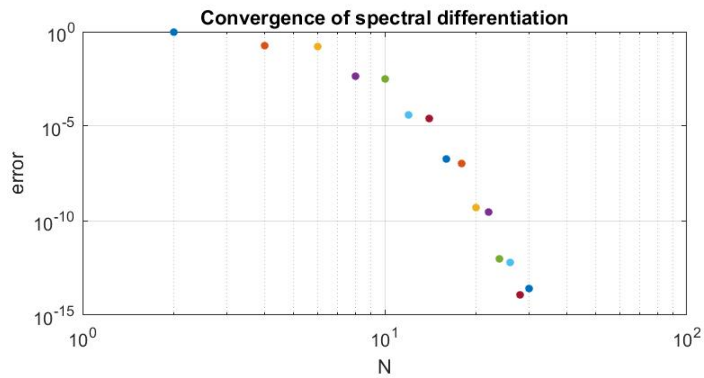

Figure 15: Problem 3a: Convergence of spectral difference

and then plotting the error vs. h we get the following plot

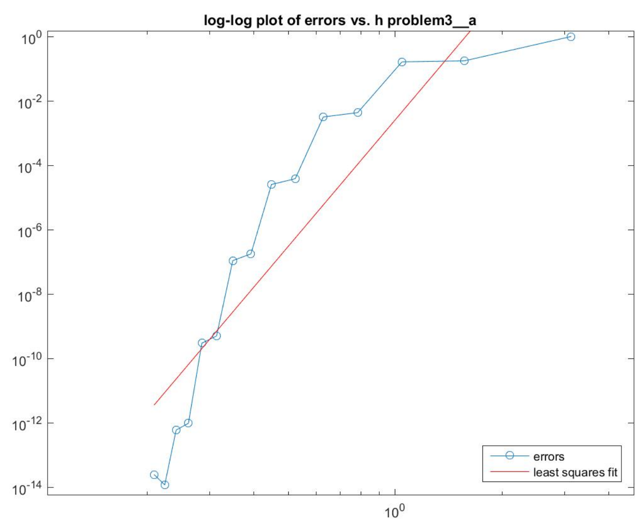

Figure 16: Problem 3a: Error vs. h

which gives us an order of accuracy of $\approx \mathcal{O}(h^{13})$.

b)

A second-order finite difference scheme was used to compute the first derivative using

$$D_0 = \frac{u(x_{i+1}) - u(x_{i-1})}{2h}, \tag{18}$$

which gives us the following plot of error vs. h for a grid refinement of

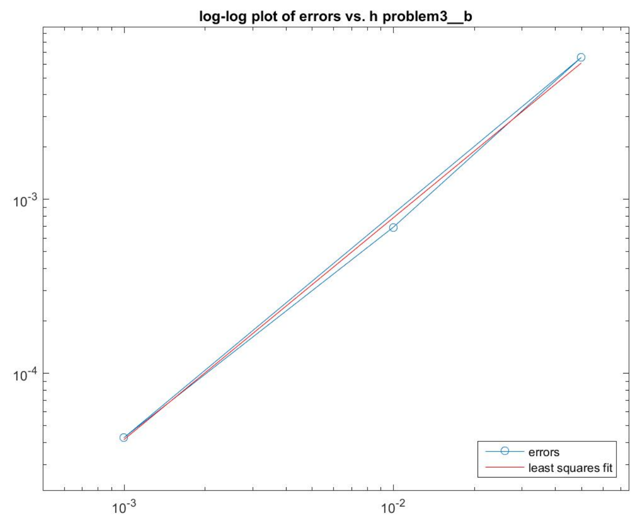

Figure 17: Problem 3b

which gives an order of accuarcy of $\mathcal{O}(h^{1.275}) \approx \mathcal{O}(h)$.

c)

The convergence of the spectral difference is plotted as

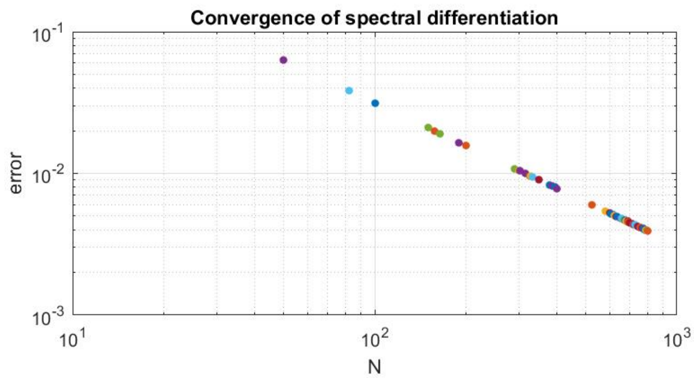

Figure 18: Problem 3c

which has a rate of convergence approimately proportional to N.

