UE TSM - Les Séries de Volterra
TD - Partie 1

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Problem 1. System $\mathcal{F}$ is described with an input $u = v_j$ and an output $y = v_{j+1}$ and follows the equation (1):

$$\frac{1}{\omega}\frac{dv_j}{dt} + \tanh(v_j) = \tanh(v_{j-1}) \tag{1}$$

Question 1. Show the Volterra Kernel of $y = \tanh(u)$ in the time domain and Laplace domain:

$$\tanh(x) = \sum_{n=1}^{\infty} T_n x^n,$$

with $T_1 = 1, T_2 = 0, T_3 = -\frac{1}{3}, T_4 = 0, \cdots$

Solution. In time domain, this is a special case, static nonlinear system described by a polynomial in the input,

$$\begin{aligned} y(t) &= \sum_{i=1}^{\infty} g_n(t_1, \cdots, t_n)u(t-t_1)\cdots u(t-t_n)dt_1\cdots dt_n \\ &= \sum_{n=1}^{\infty} T_n u^n(t) \end{aligned}$$

Therefore, the kernel

$$g_n(t_1, \cdots, t_n) = T_n\delta(t_1)\cdots\delta(t_n).$$

In Laplace domain,

$$\begin{aligned} G_n(s_1, \cdots, s_n) &= \int_{\mathbb{R}^n} g_n(t_1, \cdots, t_n)e^{-(s_1t_1+\cdots+s_nt_n)}dt_1\cdots dt_n \\ &= T_n\int \delta(t_1)e^{-s_1t_1}dt_1\cdots\int\delta(t_n)e^{-s_nt_n}dt_n \\ &= T_n \end{aligned}$$

Therefore,

$$G_n(s_1, \cdots, s_n) = T_n$$

with ROC=$\mathbb{C}^n$. $\square$

Question 2. Show a block-diagram of *Canceling system* of $\mathcal{F}$.

Solution. From equation (1), we could obtain the relation between input u and output y

$$\frac{1}{\omega}\dot{y} + \tanh(y) - \tanh(u) = 0.$$

Let $z = \frac{1}{\omega}\dot{y} + \tanh(y) - \tanh(u)$, the block diagram is shown as Figure 1. We only need to verify the output $z = 0$.

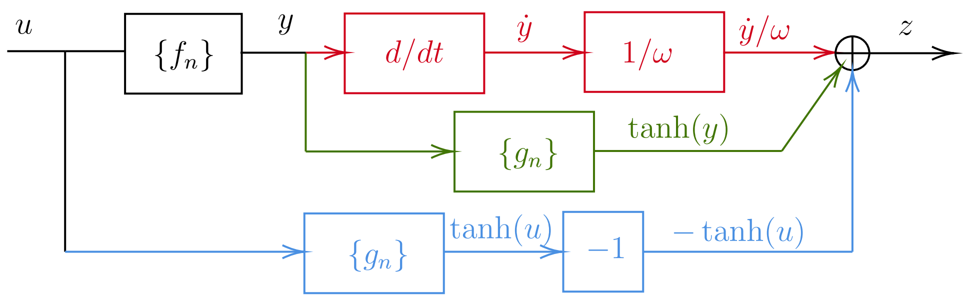

Figure 1: Canceling system schema of $\mathcal{F}$

□

Question 3. Show equations about $\{F_n\}$ of the diagram and calculate for $n = 1, 2, 3$.

Solution. $\{F_n\}$ cascades with the red part system (kernel $R(s) = \frac{s}{\omega}$), so that the equivalent kernel of these three system is

$$H_{r,n}(s_1, \cdots, s_n) = F_n(s_1, \cdots, s_n) R(s_1 + \cdots + s_n)$$

$\{F_n\}$ also cascades with green part system(kernel $G_n(s_1, \cdots, s_n) = T_n$), so that the equivalent kernel of these two system is

$$H_{g,n}(s_1, \cdots, s_n) = \sum_{p=1}^{n} \sum_{i_1+\cdots+i_p=n} F_{i_1}(s_1, \cdots, s_{i_1}) \cdots F_{i_p}(s_{i_1+\cdots+i_{p-1}+1}, \cdots, s_n)$$
$$G_p(s_1 + \cdots + s_{i_1}, \cdots, s_{i_1+\cdots+i_{p-1}+1} + \cdots + s_n)$$

The blue part equivalent kernel is $H_{b,n}(s_1, \cdots, s_n) = -G_n(s_1, \cdots, s_n)$.

