Optimizing Quadratic Functions Using Gradient Descent as a Mathematical Modeling Tool

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Abstract

Optimization is one of the most important ideas in mathematics and applied sciences. Many real-world systems rely on finding the best possible solution, whether that means minimizing cost, reducing error, or increasing efficiency. This project focuses on modeling and solving optimization problems using gradient descent. A one variable and a two variable quadratic function are examined using both analytical calculus methods and numerical gradient descent. The goal is to show how gradient descent can approximate exact solutions and how this process connects directly to artificial intelligence and machine learning. The results show that gradient descent successfully converges to the true minimum in both cases under proper conditions.

Introduction

Optimization is the process of finding the best possible outcome from a set of available choices. In mathematics, this usually means finding the minimum or maximum value of a function. Optimization appears in many real-world fields including engineering, economics, physics, and artificial intelligence. In machine learning, models improve by minimizing error through repeated adjustments to parameters.

In early calculus courses, optimization problems are solved analytically by taking derivatives and locating critical points. As systems become more complex and involve many variables, analytical solutions often become difficult or impossible to find. Because of this, numerical optimization methods such as gradient descent are widely used.

The purpose of this project is to model optimization problems using gradient descent and compare the numerical results to exact analytical solutions. A quadratic function in one variable and another in two variables are used to show how gradient descent works step by step and why it is such an important tool in modern computation.

Mathematical Formulation

One Variable Function:

The first function examined in this project is

$$f(x) = (x - 3)^2 + 4$$

This function represents a quadratic curve with one global minimum. The derivative of this function is

$$f'(x) = 2(x - 3)$$

Gradient descent updates the value of x using the formula

$$x_{n+1} = x_n - \alpha f'(x_n)$$

where α represents the learning rate and controls the size of each step.

Two Variable Function:

The second function examined is

$$f(x, y) = (x - 2)^2 + (y - 3)^2$$

This function forms a curved surface with a single minimum point. The gradient vector is

$$\partial f/\partial x = 2(x - 2)$$

$$\partial f/\partial y = 2(y - 3)$$

The gradient descent update rules become

$$x_{n+1} = x_n - \alpha\, 2(x_n - 2)$$

$$y_{n+1} = y_n - \alpha\, 2(y_n - 3)$$

These equations describe how the algorithm moves downhill toward the minimum point.

Analytical Solution Approach

One Variable Case:

To find the exact minimum of the one variable function, the derivative is set equal to zero

$$2(x - 3) = 0$$

$$x = 3$$

Substituting $x = 3$ back into the original function gives

$f(3) = 4$

So the exact minimum is $x = 3$ and the minimum value is 4.

Two Variable Case:

To find the analytical minimum of the two-variable function, both partial derivatives are set equal to zero

$2(x - 2) = 0$ gives $x = 2$

$2(y - 3) = 0$ gives $y = 3$

The exact minimum point is therefore (2, 3). Substituting into the function gives a minimum value of 0.

Numerical Solution Using Gradient Descent

One Variable Gradient Descent

The numerical method begins with an initial guess of $x = 0$. A learning rate of 0.1 is used, and the algorithm runs for twenty iterations. At each step, the derivative is calculated and subtracted from the current value of x.

After repeated updates, the value of x approaches 3 and the function value approaches 4. By the end of the iterations, the numerical solution is essentially equal to the analytical solution. This confirms that gradient descent successfully approximates the true minimum.

Two Variable Gradient Descent:

For the two-variable case, the algorithm begins at the point (0, 0) with the same learning rate of 0.1. At each step, both x and y are updated using the gradient formulas.

After only a few steps, the point moves significantly closer to the exact minimum of (2, 3). With additional iterations, the algorithm continues to move closer until it converges to the true minimum.

Interpretation of Results

The numerical solutions produced by gradient descent closely matched the analytical solutions found through calculus. This confirms that gradient descent works correctly for convex quadratic functions. The learning rate played an important role in determining how quickly the algorithm converged. Smaller learning rates gave slower but more stable convergence, while larger learning rates increased the risk of overshooting the minimum.

The results show that numerical optimization methods can effectively solve problems that are traditionally solved analytically and that both approaches support each other when validating results.

Assumptions and Limitations

Assumptions:

The functions used in this project were assumed to be smooth and differentiable. The cost surfaces were convex and contained only one global minimum. The learning rate was kept constant, and no randomness or noise was present in the system.

Limitations:

Gradient descent does not always perform well on complex or non-convex functions. Poor choices of learning rate can cause the algorithm to converge too slowly or fail entirely. In real world applications, the algorithm may become trapped in local minima. These limitations show why more advanced optimization methods are sometimes required.

Applications in Artificial Intelligence and the Real World

Gradient descent is the core learning mechanism behind most machine learning models. When a neural network is trained, it repeatedly adjusts millions of parameters by minimizing a loss function using gradient descent. Every improvement the model makes is based on the same process used in this project.

Gradient descent is also used in engineering to reduce energy usage, in economics to maximize profit, in robotics to optimize motion paths, and in physics to minimize energy systems. This shows how a simple mathematical model connects directly to powerful real-world technologies.

Conclusion

This project demonstrated how optimization problems can be modeled using both analytical calculus methods and numerical gradient descent. A one variable and a two variable quadratic function were solved using both approaches, and the numerical solutions closely matched the exact analytical results.

The results confirmed that gradient descent is a reliable and effective method for finding optimal solutions when analytical methods are difficult or impossible to apply. The project also showed how these ideas connect directly to artificial intelligence and modern computation. Overall, this project highlights the importance of optimization as a bridge between pure mathematics and real-world applications.

References

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