



Algebraic and geometrical models in computational musicology

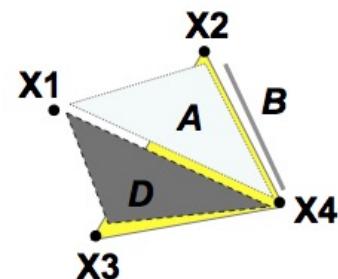
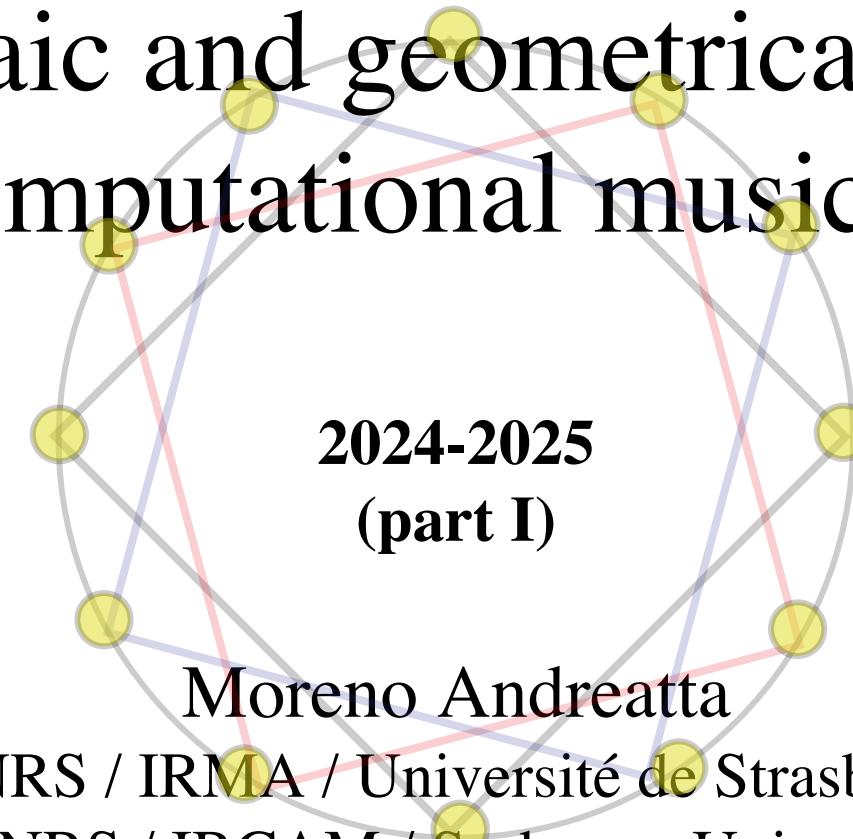
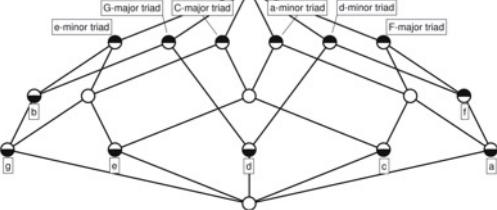
2024-2025
(part I)

Moreno Andreatta

CNRS / IRMA / Université de Strasbourg

CNRS / IRCAM / Sorbonne Université

www.morenoandreatta.com



Fondamentaux pour ATIAM – Musicologie Computationnelle

<http://www.atiam.ircam.fr/planning/>

2024-2025

9/9 : 9h30-12h30 + 14h30-16h30 (salle Stravinsky) – **François-Xavier Féron** : Musique et sciences – histoire de l'Ircam et illusions auditives (**5h**). Séance ouverte aux Ircamien.ne.s

10/9 (10h-13h) + 16/9 (14h30-17h30) + 23/9 (14h30-17h30)

Moreno Andreatta : Méthodes algébriques et topologiques en analyse musicale computationnelle (**9h**)

12/9 (14h-17h) - **Pierre Couprie** : Transcription, représentation et analyse de la musique électroacoustique (**3h**)

13/9 (14h30-17h30) - **Karim Haddad** : composition assistée par ordinateur (**3h**)
,

18/9 (9h30-12h30+14h-15h) - **Marc Chemillier** : Grammaires formelles, théorie des automates et improvisation assistée par ordinateur (**4h**)

19/9 (14h30-16h30) – **Clément Canonne** : musicologie empirique et cognitive (**2h**) -

25/9 (9h30-12h30 + 14h30-15h30) - **Angelo Orcalli** : Modèles épistémologiques en musicologie computationnelle (**4h**)

Computational Musicology in academic research

Conferences of the SMCM:

- 2007 Technische Universität (Berlin, Allemagne)
- 2009 Yale University (New Haven, USA)
- 2011 IRCAM (Paris, France)
- 2013 McGill University (Canada)
- 2015 Queen Mary University (Londres)
- 2017 UNAM (Mexico City)
- 2019 Universidad Complutense de Madrid (Spain)
- 2022 Georgia State University (Atlanta, USA)
- 2024 Coimbra



Official Journal and MC code (00A65: Mathematics and Music)

- *Journal of Mathematics and Music*, Taylor & Francis
(Editors: E. Amiot & J. Yust)



Books Series:

- *Computational Music Sciences Series*, Springer (G. Mazzola & M. Andreatta eds. – 19 books published (since 2009))
- Collection *Musique/Sciences*, Ircam-Delatour France (J.-M. Bardez & M. Andreatta dir. – 16 books published (since 2006))

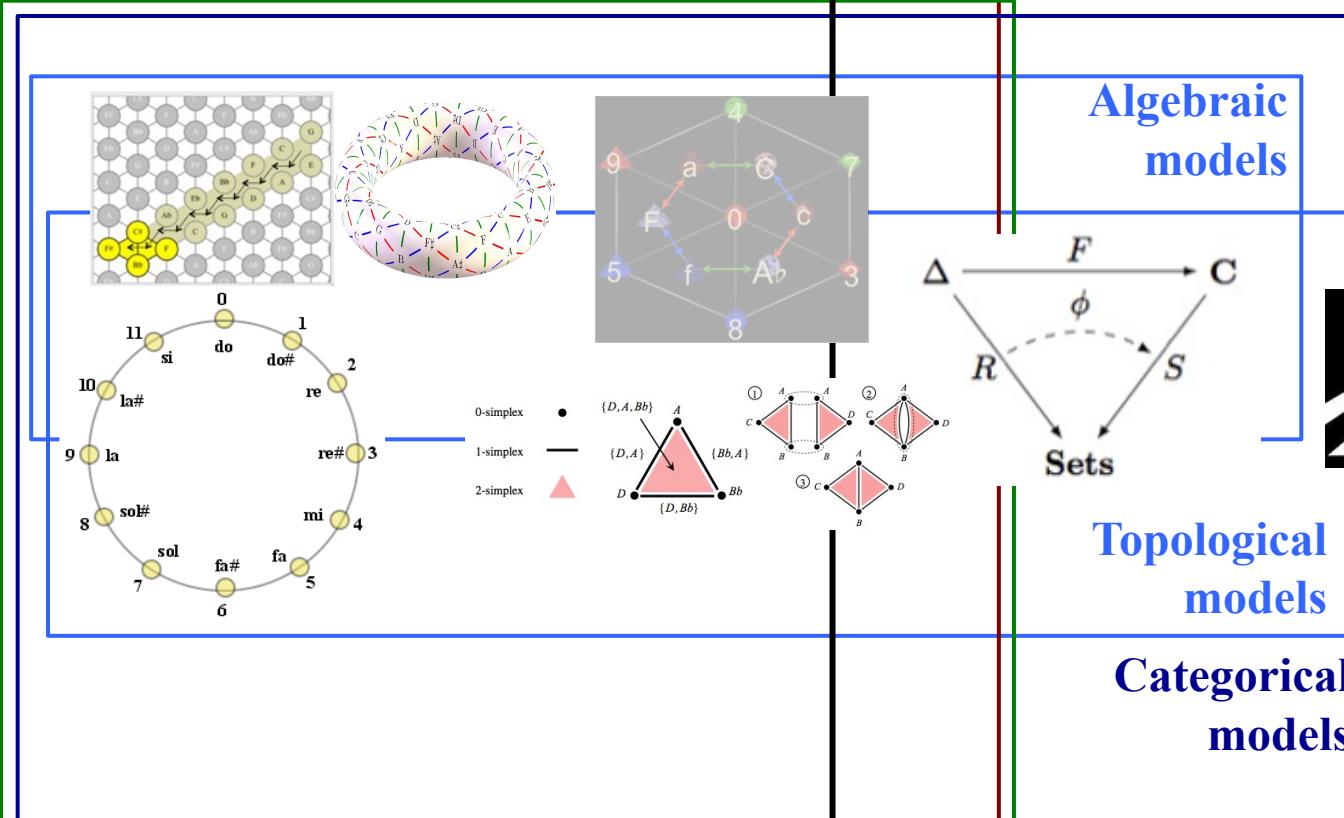


Some examples of PhD on maths & music

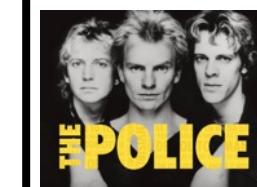
- **Christophe Weis**, *Geometric Models of Harmony as Tools for Computer Assisted Composition and Improvisation*, PhD in **music research**, Hochschule für Musik Karlsruhe (supervised by Marlon Schumacher and Moreno Andreatta).
- **Riccardo Giblas**, *Persistent homology and music analysis*, PhD in **maths** in cotutelle agreement, University of Padova (L. Fiorot & Alberto Tonolo) / Université de Strasbourg (M. Andreatta), 2024.
- **Paul Lascabettes**, *Mathematical Models for the Discovery of Musical Patterns and Structures, and for Performances Analysis*, PhD in **maths** at SU), 2023.
- **Gonzalo Romero**, *Morphologie mathématique et signal/symbolique*, PhD in **computer science** at SU), 2023.
- **Victoria Callet**, *Modélisation topologique de structures et processus musicaux*, PhD in **maths**, Université de Strasbourg (supervised by Pierre Guillot and Moreno Andreatta, IRMA), 2023.
- **Matias Fernandez Rosales**, *Mathematical models in Computer-assisted composition*, PhD in **composition and research**, HEAR/University of Strasbourg (supervision: Daniel D'Adamo, Xavier Hascher, Moreno Andreatta)
- **Greta Lanzarotto**, *Fuglede Spectral Conjecture, Musical Tilings and Homometry*, PhD in **maths** in cotutelle agreement, University of Pavia (L. Pernazza) / Université de Strasbourg (M. Andreatta), 2021.
- **Alessandro Ratoci**, Vers l'hybridation stylistique assistée par ordinateur, PhD in music **composition & research**, Sorbonne University / IRCAM (cosupervised with Laurent Cugny), ongoing
- **Sonia Cannas**, *Représentations géométriques et formalisations algébriques en musicologie computationnelle*, PhD in **maths** in cotutelle agreement, University of Pavia (L. Pernazza) / Université de Strasbourg (A. Papadopoulos & M. Andreatta), 2019.
- **Grégoire Genuys**, *Théorie de l'homométrie et musique*, PhD in **maths**, Sorbonne University / IRCAM (cosupervised with Jean-Paul Allouche), 2017.
- **Hélianthe Caure**, *Pavages en musique et conjectures ouvertes en mathématiques*, PhD in **computer science**, Sorbonne University (cosupervised with Jean-Paul Allouche), 2016.
- **Mattia Bergomi**, *Dynamical and topological tools for (modern) music analysis*, PhD in **maths** in a cotutelle agreement Sorbonne University / University of Milan (with Goffredo Haus, 2015).
- **Charles De Paiva**, *Systèmes complexes et informatique musicale*, thèse de doctorat, Programme Doctoral International « Modélisation des Systèmes Complexes », PhD in **musicology** in a cotutelle agreement, Sorbonne University / UNICAMP, Brésil, 2016.
- **Louis Bigo**, *Représentation symboliques musicales et calcul spatial*, PhD in **computer science**, University of Paris Est Créteil / IRCAM, 2013 (with Olivier Michel and Antoine Spicher)
- **Emmanuel Amiot**, *Modèles algébriques et algorithmiques pour la formalisation mathématique de structures musicales*, PhD in, Sorbonne University / IRCAM, 2010 (cosupervised with Carlos Agon) **computer science**
- **Yun-Kang Ahn**, *L'analyse musicale computationnelle*, PhD in **computer science**, Sorbonne University / IRCAM, 2009 (cosupervised with Carlos Agon)



The SMIR Project: Structural Music Information Research



Signal-based
Music
Information
Retrieval

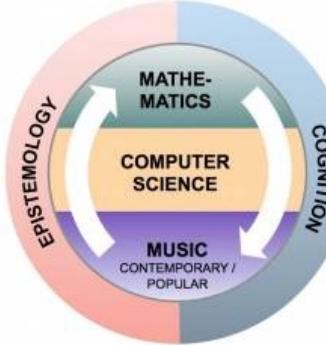


Oleg Berg



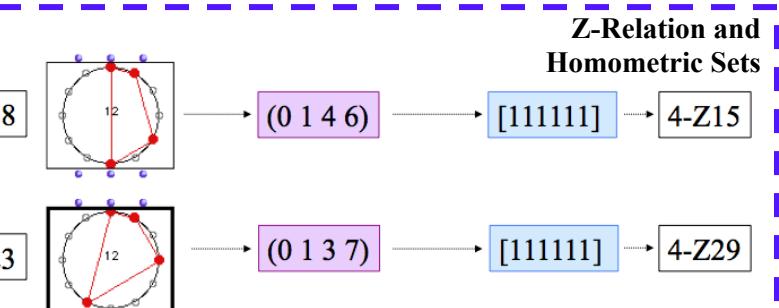
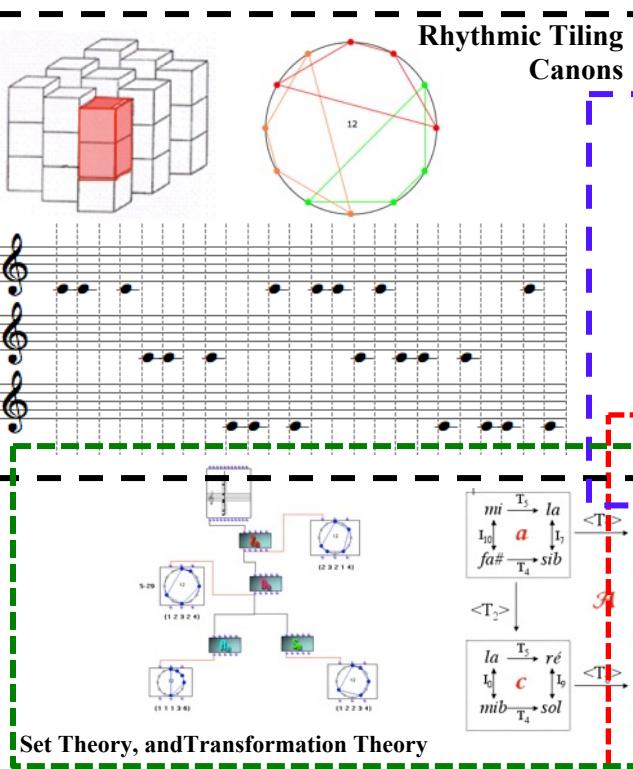
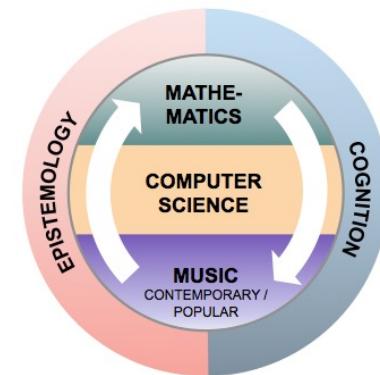
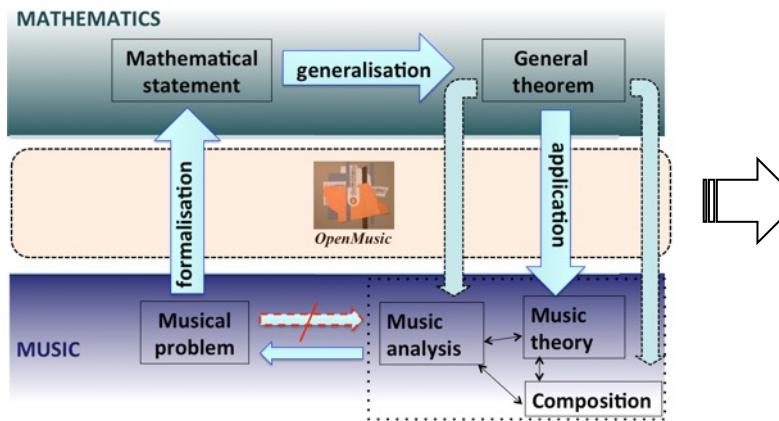
Structural Symbolic Music
Information Research

<http://repmus.ircam.fr/moreno/smir>



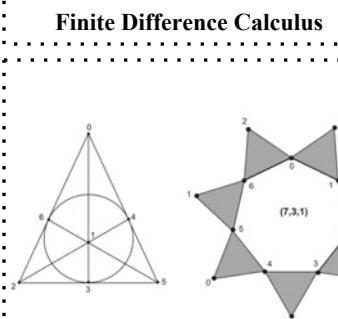
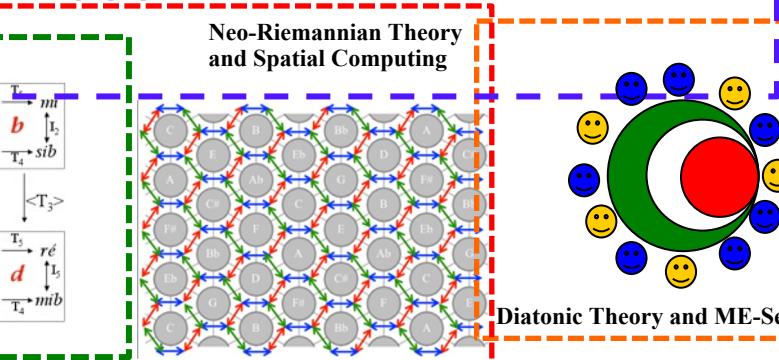
Some musically-driven mathematical problems

- Tiling Rhythmic Canons
- Z relation and homometry
- Transformational Theory
- Music Analysis, SC and FCA
- Diatonic Theory and ME-Sets
- Periodic sequences and FDC
- Block-designs in composition



$Df(x) = f(x) - f(x-1)$.

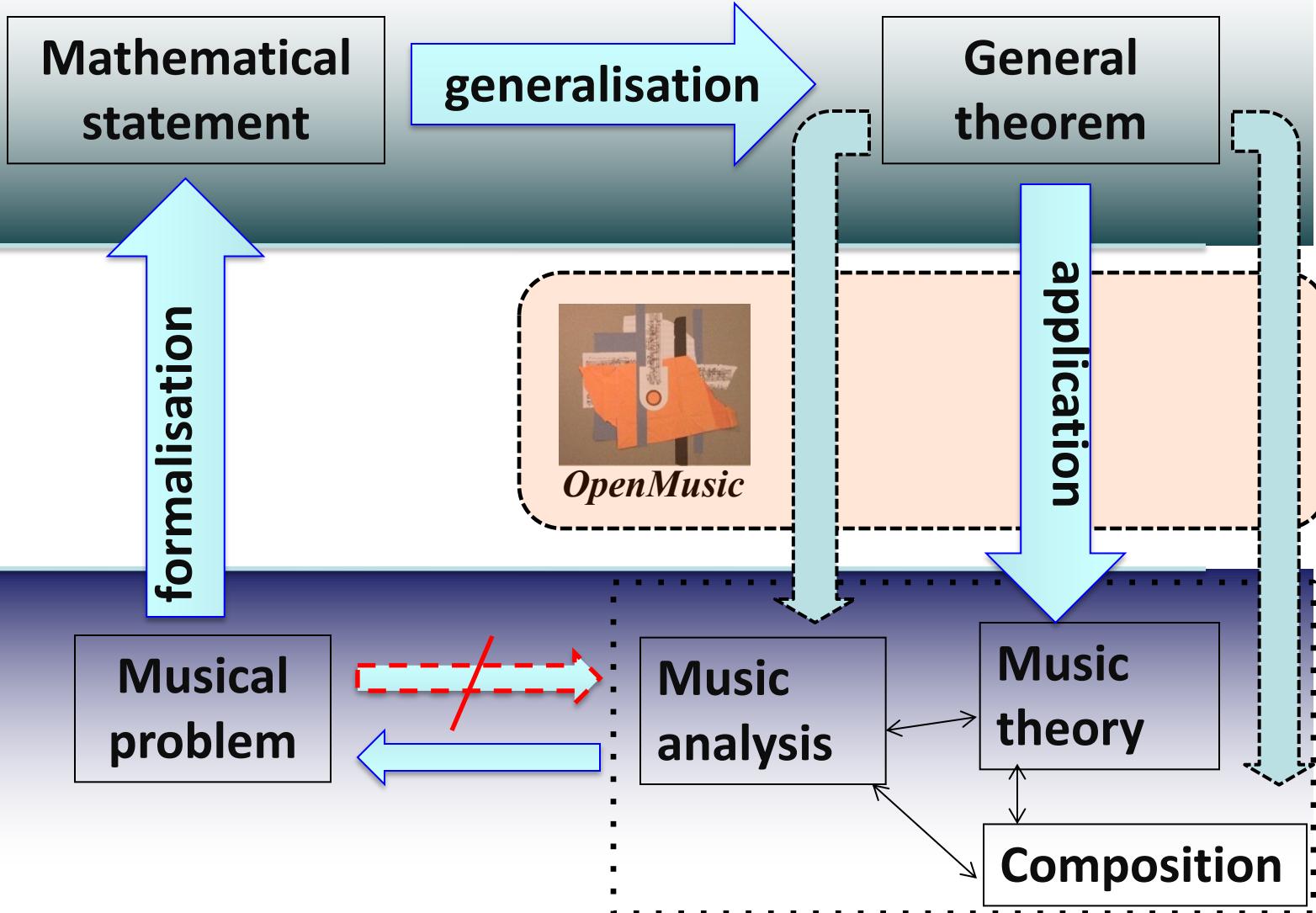
7 11 10 11 7 2 7 11 10 11 7 2 7 11...
 4 11 1 8 7 5 4 11 8 7 5 4 11...
 7 2 7 11 10 11 7 2 7 11 10 11...
 7 5 4 11 1 8 7 5 4 11 8...



Block-designs

The double movement of a ‘mathemusical’ activity

MATHEMATICS



The double movement of a ‘mathemusical’ activity

MATHEMATICS

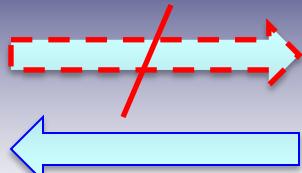


formalisation

MUSIC



Musical problem



Music analysis

Music theory

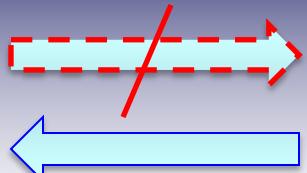
Composition

The double movement of a ‘mathemusical’ activity

MATHEMATICS



MUSIC

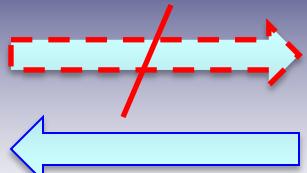


The double movement of a ‘mathemusical’ activity

MATHEMATICS



MUSIC



OpenMusic, a Visual Programming Language for computer-aided composition

www.repmus.ircam.fr/openmusic/home

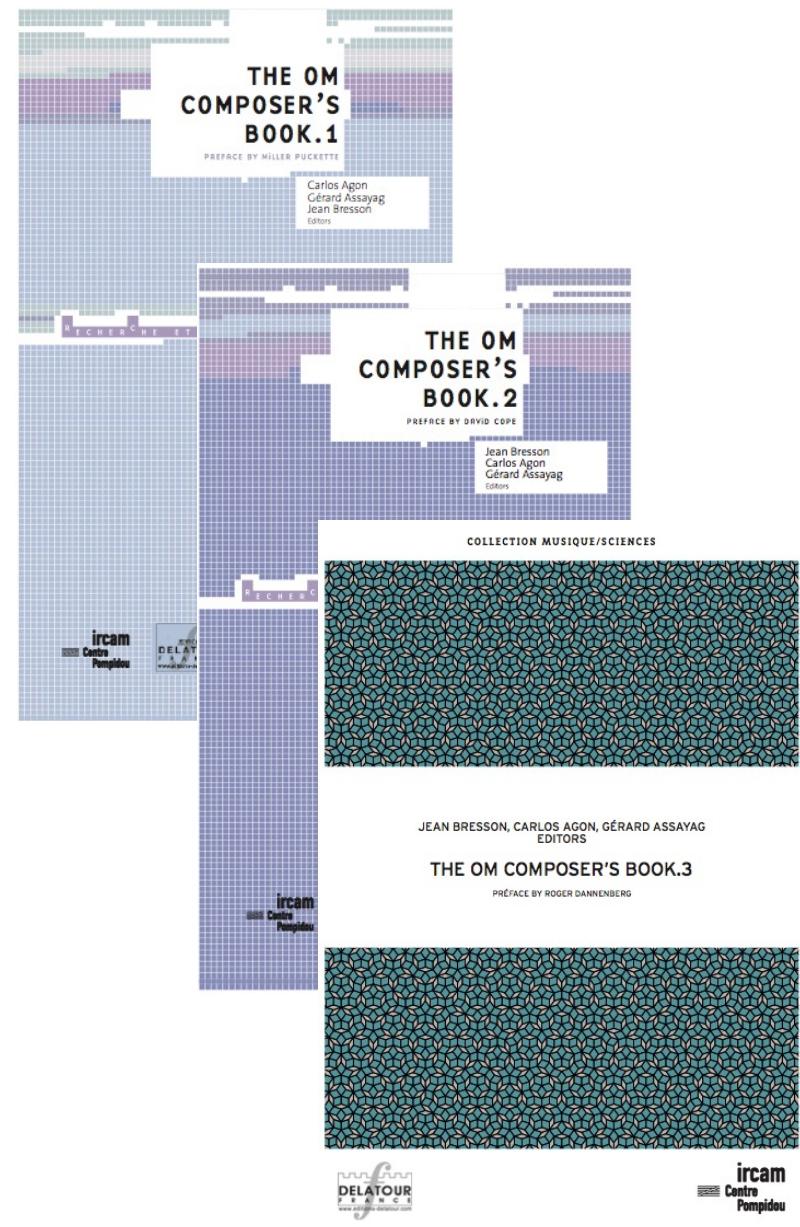
OpenMusic

(c) Ircam - Centre Pompidou



Dedicated to the memory of Gérard Grisey (French composer, 1946-1998)

Design and developpement : G. Assayag, A. Agon and J. Bresson
with help from C. Rueda, O. Delerue. Use Midishare (Grame)
Musical expertise by : M. Andreatta, J. Baboni, J. Fineberg, K. Haddad,
C. Malherbe, M. Malt, T. Murail, O. Sandred, M. Stroppa, H. Tutschku.
Artwork : A. Mohsen.



**C. Agon, G. Assayag and J. Bresson, *The OM Composer's Book* (3 volumes)
“Musique/Sciences” Series, Ircam/Delatour, 2006, 2007 and 2016**

OpenMusic, a Visual Programming Language for computer-aided composition

www.repmus.ircam.fr/openmusic/home

OpenMusic

(c) Ircam - Centre Pompidou

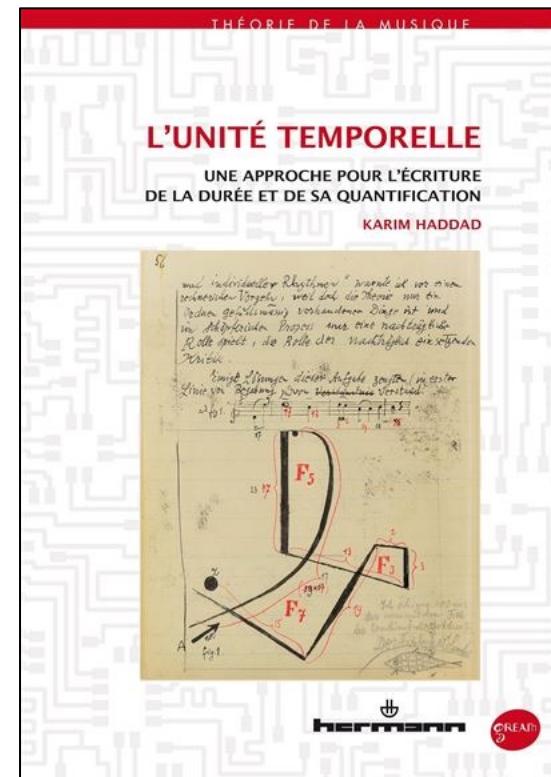


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C. Malherbe, M. Malt, T. Murail, O. Sandred, M. Stroppa, H. Tutschku.
Artwork : A. Mohsen.



Karim Haddad (né en 1962)

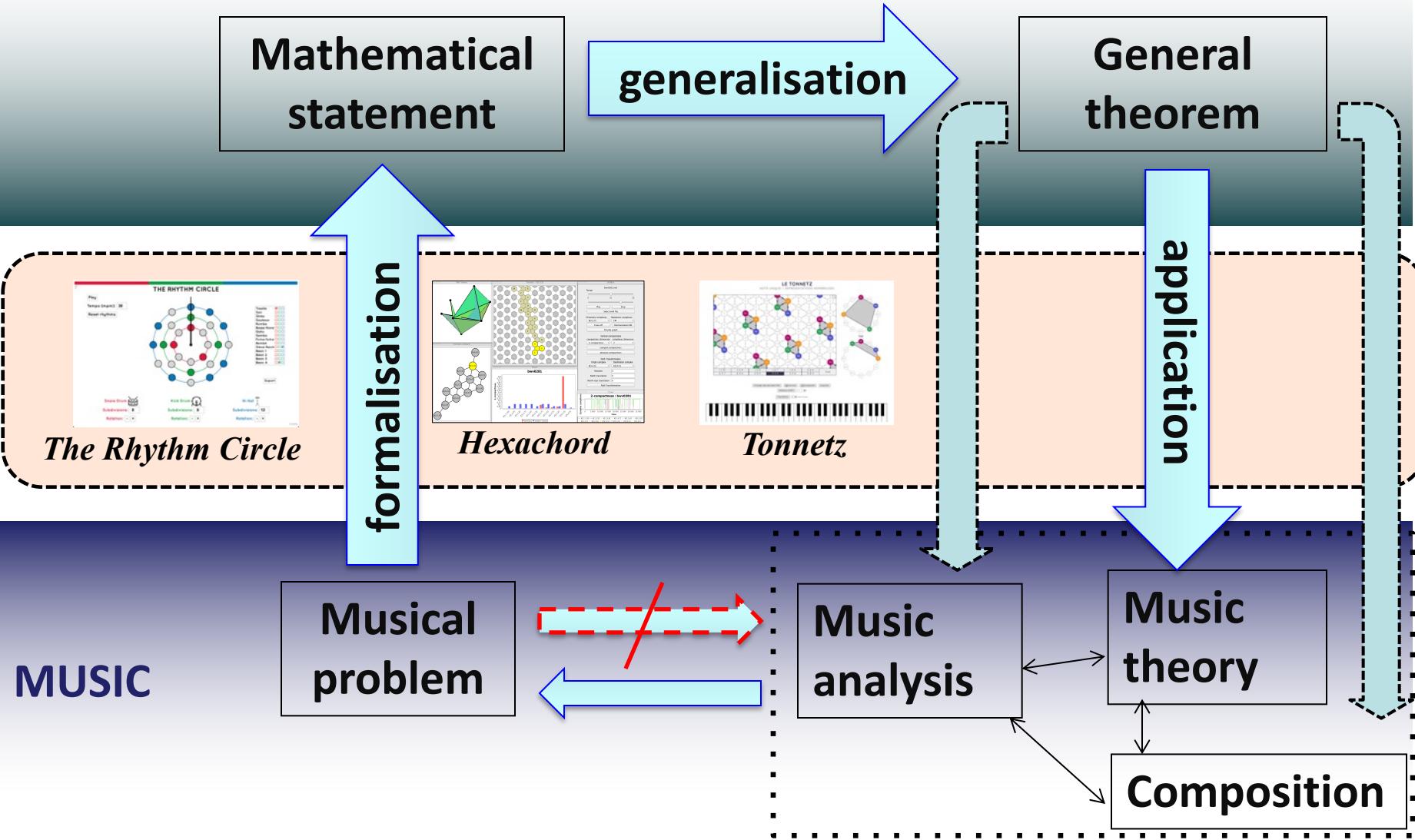


K. Haddad, *L'unité temporelle. Une approche pour l'écriture de la durée et de sa quantification*, Hermann, collection GREAM, Strasbourg, 2024.



The double movement of a ‘mathemusical’ activity

MATHEMATICS



Spatial music analysis via *Hexachord*

Plex Viewer

Tonnetz network

Computer Music Journal

Volume 39, Number 3 ISSN 0148-9267 \$18.00 Fall 2015

Dynamical Systems and Simplicial Chord Spaces

Tonnetz : K[3,4,5]

Chart bwv0281

2-compactness

K[1,1,10] K[1,2,9] K[1,3,8] K[1,4,7] K[1,5,6] K[2,2,8] K[2,3,7] K[2,4,6] K[2,5,5] K[3,3,6] K[3,4,5] K[4,4,4]

bwv0281 random chords

InfoBox

Tempo

Play Stop

Select midi file

Chromatic complexes Heptatonic complexes

K[2,3,7] CM

Trace off Harmonization ON

Display graph

Vertical compactness

compactness dimension complexes dimension

2-compactness 2

compute compactness

absolute compactness

Path Transformation

Origin complex Destination complex

K[3,4,5] K[3,4,5]

Rotation 0

North translation 0

North-east translation 0

Path Transformation

Chart

2-compactness : bwv0281

Complex compliance

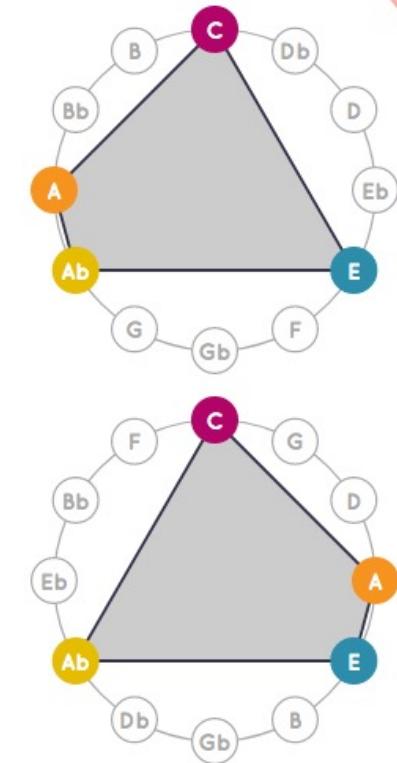
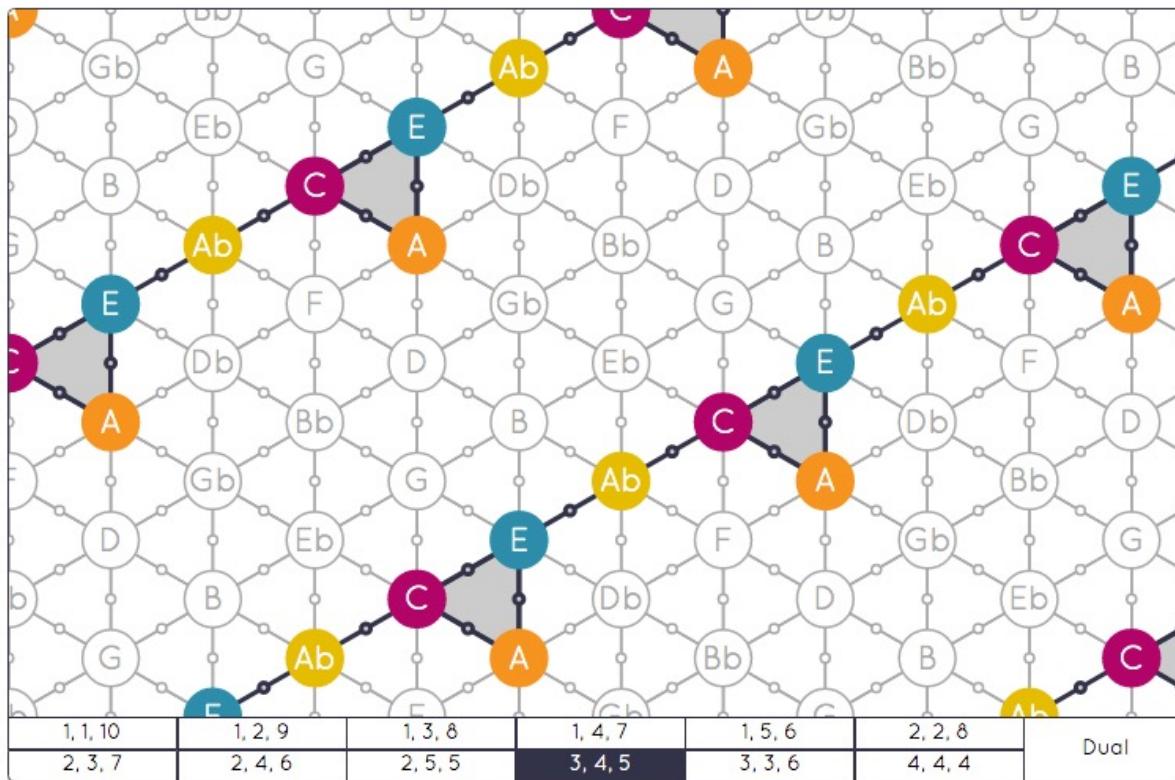
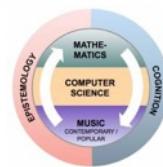
time

— K[1,1,10] — K[1,2,9] — K[1,3,8] — K[1,4,7] — K[1,5,6] — K[2,2,8]
— K[2,3,7] — K[2,4,6] — K[2,5,5] — K[3,3,6] — K[3,4,5] — K[4,4,4]

