Examination of the teaching unit *Représentations des signaux* - TSIA201

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Duration: 1:30

All documents are permitted. However electronic devices (including calculators) are forbidden.

1 Simple CQF filter bank

A two-channel filter bank is defined by the diagram in Figure 1.

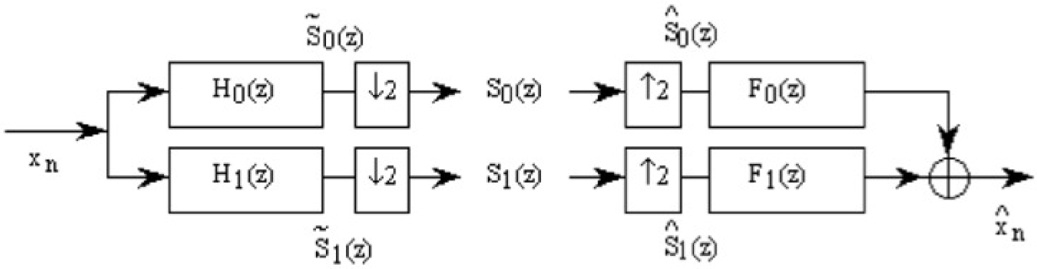

Figure 1: General diagram of a two-channel filter bank.

We remind that the two aliasing cancellation conditions of this filter bank are $F_0(z) = H_1(-z)$ and $F_1(z) = -H_0(-z)$ and that its transfer function is $T(z) = \frac{1}{2}(H_0(z)F_0(z) + H_1(z)F_1(z))$. We also remind the *conjugate quadrature filters* (CQF) constraint: $H_1(z) = -z^{-(N-1)}\widetilde{H}_0(-z)$ where $\widetilde{H}_0(z) = \overline{H_0(\frac{1}{z^*})}$ and N is even.

We now consider the very simple case of a FIR filter $H_0(z)$ of length $N = 2$: $H_0(z) = a + b\, z^{-1}$, where $a, b \in \mathbb{R}$.

1. Give the expressions of the transfer functions $H_1(z)$, $F_0(z)$ and $F_1(z)$ involving the coefficients a and b.

2. Prove that this CQF filter bank guarantees perfect reconstruction $\forall a, b \in \mathbb{R}$.

2 Recursive short-time Fourier transform

The aim of this problem is to establish recursive versions of the Short-Time Fourier Transform (STFT) based on the temporal recursivity of the windows. In general, a recursive formulation of the STFT is useful when an efficient real-time implementation is desired, or when an evaluation of the STFT at each sample is required.

The short-time Fourier transform of a signal s, analyzed with a window w, is defined as:

$$S_w(n, \nu) = \sum_{m \in \mathbb{Z}} w(n-m)s(m)e^{-2\pi j \nu m} \tag{1}$$

Exercise 1 *Recursive STFT with exponential forgetting causal window*

First, we develop a recursive STFT based on a causal window of infinite duration, which is recursive of order 1. The exponential forgetting causal window g_β with $\beta \geq 0$ is defined as:

$$\begin{cases} g_\beta(n) = \beta^n & \textit{for} \quad n \geqslant 0 \\ g_\beta(n) = 0 & \textit{for} \quad n < 0 \end{cases} \tag{2}$$

1. *In which interval β must lie, so that g_β is the impulse response of a stable filter G_β?*
2. *Write the STFT $S_g(n, \nu)$, taking into account the expression of the window g_β in equation (2).*
3. *Deduce the recursive formulation of the STFT $S_g(n, \nu)$ as a function of $S_g(n-1, \nu)$, β and $s(n)$.*
4. *Give the expression of the transfer function $G_\beta(z)$ of the filter defined by the impulse response g_β.*
5. *Deduce the temporal relationship between an input signal $x(n)$ and an output signal $y(n)$ of the filter G_β.*
6. *By using equation (1), describe the STFT $S_g(n, \nu)$ as a convolution product between g_β and a signal x deduced from the signal s.*
7. *From the answers to the two previous questions, retrieve the recursive formulation of the STFT $S_g(n, \nu)$ already established in question 3.*

Exercise 2 *Recursive STFT with rectangle window*

The recursive formulation based on the exponential forgetting causal window is simple and effective. But it relies on a window of infinite duration, which is also non-symmetrical. In this second part, we study a recursive formulation based on the simplest symmetrical window of finite duration. Let r_M be the rectangular window of duration M defined as:

$$\begin{cases} r_M(n) = 1 & \textit{for} \qquad 0 \leqslant n \leqslant M-1 \\ r_M(n) = 0 & \textit{otherwise} \end{cases} \tag{3}$$

1. *Write the STFT $S_r(n, \nu)$, taking into account the expression of the window r_M in equation (3).*
2. *Derive the recursive formulation of $S_r(n, \nu)$ as a function of $S_r(n-1, \nu)$ and the signal s.*
3. *Give the expression of the transfer function $R_M(z)$ of the filter defined by its impulse response r_M.*
4. *Deduce the temporal relationship between an input signal $x(n)$ and an output signal $y(n)$ of the filter R_M.*
5. *Using equation (1), interpret the STFT $S_r(n, \nu)$ as a convolution product between r_M and a signal x deduced from s.*
6. *From the answers to the two previous questions, find the recursive formulation of $S_r(n, \nu)$ already established in question 2.*