Therefore,

$$H_{r,n}(s_1, \cdots, s_n) + H_{g,n}(s_1, \cdots, s_n) + H_{b,n}(s_1, \cdots, s_n) = 0$$

When $n = 1$, recall that $G_1(s) = T_1 = 1$,

$$\begin{cases} H_{r,1}(s_1) = F_1(s_1)\frac{s_1}{\omega} \\ H_{g,1}(s_1) = F_1(s_1)G_1(s_1) = F_1(s_1) \\ H_{b,1}(s_1) = -1 \end{cases}$$

$$F_1(s_1) = \frac{1}{1 + s_1/\omega}$$

$$\forall s_1 \in \mathbb{D} = (-\omega, +\infty) + i\mathbb{R}$$

When $n = 2$, recall that $G_2(s) = T_2 = 0$,

$$\begin{cases} H_{r,2}(s_1, s_2) = F_2(s_1, s_2)\frac{s_1+s_2}{\omega} \\ H_{g,2}(s_1, s_2) = F_2(s_1, s_2)G_1(s_1 + s_2) + F_1(s_1)F_1(s_2)G_2(s_1, s_2) = F_2(s_1, s_2) \\ H_{b,2}(s_1, s_2) = 0 \end{cases}$$

$$F_2(s_1, s_2) = 0$$

$$\forall s_1, s_2 \in \mathbb{C}^2$$

When $n = 3$, recall that$G_3(s) = T_3 = -1/3$,

$$\begin{cases} H_{r,3}(s_1, s_2, s_3) = F_3(s_1, s_2, s_3)\frac{s_1+s_2+s_3}{\omega} \\ H_{g,3}(s_1, s_2, s_3) = \quad F_3(s_1, s_2, s_3)G_1(s_1 + s_2 + s_3) \\ \qquad\qquad +F_2(s_1, s_2)F_1(s_3)G_2(s_1 + s_2, s_3) + F_1(s_1)F_2(s_2, s_3)G_2(s_1, s_2 + s_3) \\ \qquad\qquad +F_1(s_1)F_1(s_2)F_1(s_3)G_3(s_1, s_2, s_3) \\ \qquad\quad = F_3(s_1, s_2, s_3) - \frac{1}{3}F_1(s_1)F_1(s_2)F_1(s_3) \\ H_{b,3}(s_1, s_2, s_3) = 1/3 \end{cases}$$

$$F_3(s_1, s_2, s_3) = \frac{1}{3}\frac{F_1(s_1)F_1(s_2)F_1(s_3) - 1}{1 + (s_1 + s_2 + s_3)/\omega}$$

$$\forall s_1, s_2, s_3 \in \mathbb{D}^3$$

□

Question 4. Show a diagram of implementation for 3-order Volterra system.

Solution. We could express H_1, H_2, H_3 as

$$\begin{aligned} H_1(s_1) &= A_{1,1}(s_1) \quad \text{because convolution of two linear systems is still linear} \\ H_2(s_2, s_2) &= A_{2,1}(s_1)B_{2,1}(s_2)C_{2,1}(s_1 + s_2) \\ H_3(s_1, s_2, s_3) &= A_{3,1}(s_1)B_{3,1}(s_2)C_{3,1}(s_3)D_{3,1}(s_1 + s_2 + s_3) \end{aligned}$$

So that the diagram is shown as Figure 2.

Note that this implementation is only valid for a system lower than 3-order, as it is designed to precisely reproduce the terms of the Volterra series up to this order.

□

Problem 2. System $\mathcal{H}$ is described with an input $u = v_0$ and an output $y = v_4$, which cascades 4 $\mathcal{F}$-systems.

Question 5. Show the kernel of $\mathcal{H}$, H_n, for $n = 1, 2, 3, 4$.

