Devoir Maison UE FpA - Acoustique
Partie Acoustique

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Exercise 1.

$$p(x,t) = Z_c u_s[\delta(t) - 2\delta(t - 2l/c) + 2\delta(t - 4l/c) - 2\delta(t - 6l/c) + \cdots] \tag{1}$$

Question 1. The definition of the impulse response.

Solution. The impulse response describes the react of the system when the input signal is Dirac $\delta(t)$. $\square$

Question 2. The primary wave.

Solution. The (1) shows that at $t = 0$, $t = \frac{2l}{c}$, $t = \frac{4l}{c}$, $\cdots$ the wave reflected to the start point. So that the primary waves are $p_0(x,t) = Z_c u_s \delta(t)$, $p_1(x,t) = -2Z_c u_s \delta(t - 2l/c)$, $\cdots$, $p_n(x,t) = (-1)^n \cdot 2Z_c u_s \delta(t - 2nl/c)$, which correspond to $t = 0, \frac{2l}{c}, \frac{4l}{c}, \cdots, \frac{2nl}{c}$. $\square$

Question 3. Show the reflection functions of the left and right ends of cylinder, by using the operator g_{ar} with the convolution of the primary waves.

Solution. The reflection functions at the ends of the cylinder depends on the boundary conditions: closed/open ends.

- Open end: Under this condition, the pressure p is null. So that

$$\begin{aligned} p(x,t) &= 0 \\ p^+(x,t) + p^-(x,t) &= 0 \\ p^+(x,t) &= -p^-(x,t) \end{aligned}$$

  which indicates that the reflection coefficient $R_{open} = -1$.

- Closed end: Similarly, under this condition, the velocity u is null.

$$\begin{aligned} u(x,t) &= 0 \\ \frac{1}{\rho c}(p^+(x,t) - p^-(x,t)) &= 0 \\ p^+(x,t) &= p^-(x,t) \end{aligned}$$

  which indicates that $R_{closed} = 1$.

Then the ends could be summarized as $R_{left} = R_{closed} = 1, R_{right} = R_{open} = -1$.

Consider the reflection waves. At the beginning, the wave propagates directly to the right

$$p_0(x,t) = Z_c u_s \delta(t - x/c)$$

and first reflection happens at $x = l$

$$p_r(x,t) = p_0(l,t) * R_{right} * \delta(t - (l-x)/c) = -Z_c u_s \delta(t - (2l - x)/c).$$

Then it continuous propagates to the left and reflects again at $x = 0$

$$p_l(x,t) = p_r(0,t) * R_{left} * \delta(t - x/c) = -Z_c u_s \delta(t - (2l + x)/c).$$

The first "aller-retour" reflection could be expressed as

$$\begin{aligned} p_{ar}(x,t) &= p_r(0,t) + p_l(x,t) \\ &= -2Z_c u_s \delta(t - (2l - x)/c) \\ &= 2g_{ar} * p_0(x,t) * \delta(t - x/c) \end{aligned}$$

where $g_{ar} = R_{left} * R_{right} * \delta(t - 2l/c) = -\delta(t - 2l/c)$.

Use mathematical induction, the n^{th} "aller-retour" reflection function could be expressed as

$$p_n(x,t) = 2g_{ar}^n * p_0(x,t) * \delta(t - x/c).$$

Thus, the reflection functions is

$$p(x,t) = p_0(x,t) + 2\sum_{n=1}^{\infty} p_n(x,t) = p_0(x,t) + 2\sum_{n=1}^{\infty} g_{ar}^n * p_0(x,t) * \delta(t - x/c). \tag{2}$$

□

Question 4. Show the impulse response is (1).

Solution. Initial Impulse: at $t = 0$, the input pressure (source) is $\delta(t)$ at $x = 0$, so that $p_0(0,t) = Z_c u_s \delta(t)$.

Replace $p_0(x,t)$ in (2),

$$\begin{aligned} p(x,t) &= Z_c u_s \delta(t) + 2\sum_{n=1}^{\infty} g_{ar}^n * Z_c u_s \delta\left(t - \frac{2nl}{c}\right) \\ &= Z_c u_s \left[\delta(t) + 2\sum_{n=1}^{\infty} (-1)^n \delta\left(t - \frac{2nl}{c}\right)\right] \end{aligned}$$

which is exactly (1). □

Question 5. Interpret the different terms in (1) and show what happens when source is at $t = 0, x = 0$.

Solution. First term $\delta(t)$ is the initial pressure pulse at $t = 0, x = 0$. The other terms correspond to the n^{th} reflection of the wave at $t = 2nl/c$. At $t = 0$, the pressure source has an impulse into the cylinder. The wave propagate through the cylinder, reflects at the right end (which is an open end) and then reflects again at the left end (which is a closed end). These reflections continue indefinitely. □

Exercise 2. Find the modes and natural frequencies of a cylinder tube with a closed input $(x = 0)$ and a open output $(x = l)$.