→ <http://www.lacl.fr/~lbigo/hexachord>

THE TONNETZ

ONE KEY – MANY REPRESENTATIONS



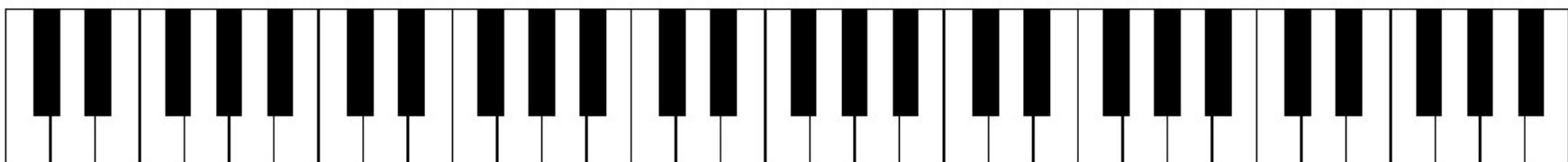
Load Midi File

Play

Start Recording

Rotate 180°

Translate



→ <https://thetonnez.com>

THE RHYTHM CIRCLE

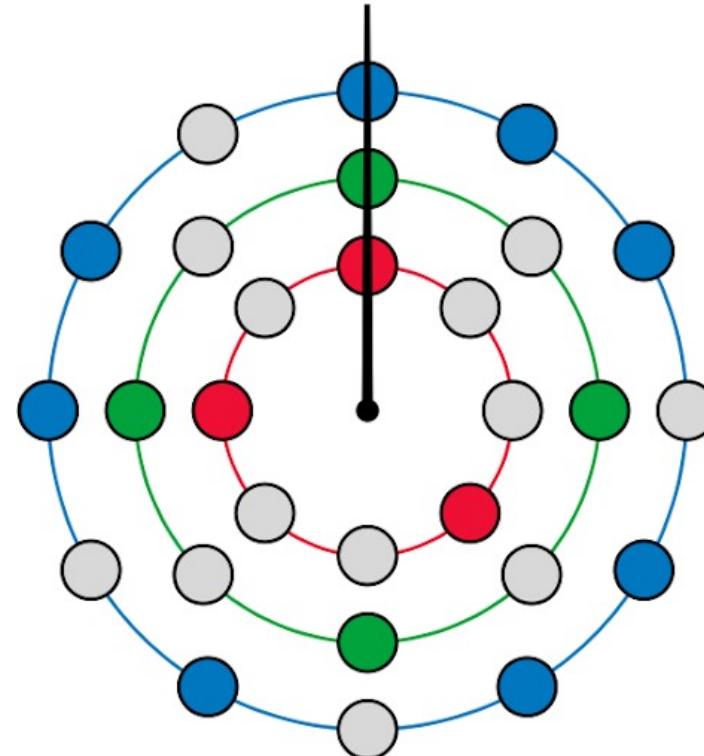
Play

Tempo (mpm): **30**

Reset rhythms



→ www.youtube.com/@mathemusique



Tresillo	<input checked="" type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>
Son	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>
Shiko	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>
Soukous	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>
Rumba	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>
Bossa Nova	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>
Gahu	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>
Samba	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>
Fume-fume	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>
Bembé	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>
Steve Reich	<input type="checkbox"/>	<input type="checkbox"/>	<input checked="" type="checkbox"/>
Basic 1	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>
Basic 2	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>
Basic 3	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>
Basic 4	<input type="checkbox"/>	<input checked="" type="checkbox"/>	<input type="checkbox"/>

Export

Snare Drum

Subdivisions: **8**

Rotation: **- +**

Kick Drum

Subdivisions: **8**

Rotation: **- +**

Hi Hat

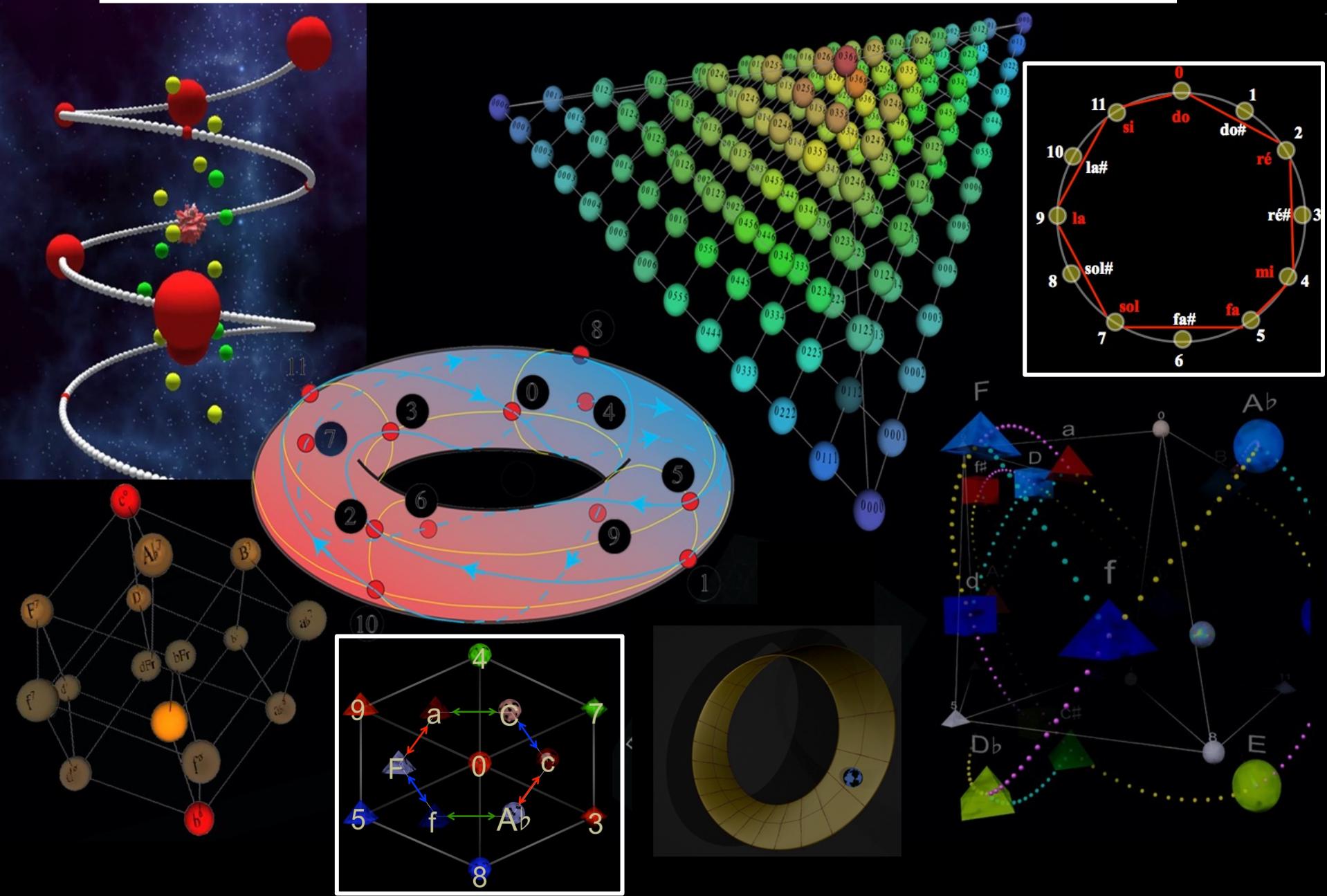
Subdivisions: **12**

Rotation: **- +**

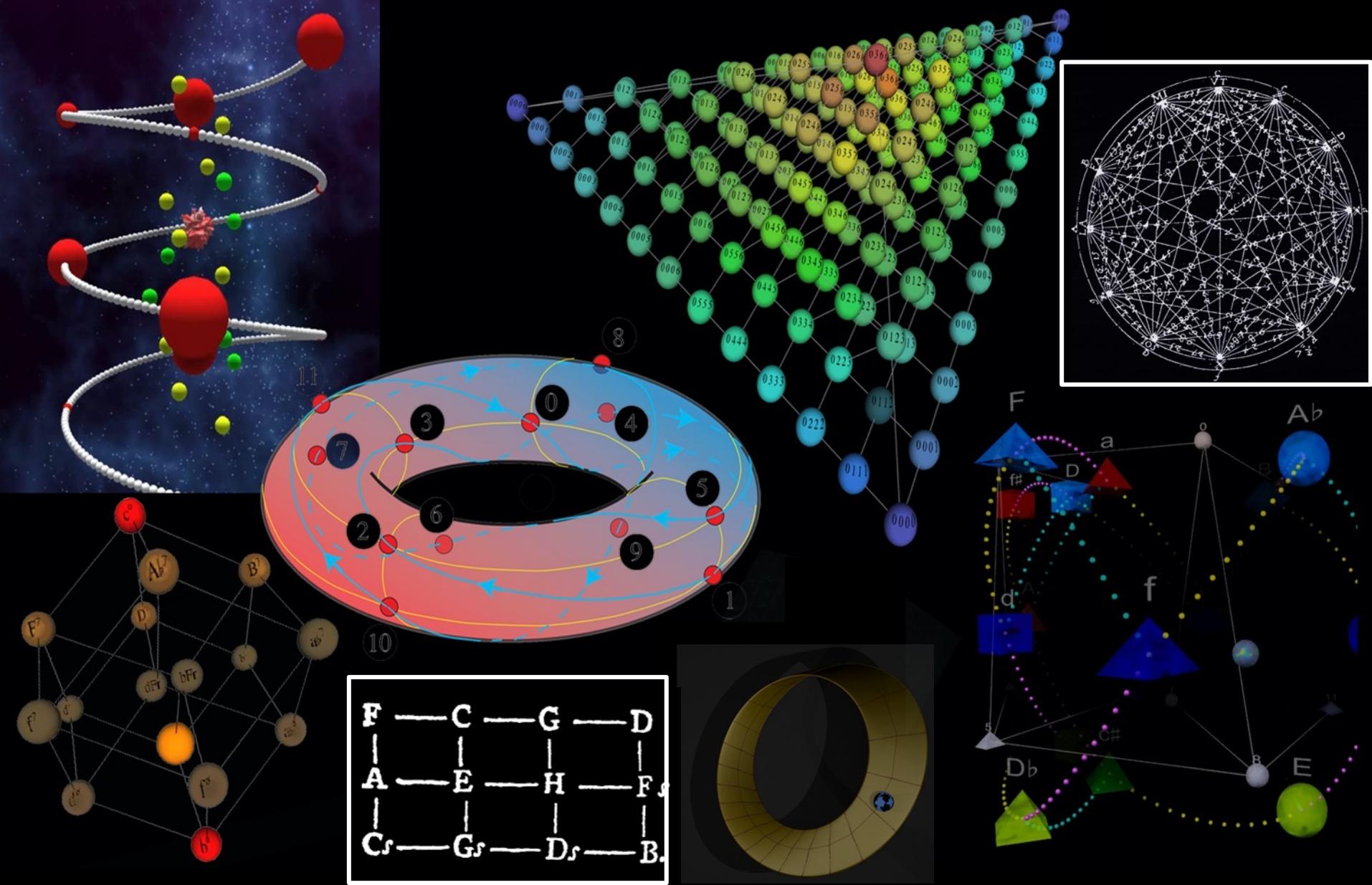
→ <https://rhythm-circle.com>



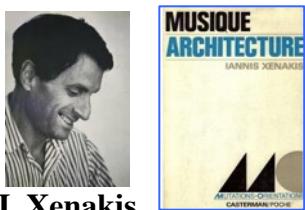
The galaxy of geometrical models at the service of music



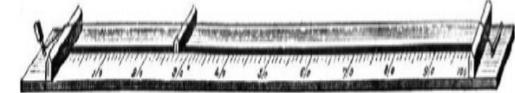
The galaxy of geometrical models at the service of music



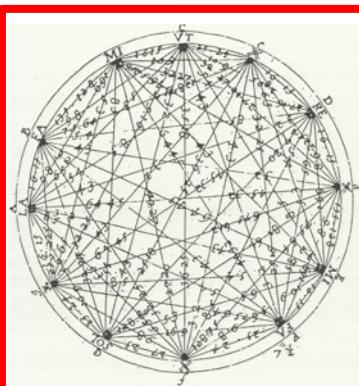
Music and mathematics: « prima la musica »!



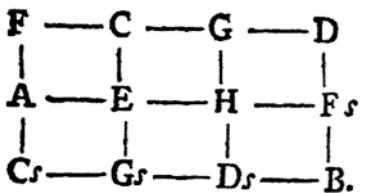
MUSIC	MATHS
500 B.C. Pitches and lengths of strings are related. Here music gives a marvelous thrust to number theory and geometry . <i>No correspondence in music.</i>	Discovery of the fundamental importance of natural numbers and the invention of fractions . Positive irrational numbers [...]
300 B.C. [...] Music theory highlights the discovery of the isomorphism between the logarithms (musical intervals) and exponentials (string lengths) more than 15 centuries before their discovery in mathematics; also a premonition of group theory is suggested by Aristoxenos.	No reaction in mathematics. [...]
1000 A.D. Invention of the two-dimensional spatial representation of pitches linked with time by means of staves and points [...] seven centuries (1635-37) before the magnificent analytical geometry of Fermat and Descartes.	<i>No parallel in mathematics.</i>
1500 No response or development of the preceding concepts.	Zero and negative numbers are adopted. Construction of the set of rationals.
1600 No equivalence, no reaction.	The sets of real numbers and of logarithms are invented.
1648 Invention of musical combinatorics by Marin Mersenne (<i>Harmonicorum Libri</i>)	Probability theory by Bernoulli (<i>Ars Conjectandi</i> , 1713)
1700 [...] The fugue, for example, is an abstract automaton used two centuries before the birth of the science of automata. Also, there is an unconscious manipulation of finite groups (Klein group) in the four variations of a melodic line used in counterpoint.	Number theory is ahead of but has no equivalent yet in temporal structures. [...]
1773 A first geometric and graph-theoretic representation of pitches (<i>Speculum Musicum</i>)	Invention of graph theory
1900 Liberation from the tonal yoke. First acceptance of the neutrality of chromatic totality (Loquin [1895], Hauer, Schoenberg).	The infinite and transfinite numbers (Cantor). Peano axiomatics. [...] The beautiful measure theory (Lebesgue, ...)
1920 First radical formalization of macrostructures through the serial system of Schoenberg.	No new development of the number theory.
1929 and 1937-1939 Susanne K. Langer and Ernst Krenek on the role of axioms in music	David Hilbert, <i>Die Grundlage der Geometrie</i> (1899)
1946 Milton Babbitt on group theory and integral serialism	Rudolf Carnap, <i>The Logical Syntax of Language</i> (1937)



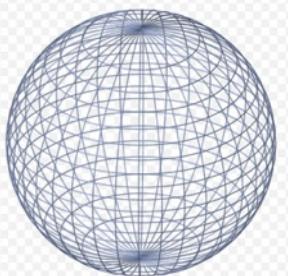
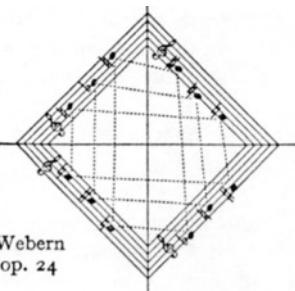
Pythagoras and the monochord,
VIth-Vth Century B.C.



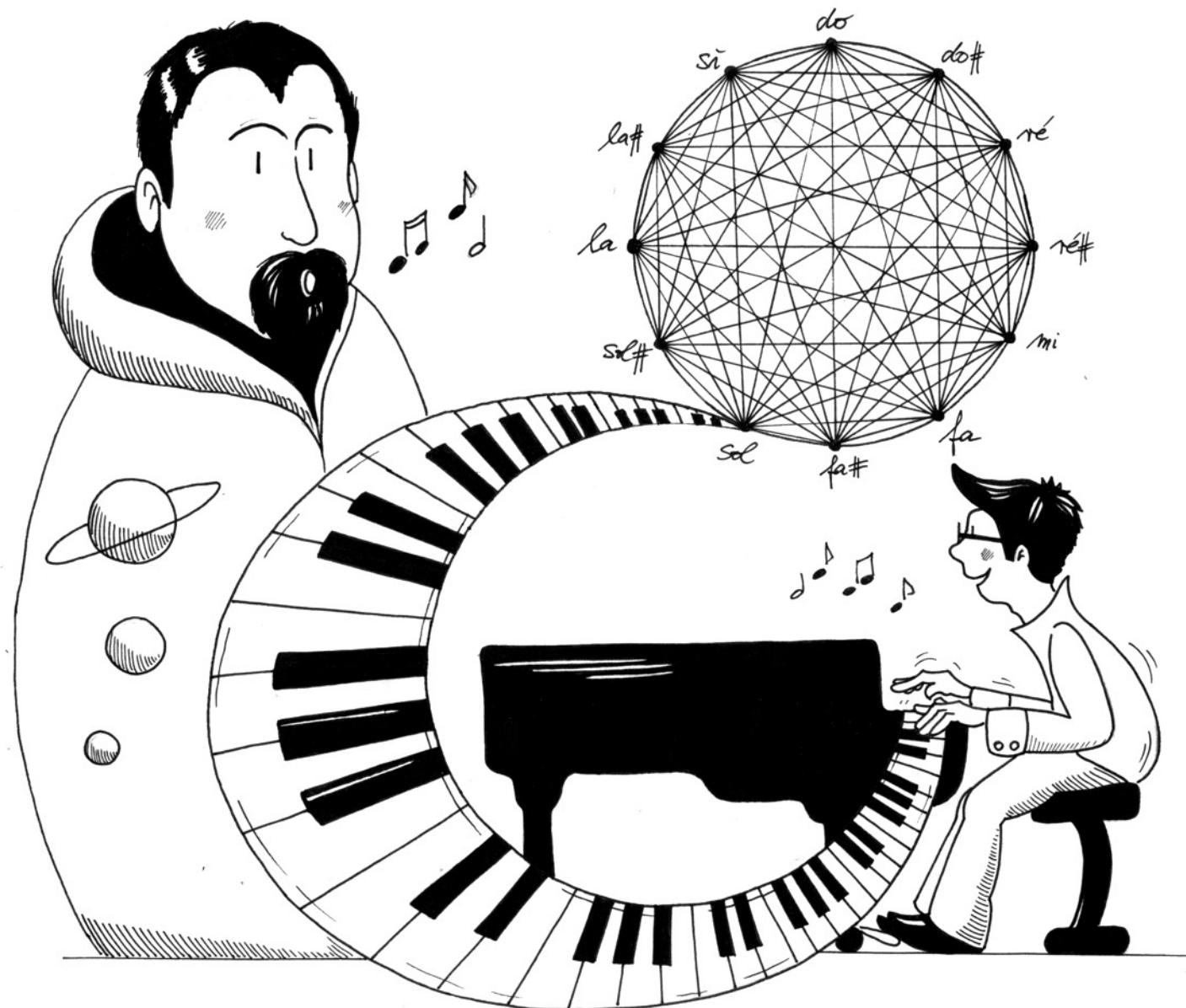
Mersenne and
the ‘musical
clock’, 1648



Euler and the
*Speculum
musicum*, 1773



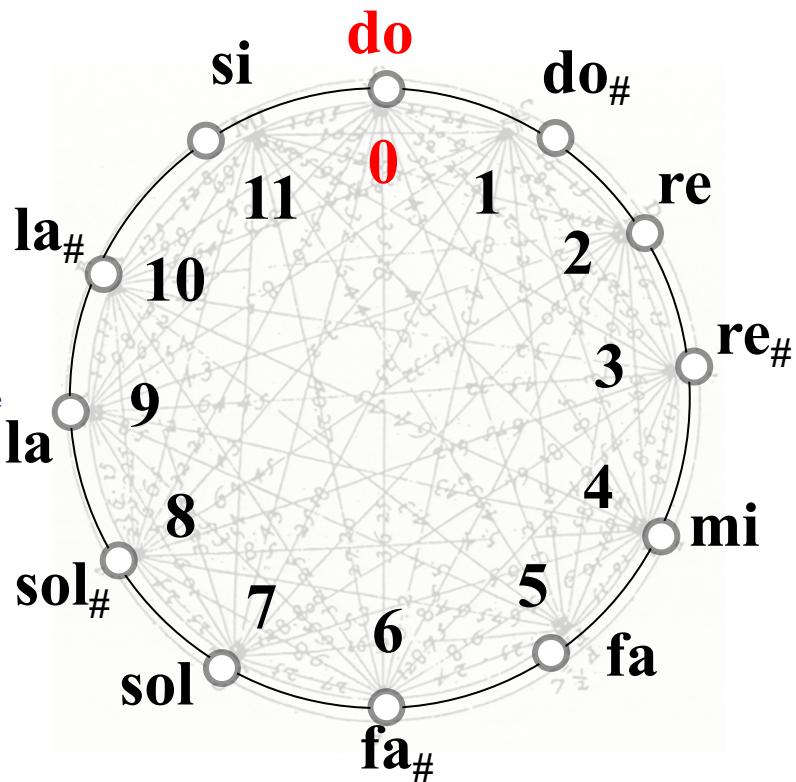
Marin Mersenne, the father of combinatorics



The circular representation of the pitch space



Marin Mersenne



Harmonicorum Libri XII, 1648



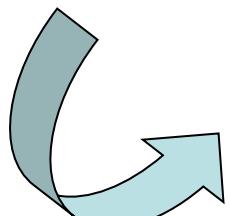
LIBER SEPTIMVS
DE CANTIBVS, SEV CANTILENIS,
EARVMQ; NVMERO, PARTIBVS, ET SPECIEBV.

Tabula Combinationis ab I ad XII.

I	1
II	2
III	6
IV	24
V	120
VI	720
VII	5040
VIII	40320
IX	361880
X	3618800
XI	39916800
XII	479001600
XIII	6178102800
XIV	87178191200
XV	1307674368000
XVI	20922789888000
XVII	335687418096000
XVIII	6402373705718000
XIX	12164100405881000
XX	2411901008176640000
XXI	51090942171709440000
XXII	11140007217777607680000

Varietas, seu Combinatio quator notarum.

Musical notation on a staff showing various ways to combine four notes. The notes are represented by small diamonds or squares. The staff has 24 numbered positions, corresponding to the 24 entries in the combination table above. Arrows point from the numbers in the table to the corresponding positions on the staff.



Musical notation on a staff with numbered positions below it. The positions are labeled 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, and 12. Arrows point from the numbers in the table above to the corresponding positions on the staff.

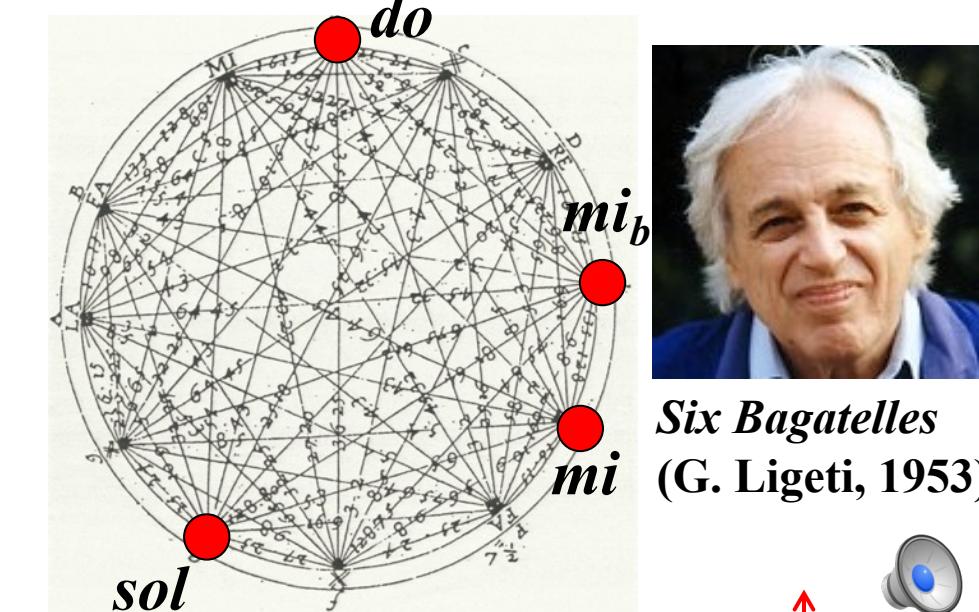
Permutational strategies in contemporary music

II.4. Marin Mersenne, *Harmonicorum Libri XII*, 1648

LIBER SEPTIMVS. DE CANTIBVS, SEV CANTILENIS, EARVMQ; NVMERO, PARTIBVS, ET SPECIEBVS.

Tabula Combinationis ab I ad 22.

I	1
II	2
III	6
IV	24
V	120
VI	720
VII	5040
VIII	40320
IX	361880
X	3618800
XI	39916800
XII	479001600
XIII	6117020800
XIV	87178191200
XV	1307674368000
XVI	20922789888000
XVII	335687418096000
XVIII	6402373705718000
XIX	121645100408831000
XX	2431901008176640000
XXI	51090942171709440000
XXII.	1114000717777607680000



Six Bagatelles
(G. Ligeti, 1953)

The musical score consists of 24 staves of music, each staff having four measures. The staves are arranged in a grid pattern. Above the grid, the title 'Varietas, seu Combinatio quathor notarum.' is written. Red numbers 1 through 24 are placed below each staff, corresponding to the numbers in the table above. An arrow points from the number 24 in the table to the first staff of the score.



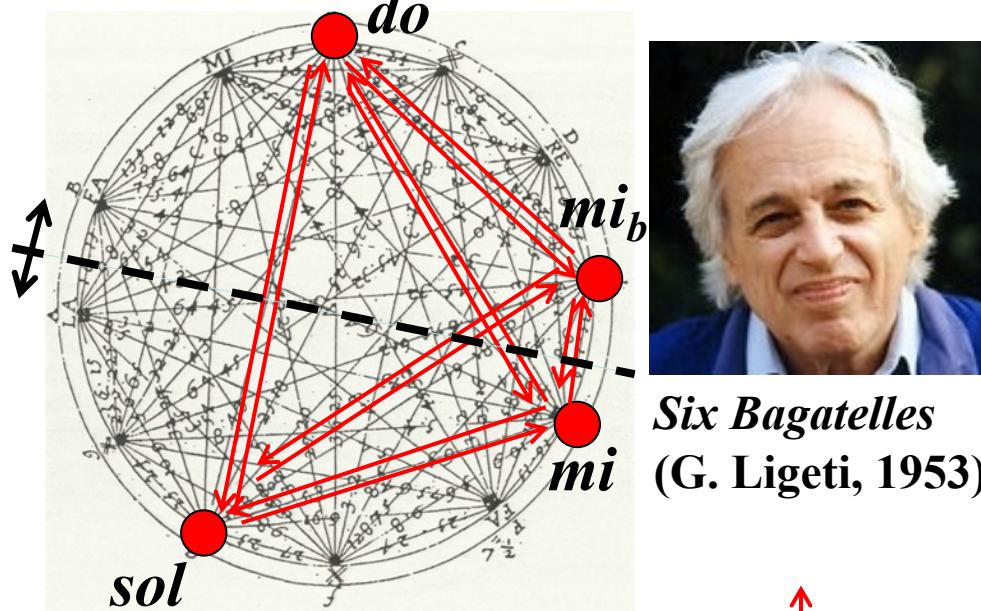
Permutational strategies in contemporary music

II.4. Marin Mersenne, *Harmonicorum Libri XII*, 1648

LIBER SEPTIMVS. DE CANTIBVS, SEV CANTILENIS, EARVMQ; NVMERO, PARTIBVS, ET SPECIEBVS.

Tabula Combinationis ab I ad 22.

I	1
II	2
III	6
IV	24
V	120
VI	720
VII	5040
VIII	40320
IX	361880
X	3618800
XI	39916800
XII	479001600
XIII	6117020800
XIV	87178191200
XV	1307674368000
XVI	20922789888000
XVII	355687418096000
XVIII	6402373705718000
XIX	121645100408831000
XX	2431901008176640000
XXI	51090942171709440000
XXII.	1114000717777607680000



Permutational melodies in song writing

Se telefonando, 1966 (Maurizio Costanzo/Ennio Morricone) / Mina

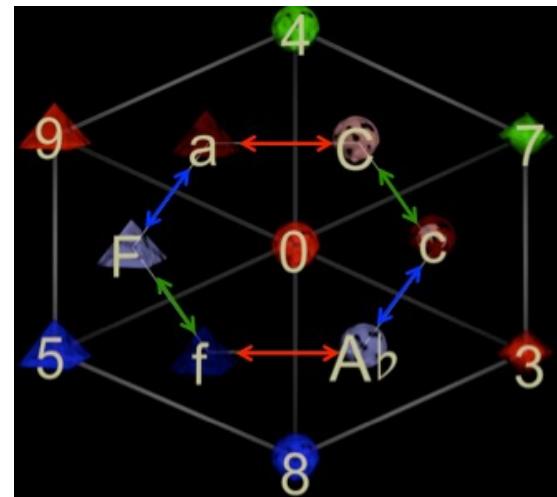


(min. 0'53")



Ennio Morricone

The harmonic space



C c C_# c_# D d

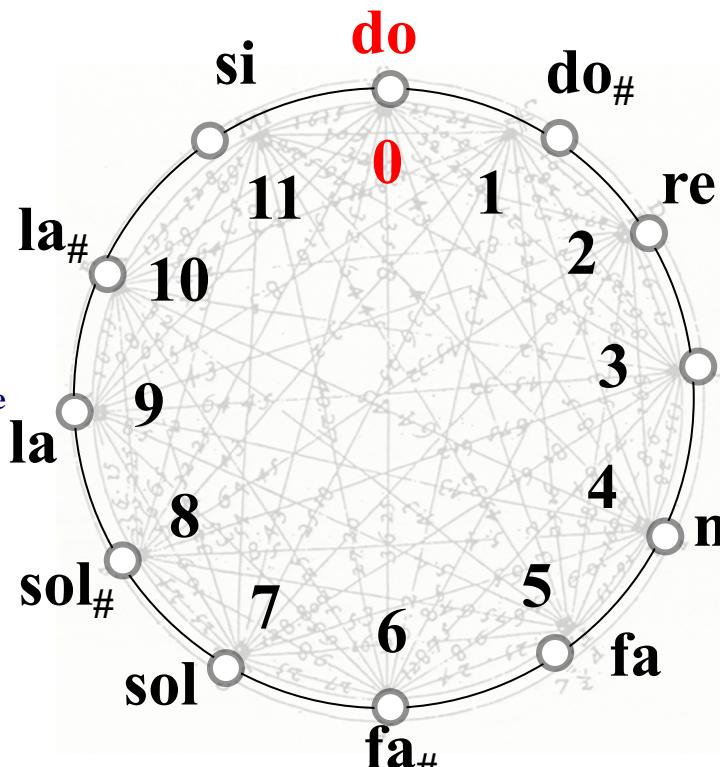
E_b e_b E e F f

F_# f_# G g G_# g_#

A a B_b b_b B b

Chord enumeration

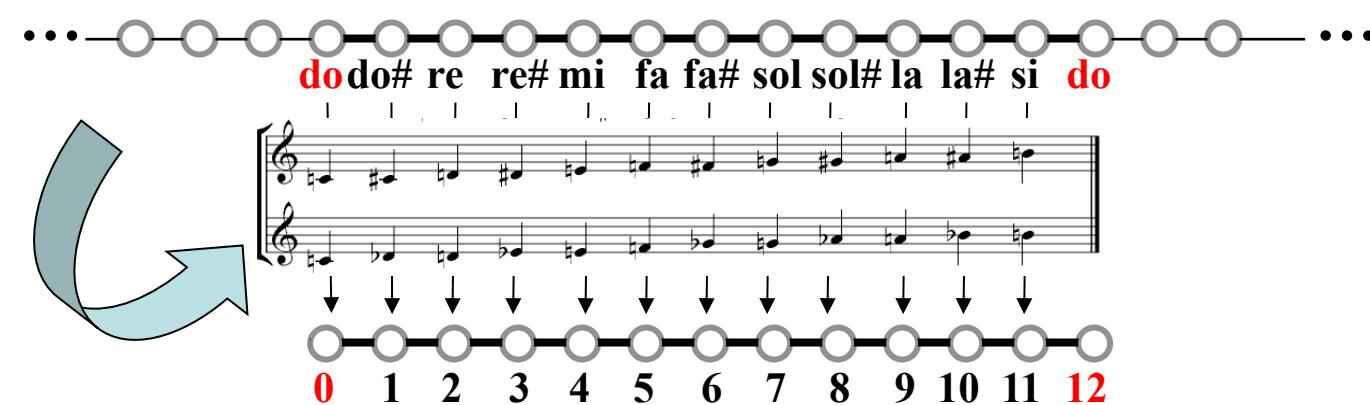
The circular representation of the pitch space



Harmonicorum Libri XII, 1648



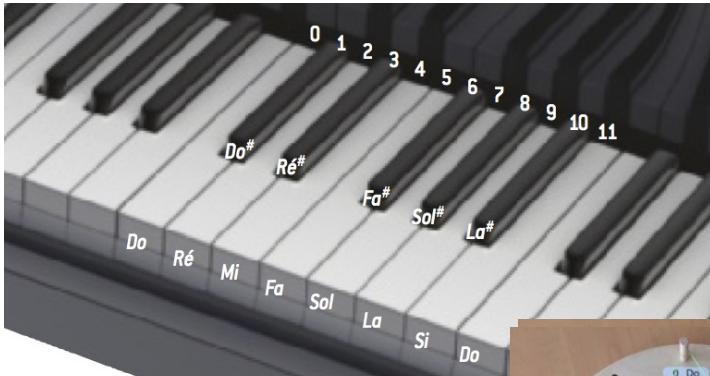
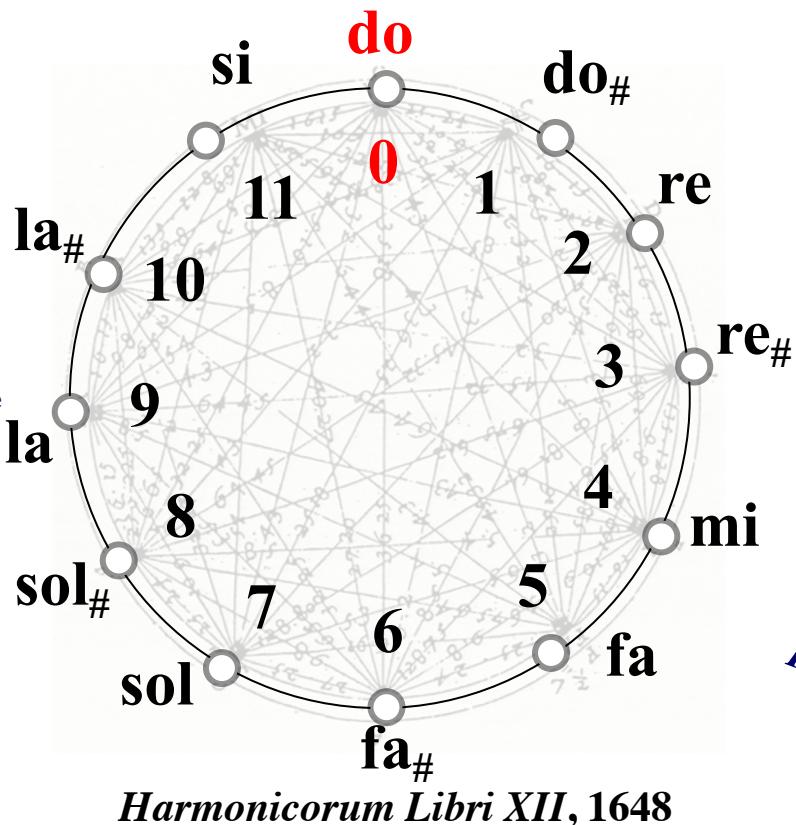
M. Andreatta, C. Agon,
«La musique mise en algèbre»,
Pour la Science, 2008



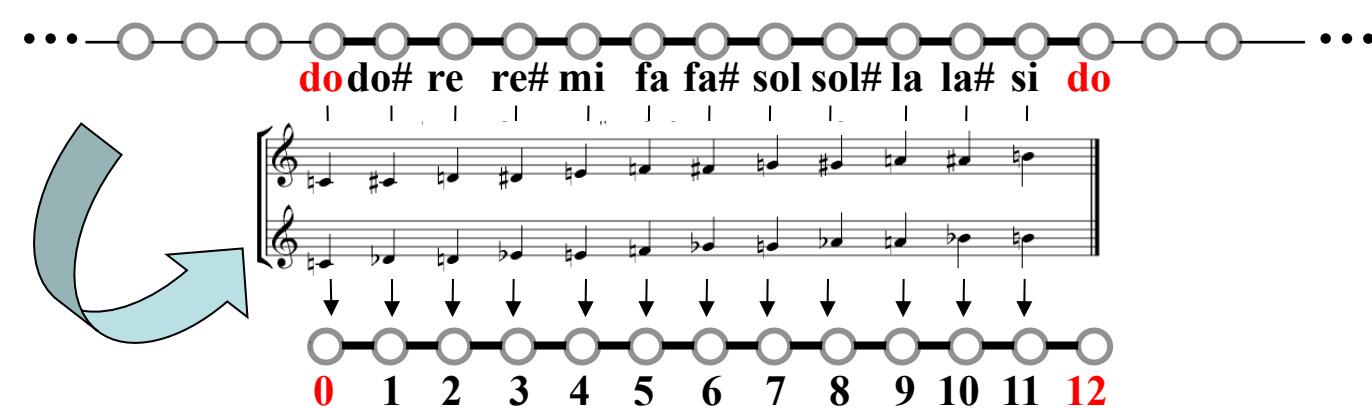
The circular representation of the pitch space



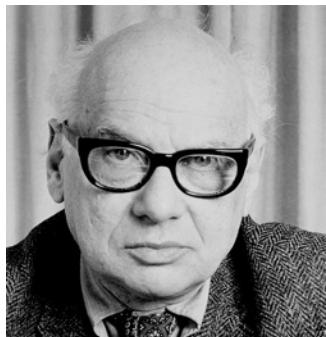
Marin Mersenne



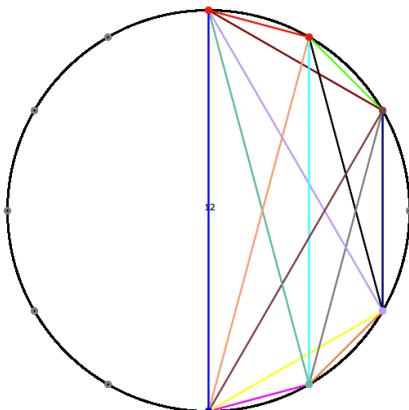
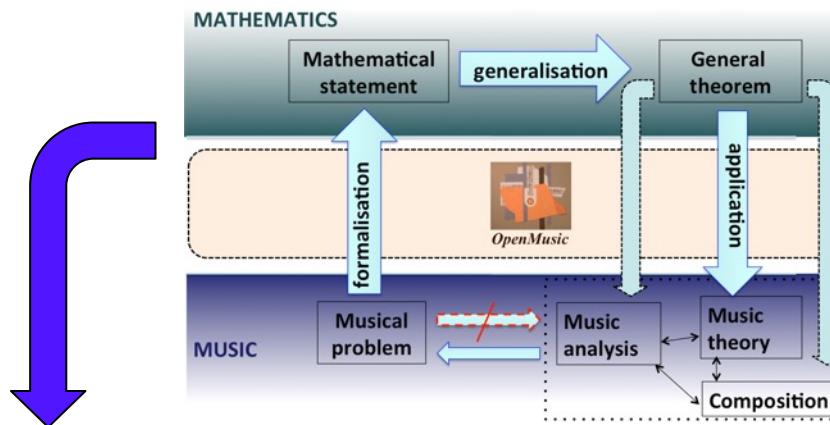
M. Andreatta, C. Agon,
«Algèbre et géométrie :
sont-elles inscrites dans le
cerveau ?»,
Pour la Science, 2018



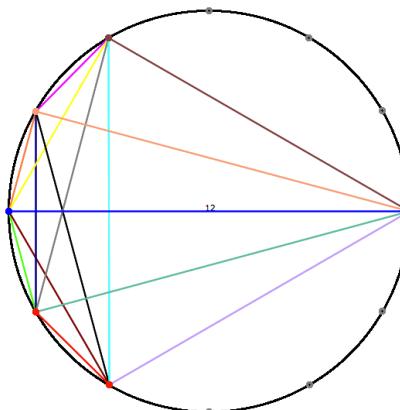
A historical example of “mathemusical” problem