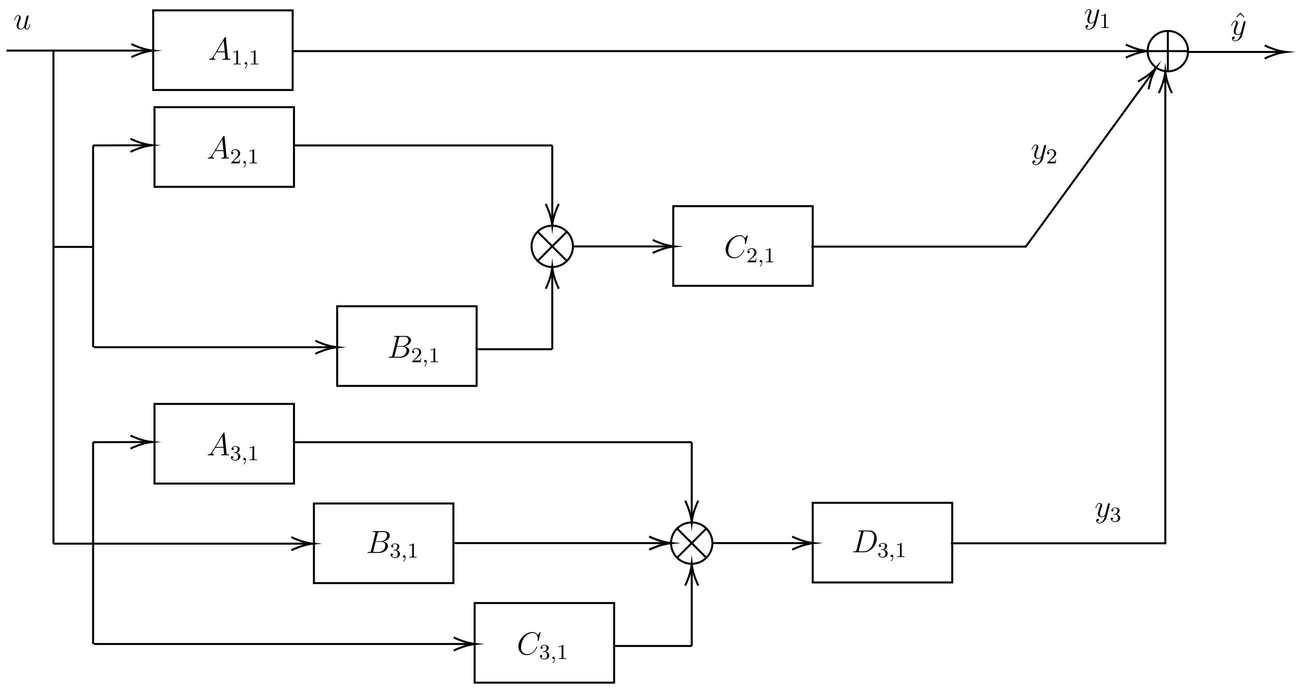

Figure 2: Diagram of Implementation for 3-order Volterra

Solution. First, consider system $\mathcal{G}$ is the cascade of two systems $\mathcal{F}$,

$$
\begin{aligned}
G_1(s_1) = &\quad [F_1(s_1)]^2 \\
G_2(s_1, s_2) = &\quad F_2(s_1, s_2)F_1(s_1 + s_2) \\
&+ F_1(s_1)F_1(s_2)F_2(s_1, s_2) \\
G_3(s_1, s_2, s_3) = &\quad F_3(s_1, s_2, s_3)F_1(s_1 + s_2 + s_3) \\
&+ F_2(s_1, s_2)F_1(s_3)F_2(s_1 + s_2, s_3) + F_1(s_1)F_2(s_2, s_3)F_2(s_1, s_2 + s_3) \\
&+ F_1(s_1)F_1(s_2)F_1(s_3)F_3(s_1, s_2, s_3) \\
G_4(s_1, s_2, s_3, s_4) = &\quad F_4(s_1, s_2, s_3, s_4)F_1(s_1 + s_2 + s_3 + s_4) \\
&+ F_3(s_1, s_2, s_3)F_1(s_4)F_2(s_1 + s_2 + s_3, s_4) \\
&+ F_2(s_1, s_2)F_2(s_3, s_4)F_2(s_1 + s_2, s_3 + s_4) \\
&+ F_1(s_1)F_3(s_2, s_3, s_4)F_2(s_1, s_2 + s_3 + s_4) \\
&+ F_2(s_1, s_2)F_1(s_3)F_1(s_4)F_3(s_1 + s_2, s_3, s_4) \\
&+ F_1(s_1)F_2(s_2, s_3)F_1(s_4)F_3(s_1, s_2 + s_3, s_4) \\
&+ F_1(s_1)F_1(s_2)F_2(s_3, s_4)F_3(s_1, s_2, s_3 + s_4) \\
&+ F_1(s_1)F_1(s_2)F_1(s_3)F_1(s_4)F_4(s_1, s_2, s_3, s_4)
\end{aligned}
$$