Question 1. Under which condition, the propagation could be considered as a linear propagation through a tube?

Solution. The linear propagation conditions are:

- Small wave amplitude.
- Homogeneous medium.
- Ignore the dissipation.
- The section of tube is a constant.
- The boundary condition is ideal (closed or open).

□

Question 2. Show the linear equation of the wave in 3D, by using Laplacian operator and partial derivation.

Solution. Wave equation in 3D

$$\nabla^2 p(\mathbf{r}, t) - \frac{1}{c^2}\frac{\partial^2 p(\mathbf{r}, t)}{\partial t^2} = \rho\left(\nabla \cdot \mathbf{F} - \frac{\partial q(\mathbf{r}, t)}{\partial t}\right) \tag{3}$$

where $\mathbf{r} = r(x, y, z)$ is the position, $p(\mathbf{r}, t)$ is the pressure, ρ is gas density, $\mathbf{F}$ is the external force per unit mass and $q(\mathbf{r}, t)$ is the particle velocity. □

Question 3. Under which condition the propagation through the tube could be considered as 1D wave equation rather than approximation?

Solution. The conditions of propagation in 1D wave equation are:

- Length of the tube is much more greater than the diameter of the tube.
- Homogeneous medium.
- Ignore the dissipation.

□

Question 4. Which boundary condition for pressure could show the characteristic of the open/closed tube?

Solution. As it shows in *Solution* of Question 3 in the previous exercises, when the tube is open, the pressure is null,

$$p(x,t) = 0$$

and when the tube is closed, the velocity is null,

$$u(x,t) = 0.$$

□

Question 5. Find the solutions of differential equation

$$\partial_x^2 p(x,t) - \frac{1}{c_0^2}\partial_t^2 p(x,t) = 0.$$

Explain why, in relation to the objective announced at the start of the exercise. What form would you suggest for the solution you're looking for?

Solution. First, there is no external force applied to the tube, so that the term of source (which is the right part of (3)) is null. And the wave propagated along the direction of x, so that the other two directions are ignored. Thus (3) could be simplified as

$$\frac{\partial^2}{\partial x^2}p(x,t) - \frac{1}{c_0^2}\frac{\partial^2}{\partial t^2}p(x,t) = 0. \tag{4}$$

Assume that $p(x,t)$ could decompose as $p(x,t) = X(x)T(t)$. Replace $p(x,t)$ into (4)

$$\frac{1}{X(x)}\frac{\partial^2}{\partial x^2}X(x) = \frac{1}{c_0^2}\frac{1}{T(t)}\frac{\partial^2}{\partial t^2}T(t) = -k^2 \tag{5}$$

where k is a constant indicated wave number. From (5) we could obtain two differential equations

$$\begin{cases} \frac{\partial^2}{\partial x^2}X(x) + k^2 X(x) &= 0 \\ \frac{\partial^2}{\partial t^2}T(t) + c_0^2 k^2 T(t) &= 0 \end{cases}$$

and we propose the solution form as

$$\begin{aligned} X(x) &= A\cos(kx) + B\sin(kx) \\ T(t) &= C\cos(\omega_n t) + D\sin(\omega_n t) \end{aligned}$$

where $\omega_n = c_0 k_n$. □

Question 6. Find the solution with boundary conditions

$$\partial_x p(x=0,t) = 0 \forall t \text{ and } p(x=l,t) = 0 \forall t.$$

Deduce that opening a hole at $x = l/3$ modifies the first mode without modifying the second mode.

Solution. Use the boundary conditions, at $x = 0$:

$$[-kA\sin(kx) + kB\cos(kx)]_{x=0} = 0$$

and at $x = l$:

$$[A\cos(kx) + B\sin(kx)]_{x=l} = 0.$$

Thus, $X(x) = A\cos(k_n x)$ where $k_n = (2n-1)\frac{\pi}{2l}$, and

$$p(x,t) = \sum_{n=1}^{\infty} [C_n \cos(\omega_n t) + D_n \sin(\omega_n t)] \cos(k_n x)$$

where $\omega_n = c_0 k_n$ and $k_n = (2n-1)\frac{\pi}{2l}$.

Note that the last term $\cos(k_n x)$ will change with the variation of x. Let $x = l/3$, and show the last terms of $n = 1, 2$:

$$\cos(k_1 x) = \cos(\frac{\pi}{2l}\frac{l}{3})$$
$$\cos(k_2 x) = \cos(\frac{3\pi}{2l}\frac{l}{3}) = 0.$$

This shows that the first mode will change but the second mode will not change. □