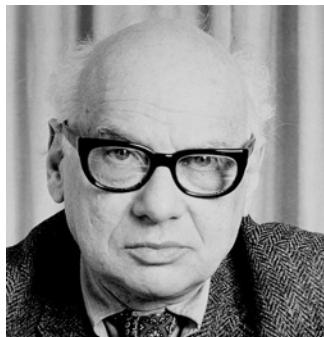
M. Babbitt



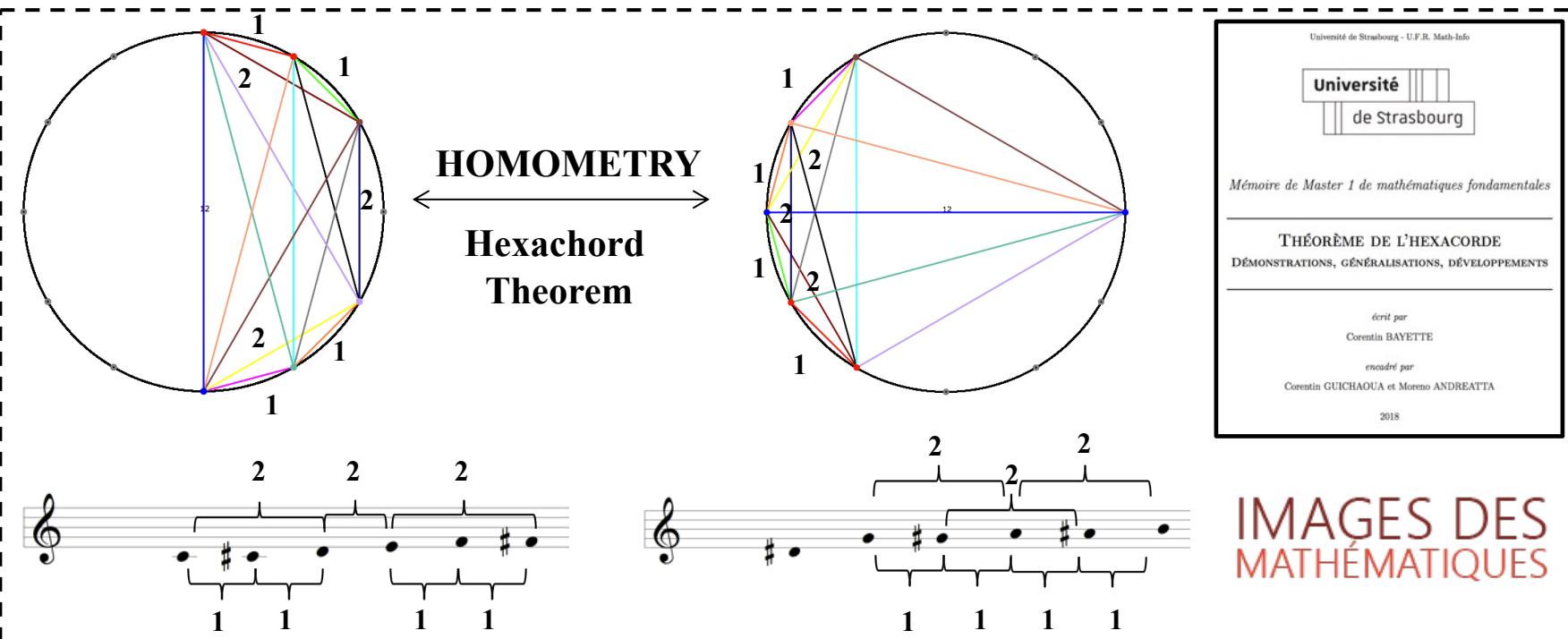
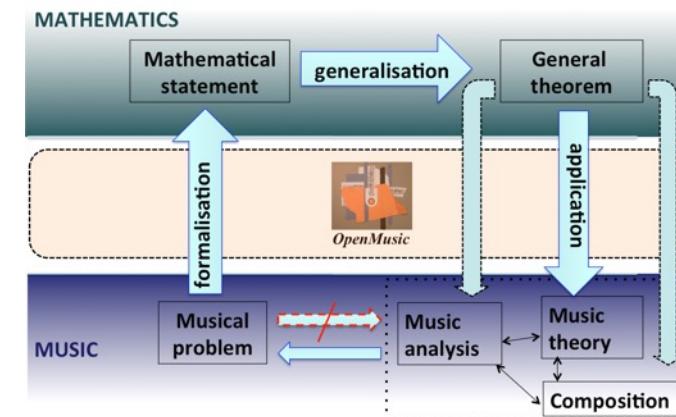
Hexachord
Theorem



A historical example of “mathemusical” problem



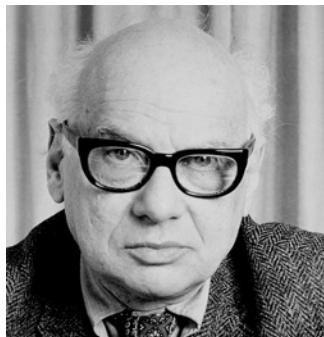
M. Babbitt



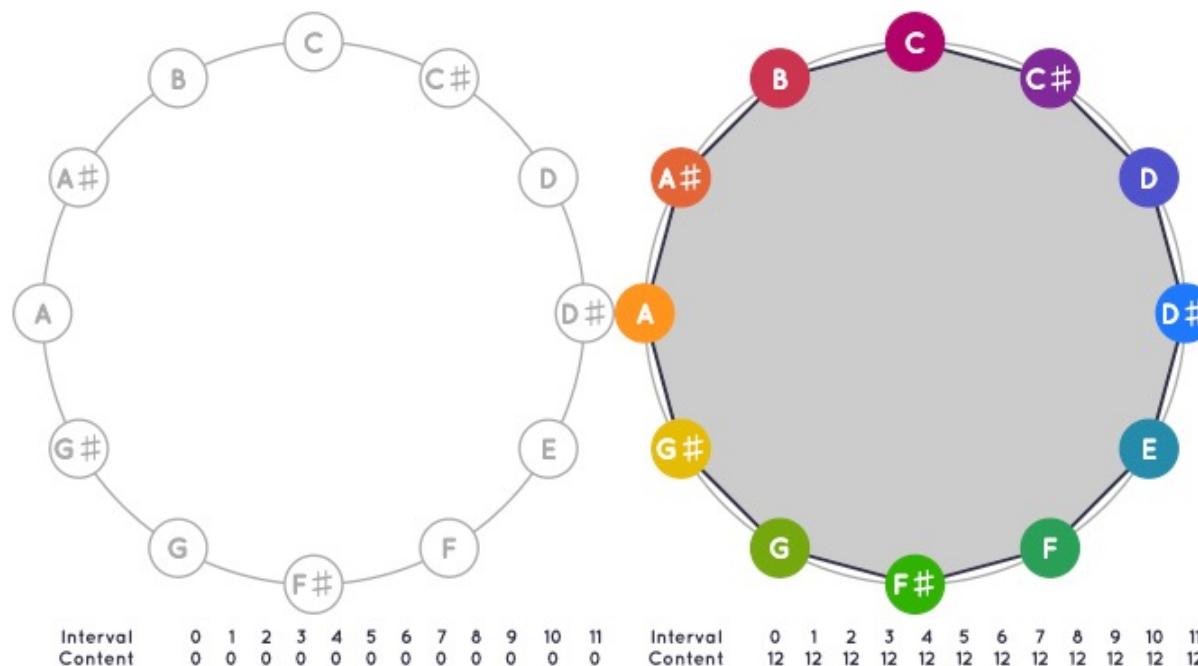
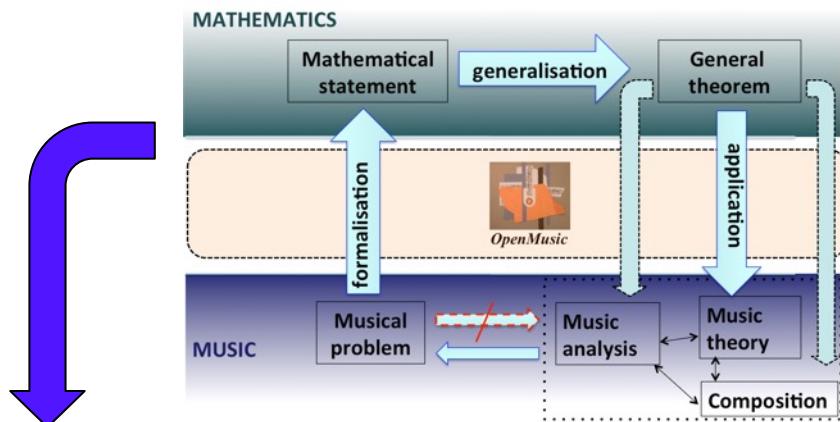
IMAGES DES
MATHÉMATIQUES

→ Corentin Bayette, Théorème de l'Hexacorde, Master 1 dissertation, University of Strasbourg, 2018

A historical example of “mathemusical” problem

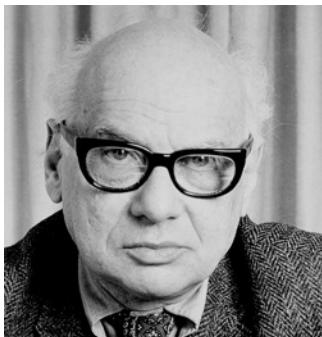


M. Babbitt

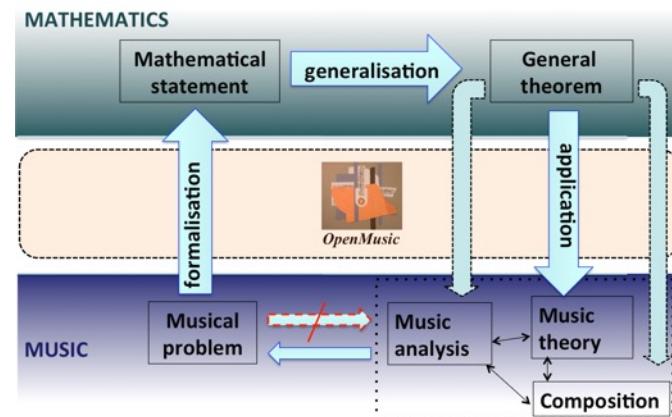
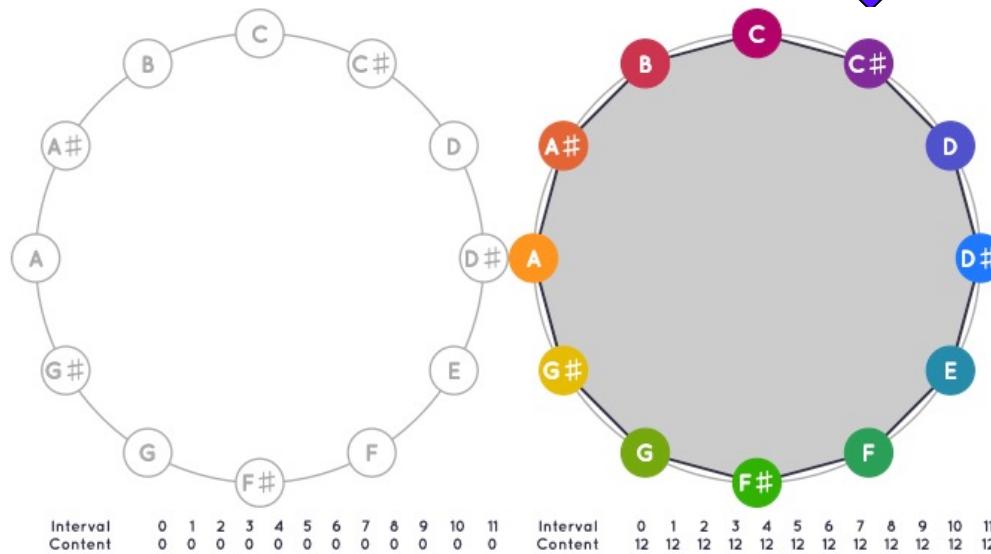


DEMO

A historical example of “mathemusical” problem



M. Babbitt

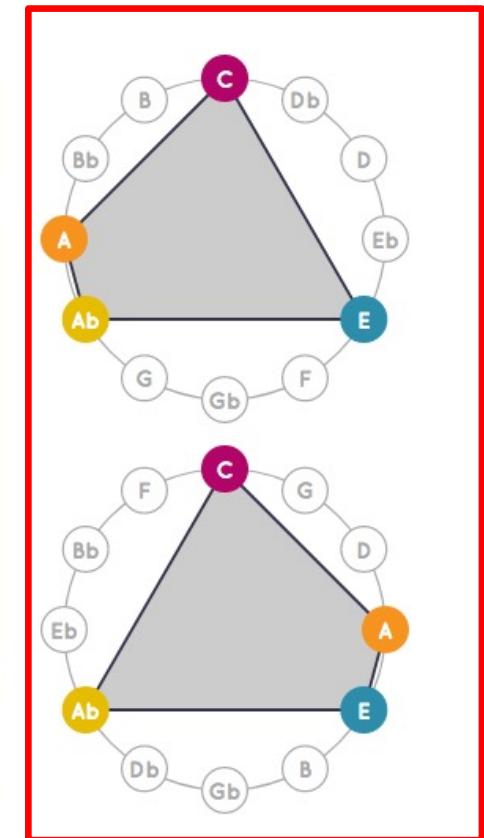
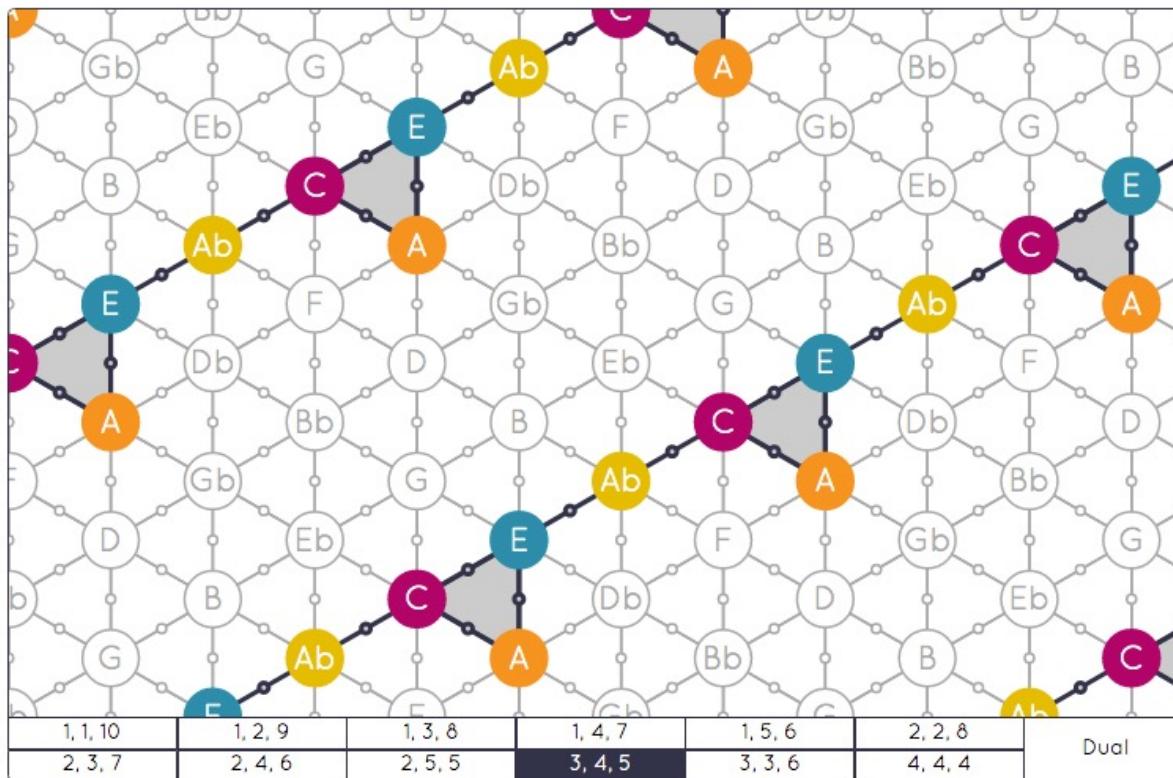


- Corentin Bayette, Théorème de l'Hexacorde, Master 1 dissertation, University of Strasbourg, 2018
- <https://guichaoua.gitlab.io/web-hexachord/hexachordTheorem>
- M. Andreatta et al., New hexachordal theorems in metric spaces with a probability measure", Rendiconti Univ Padova, 2023
- M. Andreatta et al., Taking music seriously: on the dynamics of ‘mathemusical’ research with a focus on Hexachordal Theorems", SIGMA (Symmetry, Integrability and Geometry: Methods and Applications), 2024

SYMMETRY, >>>>>
INTEGRABILITY
and **G**EOMETRY:
METHODS and
APPICATIONS >>>>>

THE TONNETZ

ONE KEY – MANY REPRESENTATIONS



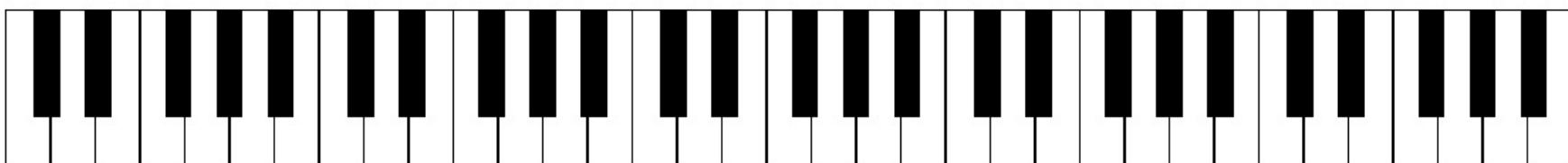
Load Midi File

Play

Start Recording

Rotate 180°

Translate



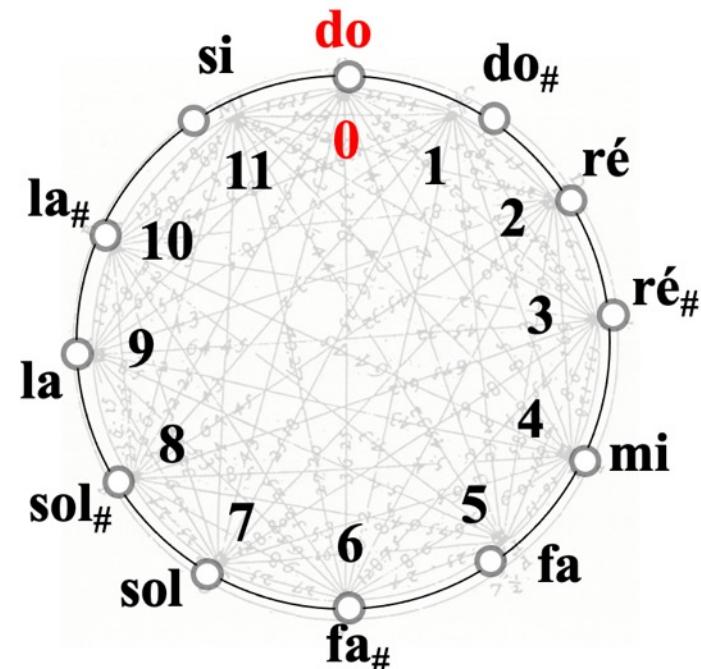
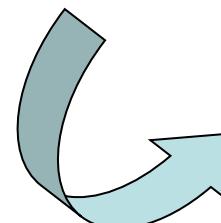
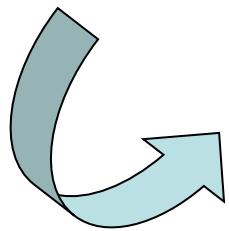
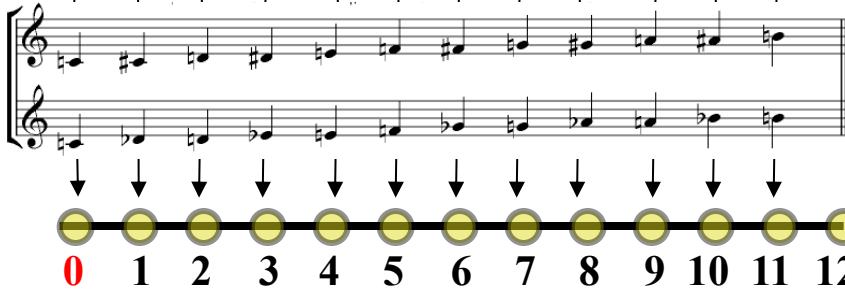
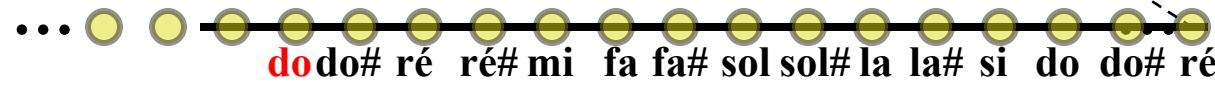
→ <https://thetonnez.com>

DEMO

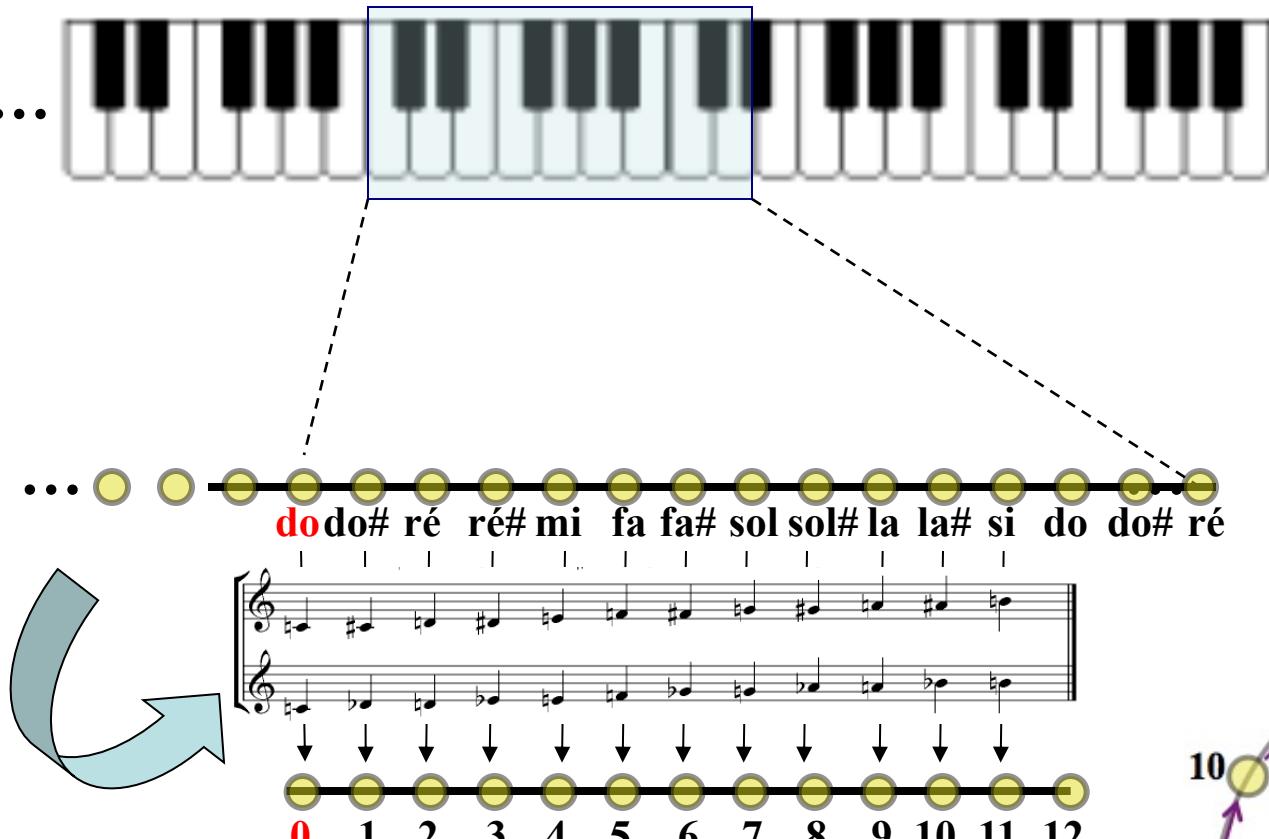
The equal tempered space is a cyclic group



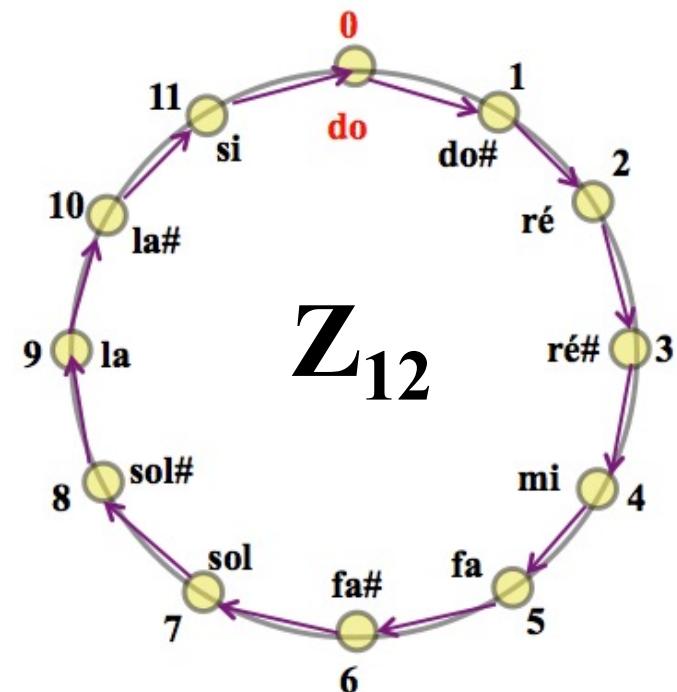
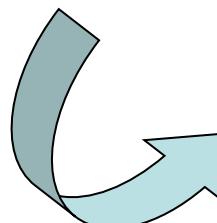
The (cyclic) group structure, its axioms and generators



The equal tempered space is a cyclic group

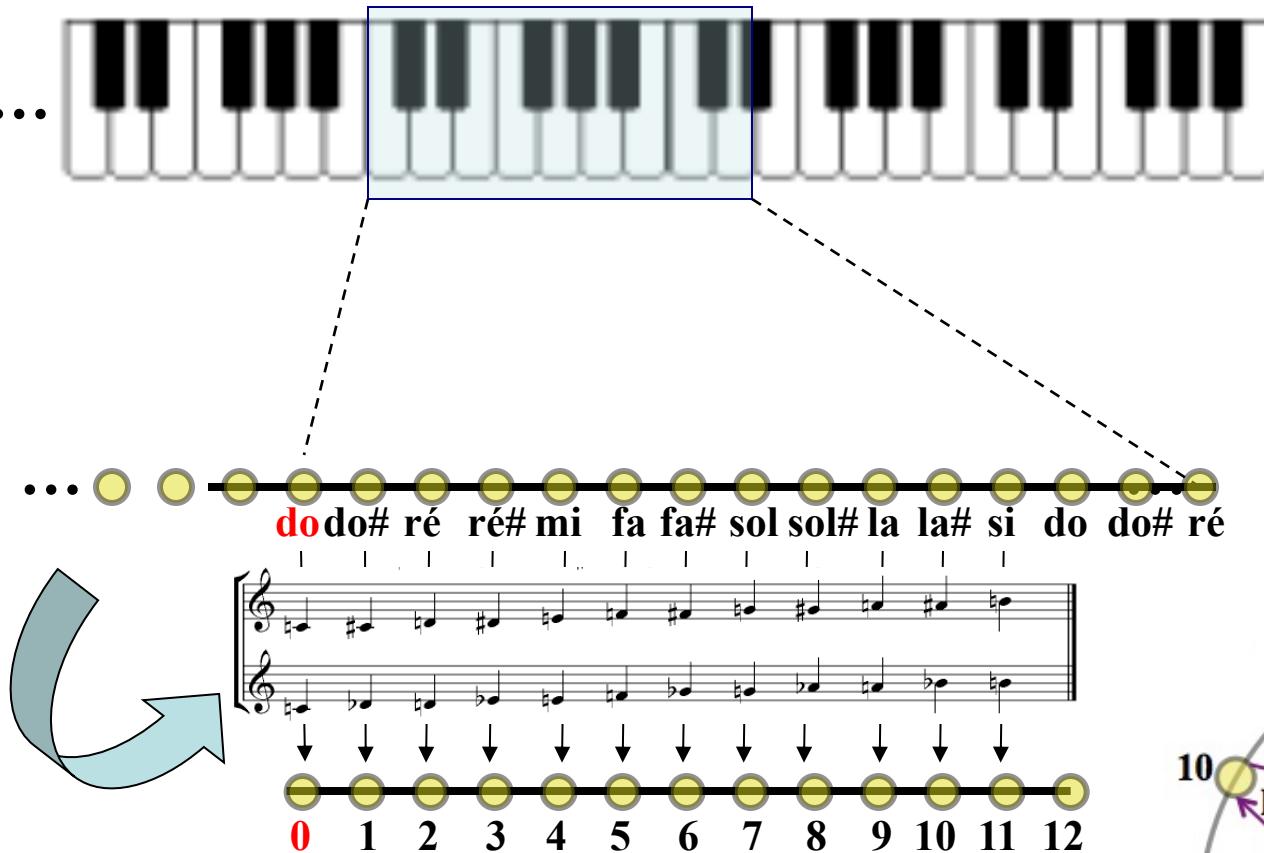


$$Z_{12} = \langle T_1 \mid (T_1)^{12} = T_0 \rangle$$

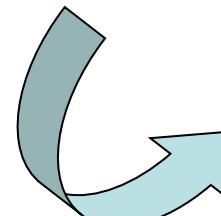


The generators of the cyclic group of order 12 are the transpositions T_1 , T_5 , T_7 et T_{11}
where
 $T_k: x \rightarrow x+k \bmod 12$

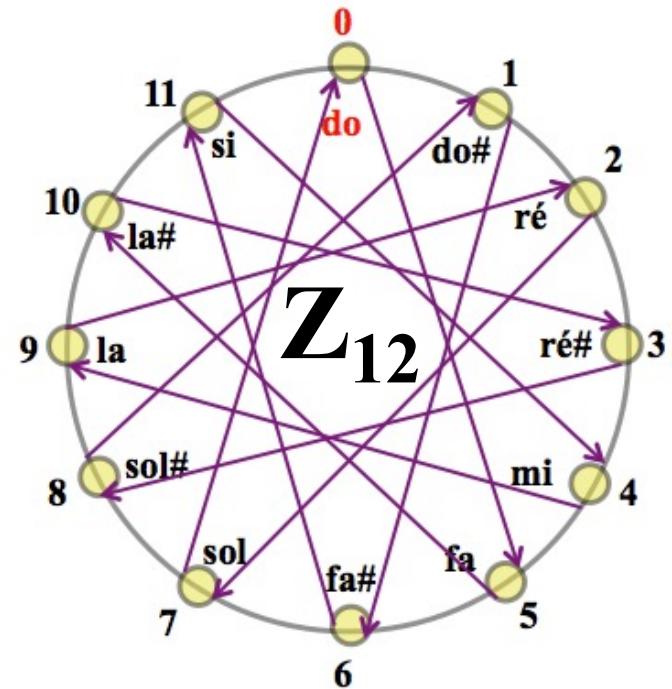
The equal tempered space is a cyclic group



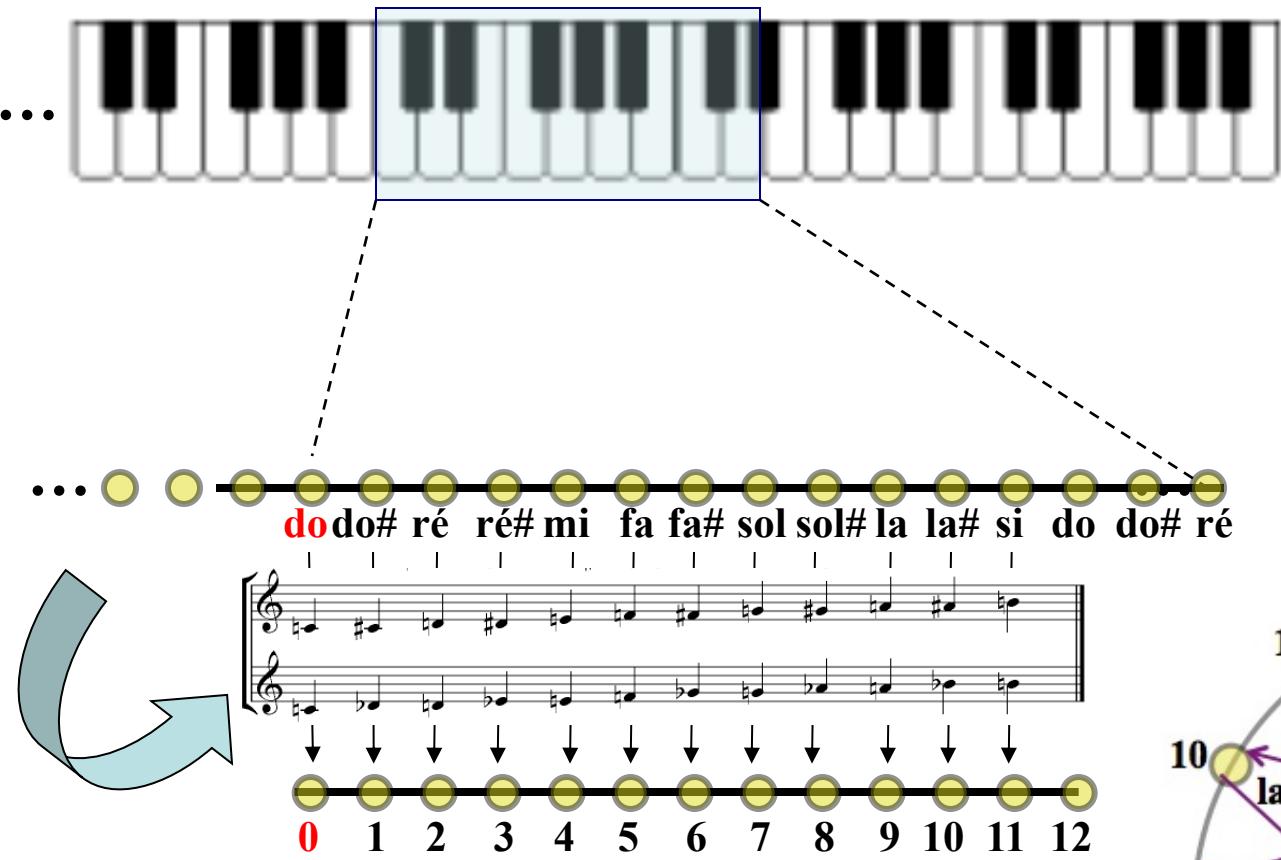
$$\begin{aligned} Z_{12} &= \langle T_1 \mid (T_1)^{12} = T_0 \rangle = \\ &= \langle T_5 \mid (T_5)^{12} = T_0 \rangle \end{aligned}$$



The generators of the cyclic group of order 12 are the transpositions T_1 , T_5 , T_7 et T_{11}
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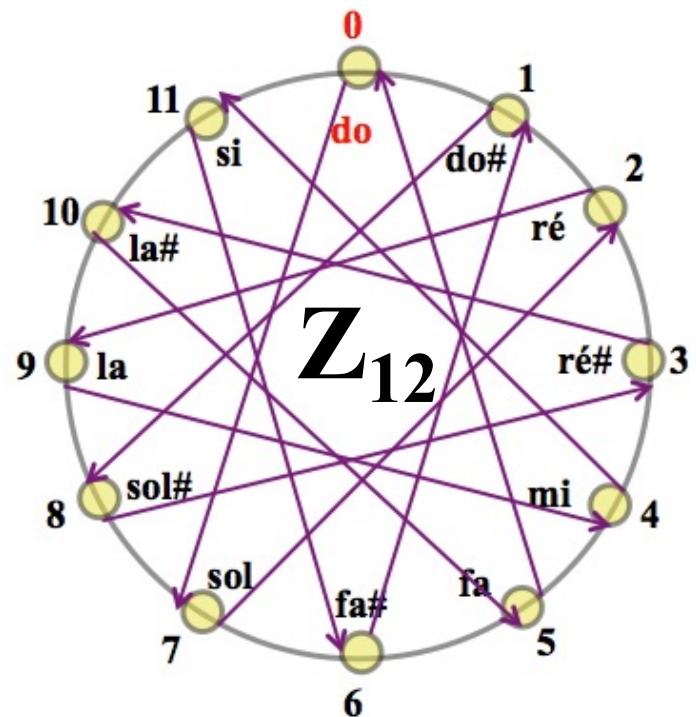


The equal tempered space is a cyclic group

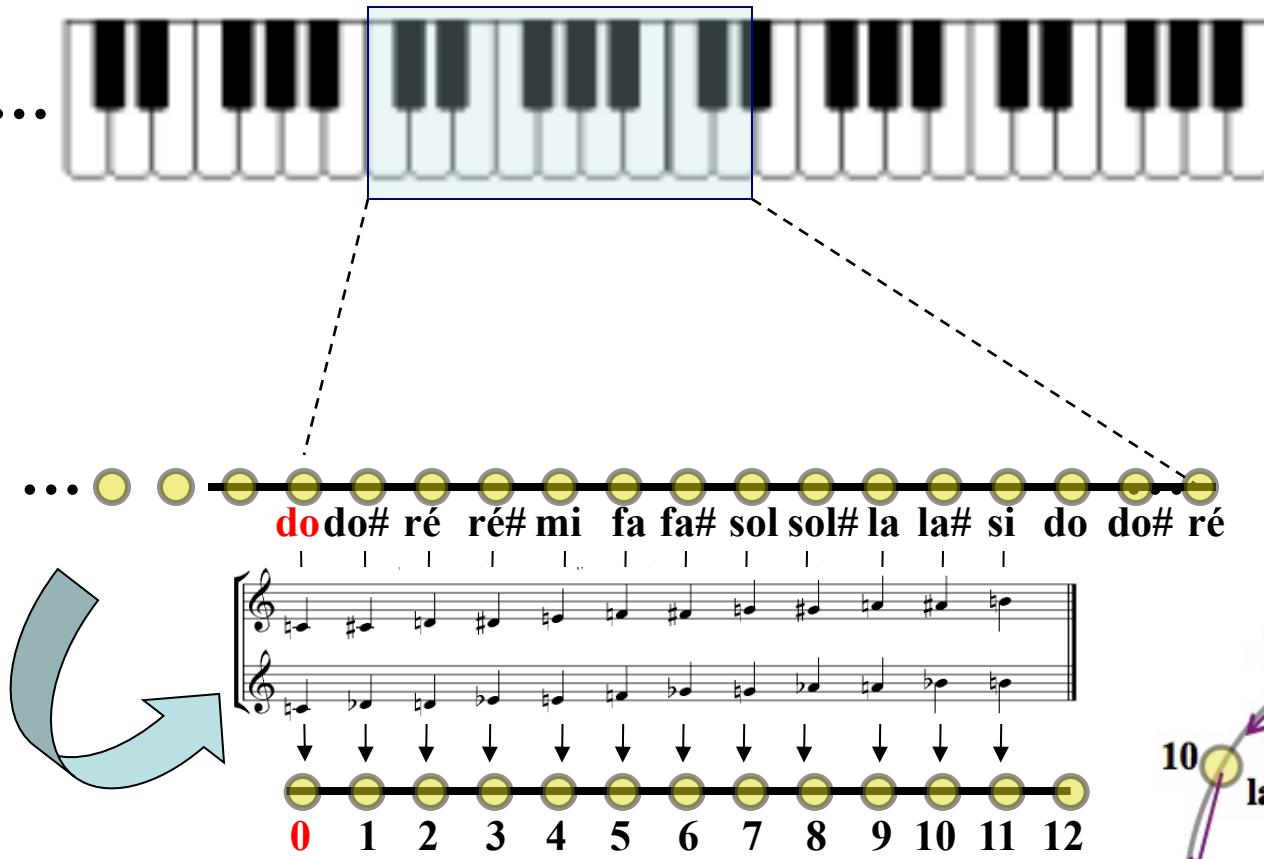


$$\begin{aligned} Z_{12} &= \langle T_1 \mid (T_1)^{12} = T_0 \rangle = \\ &= \langle T_5 \mid (T_5)^{12} = T_0 \rangle = \\ &= \langle T_7 \mid (T_7)^{12} = T_0 \rangle \end{aligned}$$