And then consider system $\mathcal{H}$ is the cascade of two systems $\mathcal{G}$,

$$
\begin{aligned}
H_1(s_1) = &\quad [G_1(s_1)]^2 \\
H_2(s_1, s_2) = &\quad G_2(s_1, s_2)G_1(s_1 + s_2) \\
&+ G_1(s_1)G_1(s_2)G_2(s_1, s_2)
\end{aligned}
$$

$$
\begin{aligned}
H_3(s_1,s_2,s_3) = &\quad G_3(s_1,s_2,s_3)G_1(s_1+s_2+s_3)\\
&+ G_2(s_1,s_2)G_1(s_3)G_2(s_1+s_2,s_3)\\
&+ G_1(s_1)G_2(s_2,s_3)G_2(s_1,s_2+s_3)\\
&+ G_1(s_1)G_1(s_2)G_1(s_3)G_3(s_1,s_2,s_3)\\
H_4(s_1,s_2,s_3,s_4) = &\quad G_4(s_1,s_2,s_3,s_4)G_1(s_1+s_2+s_3+s_4)\\
&+ G_3(s_1,s_2,s_3)G_1(s_4)G_2(s_1+s_2+s_3,s_4)\\
&+ G_2(s_1,s_2)G_2(s_3,s_4)G_2(s_1+s_2,s_3+s_4)\\
&+ G_1(s_1)G_3(s_2,s_3,s_4)G_2(s_1,s_2+s_3+s_4)\\
&+ G_2(s_1,s_2)G_1(s_3)G_1(s_4)G_3(s_1+s_2,s_3,s_4)\\
&+ G_1(s_1)G_2(s_2,s_3)G_1(s_4)G_3(s_1,s_2+s_3,s_4)\\
&+ G_1(s_1)G_1(s_2)G_2(s_3,s_4)G_3(s_1,s_2,s_3+s_4)\\
&+ G_1(s_1)G_1(s_2)G_1(s_3)G_1(s_4)G_4(s_1,s_2,s_3,s_4)
\end{aligned}
$$

We develop each term,

$$
H_1(s_1) = [F_1(s_1)]^4
$$

$$
\begin{aligned}
H_2(s_1,s_2) = &\quad (F_2(s_1,s_2)F_1(s_1+s_2) + F_1(s_1)F_1(s_2)F_2(s_1,s_2)) \cdot [F_1(s_1+s_2)]^2\\
&+ [F_1(s_1)]^2 \cdot [F_1(s_2)]^2 \cdot (F_2(s_1,s_2)F_1(s_1+s_2) + F_1(s_1)F_1(s_2)F_2(s_1,s_2))
\end{aligned}
$$