Problem #4

a)

Starting from the 9-point Laplacian

$$\Delta_9^h u_{ij} = \frac{1}{6h^2}\Big[4(u_{i-1,j} + u_{i+1,j} + u_{i,j-1} + u_{i,j+1})$$

$$+ u_{i-1,j-1} + u_{i-1,j+1} + u_{i+1,j-1} + u_{i+1,j+1} - 20u_{ij}\Big], \tag{19}$$

and the 5-point Laplacian is

$$\Delta_5^h u_{ij} = \frac{1}{h^2}\Big[u_{i-1,j} + u_{i+1,j} + u_{i,j-1} + u_{i,j+1} - 4u_{ij}\Big]. \tag{20}$$

Substitute Equation 20 into Equation 19, we get

$$\Delta_9^h u_{ij} = \frac{1}{6h^2}\Big[u_{i-1,j-1} + u_{i-1,j+1} + u_{i+1,j-1} + u_{i+1,j+1} - 4u_{ij}\Big] + \frac{4}{6}\Delta_5^h u_{ij}. \tag{21}$$

The truncation error can be found by

$$\tau = \Delta_9^h u_{ij} - \Delta u \tag{22}$$

$$\tau = \frac{1}{6h^2}\Big[u_{i-1,j-1} + u_{i-1,j+1} + u_{i+1,j-1} + u_{i+1,j+1} - 4u_{ij}\Big] + \frac{4}{6}\Delta_5^h u_{ij} - \Delta u, \tag{23}$$

where

$$\Delta u = u_{xx} + u_{yy}. \tag{24}$$

Take the Taylor expansions of the terms in Equation 23,

$$\begin{aligned} u_{i-1,j-1} = u + h\left(-u_x - u_y\right) + \frac{1}{2}h^2\left(u_{xx} + 2_{xy} + u_{yy}\right) + \frac{1}{6}h^3\left(-u_{xxx} - 3u_{xxy} - 3u_{xyy} - u_{yyy}\right) \\ + \frac{1}{24}h^4\left(u_{xxxx} + 4u_{xxxy} + 6u_{xxyy} + 4u_{xyyy} + u_{yyyy}\right) \end{aligned} \tag{25}$$

$$\begin{aligned} u_{i-1,j+1} = u + h\left(-u_x + u_y\right) + \frac{1}{2}h^2\left(u_{xx} - 2_{xy} + u_{yy}\right) + \frac{1}{6}h^3\left(-u_{xxx} + 3u_{xxy} - 3u_{xyy} + u_{yyy}\right) \\ + \frac{1}{24}h^4\left(u_{xxxx} - 4u_{xxxy} + 6u_{xxyy} - 4u_{xyyy} + u_{yyyy}\right) \end{aligned} \tag{26}$$

$$\begin{aligned} u_{i+1,j-1} = u + h\left(+u_x - u_y\right) + \frac{1}{2}h^2\left(u_{xx} - 2_{xy} + u_{yy}\right) + \frac{1}{6}h^3\left(+u_{xxx} - 3u_{xxy} + 3u_{xyy} - u_{yyy}\right) \\ + \frac{1}{24}h^4\left(u_{xxxx} - 4u_{xxxy} + 6u_{xxyy} - 4u_{xyyy} + u_{yyyy}\right) \end{aligned} \tag{27}$$

$$\begin{aligned} u_{i+1,j-1} = u + h\left(+u_x + u_y\right) + \frac{1}{2}h^2\left(u_{xx} + 2_{xy} + u_{yy}\right) + \frac{1}{6}h^3\left(+u_{xxx} + 3u_{xxy} + 3u_{xyy} + u_{yyy}\right) \\ + \frac{1}{24}h^4\left(u_{xxxx} + 4u_{xxxy} + 6u_{xxyy} + 4u_{xyyy} + u_{yyyy}\right) \end{aligned} \tag{28}$$