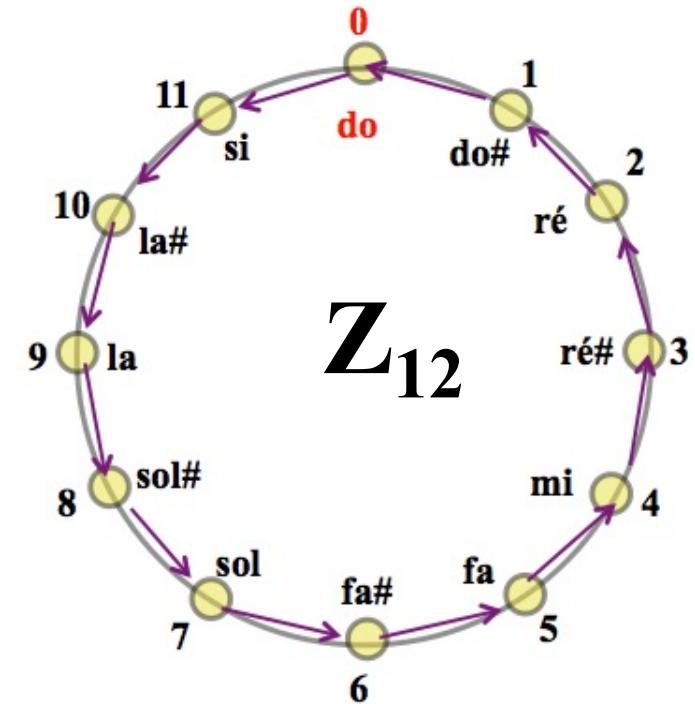
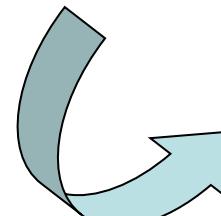
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The equal tempered space is a cyclic group

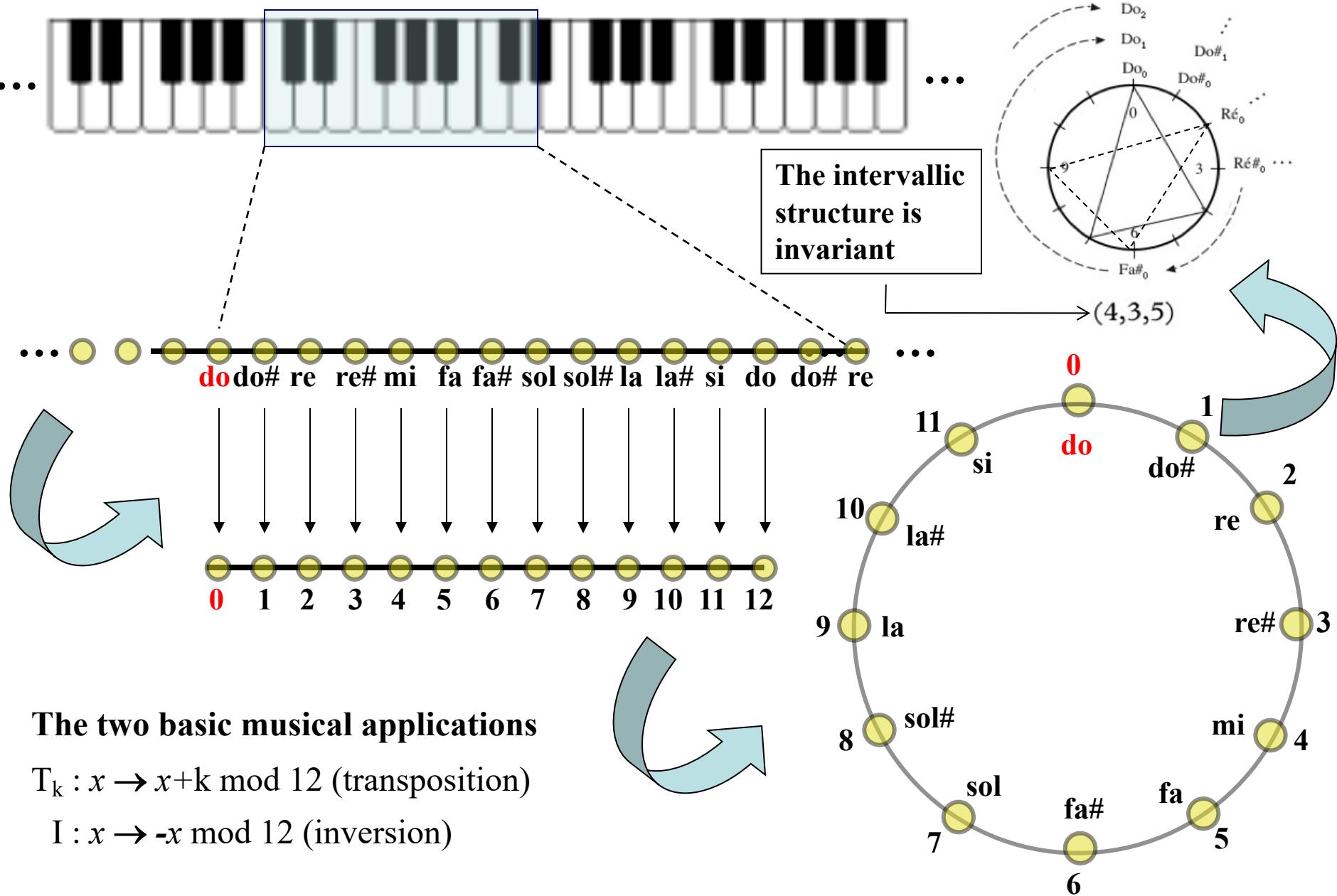


$$\begin{aligned}
 Z_{12} &= \langle T_1 \mid (T_1)^{12} = T_0 \rangle = \\
 &= \langle T_5 \mid (T_5)^{12} = T_0 \rangle = \\
 &= \langle T_7 \mid (T_7)^{12} = T_0 \rangle = \\
 &= \langle T_{11} \mid (T_{11})^{12} = T_0 \rangle
 \end{aligned}$$

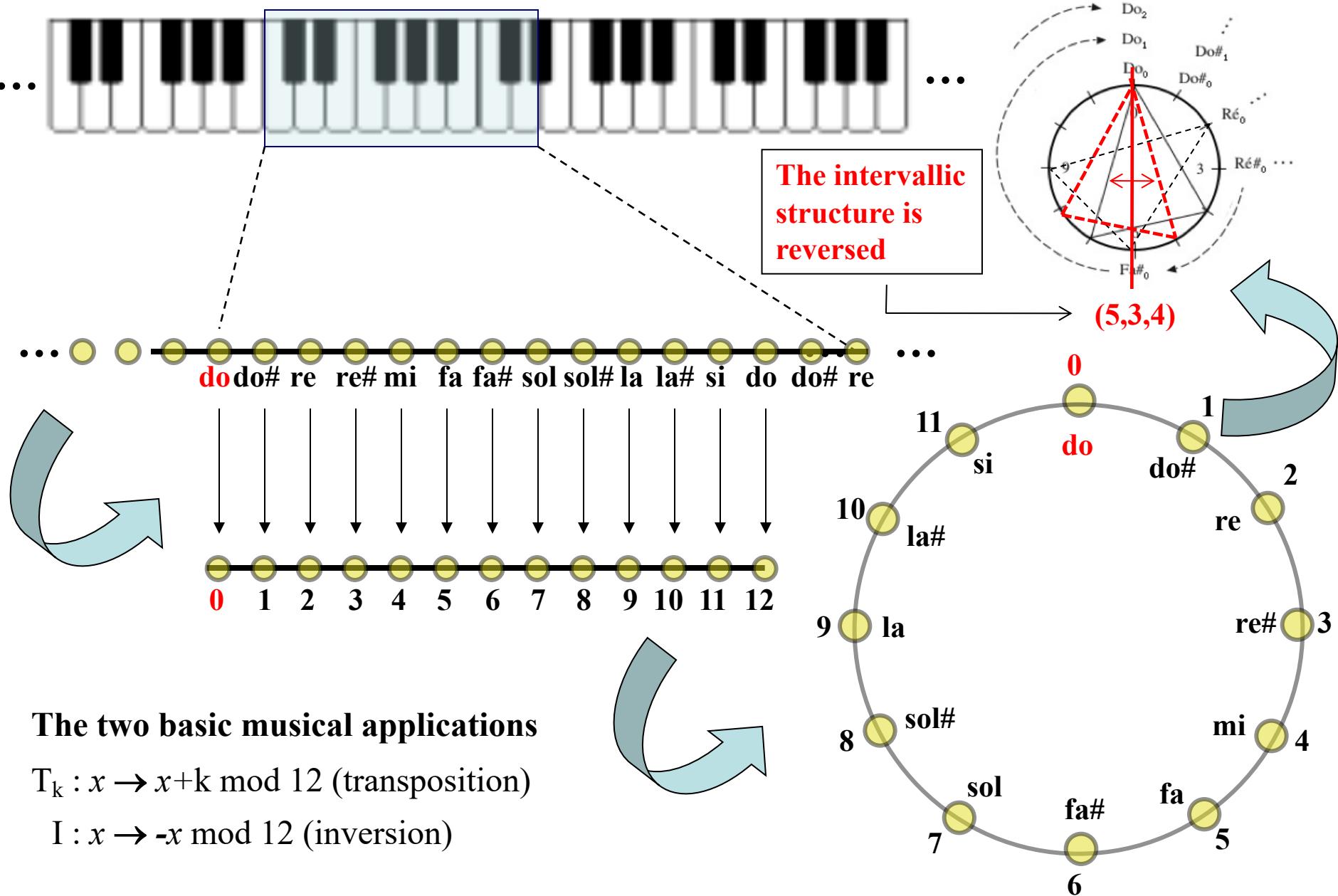


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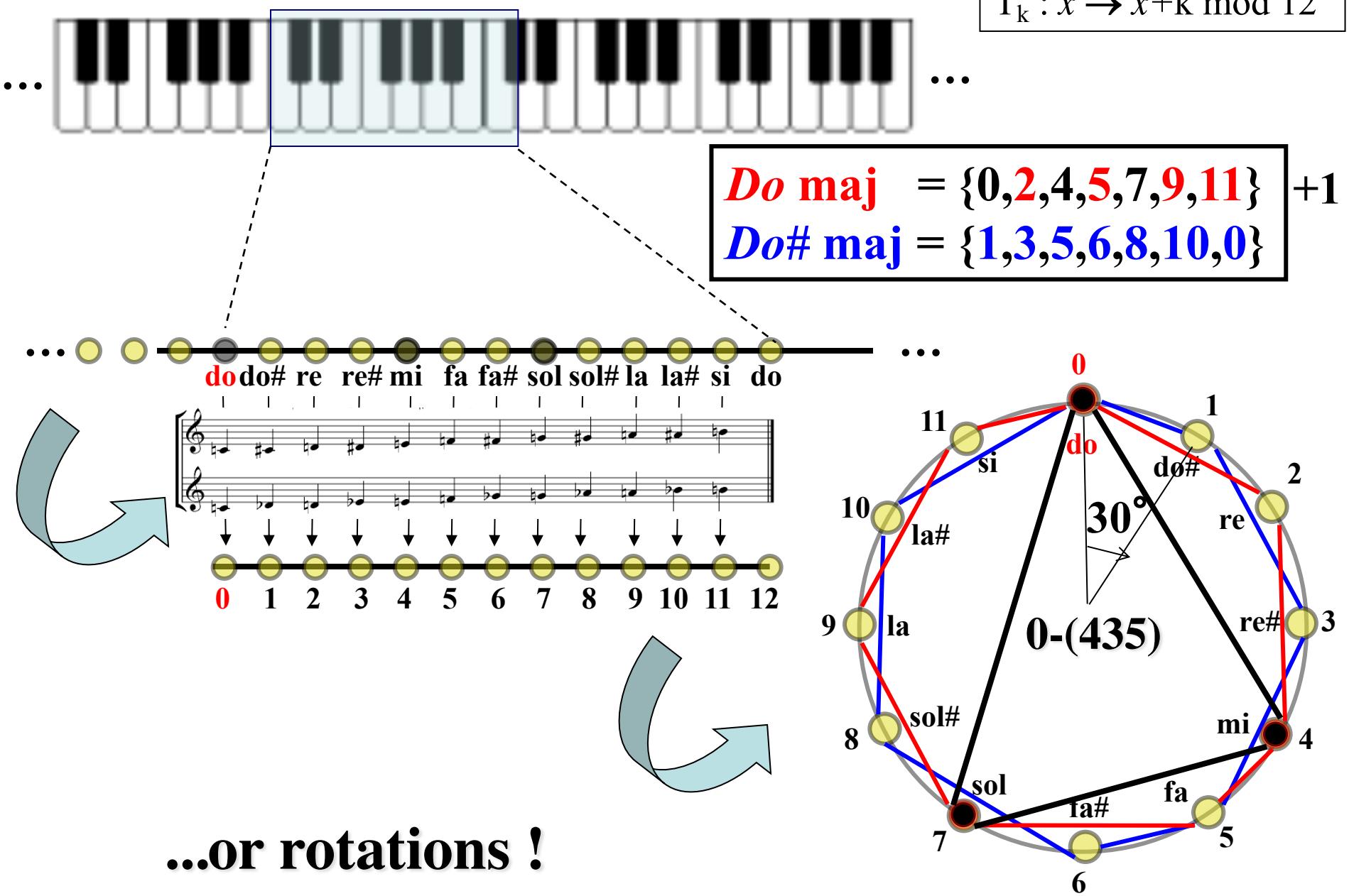
Circular representation and intervallic structure



Circular representation and intervallic structure



Musical transpositions are additions...



Musical transpositions are additions...

$T_k : x \rightarrow x+k \bmod 12$

The diagram illustrates musical transpositions through various representations:

- Piano Keyboard:** A horizontal piano keyboard with black and white keys. A blue rectangular box highlights a segment of six keys.
- Number Line:** A horizontal number line from 0 to 12. Yellow circles mark the integers. Below the line, note names are written above the numbers:
 - 0: do
 - 1: do#
 - 2: re
 - 3: re#
 - 4: mi
 - 5: fa
 - 6: fa#
 - 7: sol
 - 8: sol#
 - 9: la
 - 10: la#
 - 11: si
 - 12: do
- Music Staff:** Two staves of music notation in G major (one treble clef, one bass clef) with sharp signs. Arrows point from the notes in the staff to their corresponding positions on the number line below.
- Graph:** A circular graph with 12 nodes numbered 0 to 11. Nodes 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11 are yellow, while 12 is black. Edges connect adjacent nodes. Labels around the perimeter include:
 - 0: do
 - 1: do#
 - 2: re
 - 3: re#
 - 4: mi
 - 5: fa
 - 6: fa#
 - 7: sol
 - 8: sol#
 - 9: la
 - 10: la#
 - 11: si
 - 12: doA red path connects nodes 0, 11, 1, 10, 2, 9, 3, 8, 4, 7, 5, 6, 0. A blue path connects nodes 0, 11, 1, 10, 2, 9, 3, 8, 4, 7, 5, 12. A black path connects nodes 0, 11, 1, 10, 2, 9, 3, 8, 4, 7, 5, 6. A dashed black path connects nodes 0, 11, 1, 10, 2, 9, 3, 8, 4, 7, 5, 12. A label "1-(435)" is placed near the bottom right of the graph.
- Text:** At the bottom left, the text "...or rotations!" is written.

Musical inversions are differences...

... or axial symmetries!

The diagram illustrates musical inversions and axial symmetries using a piano keyboard, a circle of fifths, and musical notation.

Piano Keyboard: A horizontal piano keyboard is shown with a blue box highlighting a segment of keys. Dashed lines connect this segment to a circle of fifths and a musical staff.

Circle of Fifths: A circular diagram showing the 12 notes of the chromatic scale. Notes are labeled with their corresponding numbers (0-11) and names: do, do#, re, re#, mi, fa, fa#, sol, sol#, la, la#, si, do. A red line connects notes 0, 4, and 7. A blue line connects notes 0, 4, and 9. An arrow labeled I_4 indicates a 4-note cycle. The formula $I : x \rightarrow -x \bmod 12$ is given, and the mapping $I_4(x) = 4 - x$ is shown for the notes 0, 4, and 7.

Musical Staff: A musical staff with two staves is shown. The top staff uses a treble clef and the bottom staff uses a bass clef. Arrows point from the notes on the staff to the corresponding notes on the circle of fifths. The notes are labeled: do, do#, re, re#, mi, fa, fa#, sol, sol#, la, la#, si, do.

Text: The text "Do maj = {0,4,7}" is written in red, and "La min = {0,4,9}" is written in blue, both referring to the sets of notes highlighted on the circle of fifths.

Musical inversions are differences...

... or axial symmetries!

The diagram illustrates musical inversions and axial symmetries using a piano keyboard, a circle of fifths, and musical notation.

Piano Keyboard: A horizontal piano keyboard is shown with black and white keys. A blue dashed rectangle highlights a segment of the keyboard. Dashed arrows point from this segment to two boxes below.

Do maj and Do min sets: Two boxes define sets of notes:

- Do maj** = {0, 4, 7}
- Do min** = {0, 3, 7}

A formula $I_7(x) = 7 - x$ is shown next to the Do min set, indicating the inversion of the major scale.

Circle of Fifths: A circular diagram shows the 12 notes of the chromatic scale (do, do#, re, re#, mi, fa, fa#, sol, sol#, la, la#, si, do) arranged clockwise. Arrows indicate a clockwise cycle. The notes are labeled with numbers 0 through 11, with 0 at the top and 11 at the bottom. A red arrow connects note 0 to note 7 (sol). A blue arrow connects note 0 to note 4 (fa).

Music Notation: Below the circle, a treble clef staff shows musical notes corresponding to the notes on the circle. Arrows point from the notes on the circle to the staff. The notes are labeled with their corresponding numbers (0-11) below the staff.

Musical inversions are differences...

... or axial symmetries!

The diagram illustrates musical inversions and axial symmetries using a piano keyboard, a circle of fifths, and musical notation.

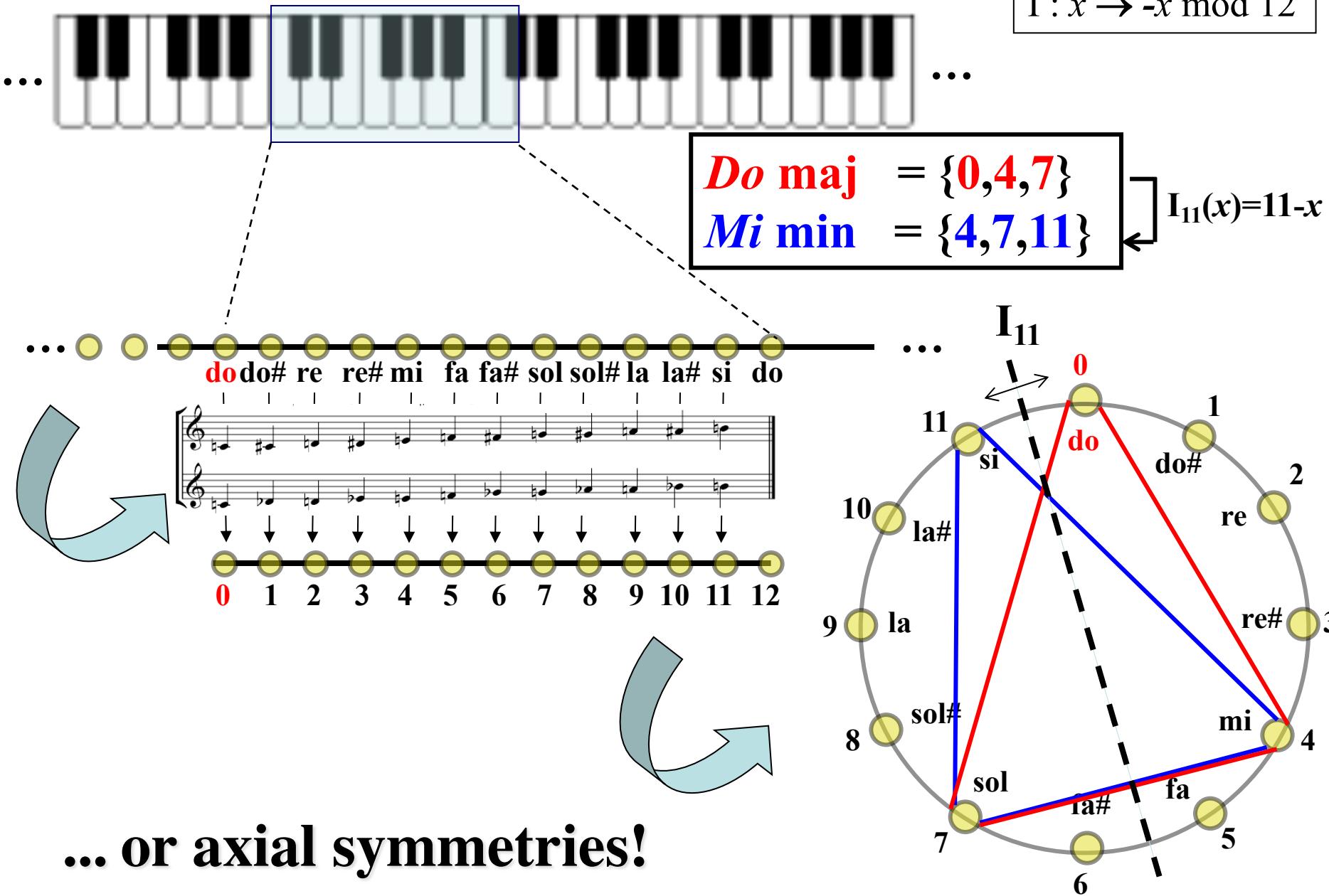
Piano Keyboard: A horizontal piano keyboard is shown with black and white keys. A blue dashed rectangle highlights a segment of the keyboard, which is magnified below to show note names: **Do maj** = {0, 4, 7} and **Do min** = {0, 3, 7}. A formula $I : x \rightarrow -x \bmod 12$ is given, and the mapping $I_7(x) = 7 - x$ is shown for the Do min set.

Circle of Fifths: A circular diagram shows the 12 notes of the chromatic scale (C, C#, D, D#, E, F, F#, G, G#, A, A#, B) arranged clockwise. Arrows indicate the direction of a fifth (e.g., from C to G). The notes are labeled with their corresponding numbers (0 to 11) and names (do, do#, re, re#, mi, fa, fa#, sol, sol#, la, la#, si, do). A red arrow points from the top note (0/do) to the bottom note (7/sol), and a blue arrow points from the bottom note (7/sol) to the top note (0/do).

Musical Notation: Below the circle of fifths is a musical staff with two staves. The top staff uses a treble clef and the bottom staff uses a bass clef. Notes are placed on the staff according to the note numbers (0 to 11) indicated by arrows from the circle of fifths. A large blue curved arrow indicates a 12-note cycle.

Axes of Symmetry: A vertical axis of symmetry is shown through the center of the circle of fifths, with points 0 and 7 at the top and bottom. A horizontal axis of symmetry is shown through the center, with points 0 and 6 at the left and right ends. The formula I_7 is associated with the vertical axis.

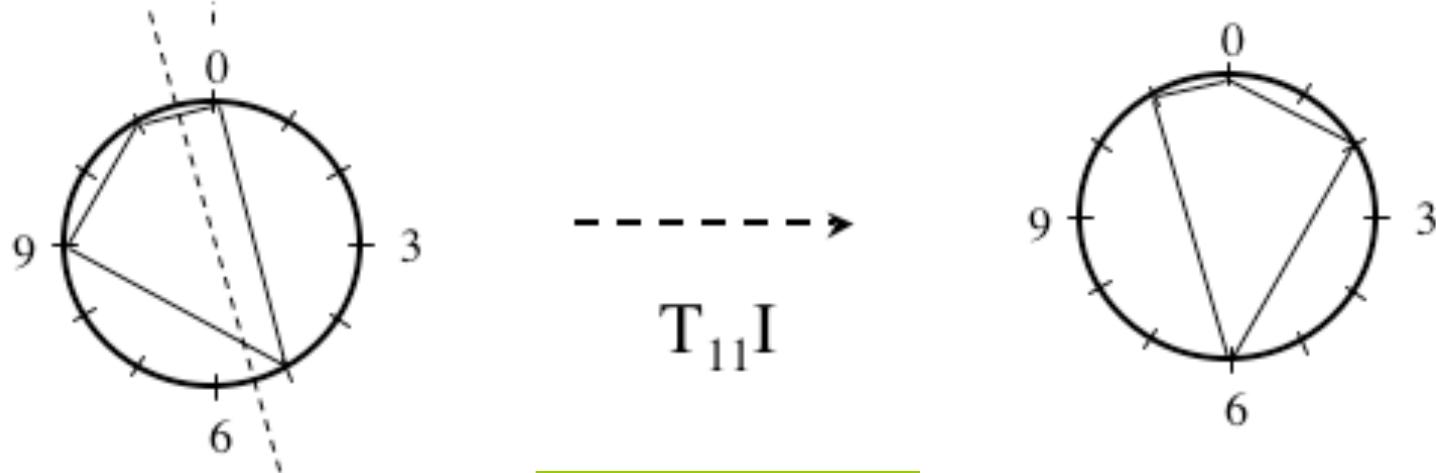
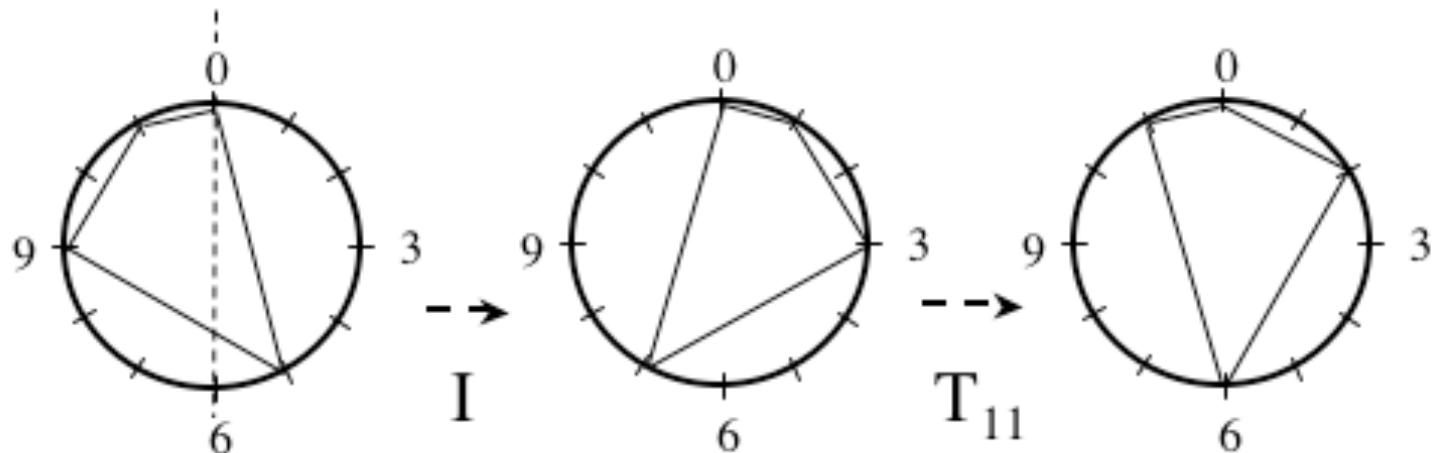
Musical inversions are differences...



Transposition and Inversion

I: $x \rightarrow 12-x$

$T_k: x \rightarrow k+x$



$\{0, 5, 9, 11\}$

$T_{11}I: x \rightarrow 11-x$

$\{11, 6, 3, 0\}$

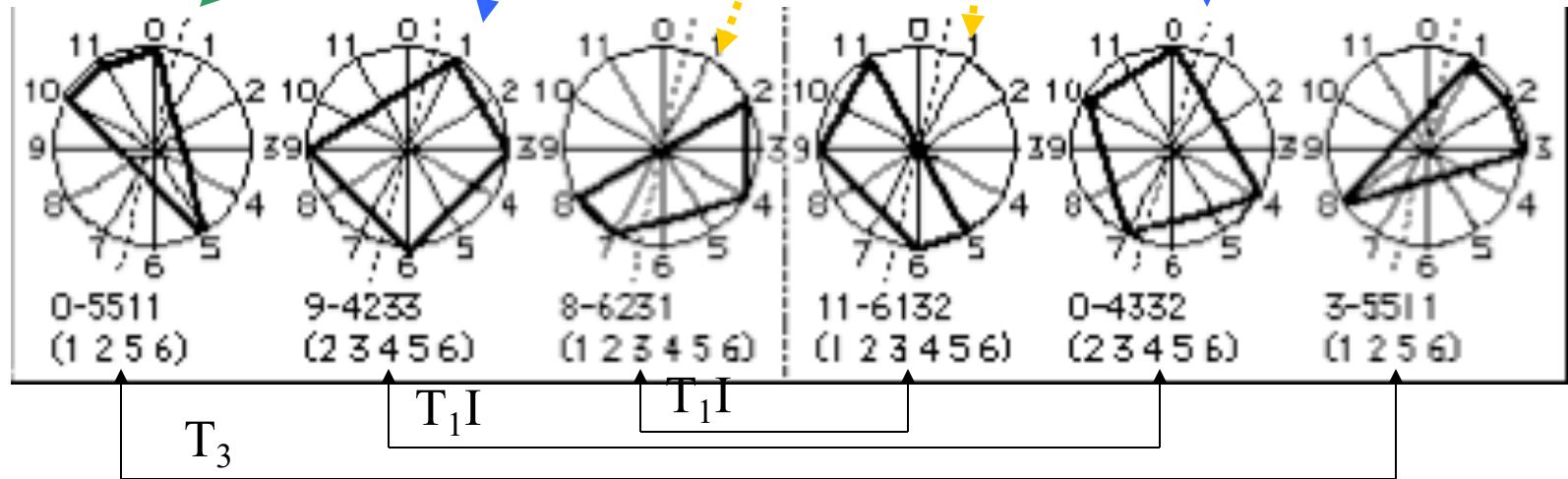


« Entités formelles pour l'analyse musicale »

Marcel Mesnage (1998)

A. Schoenberg : *Klavierstück Op. 33a*, 1929

The musical score consists of two staves: treble and bass. Several musical entities are highlighted with colored boxes and arrows pointing to corresponding circle graphs below.



« Making and Using a Pcset Network for Stockhausen's Klavierstück III »

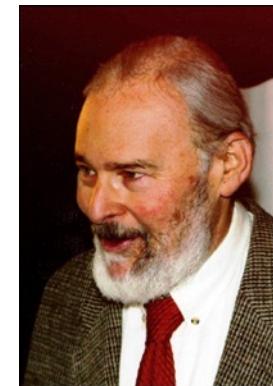
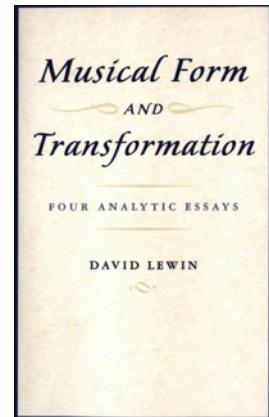
Musical score for Klavierstück III, page 5, measures 4-8. The score is in 4/8 time. Measure 4 starts with a piano dynamic (p) and includes a tempo marking of 8. Measures 5 and 6 show a transition with dynamics f and mf. Measure 7 is in 3/8 time with dynamics p and mf. Measure 8 concludes with a forte dynamic (f). Various performance markings like grace notes and slurs are present.

Three interpretations:

Henck

Kontarsky

Tudor

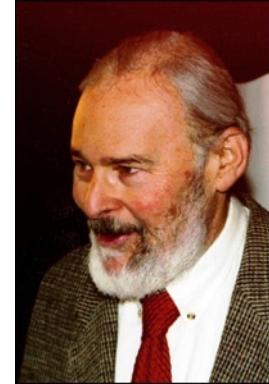
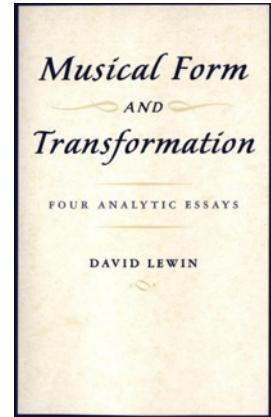


Musical score for Klavierstück III, page 5, measures 5-8. The score is in 4/8 time. Measure 5 starts with a dynamic f. Measures 6 and 7 show a transition with dynamics p and mf. Measure 8 concludes with a forte dynamic (f). Various performance markings like grace notes and slurs are present.

Musical score for Klavierstück III, page 11, measures 11-15. The score is in 3/8 time. Measure 11 starts with a dynamic mf. Measures 12 and 13 show a transition with dynamics f and p. Measure 14 is in 3/8 time with dynamics p and ff. Measure 15 concludes with a forte dynamic (ff). Various performance markings like grace notes and slurs are present.

« Making and Using a Pcset Network for Stockhausen's Klavierstück III »

A musical score for Klavierstück III by Stockhausen. The score consists of two staves. The top staff has a treble clef, a key signature of one sharp, and common time (4). The bottom staff has a bass clef and common time (8). The score includes various dynamics like *p*, *mf*, *f*, and *mf p*. Three specific regions are highlighted with colored boxes: a red box on the first measure, a green box on the second measure, and a blue box on the third measure. Arrows from each box point to one of three 12-note circles below the score, labeled '12'.