$$
\begin{aligned}
H_3(s_1,s_2,s_3) = &\quad (\quad F_3(s_1,s_2,s_3)F_1(s_1+s_2+s_3) + F_2(s_1,s_2)F_1(s_3)F_2(s_1+s_2,s_3)\\
&\qquad + F_1(s_1)F_2(s_2,s_3)F_2(s_1,s_2+s_3) + F_1(s_1)F_1(s_2)F_1(s_3)F_3(s_1,s_2,s_3))\\
&\quad \cdot [F_1(s_1+s_2+s_3)]^2\\
&+ (F_2(s_1,s_2)F_1(s_1+s_2) + F_1(s_1)F_1(s_2)F_2(s_1,s_2))\\
&\quad \cdot [F_1(s_3)]^2\\
&\quad \cdot (F_2(s_1+s_2,s_3)F_1(s_1+s_2+s_3) + F_1(s_1+s_2)F_1(s_3)F_2(s_1+s_2,s_3))\\
&+ [F_1(s_1)]^2\\
&\quad \cdot (F_2(s_2,s_3)F_1(s_2+s_3) + F_1(s_2)F_1(s_3)F_2(s_2,s_3))\\
&\quad \cdot (F_2(s_1,s_2+s_3)F_1(s_1+s_2+s_3) + F_1(s_1)F_1(s_2+s_3)F_2(s_1,s_2+s_3))\\
&+ [F_1(s_1)]^2\\
&\quad \cdot [F_1(s_2)]^2\\
&\quad \cdot [F_1(s_3)]^2\\
&\quad \cdot (\quad F_3(s_1,s_2,s_3)F_1(s_1+s_2+s_3) + F_2(s_1,s_2)F_1(s_3)F_2(s_1+s_2,s_3)\\
&\qquad + F_1(s_1)F_2(s_2,s_3)F_2(s_1,s_2+s_3) + F_1(s_1)F_1(s_2)F_1(s_3)F_3(s_1,s_2,s_3))
\end{aligned}
$$

$$
\begin{aligned}
& H_4(s_1, s_2, s_3, s_4) = \\
& (\quad F_4(s_1, s_2, s_3, s_4) F_1(s_1 + s_2 + s_3 + s_4) \\
& \quad + F_3(s_1, s_2, s_3) F_1(s_4) F_2(s_1 + s_2 + s_3, s_4) \\
& \quad + F_2(s_1, s_2) F_2(s_3, s_4) F_2(s_1 + s_2, s_3 + s_4) \\
& \quad + F_1(s_1) F_3(s_2, s_3, s_4) F_2(s_1, s_2 + s_3 + s_4) \\
& \quad + F_2(s_1, s_2) F_1(s_3) F_1(s_4) F_3(s_1 + s_2, s_3, s_4) \\
& \quad + F_1(s_1) F_2(s_2, s_3) F_1(s_4) F_3(s_1, s_2 + s_3, s_4) \\
& \quad + F_1(s_1) F_1(s_2) F_2(s_3, s_4) F_3(s_1, s_2, s_4 + s_4) \\
& \quad + F_1(s_1) F_1(s_2) F_1(s_3) F_1(s_4) F_4(s_1, s_2, s_3, s_4)) \\
& \cdot [F_1(s_1 + s_2 + s_3 + s_4)]^2 \\
+ & (\quad F_3(s_1, s_2, s_3) F_1(s_1 + s_2 + s_3) \\
& \quad + F_2(s_1, s_2) F_1(s_3) F_2(s_1 + s_2, s_3) \\
& \quad + F_1(s_1) F_2(s_2, s_3) F_2(s_1, s_2 + s_3) \\
& \quad + F_1(s_1) F_1(s_2) F_1(s_3) F_3(s_1, s_2, s_3)) \\
& \cdot [F_1(s_4)]^2 \\
& \cdot (\quad F_2(s_1 + s_2 + s_3, s_4) F_1(s_1 + s_2 + s_3 + s_4) \\
& \quad + F_1(s_1 + s_2 + s_3) F_1(s_4) F_2(s_1 + s_2 + s_3, s_4)) \\
+ & (\quad F_2(s_1, s_2) F_1(s_1 + s_2) \\
& \quad + F_1(s_1) F_1(s_2) F_2(s_1, s_2)) \\
& \cdot (\quad F_2(s_3, s_4) F_1(s_3 + s_4) \\
& \quad + F_1(s_3) F_1(s_4) F_2(s_3, s_4)) \\
& \cdot (\quad F_2(s_1 + s_2, s_3 + s_4) F_1(s_1 + s_2 + s_3 + s_4) \\
& \quad + F_1(s_1 + s_2) F_1(s_3 + s_4) F_2 s_1 + s_2, s_3 + s_4)) \\
+ & [F_1(s_1)]^2 \\
& \cdot (\quad F_3(s_2, s_3, s_4) F_1(s_2 + s_3 + s_4) \\
& \quad + F_2(s_2, s_3) F_1(s_4) F_2(s_2 + s_3, s_4) \\
& \quad + F_1(s_2) F_2(s_3, s_4) F_2(s_2, s_3 + s_4) \\
& \quad + F_1(s_2) F_1(s_3) F_1(s_4) F_3(s_2, s_3, s_4)) \\
& \cdot (\quad F_2(s_1, s_2 + s_3 + s_4) F_1(s_1 + s_2 + s_3 + s_4) \\
& \quad + F_1(s_1) F_1(s_2 + s_3 + s_4) F_2(s_1, s_2 + s_3 + s_4)) \\
+ & (\quad F_2(s_1, s_2) F_1(s_1 + s_2) \\
& \quad + F_1(s_1) F_1(s_2) F_2(s_1, s_2))
\end{aligned}
$$