From this we have $-4u_{ij}$ in Eqn. 23 cancles out with the first terms of the taylor expansions. Collecting the second terms we get

$$h\left(-u_x - u_y\right) + h\left(-u_x + u_y\right) + h\left(+u_x - u_y\right) + h\left(+u_x + u_y\right) = 0. \tag{29}$$

For the second terms we get

$$\frac{1}{2}h^2\left(u_{xx} + 2_{xy} + u_{yy}\right) + \frac{1}{2}h^2\left(u_{xx} - 2_{xy} + u_{yy}\right) + \frac{1}{2}h^2\left(u_{xx} - 2_{xy} + u_{yy}\right) + \frac{1}{2}h^2\left(u_{xx} + 2_{xy} + u_{yy}\right) - \Delta u = 0 \tag{30}$$

Collecting the fourth terms we get

$$\begin{aligned} \frac{1}{6}h^3\left(-u_{xxx} - 3u_{xxy} - 3u_{xyy} - u_{yyy}\right) + \frac{1}{6}h^3\left(-u_{xxx} + 3u_{xxy} - 3u_{xyy} + u_{yyy}\right) \\ + \frac{1}{6}h^3\left(+u_{xxx} - 3u_{xxy} + 3u_{xyy} - u_{yyy}\right) + + \frac{1}{6}h^3\left(+u_{xxx} + 3u_{xxy} + 3u_{xyy} + u_{yyy}\right) = 0. \end{aligned} \tag{31}$$

Finally we collect the last terms

$$\begin{aligned} + \frac{1}{24}h^4\left(u_{xxxx} + 4u_{xxxy} + 6u_{xxyy} + 4u_{xyyy} + u_{yyyy}\right) + \frac{1}{24}h^4\left(u_{xxxx} - 4u_{xxxy} + 6u_{xxyy} - 4u_{xyyy} + u_{yyyy}\right) \\ + \frac{1}{24}h^4\left(u_{xxxx} - 4u_{xxxy} + 6u_{xxyy} - 4u_{xyyy} + u_{yyyy}\right) + \frac{1}{24}h^4\left(u_{xxxx} + 4u_{xxxy} + 6u_{xxyy} + 4u_{xyyy} + u_{yyyy}\right) - \Delta u \\ = h^4 u_{xxyy}. \end{aligned} \tag{32}$$

We then note that the next preceding term will go to zero, like the other odd terms of h, and the next term is $\mathcal{O}(h^6)$. Now we plug this back into Eqn.23 to get

$$\tau = \frac{1}{6h^2}\left[h^4 u_{xxyy} + \mathcal{O}(h^6)\right] + \frac{4}{6}\Delta_5^h u_{ij}, \tag{33}$$

which gives us the truncation error of

$$\tau = \frac{1}{6}h^2 u_{xxyy} + \mathcal{O}(h^4) + \frac{4}{6}\Delta_5^h u. \tag{34}$$

b)

To get the A matrix in the form for the 9 point stencil we change the form of it in the matlab code to

```
e = ones(m,1);
S=spdiags([e 10*e e], [-1 0 1], m,m);
I=spdiags([-1/2*e e -1/2*e].[-1 0 1],m,m);
A = (kron(I,S) + kron(S,I))/h^2;
A = -1*A;
```

Which gives us a matrix in the form of

$$\begin{bmatrix} -20 & 4 & & & & & & & \\ 4 & -20 & 4 & & & & & & \\ & 4 & -20 & & & & & & \\ & & & \ddots & & & & & \\ & & & & \ddots & & & & \\ 4 & 1 & & & & \ddots & & & \\ 1 & 4 & 1 & & & & \ddots & & \\ & 1 & 4 & & & & & \ddots \end{bmatrix}, \tag{35}$$

The error of this was giving is in order of 10^0; so there are some errors in the alogrithm for the 9-point that I was not able to figure out.