« The most ‘theoretical’ of the four essays, it focuses on the forms of one pentachord reasonably ubiquitous in the piece. A special **group of transformations** is developed, one suggested by the musical interrelations of the pentachord forms. Using that group, the essay arranges **all pentachord forms** of the music into a **spatial configuration** that illustrates network structure, for this particular phenomenon, over the entire piece. »

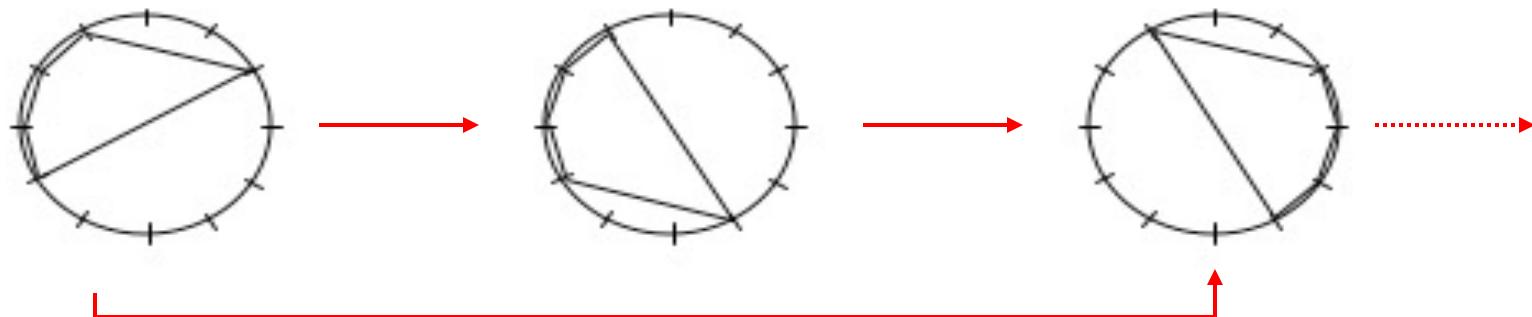
« Making and Using a Pcset Network for Stockhausen's Klavierstück III »

Lewin 1993

SI: (1, 1, 1, 3, 6) (6, 3, 1, 1, 1) (6, 3, 1, 1, 1)

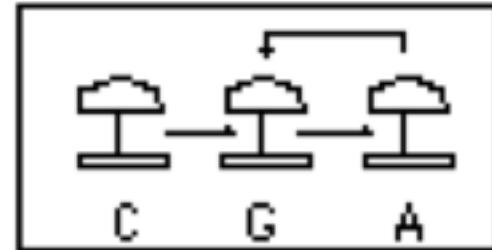
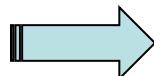
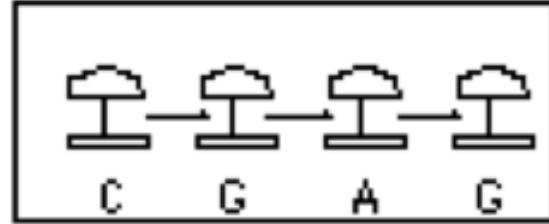
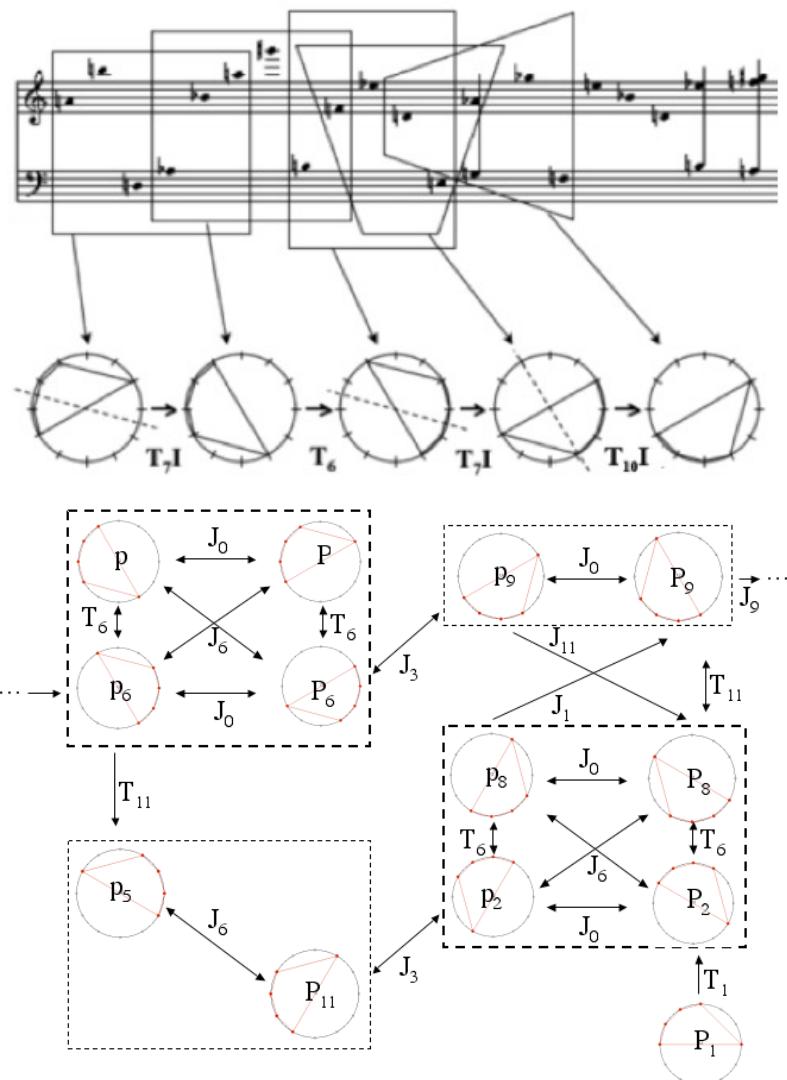
IFUNC: [5 3 2 2 1 1 1 1 1 2 2 3] [5 3 2 2 1 1 1 1 1 2 2 3] [5 3 2 2 1 1 1 1 1 1 2 2 3]

VI: [3 2 2 1 1 1] [3 2 2 1 1 1] [3 2 2 1 1 1]



Transformational Networks and Music Cognition

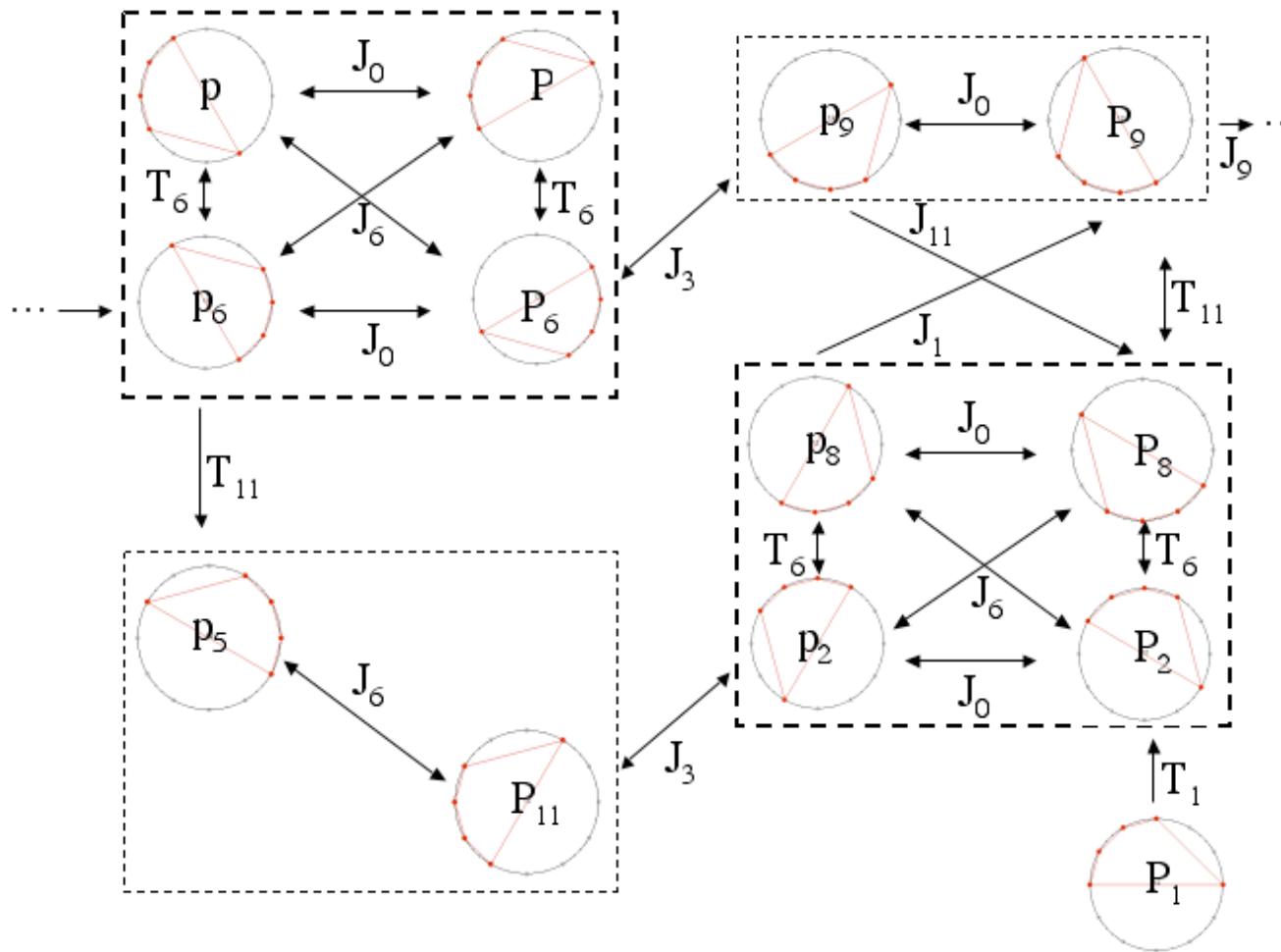
Bamberger, J. (1986). Cognitive issues in the development of musically gifted children. In *Conceptions of giftedness* (eds., R. J. Sternberg, & J. E. Davidson), pp. 388-413. Cambridge University Press, Cambridge



Bamberger, J. (2006). "What develops in musical development?" In G. MacPherson (ed.) *The child as musician: Musical development from conception to adolescence*. Oxford, U.K. Oxford University Press.

Transformational Networks and isographies

Stockhausen: *Klavierstück III* (Analyse de D. Lewin)

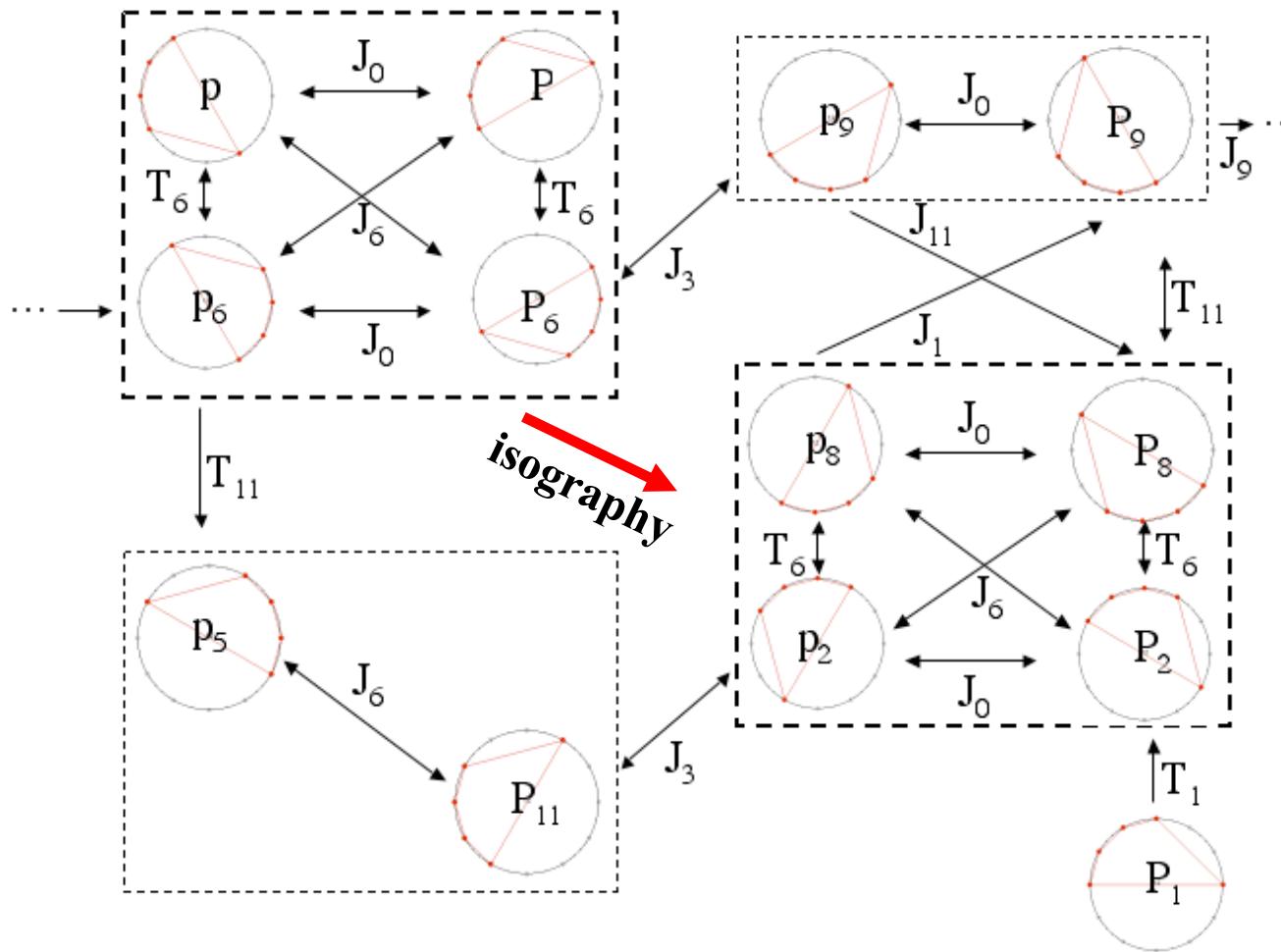


« Rather than asserting a network that follows pentachord relations one at a time, according to the chronology of the piece, I shall assert instead a network that displays all the pentachord forms used and all their **potentially functional interrelationships**, in a very compactly organized little **spatial configuration**. »

« [...] the sequence of events moves within a clearly defined world of possible relationships, and because - in so moving - it makes the abstract space of such a world accessible to our sensibilities. That is to say that the story projects what one would traditionally call *form*. »

Transformational Networks as spatial configurations

Stockhausen: *Klavierstück III* (Analyse de D. Lewin)



« Rather than asserting a network that follows pentachord relations one at a time, according to the chronology of the piece, I shall assert instead a network that displays all the pentachord forms used and all their **potentially functional interrelationships**, in a very compactly organized little **spatial configuration**. »

« [...] the sequence of events moves within a clearly defined world of possible relationships, and because - in so moving - it makes the abstract space of such a world accessible to our sensibilities. That is to say that the story projects what one would traditionally call *form*. »

K-nets as a transformational construction

D. Lewin, "A Tutorial on K-nets using the Chorale in Schoenberg's Op.11, N°2 », JMT, 1994

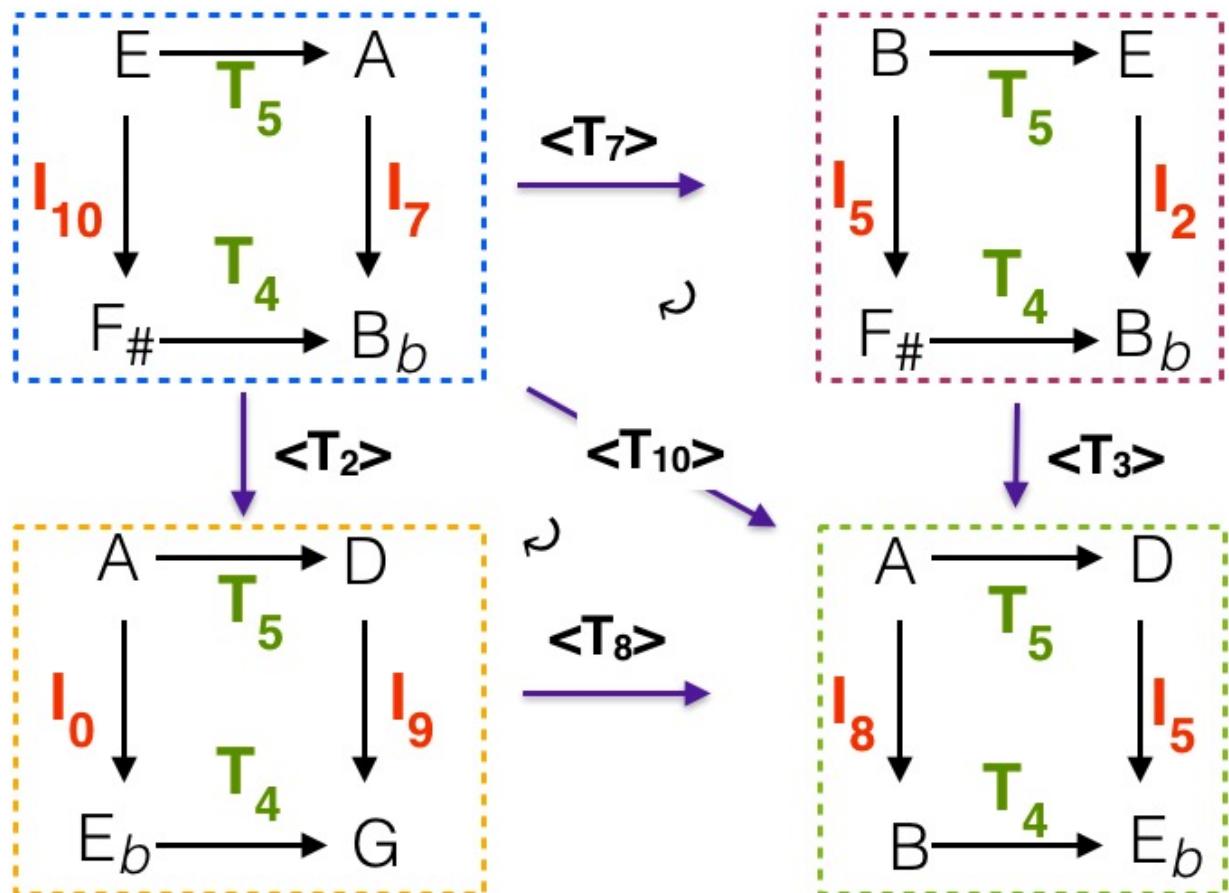


D. Lewin

H. Klumpenhouwer



$$\begin{aligned} <\mathbf{T}_k> : \mathbf{T}_m &\rightarrow \mathbf{T}_m \\ \mathbf{I}_m &\rightarrow \mathbf{I}_{k+m} \end{aligned}$$



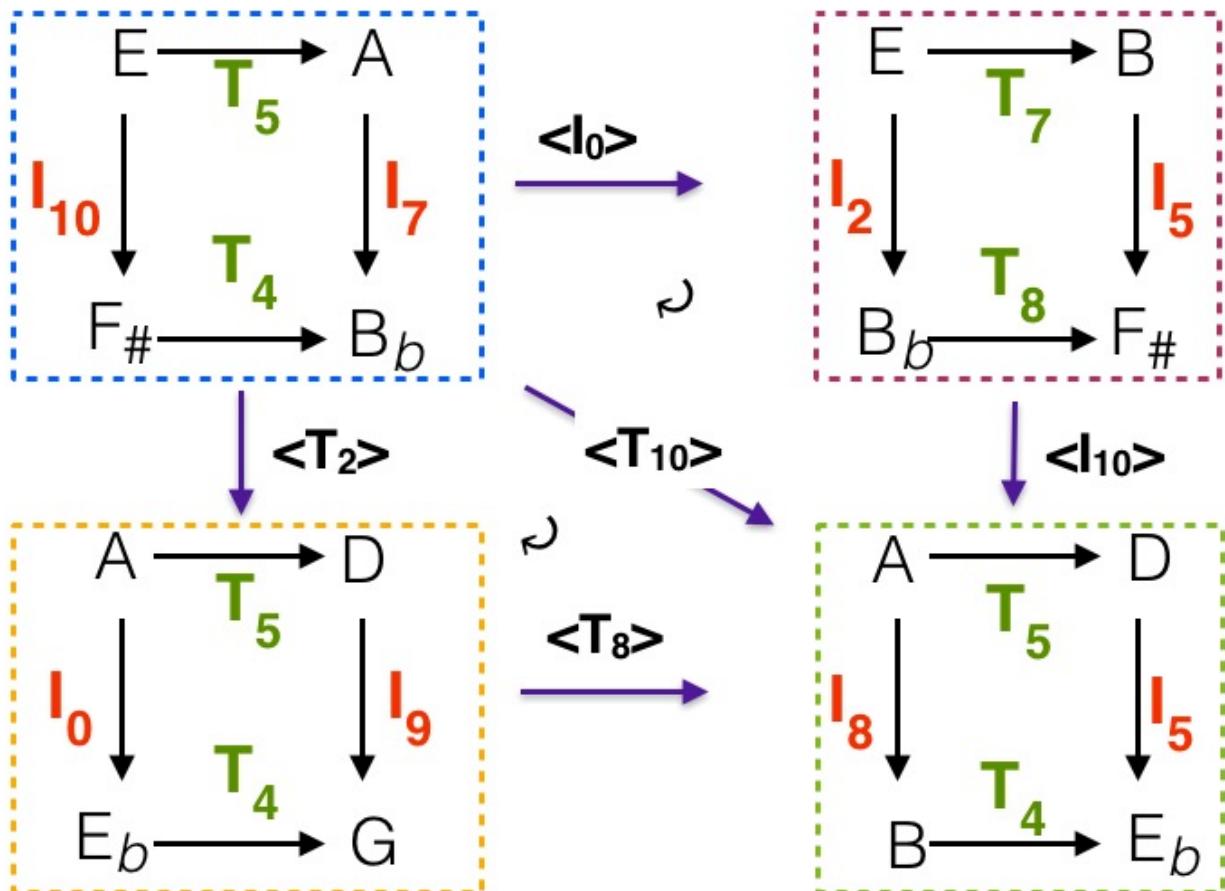
K-nets as a transformational construction

D. Lewin, "A Tutorial on K-nets using the Chorale in Schoenberg's Op.11, N°2 », JMT, 1994



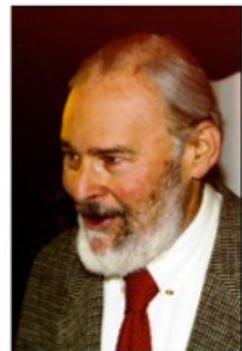
$$\langle T_k \rangle : T_m \rightarrow T_m \\ I_m \rightarrow I_{k+m}$$

$$\langle I_k \rangle : T_m \rightarrow T_{-m} \\ I_m \rightarrow I_{k-m}$$



K-nets as a transformational construction

D. Lewin, "A Tutorial on K-nets using the Chorale in Schoenberg's Op.11, N°2 », JMT, 1994

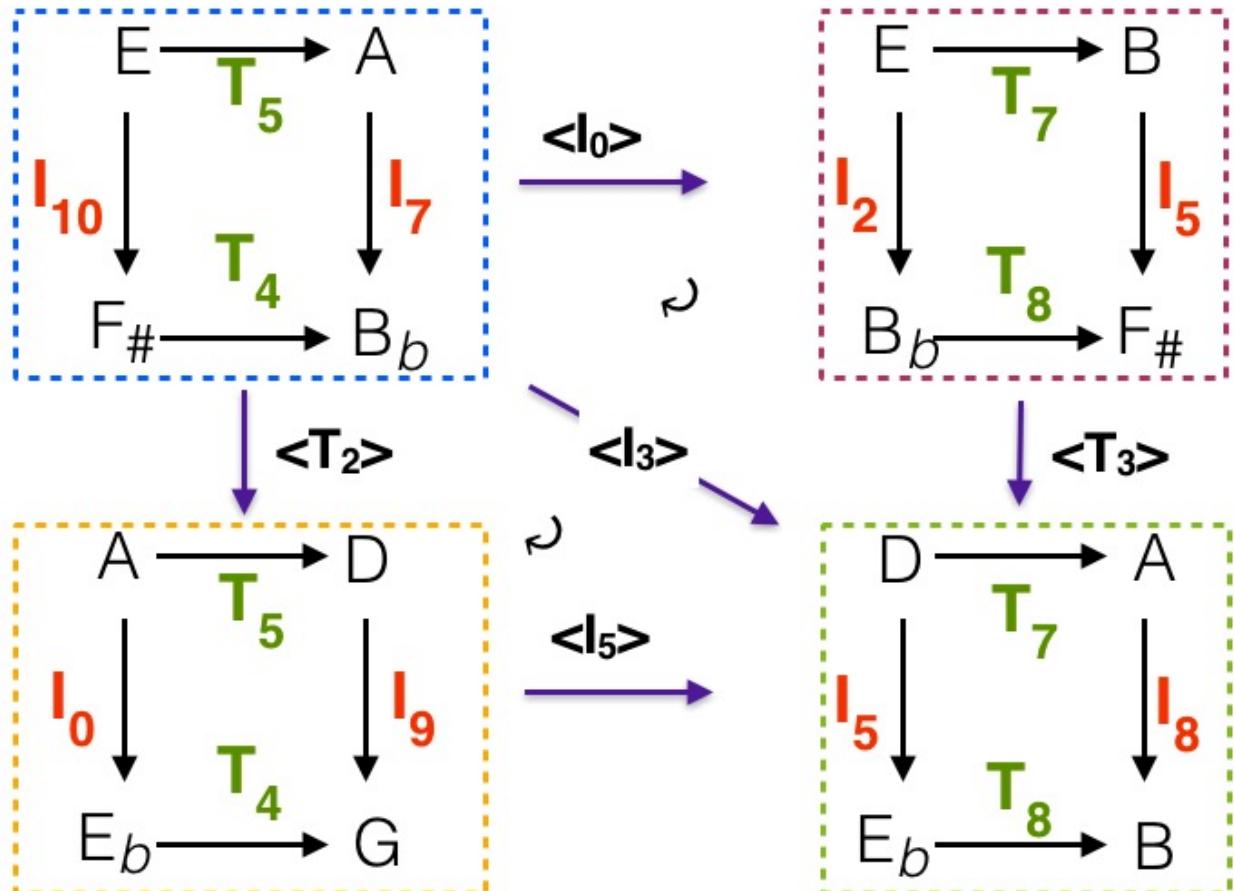


D. Lewin

H. Klumpenhouwer

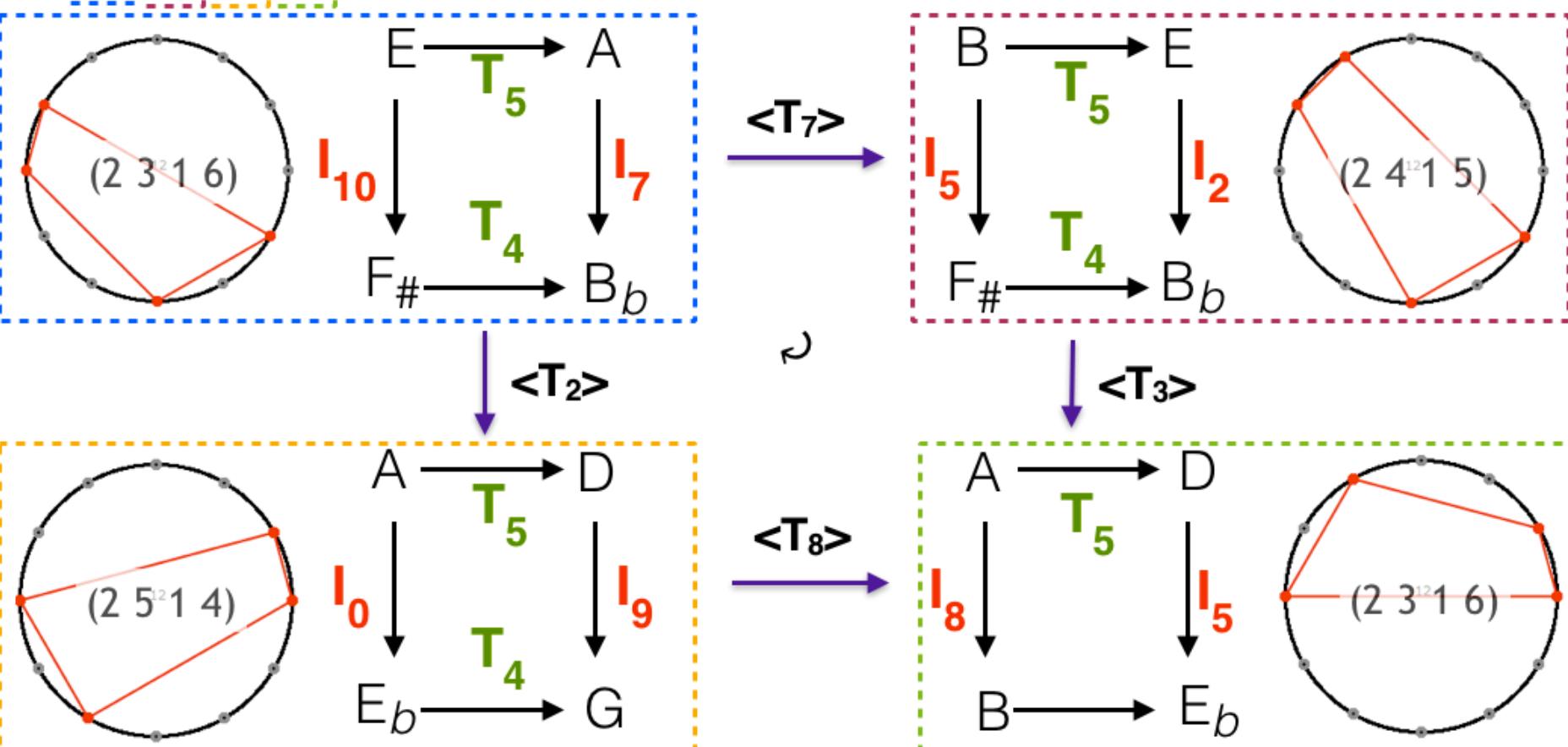
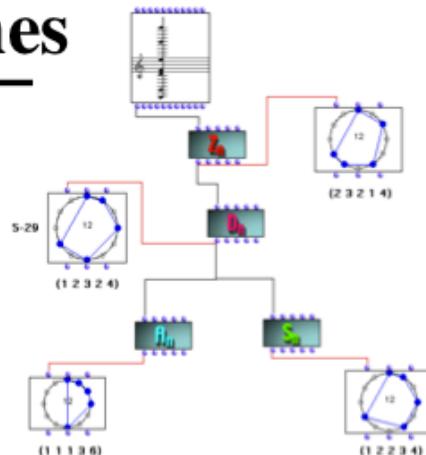
$$\begin{aligned} <\mathbf{T}_k> : \mathbf{T}_m &\rightarrow \mathbf{T}_m \\ \mathbf{I}_m &\rightarrow \mathbf{I}_{k+m} \end{aligned}$$

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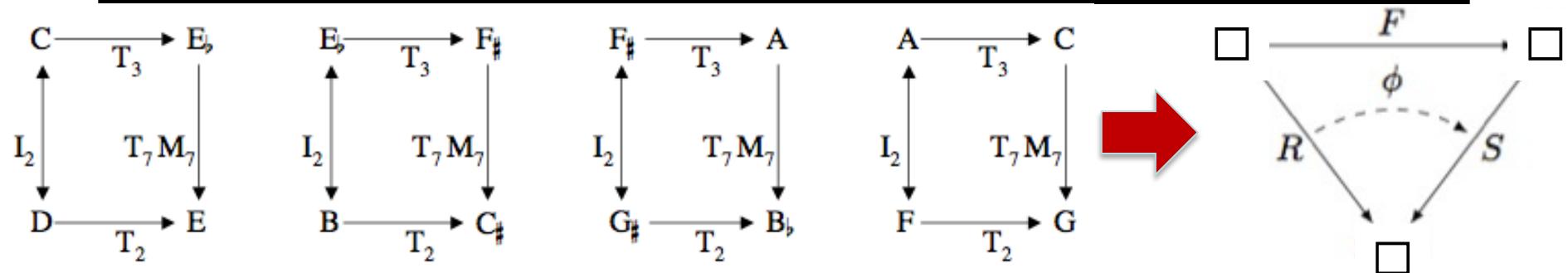


Transformational vs set-theoretical approaches

D. Lewin, "A Tutorial on K-nets using the Chorale in Schoenberg's Op.11, N°2 », JMT, 1994



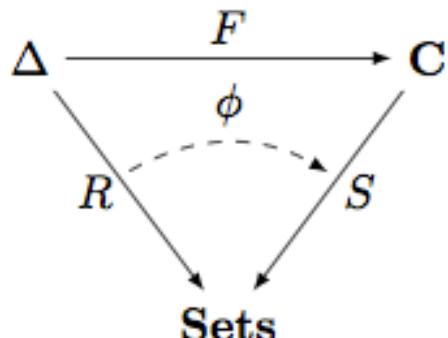
From K-Nets to category-based PK-Nets



Definition 1 Let \mathbf{C} be a category, and S a functor from \mathbf{C} to the category \mathbf{Sets} . Let Δ be a small category and R a functor from Δ to \mathbf{Sets} . A PK-net of form R and of support S is a 4-tuple (R, S, F, ϕ) , in which

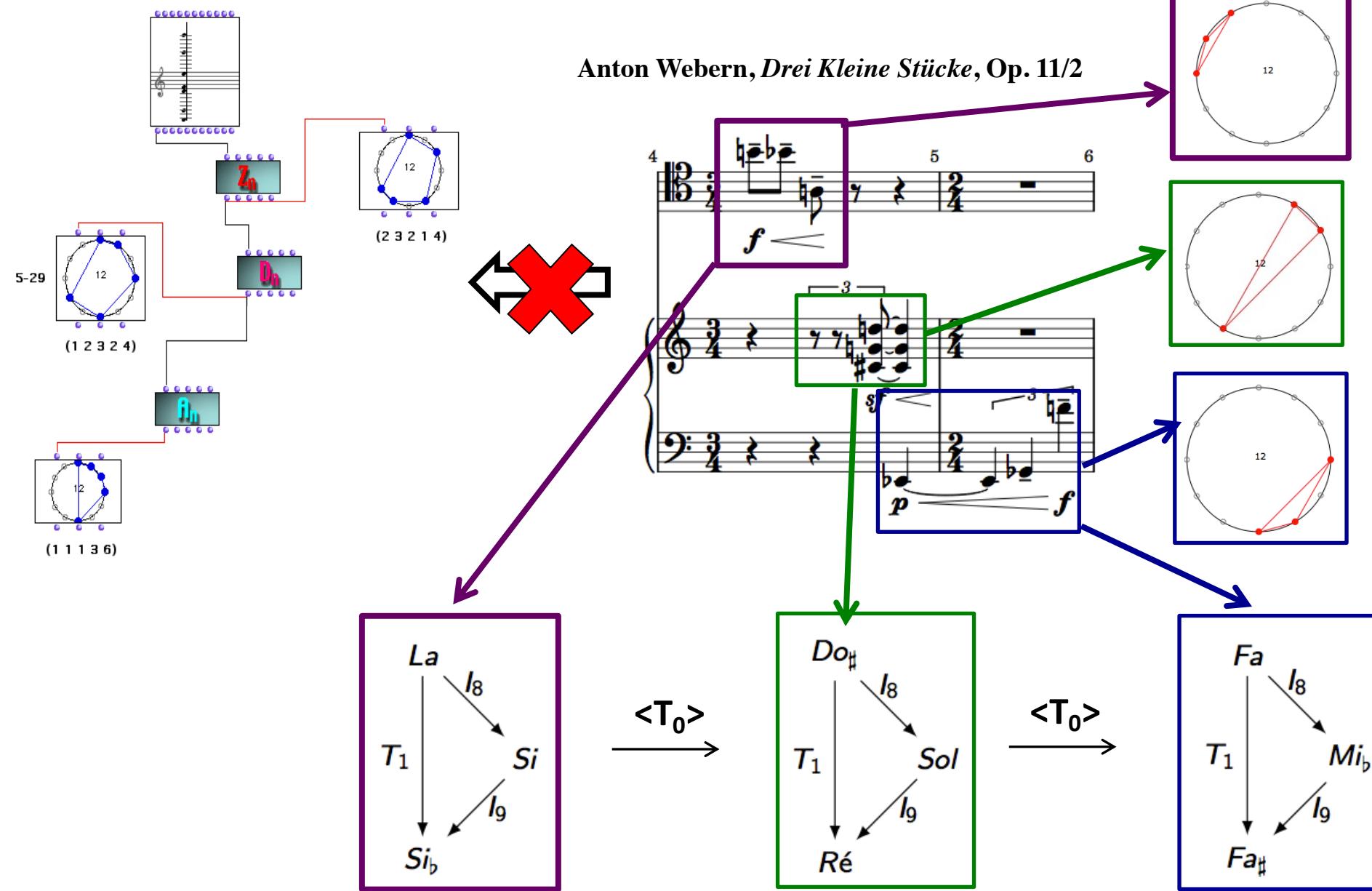
- F is a functor from Δ to \mathbf{C} ,
- and ϕ is a natural transformation from R to SF .

The definition of a PK-net is summed up by the following diagram:

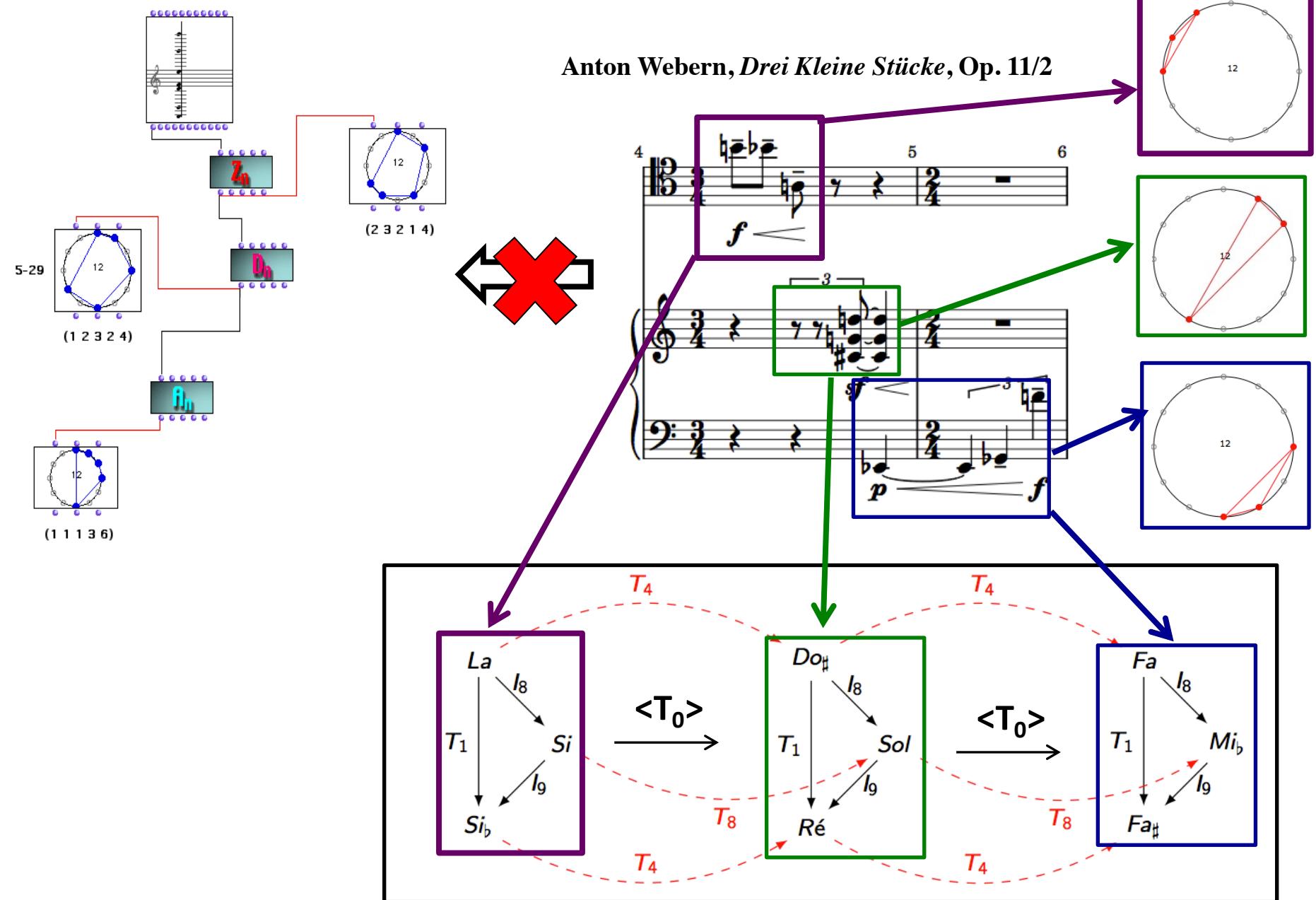


Popoff A., M. Andreatta, A. Ehresmann,
« A Categorical Generalization of
Klumpenhouwer Networks », MCM 2015,
Queen Mary University, Springer, p. 303-314

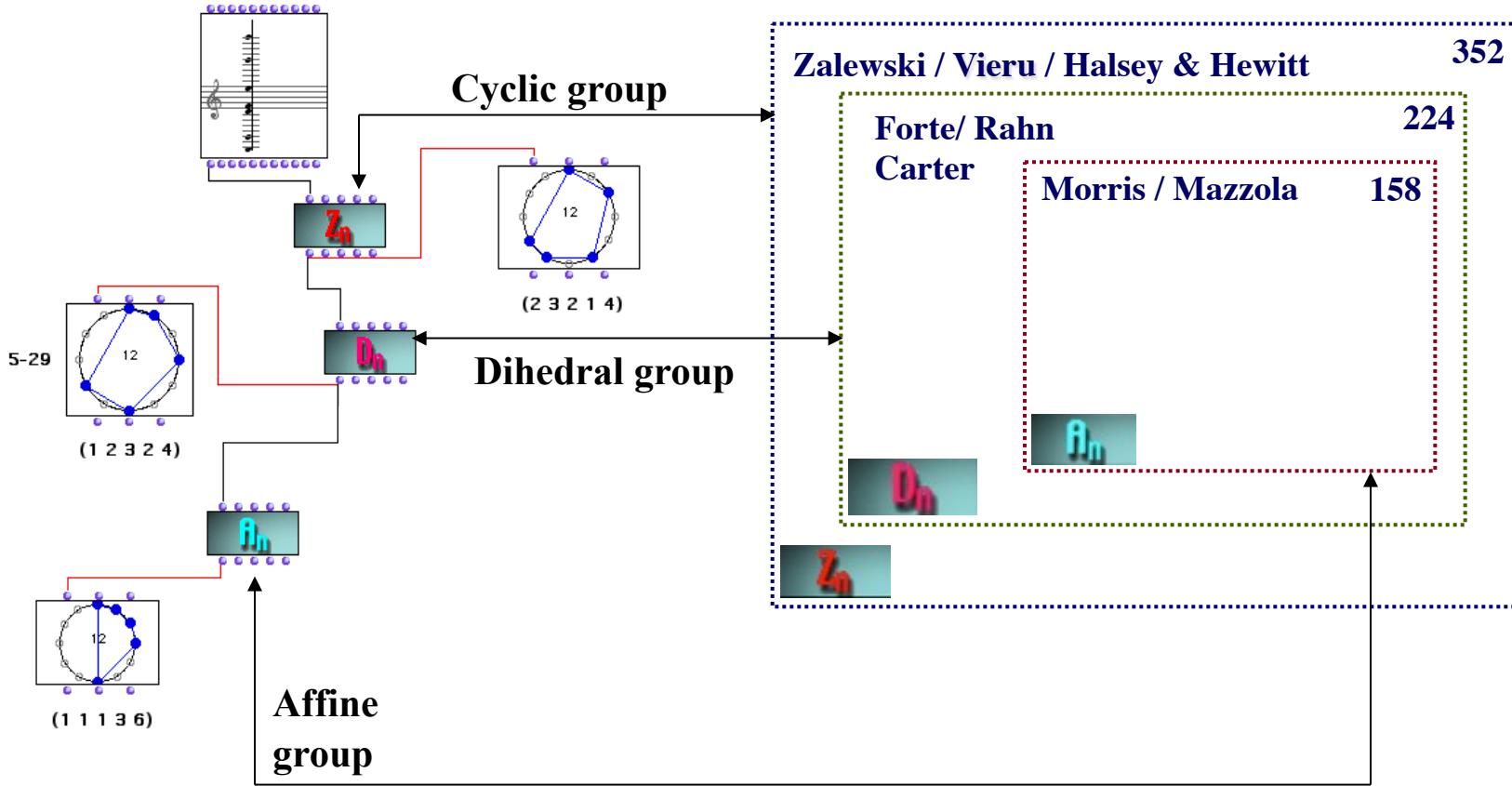
K-Nets and the paradigmatic approach



The categorical vs paradigmatic action-based approach



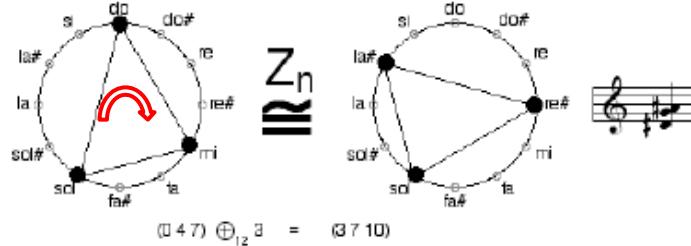
The interplay between Algebra and Geometry



The nature of a given geometry is [...] defined by the *reference* to a determinate group and the way in which spatial forms are related within that type of geometry. [Cf. Felix Klein Erlangen Program - 1872][...] We may raise the question whether there are any concepts and principles that are, although in different ways and different degrees of distinctness, necessary conditions for both the *constitution* of the **perceptual world** and the construction of the universe of **geometrical thought**. It seems to me that the **concept of group** and the **concept of invariance** are such principles.