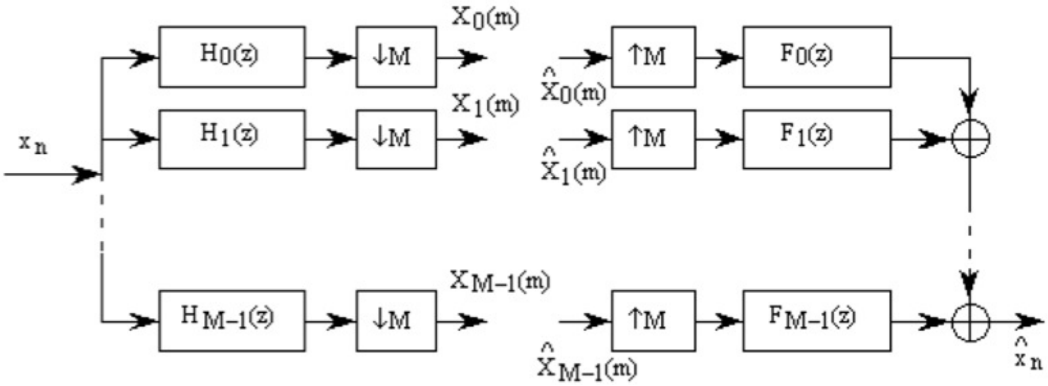

Figure 2: General diagram of an M-channel filterbank

3 M-channel filterbank

The M-channel filter bank is represented in Figure 2.

1. Give the expression of the Z-transform $\hat{X}(z)$ of the output signal as a function of the Z-transform $X(z)$ of the input signal, in the form $\hat{X}(z) = \sum_{l=0}^{M-1} A_l(z)X(zW_M^l)$ where $W_M = e^{-\frac{2i\pi}{M}}$. The terms $A_l(z)$ will be expressed as functions of the transfer functions $H_k(z)$ and $F_k(z)$ of the analysis and synthesis filters.

2. What are the required conditions for aliasing cancellation and perfect reconstruction?

3. We note $\boldsymbol{h}^\top(z) = [H_0(z) \dots H_{M-1}(z)]$ and $\boldsymbol{f}^\top(z) = [F_0(z) \dots F_{M-1}(z)]$. Express in terms of $\boldsymbol{h}(z)$ and $\boldsymbol{f}(z)$ the relationship between $\hat{X}(z)$ and $X(z)$ when the aliasing cancellation conditions are fulfilled.

4. Let $\boldsymbol{E}(z)$ be the matrix of type I polyphase components of the analysis filterbank, and $\boldsymbol{R}(z)$ be the matrix of type II polyphase components of the synthesis filterbank. We can write $\boldsymbol{h}(z) = \boldsymbol{E}(z^M)\boldsymbol{e}(z)$, where $\mathbf{e}(z) = [1, z^{-1}, \dots, z^{-(M-1)}]^\top$, and $\boldsymbol{f}^\top(z) = \widetilde{\mathbf{e}}(z)^\top \boldsymbol{R}(z^M)$, where $\widetilde{\mathbf{e}}(z) = [z^{-(M-1)}, z^{-(M-2)}, \dots, 1]^\top$. Deduce a new diagram for the implementation of the filter bank involving the matrix $\boldsymbol{P}(z) = \boldsymbol{R}(z)\boldsymbol{E}(z)$.

5. Express $\hat{x}(n)$ as a function of $x(n)$ when $\boldsymbol{P}(z) = \boldsymbol{I}_M$ and when $\boldsymbol{P}(z) = Cz^{-n_0}\boldsymbol{I}_M$ ($\boldsymbol{I}_M$ is the $M \times M$ identity matrix).

6. Construction of a solution: we assume that $\boldsymbol{E}(z)$ is factorized in the form $\boldsymbol{E}(z) = \boldsymbol{E}_0\boldsymbol{\Lambda}(z)\boldsymbol{E}_1\boldsymbol{\Lambda}(z)\dots\boldsymbol{\Lambda}(z)\boldsymbol{E}_{J-1}$ where the $\boldsymbol{E}_i$ are $M \times M$ non-singular constant matrices and $\boldsymbol{\Lambda}(z) = \begin{bmatrix} \boldsymbol{I}_{M-1} & 0 \\ 0 & z^{-1} \end{bmatrix}$. We also assume that $\boldsymbol{R}(z)$ is factorized in the form $\boldsymbol{R}(z) = \boldsymbol{R}_{J-1}\boldsymbol{\Gamma}(z)\boldsymbol{R}_{J-2}\boldsymbol{\Gamma}(z)\dots\boldsymbol{\Gamma}(z)\boldsymbol{R}_0$ where the $\boldsymbol{R}_i$ are $M \times M$ non-singular constant matrices and $\boldsymbol{\Gamma}(z) = \begin{bmatrix} z^{-1}\boldsymbol{I}_{M-1} & 0 \\ 0 & 1 \end{bmatrix}$. Express the matrices $\boldsymbol{R}_i$ as functions of $\boldsymbol{E}_i$, such that the product $\boldsymbol{P}(z) = \boldsymbol{R}(z)\boldsymbol{E}(z)$ yields $\boldsymbol{P}(z) = z^{-J}\boldsymbol{I}_M$.

7. *Application* : we choose $M = 2$, $J = 2$, and $\boldsymbol{E}_0 = \boldsymbol{E}_1^\top = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$. Deduce the transfer functions $H_0(z)$, $H_1(z)$, $F_0(z)$ and $F_1(z)$ which lead to perfect reconstruction.