$$
\begin{aligned}
& \cdot [F_1(s_3)]^2 \\
& \cdot [F_1(s_4)]^2 \\
& \cdot (\quad F_3(s_1 + s_2, s_3, s_4) F_1(s_1 + s_2 + s_3 + s_4) \\
& \quad + F_2(s_1 + s_2, s_3) F_1(s_4) F_2(s_1 + s_2 + s_3, s_4) \\
& \quad + F_1(s_1 + s_2) F_2(s_3, s_4) F_2(s_1 + s_2, s_3 + s_4) \\
& \quad + F_1(s_1 + s_2) F_1(s_3) F_1(s_4) F_3(s_1 + s_2, s_3, s_4)) \\
+ & [F_1(s_1)]^2 \\
& \cdot (\quad F_2(s_2, s_3) F_1(s_2 + s_3) \\
& \quad + F_1(s_2) F_1(s_3) F_2(s_2, s_3)) \\
& \cdot [F_1(s_4)]^2 \qquad \cdot (\\
& \quad + F_2(s_1, s_2 + s_3) F_1(s_4) F_2(s_1 + s_2 + s_3, s_4) \\
& \quad + F_1(s_1) F_2(s_2 + s_3, s_4) F_2(s_1, s_2 + s_3 + s_4) \\
& \quad + F_1(s_1) F_1(s_2 + s_3) F_1(s_4) F_3(s_1, s_2 + s_3, s_4)) \\
+ & [F_1(s_1)]^2 \\
& \cdot [F_1(s_2)]^2 \\
& \cdot (\quad F_2(s_3, s_4) F_1(s_3 + s_4) \\
& \quad + F_1(s_3) F_1(s_4) F_2(s_3, s_4)) \\
& \cdot (\quad F_3(x_1, x_2, x_3 + x_4) F_1(x_1 + x_2 + x_3 + x_4) \\
& \quad + F_2(x_1, x_2) F_1(x_3 + x_4) F_2(x_1 + x_2, x_3 + x_4) \\
& \quad + F_1(x_1) F_2(x_2, x_3 + x_4) F_2(x_1, x_2 + x_3 + x_4) \\
& \quad + F_1(x_1) F_1(x_2) F_1(x_3 + x_4) F_3(x_1, x_2, x_3 + x_4)) \\
+ & [F_1(s_1)]^2 \\
& \cdot [F_1(s_2)]^2 \\
& \cdot [F_1(s_3)]^2 \\
& \cdot [F_1(s_4)]^2 \\
& \cdot (\quad F_4(s_1, s_2, s_3, s_4) F_1(s_1 + s_2 + s_3 + s_4) \\
& \quad + F_3(s_1, s_2, s_3) F_1(s_4) F_2(s_1 + s_2 + s_3, s_4) \\
& \quad + F_2(s_1, s_2) F_2(s_3, s_4) F_2(s_1 + s_2, s_3 + s_4) \\
& \quad + F_1(s_1) F_3(s_2, s_3, s_4) F_2(s_1, s_2 + s_3 + s_4) \\
& \quad + F_2(s_1, s_2) F_1(s_3) F_1(s_4) F_3(s_1 + s_2, s_3, s_4) \\
& \quad + F_1(s_1) F_2(s_2, s_3) F_1(s_4) F_3(s_1, s_2 + s_3, s_4) \\
& \quad + F_1(s_1) F_1(s_2) F_2(s_3, s_4) F_3(s_1, s_2, s_4 + s_4) \\
& \quad + F_1(s_1) F_1(s_2) F_1(s_3) F_1(s_4) F_4(s_1, s_2, s_3, s_4))
\end{aligned}
$$