E. Cassirer, "The concept of group and the theory of perception", 1944

Equivalence classes of musical structures (up to a group action)

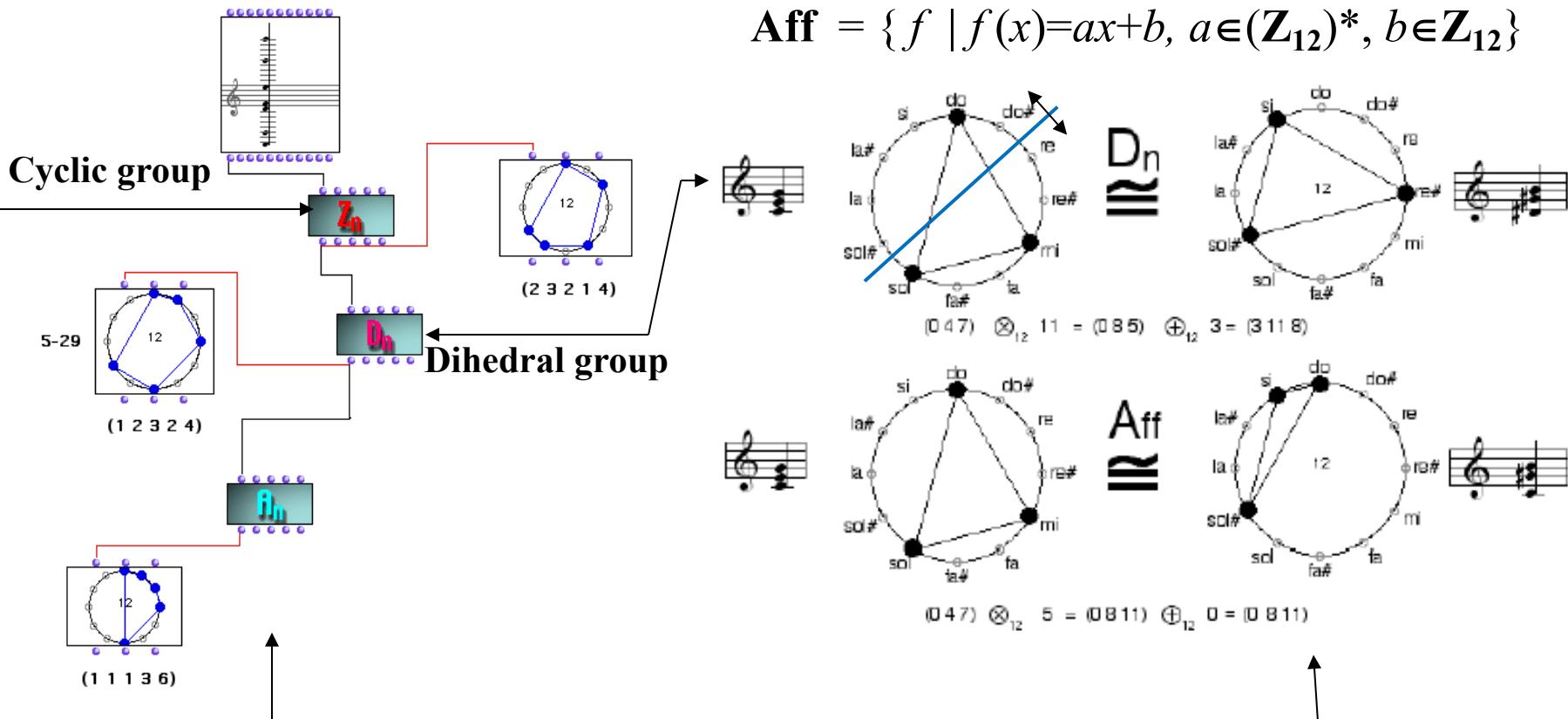


$$Z_{12} = \langle T_k \mid (T_k)^{12} = T_0 \rangle \text{ where } T_k(x) = x+1$$

$$D_{12} = \langle T_k, I \mid (T_k)^{12} = I^2 = T_0, ITI = I(IT)^{-1} \rangle$$

where $I(x) = -x$

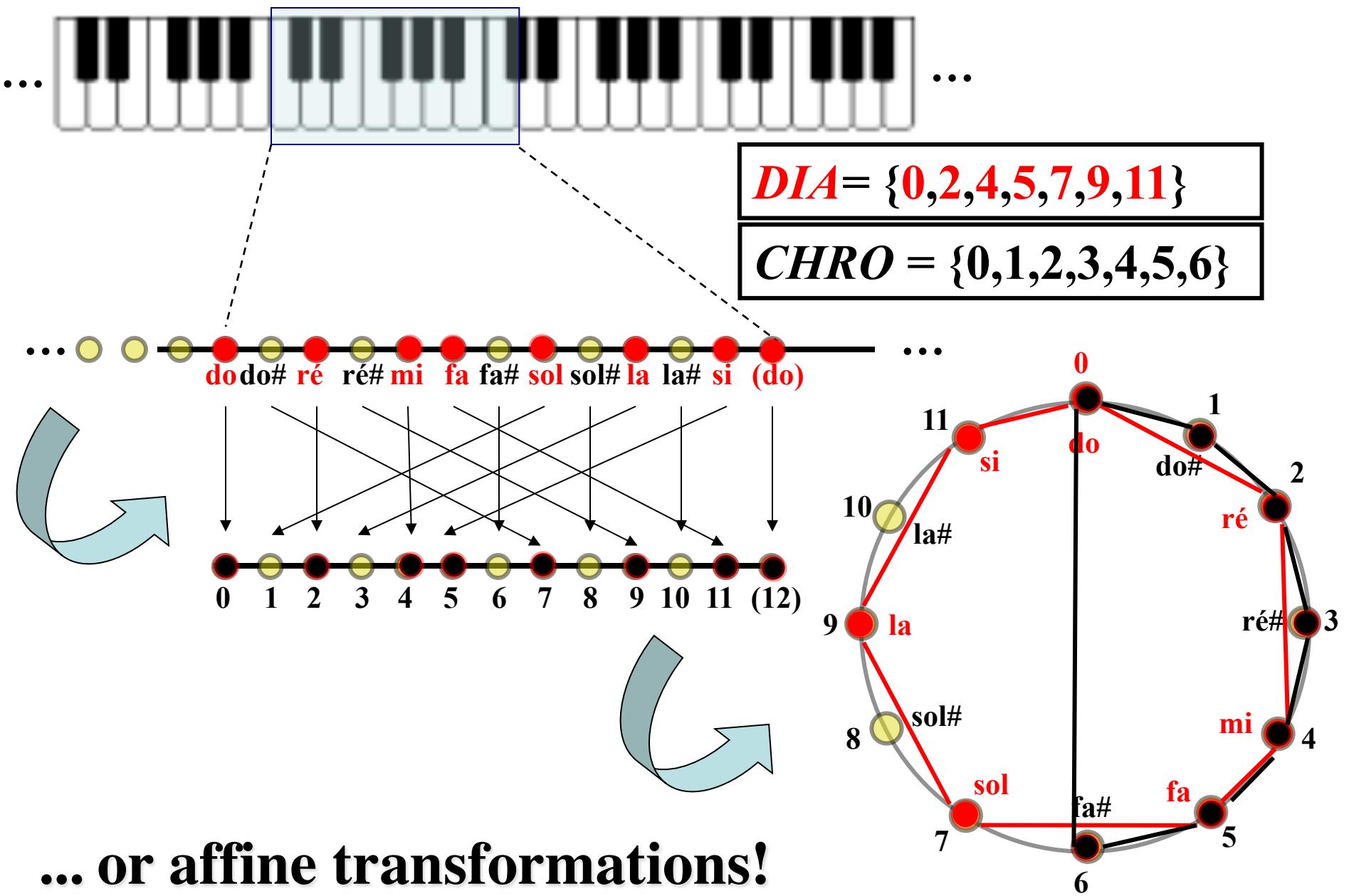
$$\text{Aff} = \{f \mid f(x) = ax + b, a \in (Z_{12})^*, b \in Z_{12}\}$$



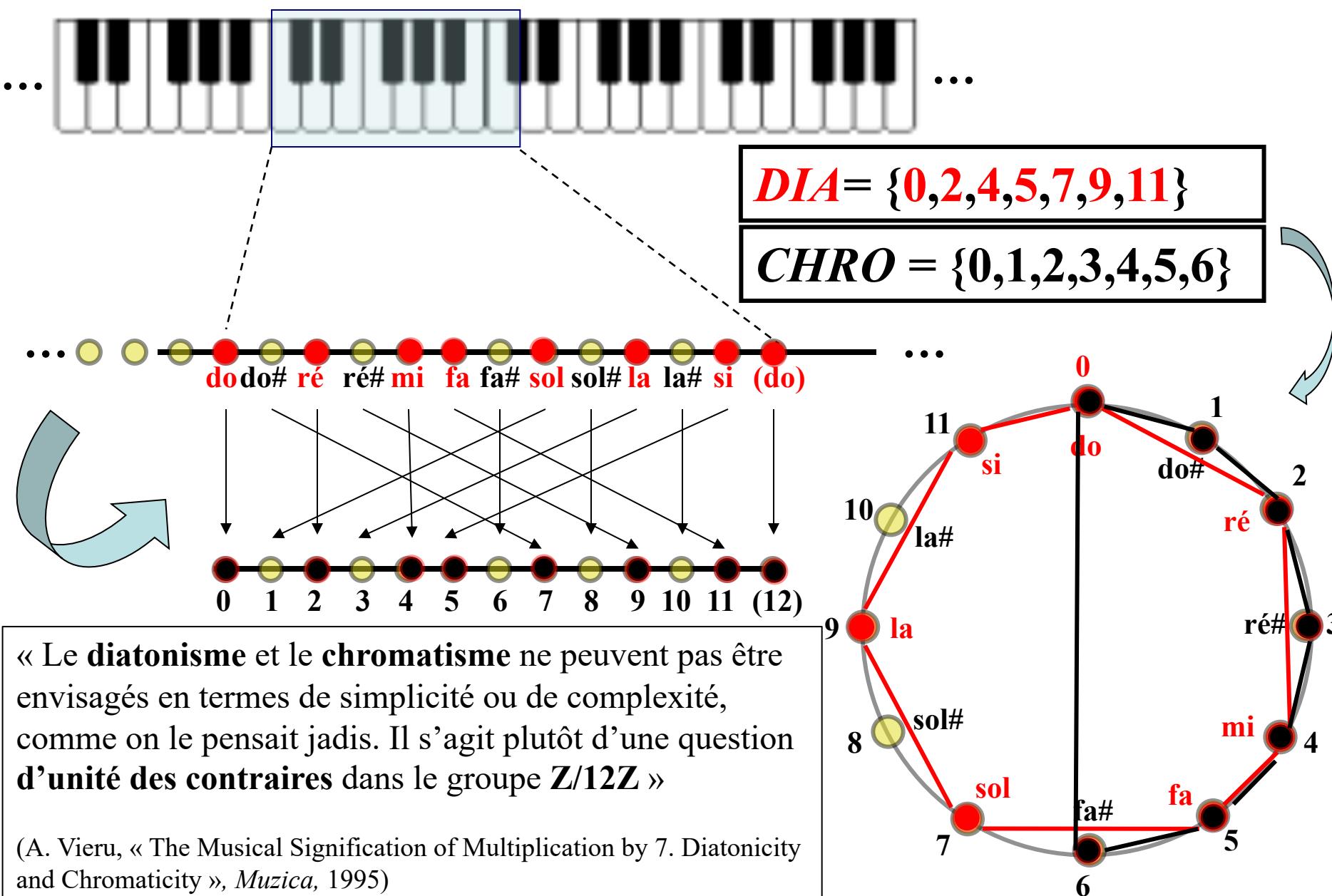
Paradigmatic architecture

Affine group

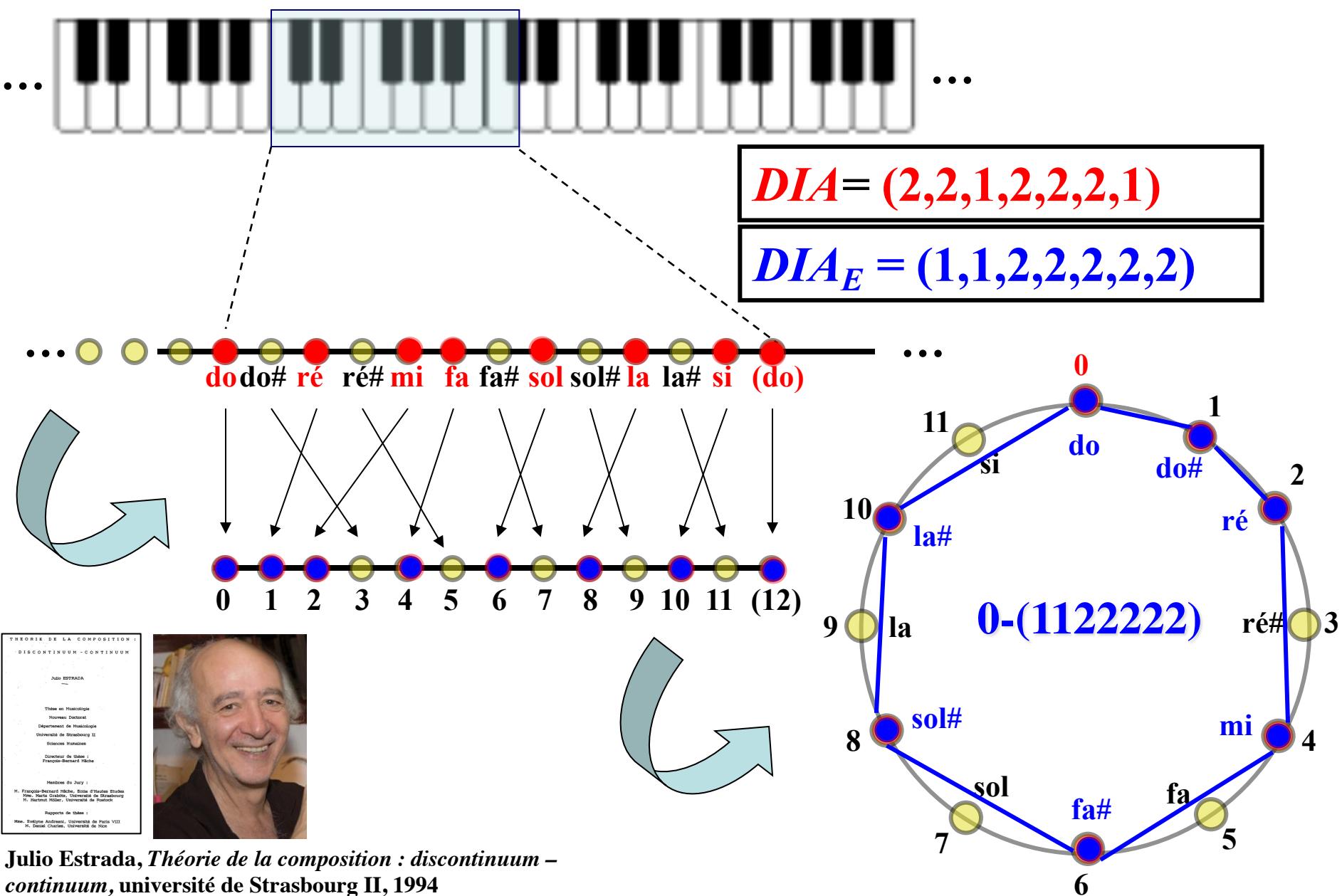
Augmentations are multiplications...



Affine transformations and DIA/CHRO duality



Classifying chords up to permutations of intervals



The permutohedron as a combinatorial space

Julio Estrada, *Théorie de la composition : discontinuum – continuum*, université de Strasbourg II, 1994

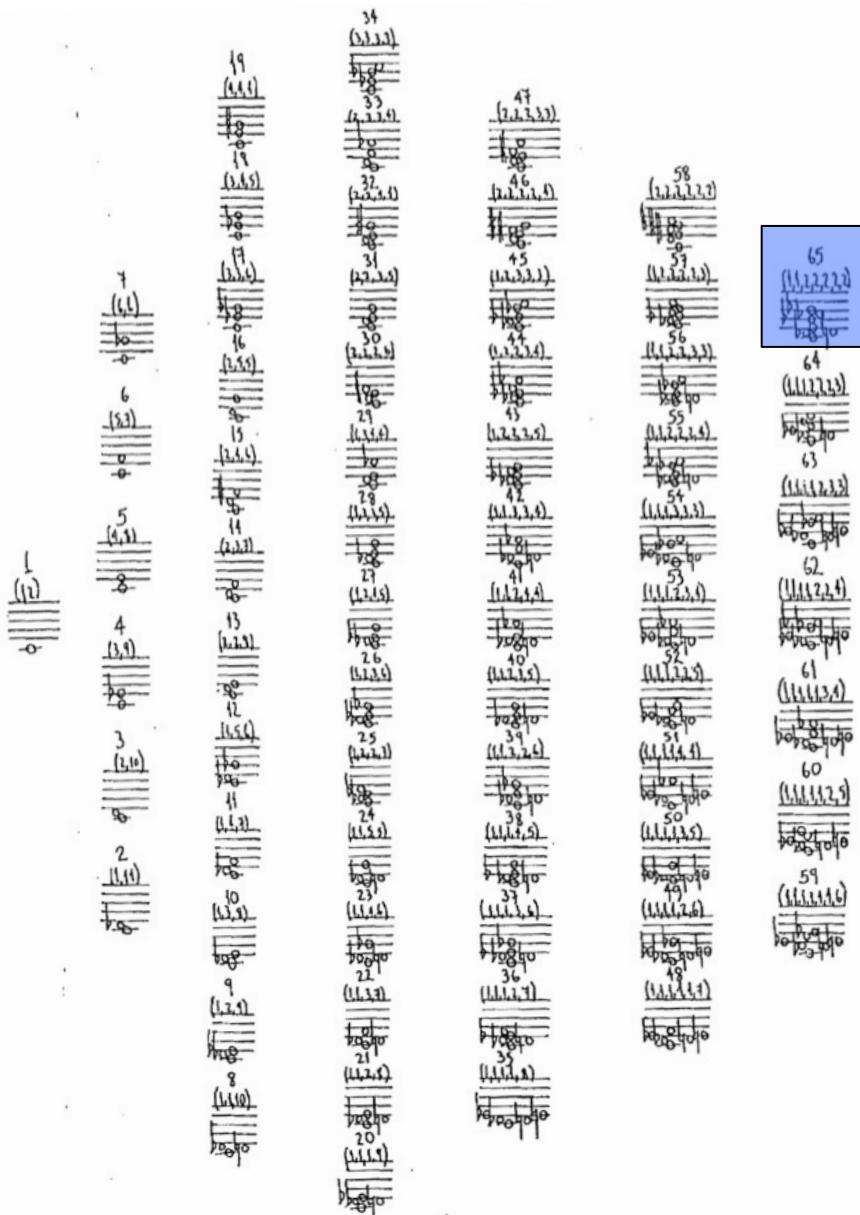
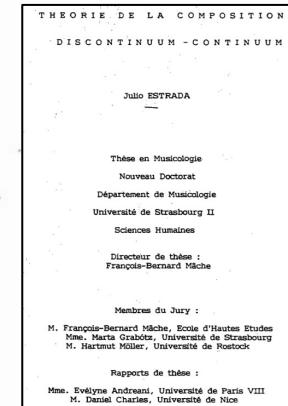
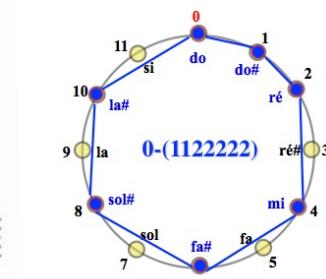
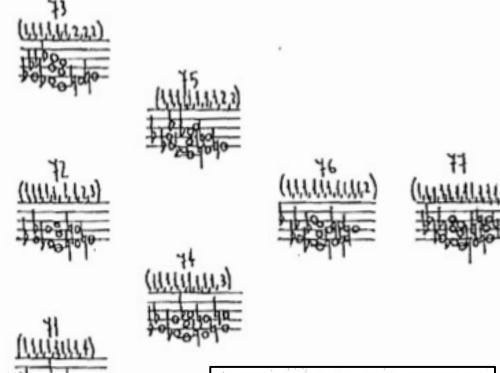


ILLUSTRATION III. REPRESENTATION EN NOTATION MUSICALE DE L'ENSEMBLE DE PARTITIONS DE L'ECHELLE DE HAUTEURS D12 :
12 NIVEAUX DE DENSITE, 77 IDENTITES.

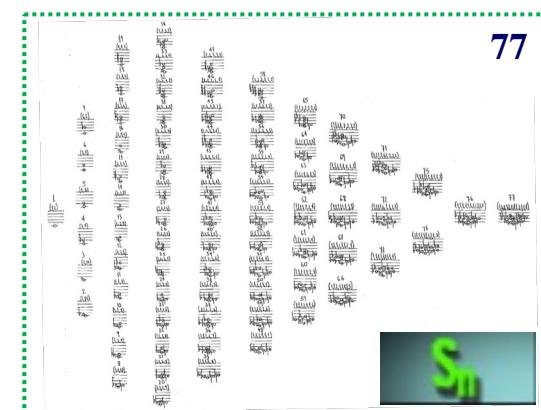
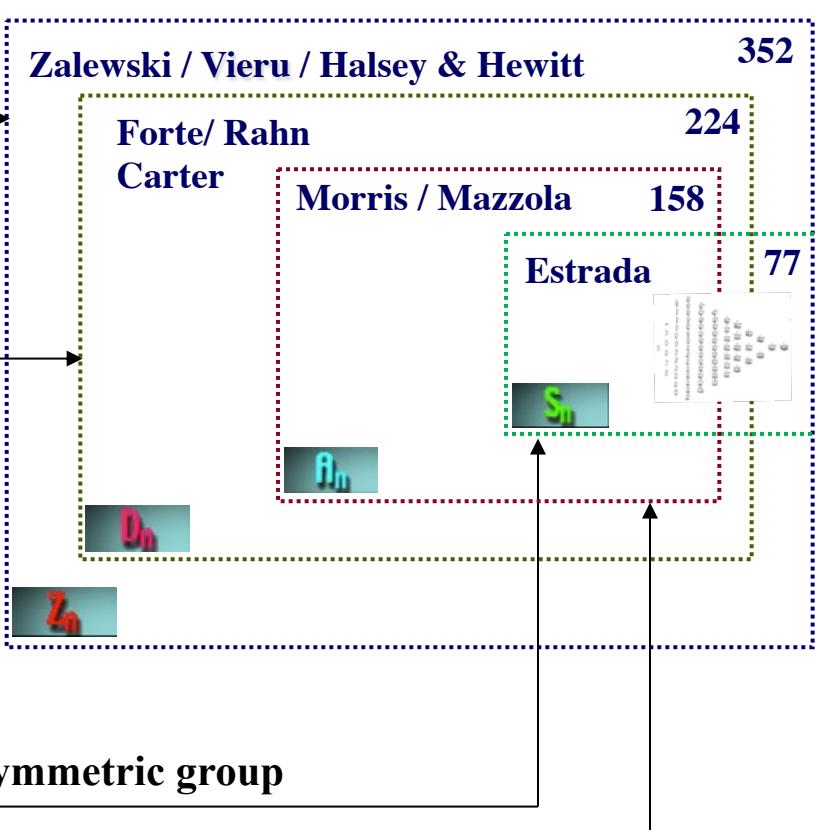
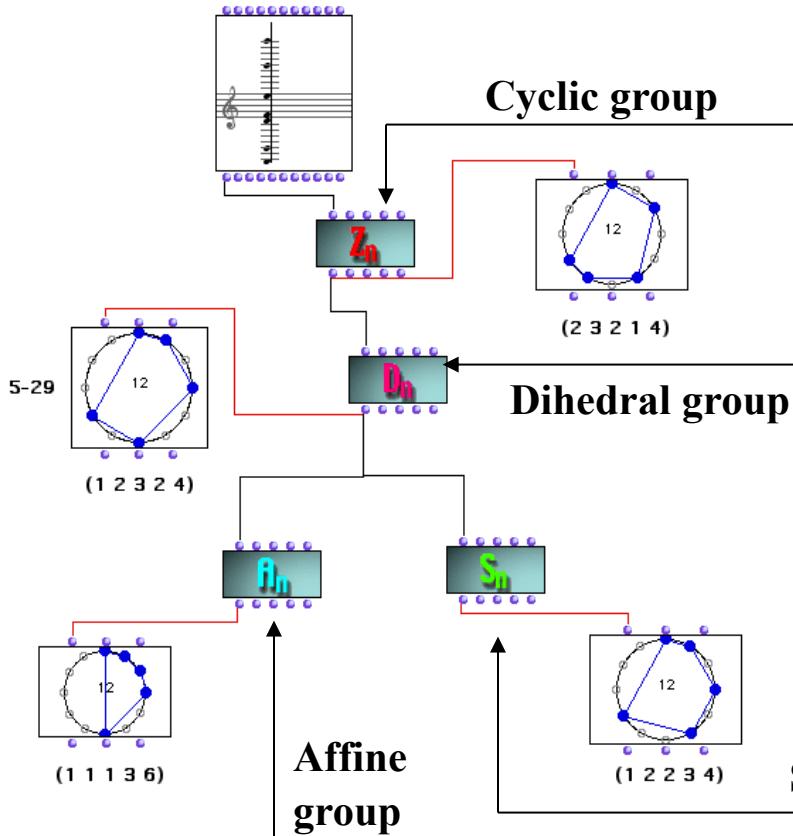


$$DIA_E = (1,1,2,2,2,2,2,2)$$

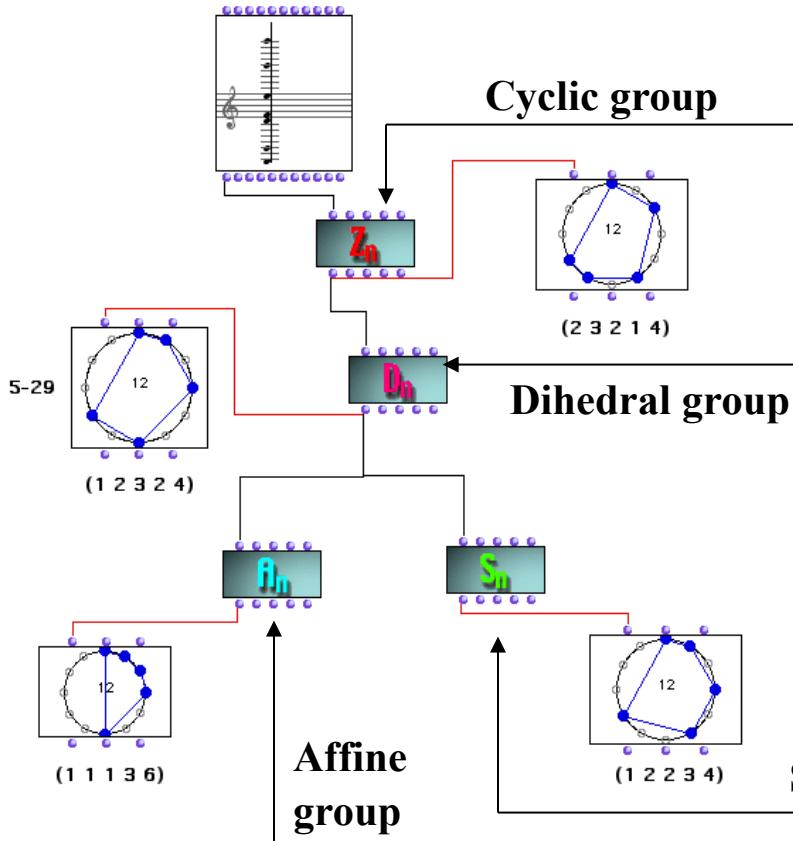


J. Estrada

Algebraic Combinatorics and Music Enumeration



Algebraic Combinatorics and Music Enumeration



Zalewski / Vieru / Halsey & Hewitt

352

Forte/ Rahn
Carter

224

Morris / Mazzola

158

Estrada

77



W. Burnside

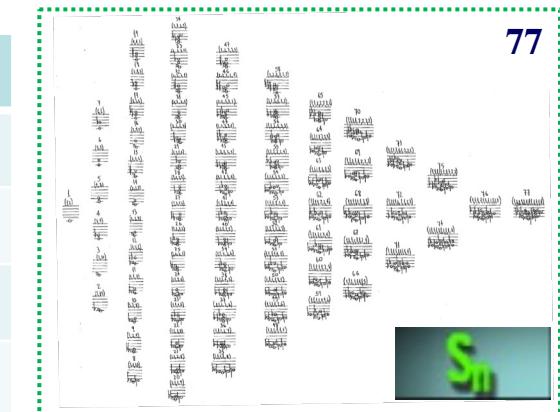


G. Polya



Julior Estrada

1	2	3	4	5	6	7	8	9	10	11	12
Z_n	1	6	19	43	66	80	66	43	19	6	1
D_n	1	6	12	29	38	50	38	29	12	6	1
A_n	1	5	9	21	25	34	25	21	9	5	1
S_n	1	6	12	15	13	11	7	5	3	2	1



The pitch-rhythm cognitive isomorphic correspondence

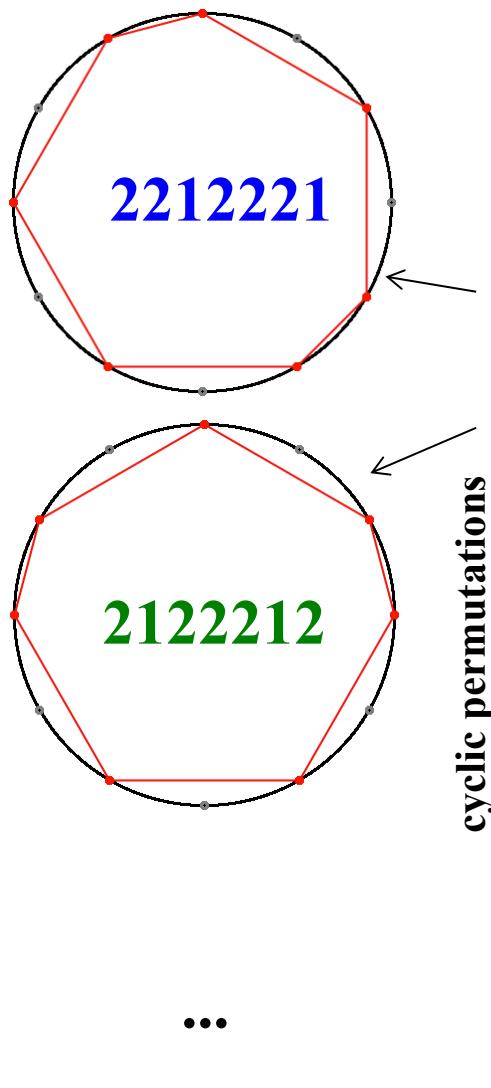


TABLE I

Comparison of $M = 7$, $L = 12$ patterns for pitch (scales) and rhythm (time-lines)

pattern	pitch domain name and notation (in C)	rhythm domain notation	examples from West Africa	references
1. 2212221	major scale (Ionian) CDEFGAB	↓ ↓ ↓ ↓ ↓	Ewe (Atsiabek, Sogba, Atsia) also Yoruba	Jones (1959), C. K. Ladzekpo, S. K. Ladzekpo and Pantaleoni, Locke
2. 2122212	Dorian CDE↑ FGAB↑	↓ ↓ ↓ ↓ ↓	Bemba—Northern Rhodesia	Jones (1965), (Ekwueme)
3. 1222122	Phrygian CD↑ E↑ FGA↑ B↑	↓↓↓↓↓↓	—	—
4. 2221221	Lydian CDEF#GAB	↓↓↓↓↓↓	Ga-Adangme (common) also common Haitian pattern, Akan (Ab fo)	C. K. Ladzekpo, Combs (1974), R. Hill, Asiamah
5. 2212212	Mixolydian CDEFGAB	↓↓↓↑↓↓	Yoruba sacred music from Ekiti	King
6. 2122122	Aeolian CDE↑ FG↑ B↑	↓↓↓↓↓↓	Ashanti (Ab fo , Mpre)	Koetting
7. 1221222	Locrian CD↑ E↑ FG↑ A↑ B↑	↓↓↓↑↓↓	Ghana*	Nketia (1963a)
8. 2121222	(#2 Locrian) CDE↑ FG↑ A↑ B↑	↓↓↓↓↓↓	Ashanti (Asedua)	C. K. Ladzekpo
9. 2112123	— CDD#EF#GA	↓↓↑↓↓↓	Akan (juvenile song)	Nketia (1963b)

* clap pattern

† mute stroke on bell

J. Pressing, "Cognitive isomorphisms between pitch and rhythm in world musics: West Africa, the Balkans and Western tonality", *Studies in Music*, 17, p. 38-61

The pitch-rhythm isomorphic correspondence

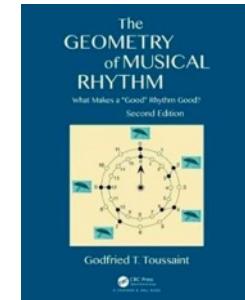
“It is well known that there exists an isomorphic relation between pitch and rhythm, which several authors have pointed out from time to time.”

(G. Toussaint, *The geometry of musical rhythm. What makes a “Good” Rhythm Good?* CRC Press, 2013, p. xiii)

The *Euclidean* Algorithm Generates Traditional Musical Rhythms



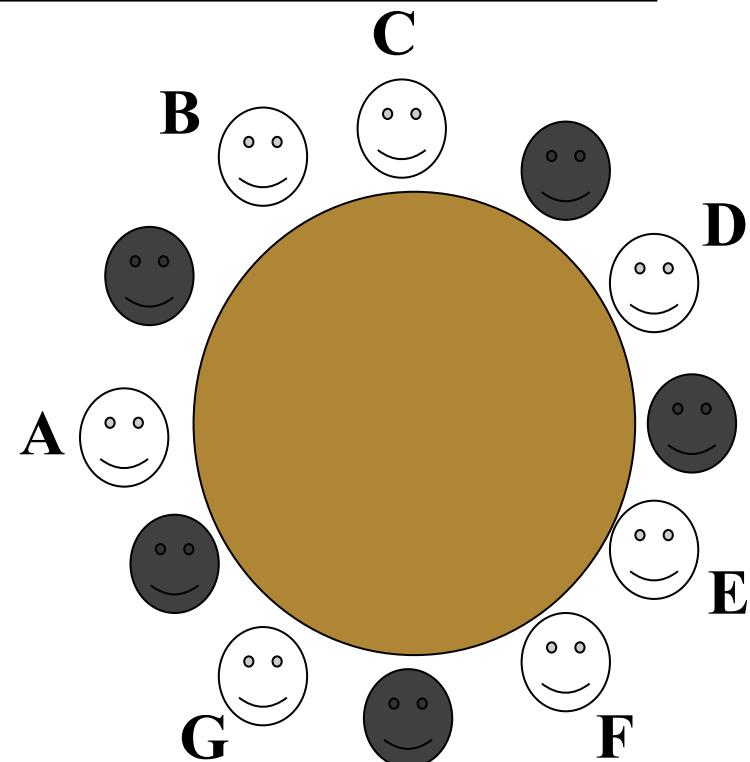
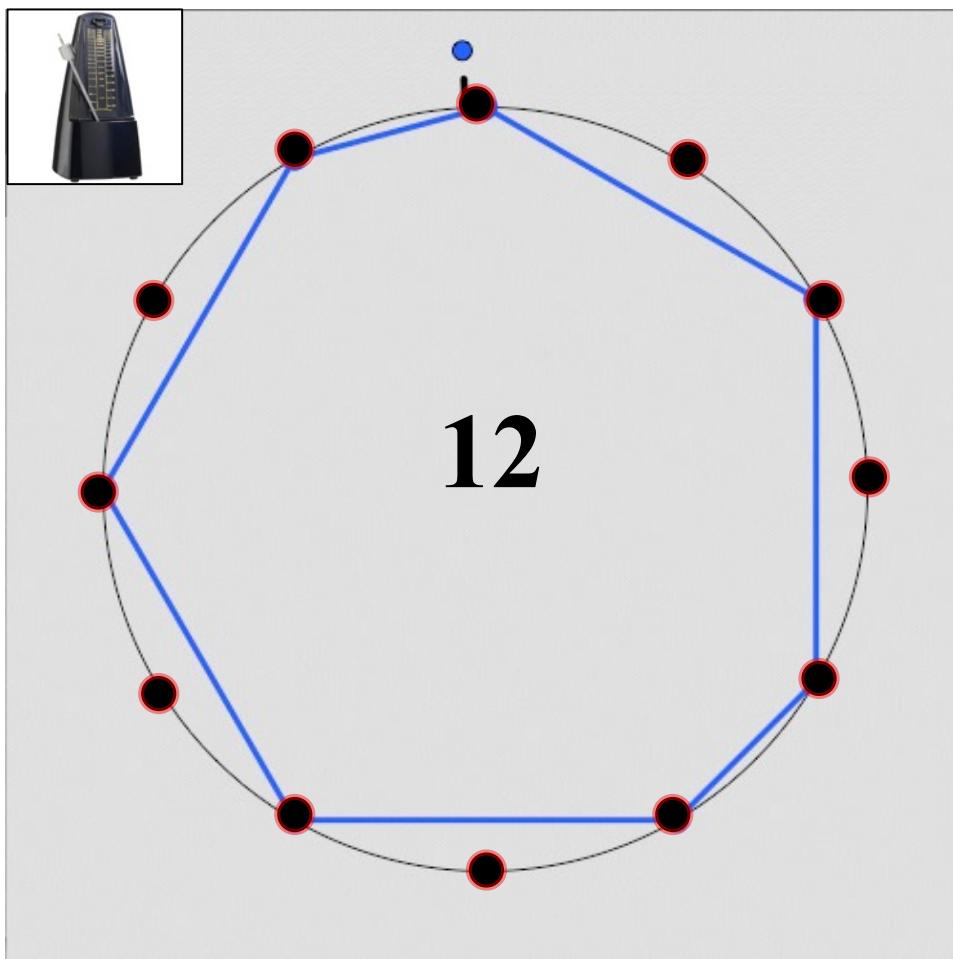
Godfried Toussaint*
School of Computer Science, McGill University
Montréal, Québec, Canada
godfried@cs.mcgill.ca



Abstract

The *Euclidean* algorithm (which comes down to us from Euclid’s *Elements*) computes the greatest common divisor of two given integers. It is shown here that the structure of the Euclidean algorithm may be used to generate, very efficiently, a large family of rhythms used as timelines (*ostinatos*), in sub-Saharan African music in particular, and world music in general. These rhythms, here dubbed *Euclidean* rhythms, have the property that their onset patterns are distributed as evenly as possible. *Euclidean* rhythms also find application in nuclear physics accelerators and in computer science, and are closely related to several families of words and sequences of interest in the study of the combinatorics of words, such as Euclidean strings, to which the *Euclidean* rhythms are compared.