In summary:

$$H_1(s_1) = F_1(s_1)^4$$

$$
\begin{aligned}
H_2(s_1, s_2) &= 0 \\
H_3(s_1, s_2, s_3) &= \sum_{k=0}^{3} F_1(s_1)^k F_1(s_2)^k F_1(s_3)^k F_3(s_1, s_2, s_3) F_1(s_1 + s_2 + s_3)^{3-k} \\
H_4(s_1, s_2, s_3, s_4) &= 0
\end{aligned}
$$

□

Question 6. Show a diagram of implementation for 3-order Volterra of system $\mathcal{F}$.

Solution. For $F_1(s) = \frac{1}{1+s_1/\omega}$, it is a first order of low pass filter.

For $F_3(s_1, s_2, s_3) = \frac{1}{3}\frac{F_1(s_1)F_1(s_2)F_1(s_3)-1}{1+(s_1+s_2+s_3)/\omega}$, it can be re-written by

$$
F_3(s_1, s_2, s_3) = A_1(s_1)B_1(s_2)C_1(s_3)D_1(s_1 + s_2 + s_3) - A_2(s_1)B_2(s_2)C_2(s_3)D_1(s_1 + s_2 + s_3)
$$

with

$$
\begin{cases}
A_1(s_1) = F_1(s_1) \quad A_2(s_1) = 1 \\
B_1(s_2) = F_1(s_2) \quad B_2(s_1) = 1 \\
C_1(s_3) = F_1(s_3) \quad C_2(s_1) = 1 \\
D_1(s_1 + s_2 + s_3) = \frac{1}{3}F_1(s_1 + s_2 + s_3)
\end{cases}
$$

We could simplify the branch $A_1(s_1)B_1(s_2)C_1(s_3)$ and $A_2(s_1)B_2(s_2)C_2(s_3)$ by a cubic function. Therefore, the diagram of implementation is shown as Figure 3.

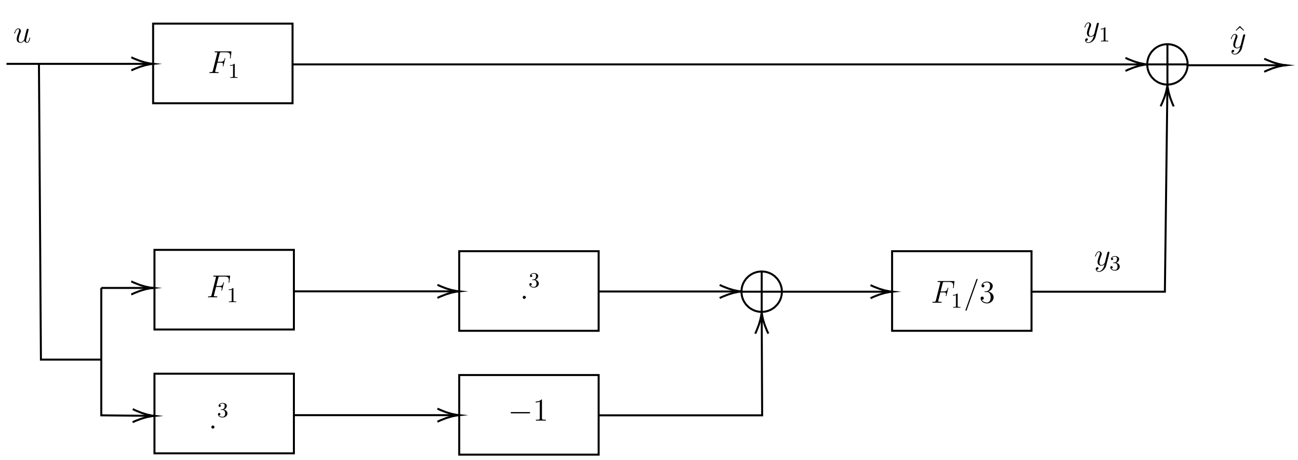

Figure 3: Diagram of Implementation of $\mathcal{F}$

□