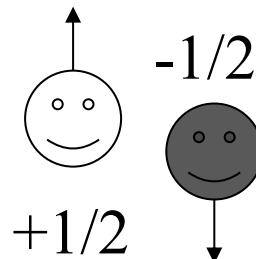
African-Cuban Maximally-Even rhythms



Dinner Table Problem



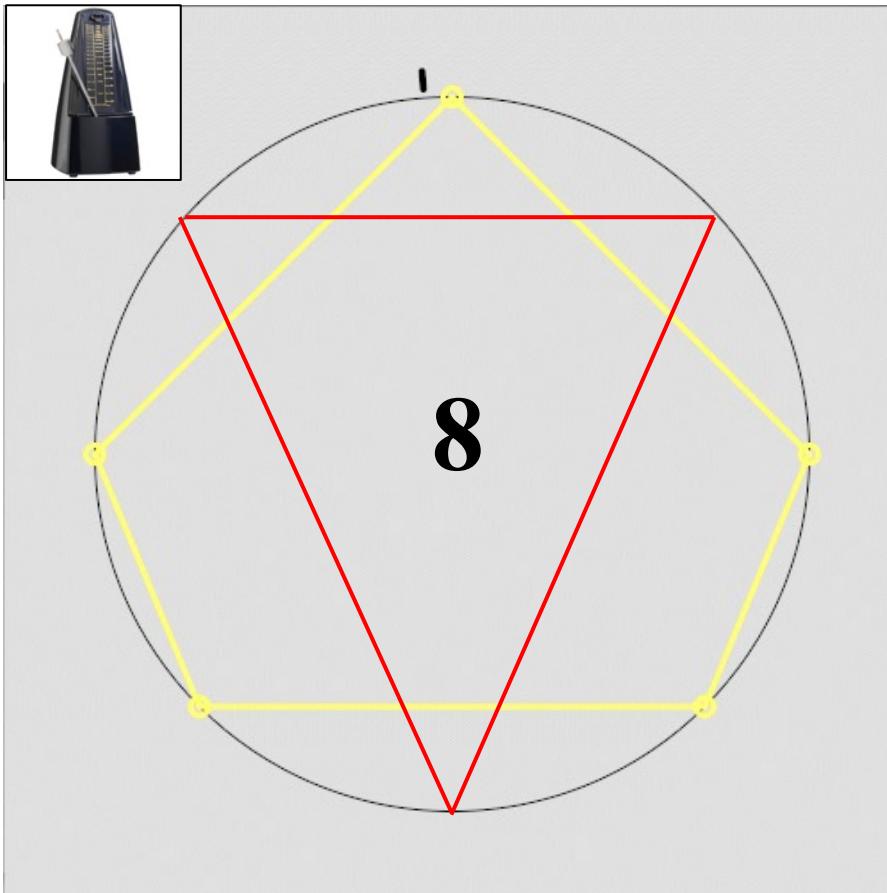
Abadja ou Bembé



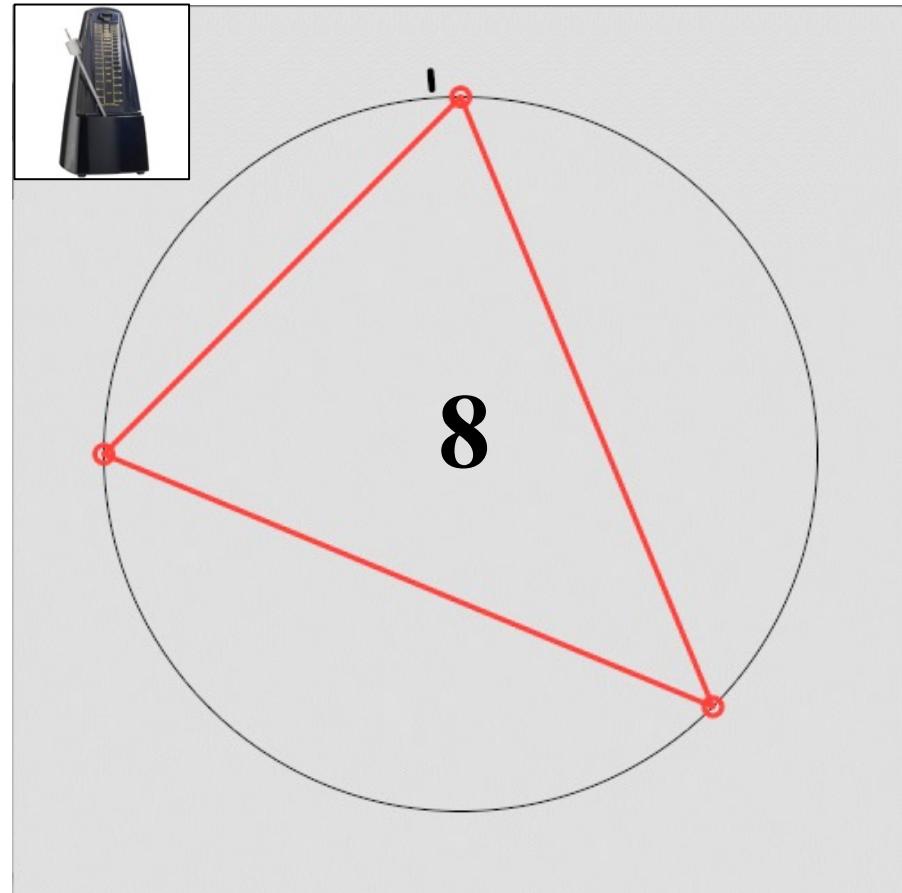
J. Douthett & R. Krantz, "Energy extremes and spin configurations for the one-dimensional antiferromagnetic Ising model with arbitrary-range interaction", *J. Math. Phys.* 37 (7), July 1996

African-cuban ME-rhythms

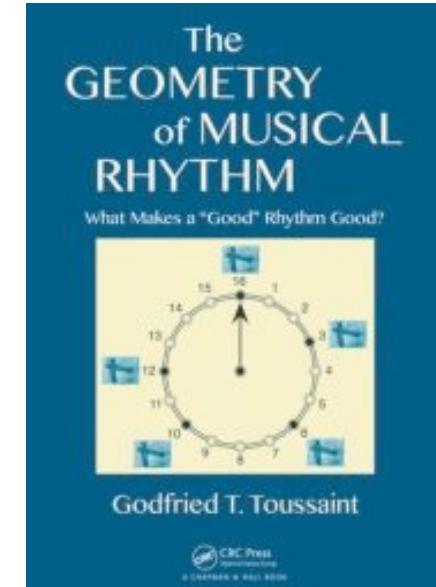
El cinquillo



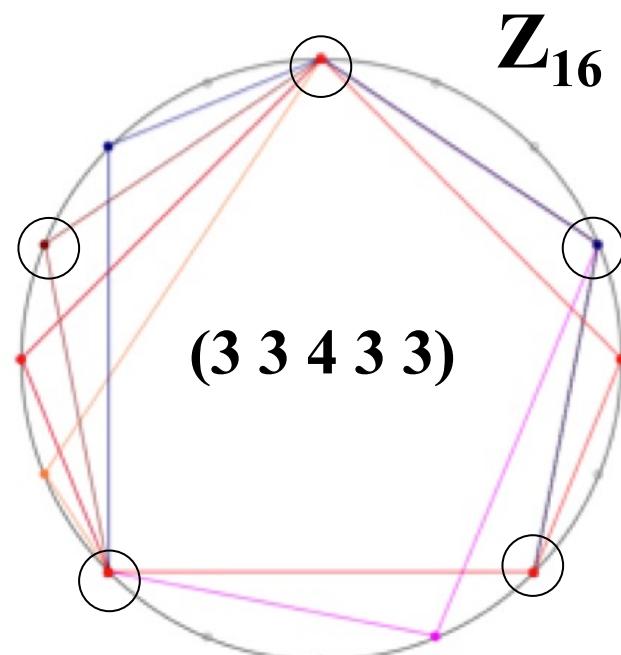
El trecillo



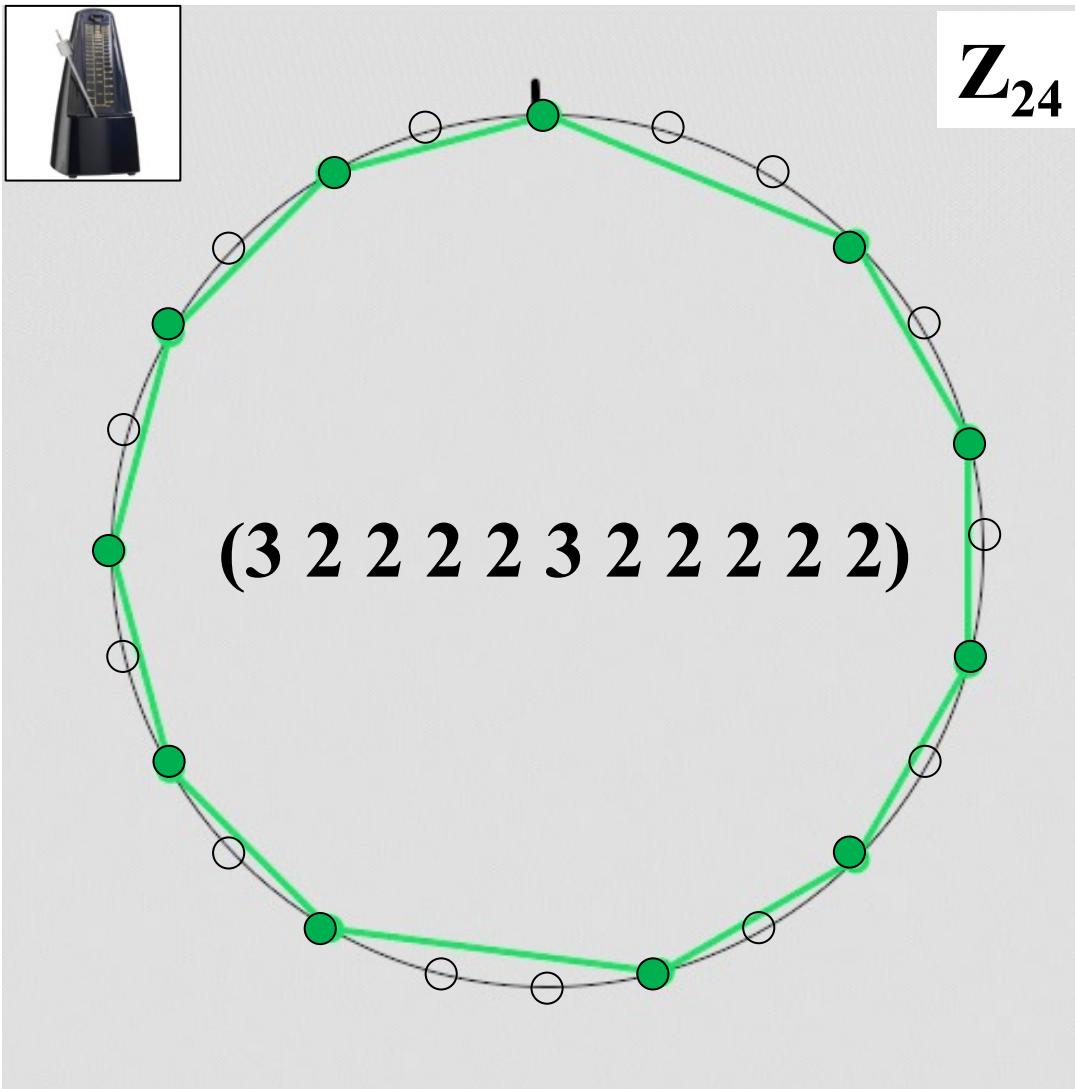
The geometry of African-Cuban rhythms



Shiko	● ● ●
Son	● ● ●
Soukous	● ● ●
Rumba	● ● ● ●
Bossa	● ● ●
Gahu	● ● ●



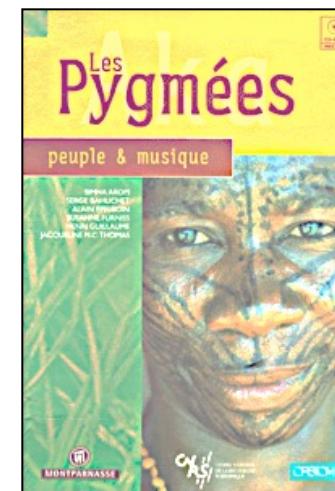
Odditive property of orally-trasmitted practices



Simha Arom



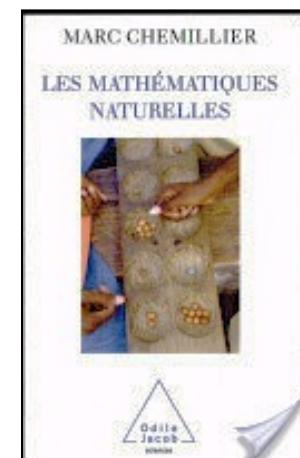
Marc Chemillier



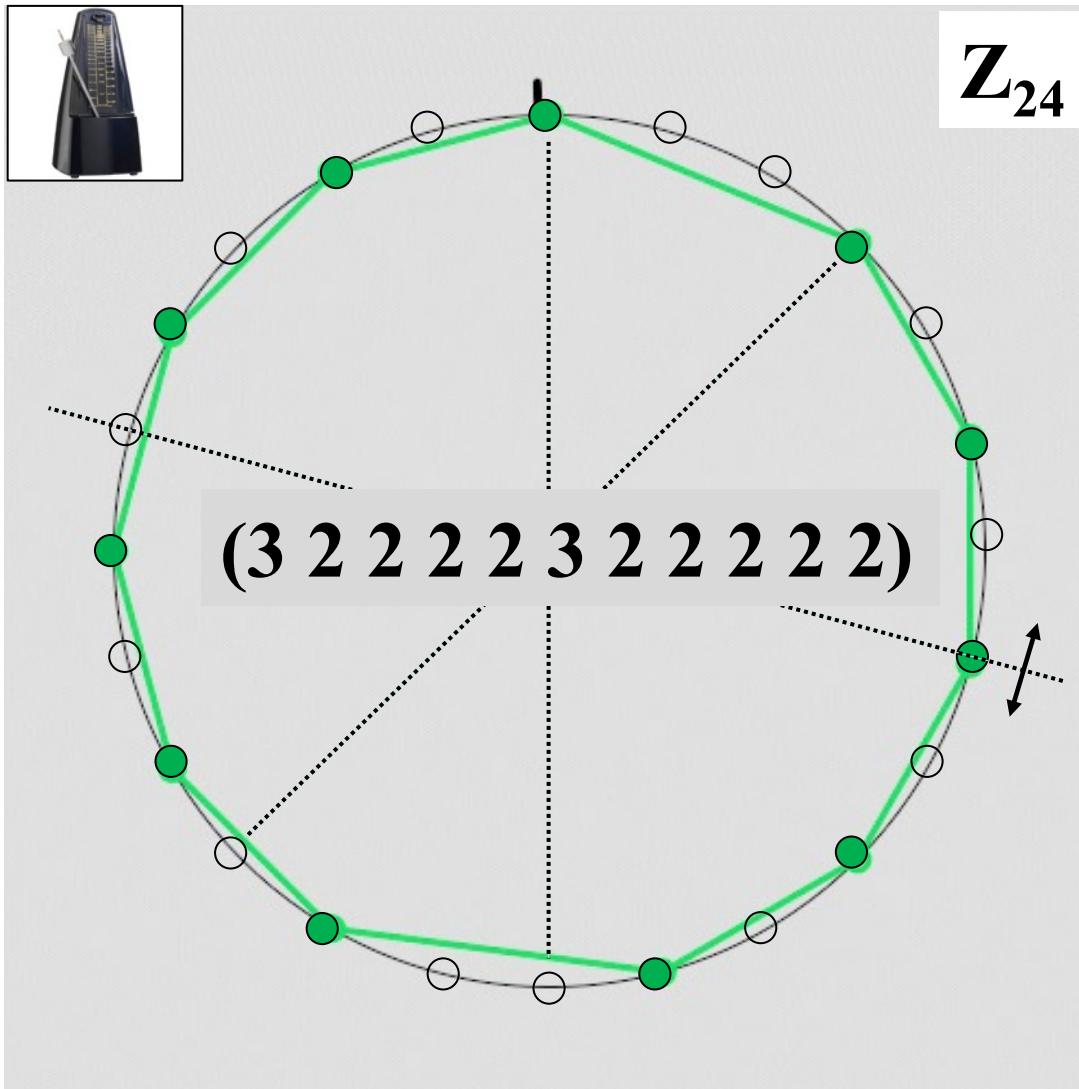
musimédiane

publiée avec le concours de la SFAM

revue audiovisuelle et multimédia d'analyse musicale



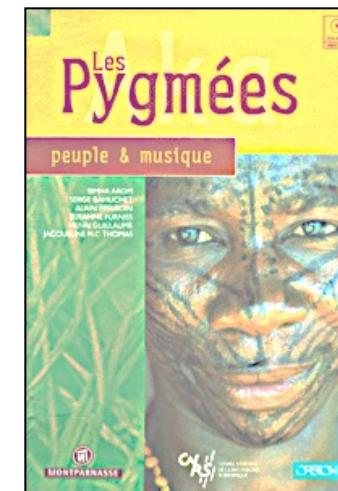
Odditive property of orally-trasmitted practices



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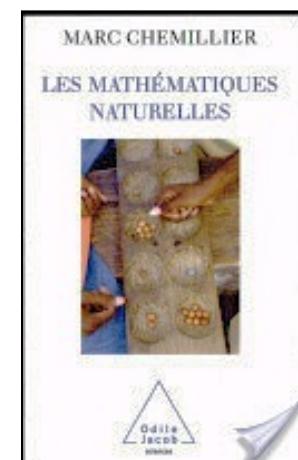
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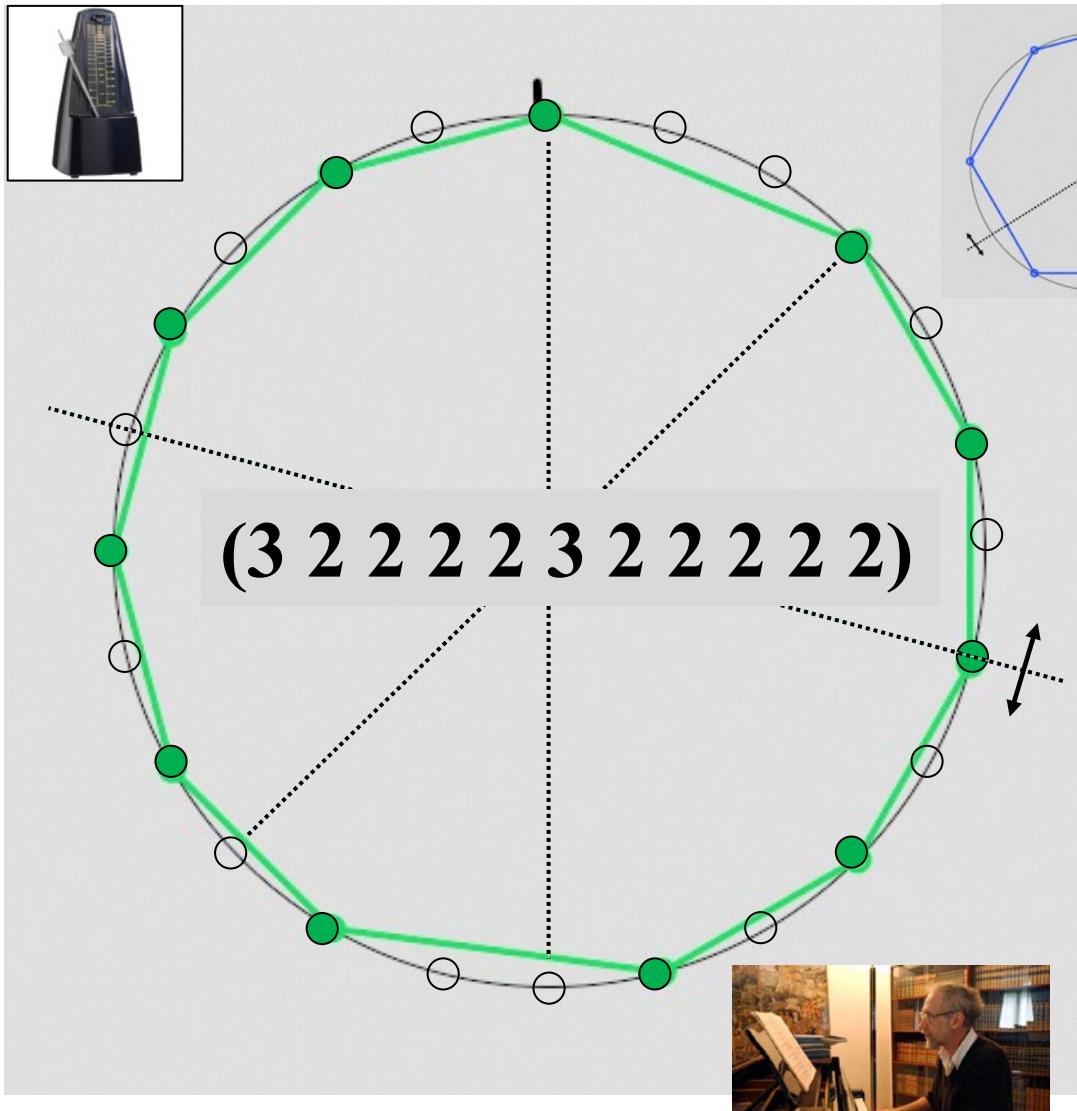
musimédiane

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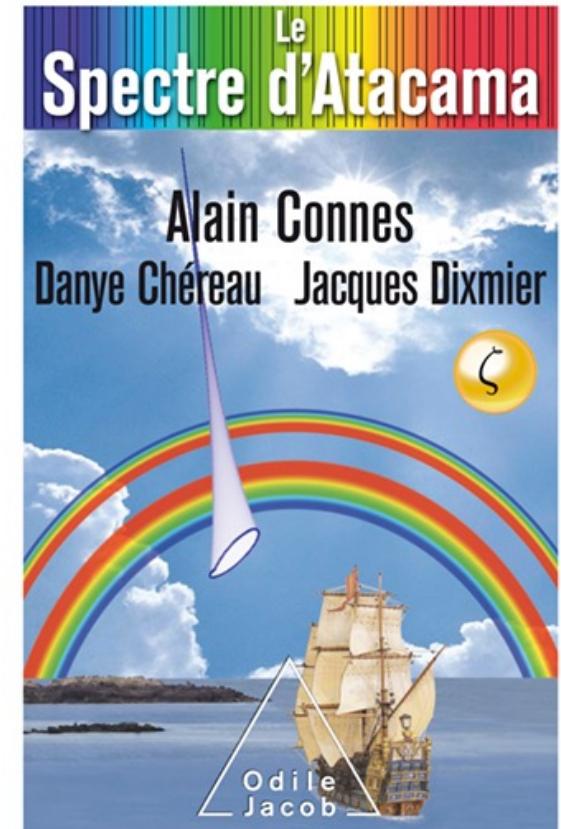
revue audiovisuelle et multimédia d'analyse musicale



Olivier Messiaen's non-invertible rhythms



Alain Connes



Olivier Messiaen

Palindromic structures in Steve Reich's music

CLAPPING MUSIC

FOR TWO PERFORMERS

J = 144-168

CLAPS CLAP2

① ② ③ ④ ⑤ ⑥ ⑦ ⑧ ⑨ ⑩ ⑪ ⑫

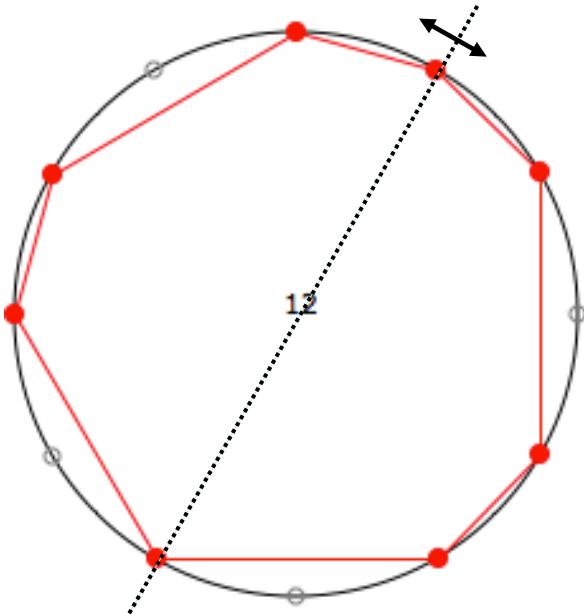
Repeat bar ⑪, then end.

The performance begins and ends with both performers in unison at bar ①. The number of repeats of each bar should be fixed at twelve repeats per bar. Since the first performer's part does not change, it is up to the second performer to move from one bar to the next. The second performer should try to keep his or her downbeat where it is written, i.e., on the first beat of each measure (not on the first beat of the group of three claps), so that his downbeat always falls on a new beat of his or her anchoring pattern.

The choice of a particular clapping sound, i.e., with clipped or flat heads, is left up to the performers. Whichever take is chosen, both performers should try to get the same one so that their two parts will blend to produce one overall resulting pattern.

Clapping Music de Steve Reich (1972)

Steve Reich /7/72
rec. copied 1/78



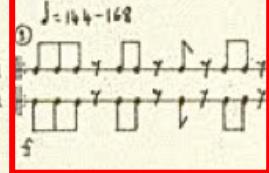
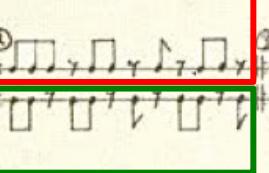
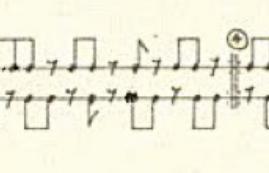
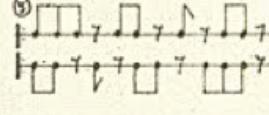
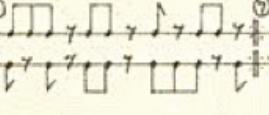
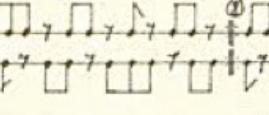
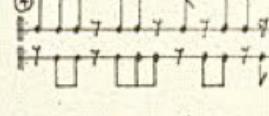
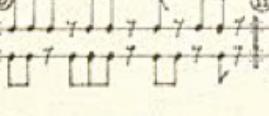
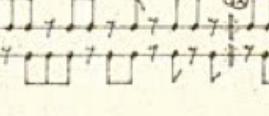
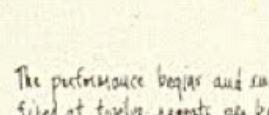
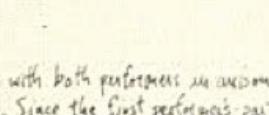
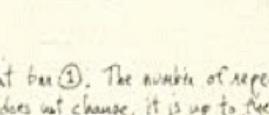
The circle and its ‘canonic’ rotations

CLAPPING MUSIC

FOR TWO PERFORMERS

J = 144-168

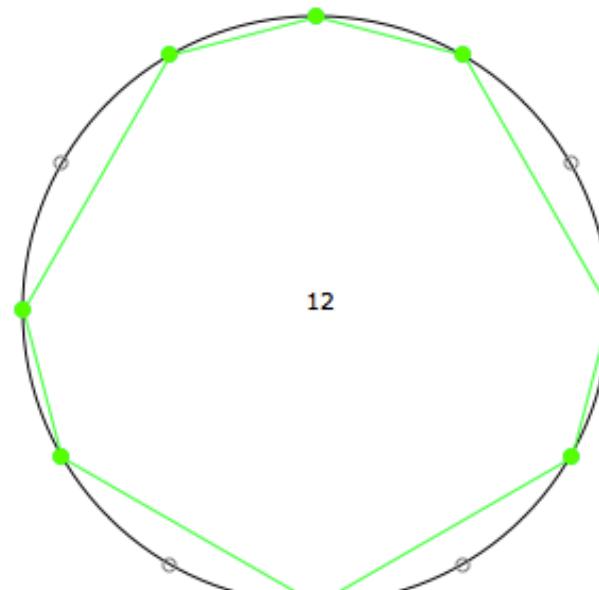
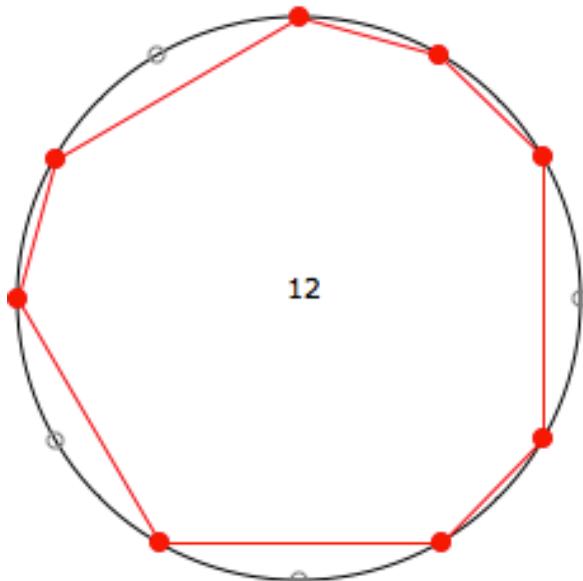
CLAPS 1
CLAP2

①   
②   
⑦   
⑩   

Repeat bar ①, then end.

The performance begins and ends with both performers in unison at bar ①. The number of repeats of each bar should be fixed at twelve repeats per bar. Since the first performer's part does not change, it is up to the second performer to move from one bar to the next. The second performer should try to keep his or her downbeat where it is written, i.e. on the first beat of each measure (not on the first beat of the group of three claps), so that his downbeat always falls on a new beat of his or her anchoring pattern.

The choice of a particular clapping sound, i.e., with clipped or flat heads, is left up to the performers. Whichever take is chosen, both performers should try to get the same one so that their two parts will blend to produce one overall resulting pattern.



The circle and its ‘canonic’ rotations

CLAPPING MUSIC

FOR TWO PERFORMERS

$J = 144-168$

CLAP1
CLAP2

① ② ③ ④ ⑤ ⑥ ⑦ ⑧ ⑨ ⑩ ⑪ ⑫

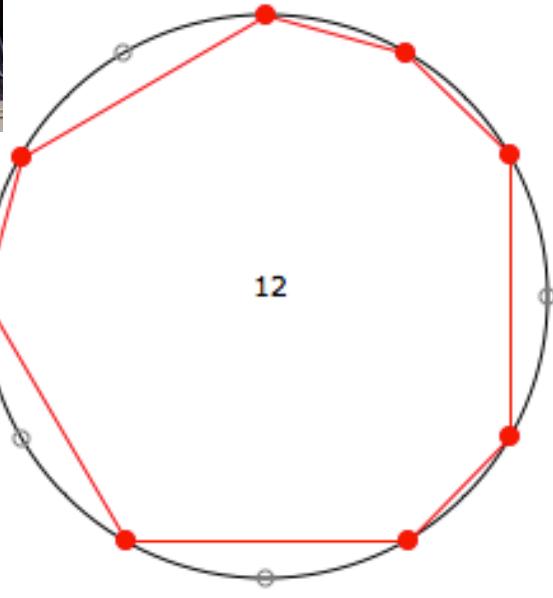
Repeat bar ⑪, then end.

The performance begins and ends with both performers in unison at bar ①. The number of repeats of each bar should be fixed at twelve repeats per bar. Since the first performer's part does not change, it is up to the second performer to move from one bar to the next. The second performer should try to keep his or her downbeat where it is written, i.e., on the first beat of each measure (not on the first beat of the group of three claps), so that his downbeat always falls on a new beat of his or her anchoring pattern.

The choice of a particular clapping sound, i.e., with cupped or flat hands, is left up to the performers. Whichever technique is chosen, both performers should try to get the same one so that their two parts will blend to produce one overall resulting pattern.

Clapping Music (1972)

Alex Rech
rec. 1972
rec. copied 1978



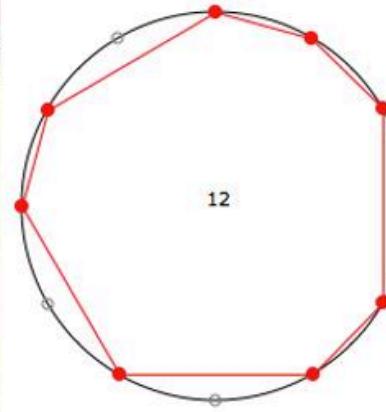
YOUTUBE.COM/GERUBACH

YOUTUBE.COM/GERUBACH

Gerubach's Scrolling Score Project
<http://www.gerubach.com>

En vous appuyant sur les outils présentés pendant le cours – ainsi que sur les slides additionnels – discuter le problème musical suivant. Dans la pièce *Clapping Music* de Steve Reich, les deux « clappeurs » commencent à l'unisson. Autrement dit, au début de la pièce, tous les points d'attaque entre les deux musiciens coïncident. Dans la deuxième mesure, les deux « clappeurs » n'ont que quatre instants où ils jouent à l'unisson, c'est-à-dire quatre points d'attaque qui coïncident. Ces points d'attaque sont cinq dans la troisième mesure, etc. (cf. figure ci-dessous).

The diagram shows the score for 'CLAPPING MUSIC FOR TWO PERFORMERS'. It includes a title, a photo of two performers, and a rhythmic grid. The grid has two rows: 'CLAP1' and 'CLAP2'. The tempo is indicated as $J=144-168$. The first measure shows 7 vertical bars. The second measure shows 5 vertical bars. The third measure shows 7 vertical bars. The fourth measure shows 5 vertical bars. The fifth measure shows 7 vertical bars. The sixth measure shows 5 vertical bars. The seventh measure shows 7 vertical bars. The eighth measure shows 5 vertical bars. The ninth measure shows 7 vertical bars. The tenth measure shows 5 vertical bars. The eleventh measure shows 7 vertical bars. The twelfth measure shows 5 vertical bars. A green box highlights the first four measures, and an orange box highlights the next four measures. Below the grid, it says 'Repeat from (1), then end.'



1. Discuter le comportement du nombre d'unissons tout au long des douze mesures de la pièce
2. Toujours en gardant la périodicité du pattern rythmique égale à 12, appliquer le même procédé compositionnel aux deux rythmes euclidiens (ou ME-sets ou ensembles « bien repartis ») de cardinalité respectivement 7 et 5 et caractériser le nombre d'unissons dans les deux nouvelles pièces.
3. Quel lien y a-t-il entre le nombre d'unissons des deux nouvelles pièces ?
4. (Optionnel) En utilisant un logiciel de notation (Musescore, Finale, Sibelius ou autres logiciels à votre gré), produire les partitions des deux nouvelles variantes de la composition originale. Ca sera votre double composition (à jouer au plaisir entre deux « clappeurs »... ;-)

En vous appuyant sur les outils présentés pendant le cours – ainsi que sur les slides additionnels – discuter le problème musical suivant. Dans la pièce *Clapping Music* de Steve Reich, les deux « clappeurs » commencent à l'unisson. Autrement dit, au début de la pièce, tous les points d'attaque entre les deux musiciens coïncident. Dans la deuxième mesure, les deux « clappeurs » n'ont que quatre instants où ils jouent à l'unisson, c'est-à-dire quatre points d'attaque qui coïncident. Ces points d'attaque sont cinq dans la troisième mesure, etc. (cf. figure ci-dessous).

The image shows the musical score for Steve Reich's 'Clapping Music' and a circular diagram. The score is for two performers, CLAP1 and CLAP2, at a tempo of $J=144-168$. It consists of four staves of music. The first staff shows measures 1-4, with cardinalities 8, 4, 5, and 6 respectively. The second staff shows measures 5-8, with cardinalities 5, 6, 4, and 6 respectively. The third staff shows measures 9-12, with cardinalities 5, 6, 5, and 4 respectively. The fourth staff is a repeat of the first. The circular diagram to the right represents a 12-fold circle, with red dots at each of the 12 positions, indicating the periodicity of 12.

1. Discuter le comportement du nombre d'unissons tout au long des douze mesures de la pièce
2. Toujours en gardant la périodicité du pattern rythmique égale à 12, appliquer le même procédé compositionnel aux deux rythmes euclidiens (ou ME-sets ou ensembles « bien repartis ») de cardinalité respectivement 7 et 5 et caractériser le nombre d'unissons dans les deux nouvelles pièces.
3. Quel lien y a-t-il entre le nombre d'unissons des deux nouvelles pièces ?
4. (Optionnel) En utilisant un logiciel de notation (Musescore, Finale, Sibelius ou autres logiciels à votre gré), produire les partitions des deux nouvelles variantes de la composition originale. Ca sera votre double composition (à jouer au plaisir entre deux « clappeurs »... ;-)

En vous appuyant sur les outils présentés pendant le cours – ainsi que sur les slides additionnels – discuter le problème musical suivant. Dans la pièce *Clapping Music* de Steve Reich, les deux « clappeurs » commencent à l'unisson. Autrement dit, au début de la pièce, tous les points d'attaque entre les deux musiciens coïncident. Dans la deuxième mesure, les deux « clappeurs » n'ont que quatre instants où ils jouent à l'unisson, c'est-à-dire quatre points d'attaque qui coïncident. Ces points d'attaque sont cinq dans la troisième mesure, etc. (cf. figure ci-dessous).

The image shows the musical score for Steve Reich's *Clapping Music*. The title "CLAPPING MUSIC FOR TWO PERFORMERS" is at the top. The score consists of two staves: "CLAP1" and "CLAP2". The tempo is indicated as $\text{♩} = 144-168$. The score is divided into measures by vertical bar lines. Above the staff, large numbers indicate the count of unisons: 8, 4, 5, 6 in the first measure; 5, 6, 4, 6 in the second; 5, 6, 5, 4 in the third; and so on. The score ends with the instruction "Repeat from (1), then end". To the right of the score is a circular diagram with 12 points on its circumference. Red lines connect the points in a specific pattern: (1,2), (2,3), (3,4), (4,5), (5,6), (6,7), (7,8), (8,9), (9,10), (10,11), (11,12), (12,1). Below the circle is the sequence of counts: [8,4,5,6,5,6,4,6,5,6,5,4].

1. Discuter le comportement du nombre d'unissons tout au long des douze mesures de la pièce
2. Toujours en gardant la périodicité du pattern rythmique égale à 12, appliquer le même procédé compositionnel aux deux rythmes euclidiens (ou ME-sets ou ensembles « bien repartis ») de cardinalité respectivement 7 et 5 et caractériser le nombre d'unissons dans les deux nouvelles pièces.
3. Quel lien y a-t-il entre le nombre d'unissons des deux nouvelles pièces ?
4. (Optionnel) En utilisant un logiciel de notation (Musescore, Finale, Sibelius ou autres logiciels à votre gré), produire les partitions des deux nouvelles variantes de la composition originale. Ca sera votre double composition (à jouer au plaisir entre deux « clappeurs »... ;-)

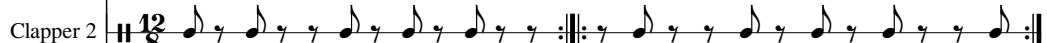
Clapping Music Euclidian 5

for 2 Performers

(Quentin Le Gall, 2023)

1

Clapper 1 |  Fine **2**

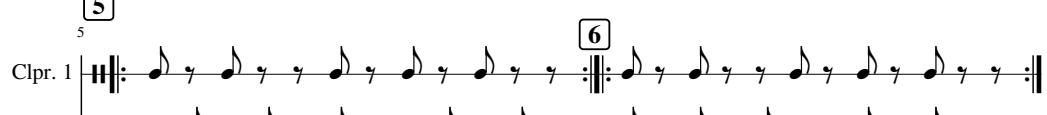
Clapper 2 | 

3

Clpr. 1 |  **4**

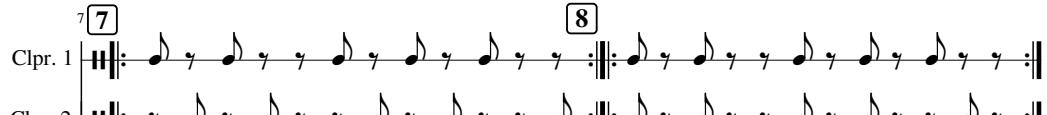
Clpr. 2 | 

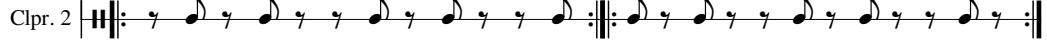
5

Clpr. 1 |  **6**

Clpr. 2 | 

7

Clpr. 1 |  **8**

Clpr. 2 | 

9

Clpr. 1 |  **10**

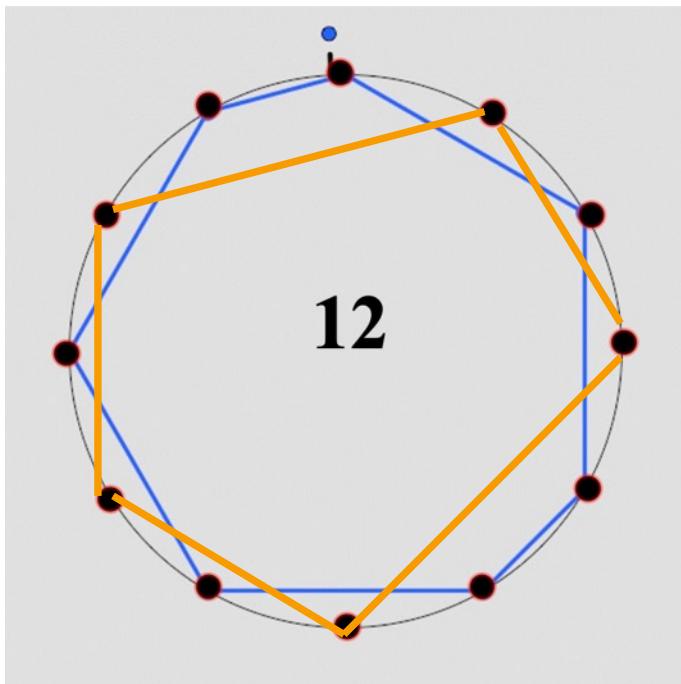
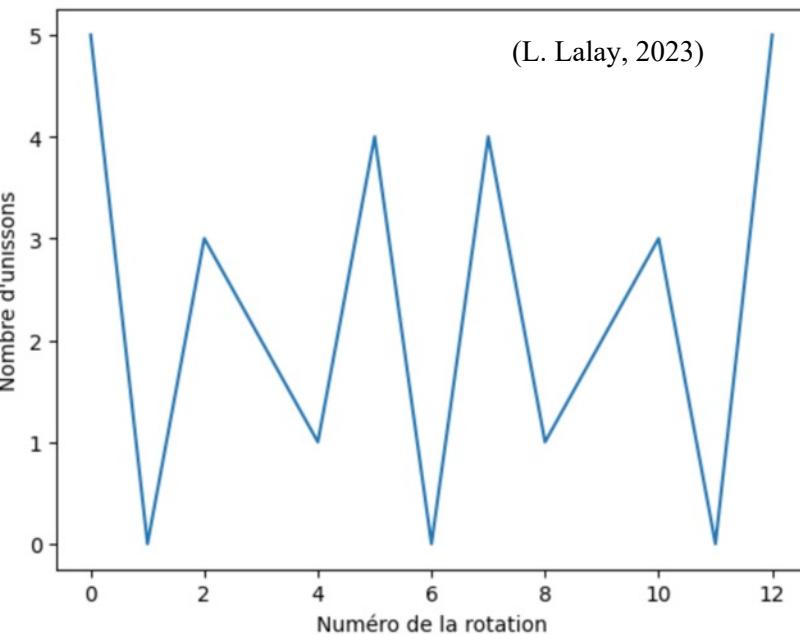
Clpr. 2 | 

11

Clpr. 1 |  **12**

D.C. al Fine

Clpr. 2 | 



Clapping Music sur le rythme Euclidien

(E,7,12)

(Pierre Chouteau, 2023)

Clapper 1 |  **1**

Clapper 2 |  **2**

Clpr. 1 |  **3**

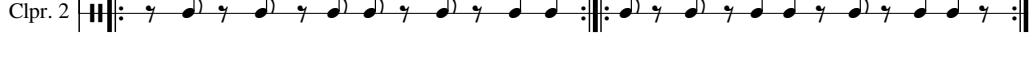
Clpr. 2 |  **4**

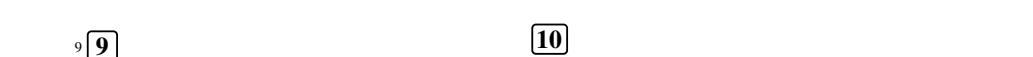
Clpr. 1 |  **5**

Clpr. 2 |  **6**

Clpr. 1 |  **7**

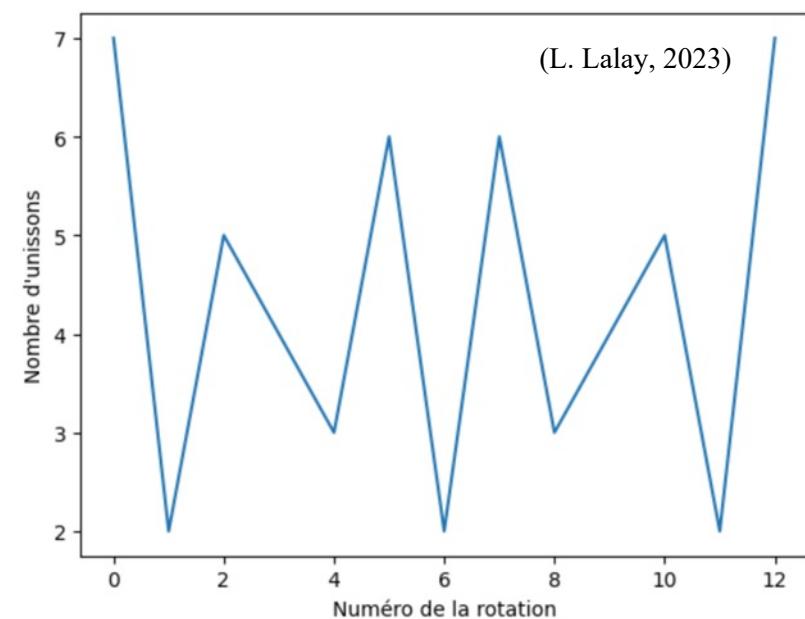
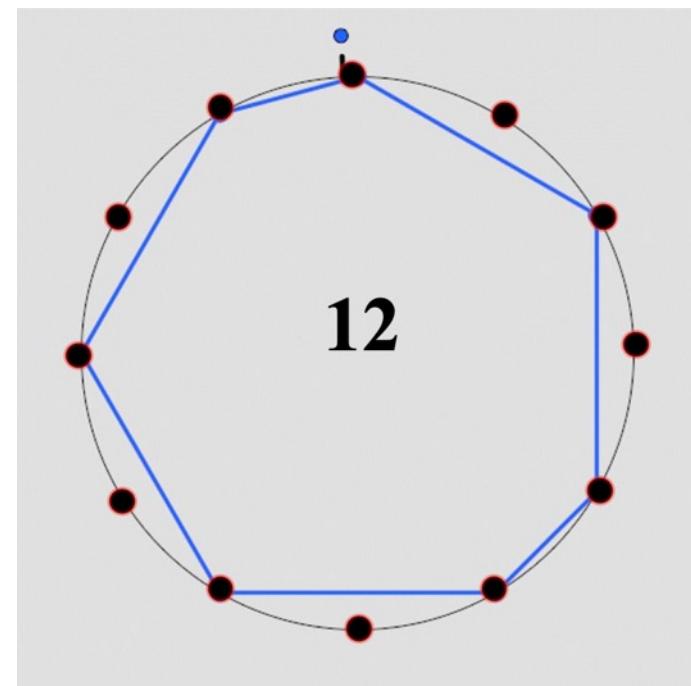
Clpr. 2 |  **8**

Clpr. 1 |  **9**

Clpr. 2 |  **10**

Clpr. 1 |  **11**

Clpr. 2 |  **12**



Clapping music à 5 temps

Wood Blocks 1

Wood Blocks 2

Wd. Bl. 1

Wd. Bl. 2

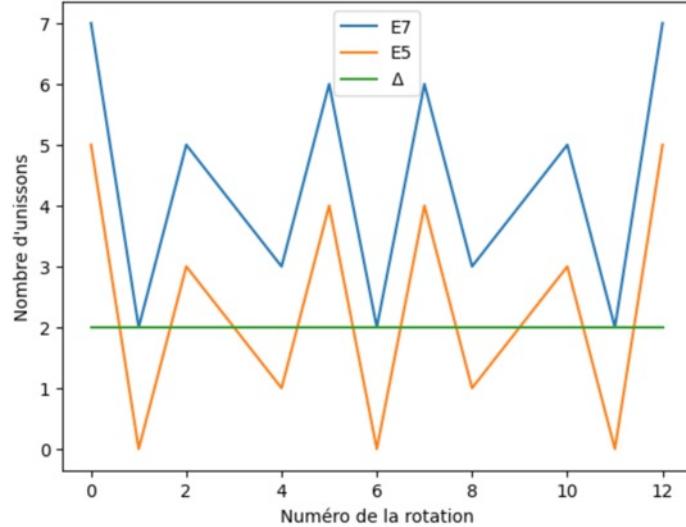
Clapping music à 7 temps

Wood Blocks 1

Wood Blocks 2

Wd. Bl. 1

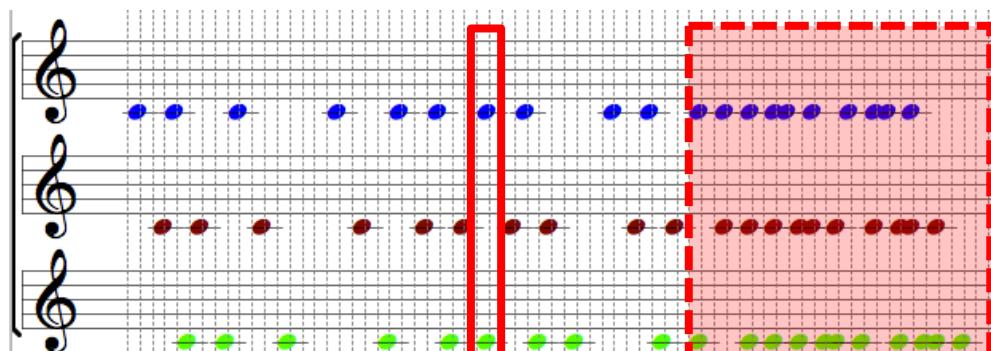
Wd. Bl. 2



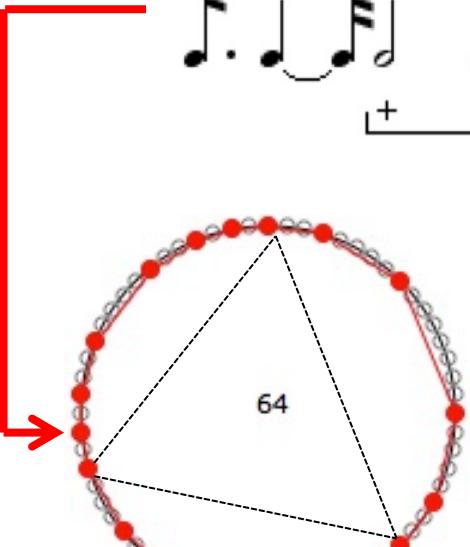
Periodic rhythmic sequences and tiling canons



Harawi (1945)



Harawi: rhythmic reduction



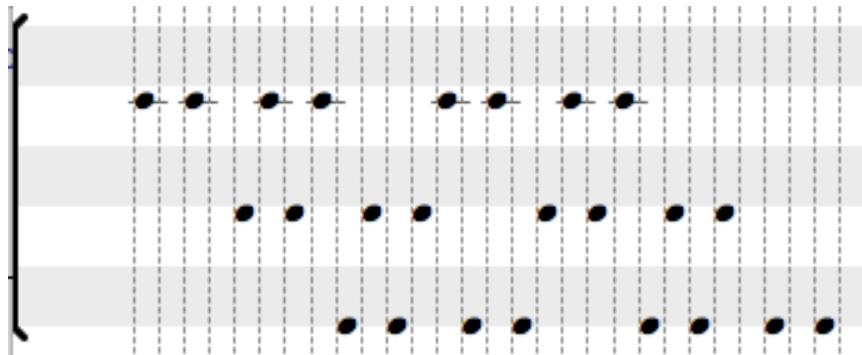
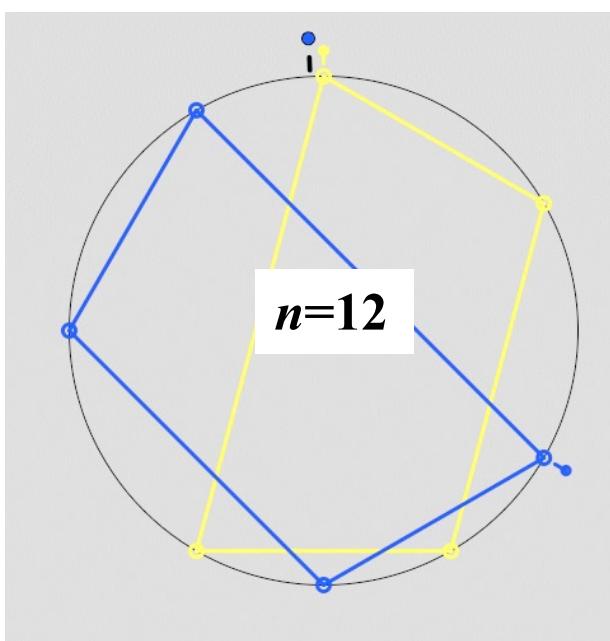
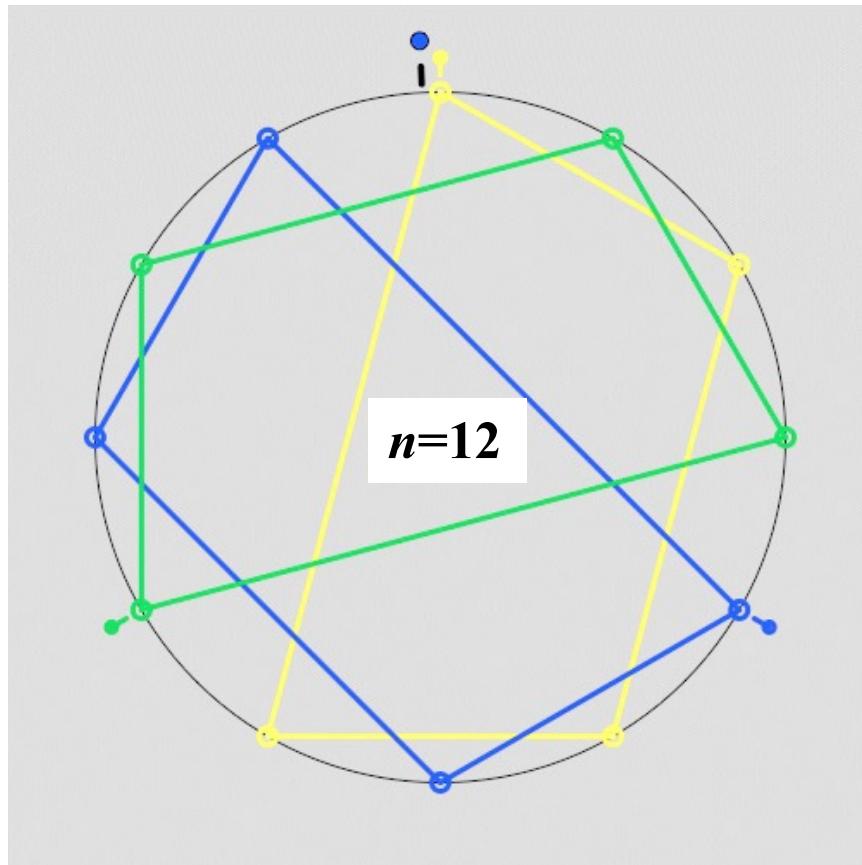
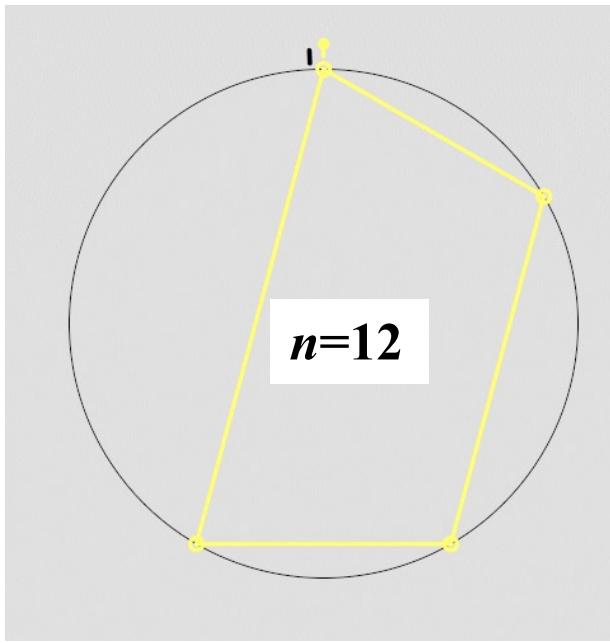
« ...il résulte de tout cela que les différentes sonorités se mélagent ou s'opposent de manières très diverses, jamais au même moment ni au même endroit [...]. C'est du désordre organisé »

O. Messiaen: *Traité de Rythme, de Couleur et d'Ornithologie*, tome 2, Alphonse Leduc, 1992.

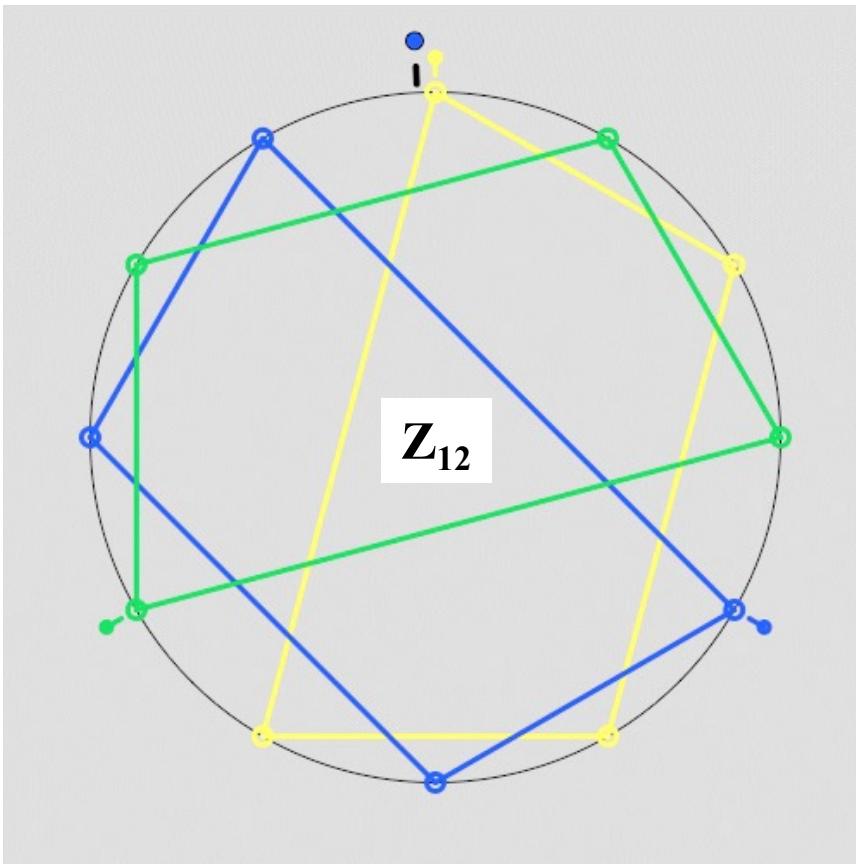


Olivier Messiaen

Tiling the time axis with translates of one tile



Formalizing the tiling process as set-theoretical operations



$$A_1 = \{0, 2, 5, 7\}$$

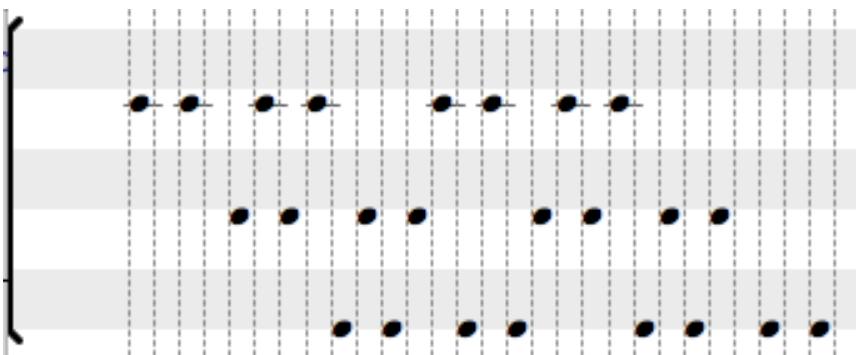
$$T_4 \downarrow \boxed{T_k : x \rightarrow x+k \bmod 12}$$

$$A_2 = \{4, 6, 9, 11\}$$

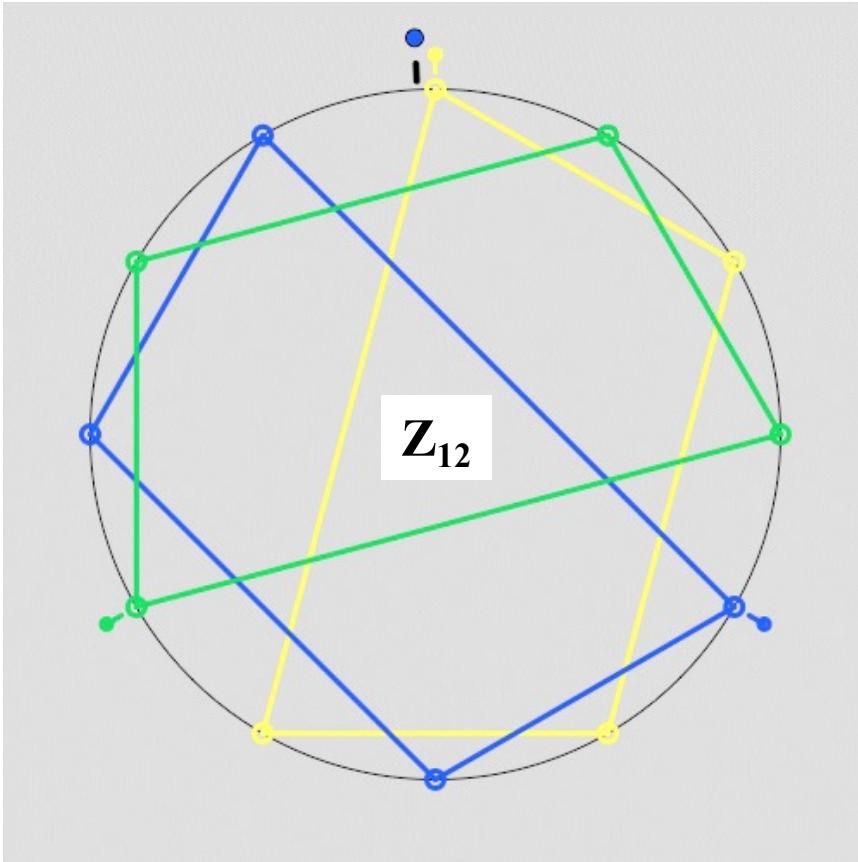
$$T_4 \downarrow$$

$$A_3 = \{8, 10, 1, 3\}$$

$$Z_{12} = A_1 \cup A_2 \cup A_3$$



Formalizing the tiling process as a direct sum of subsets



$$A_1 = \{0, 2, 5, 7\}$$

T_4
↓

$$A_2 = \{4, 6, 9, 11\}$$

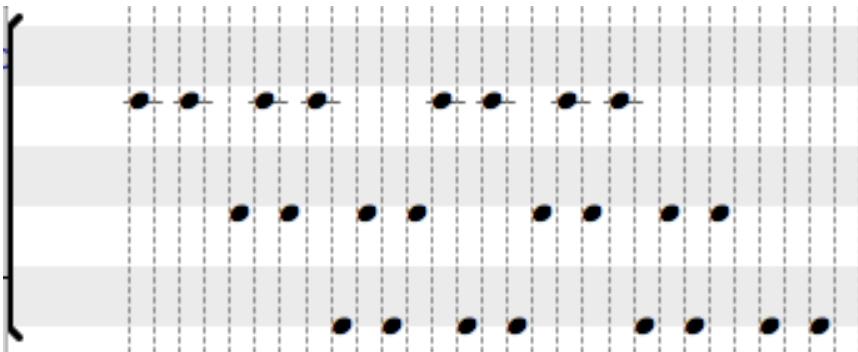
T_4
↓

$$A_3 = \{8, 10, 1, 3\}$$

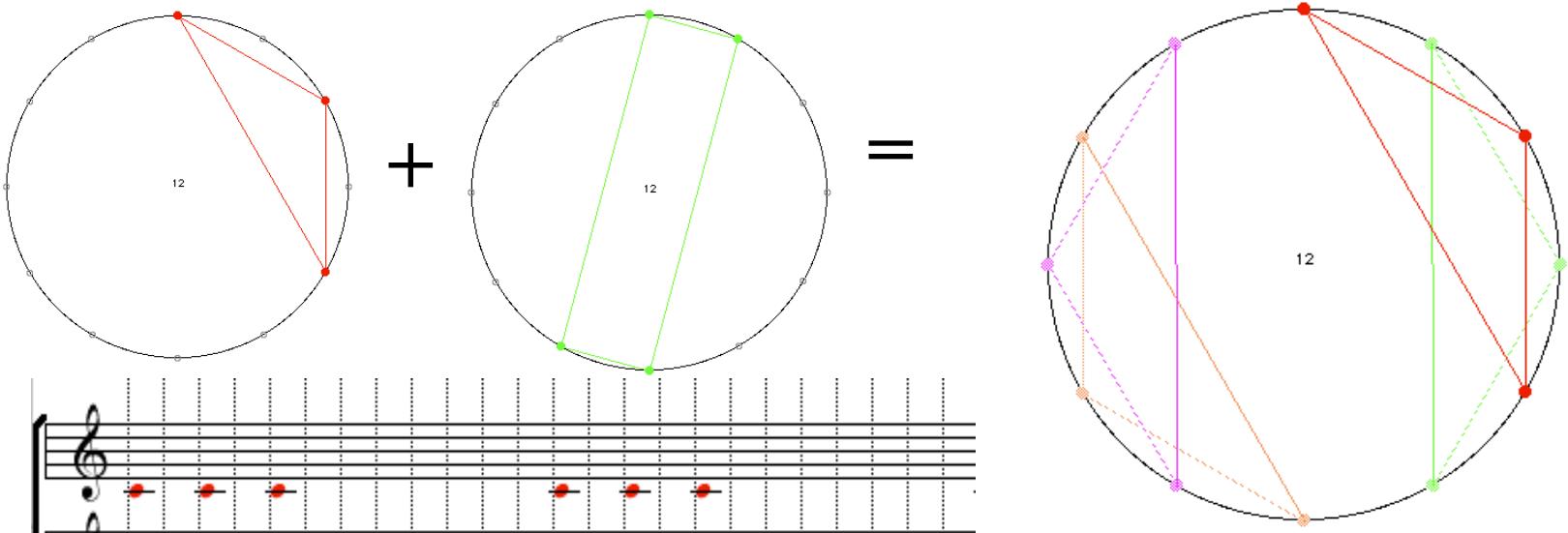
$$Z_{12} = A_1 \cup A_2 \cup A_3$$

$$Z_{12} = A \oplus B$$

$$\begin{aligned} A &= \{0, 2, 5, 7\} \\ B &= \{0, 4, 8\} \end{aligned}$$



Rhythmic tiling canons with no regular entries

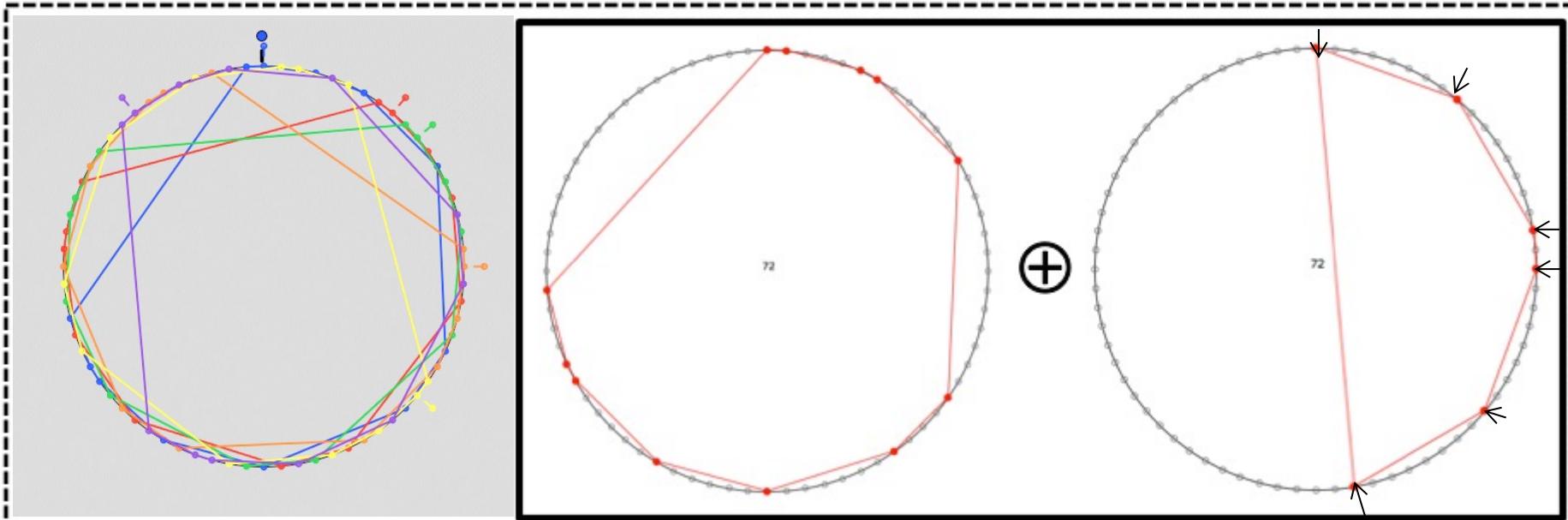


transpositional
combination

$$\{0,2,4\} \oplus \{0,1,6,7\} = Z_{12} = (2 \ 2 \ 8) \bullet (1 \ 5 \ 1 \ 5)$$

One of the two factors is a Messiaen's mode of limited transposition

Aperiodic Rhythmic Tiling Canons (Vuza Canons)



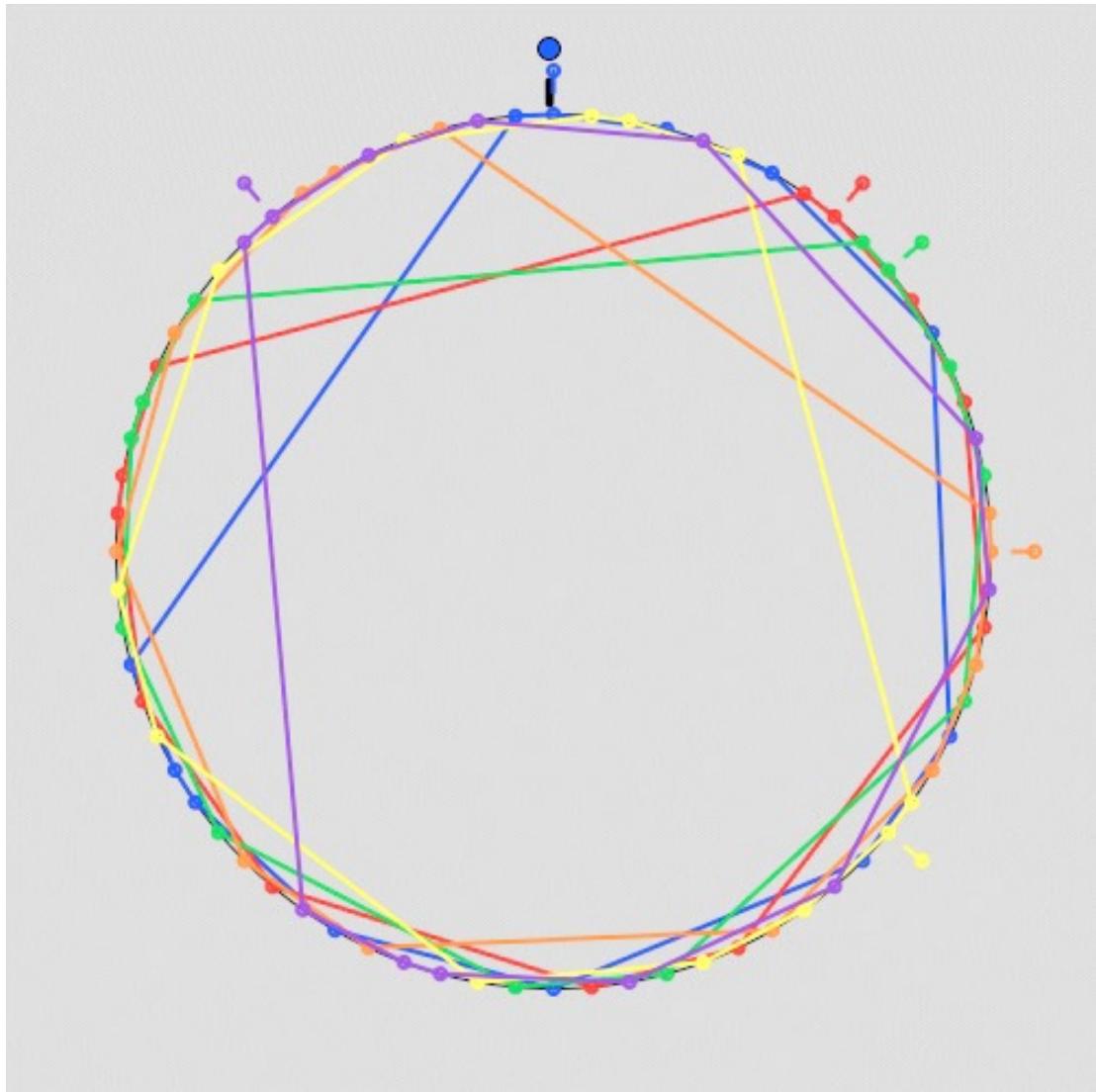
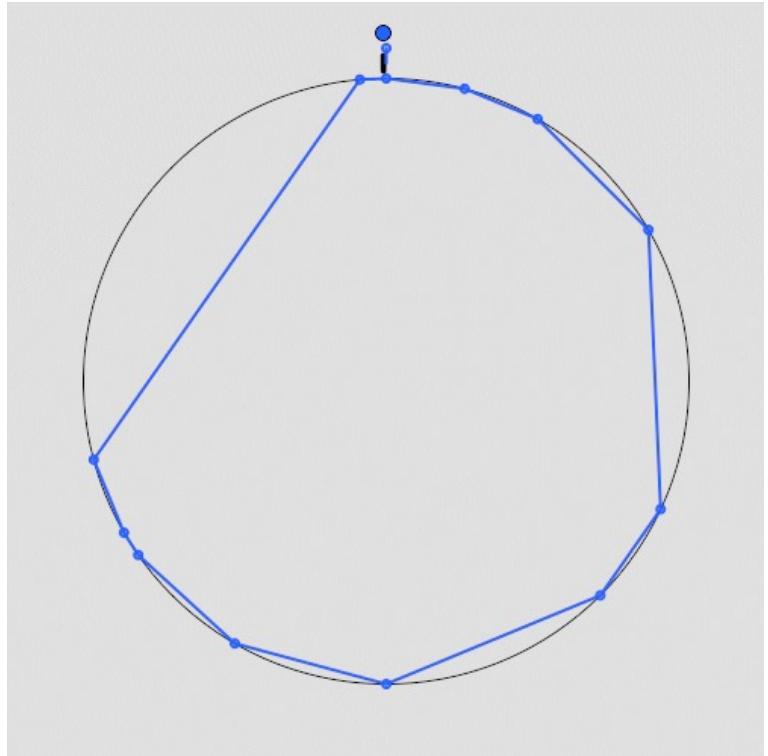
Dan Vuza



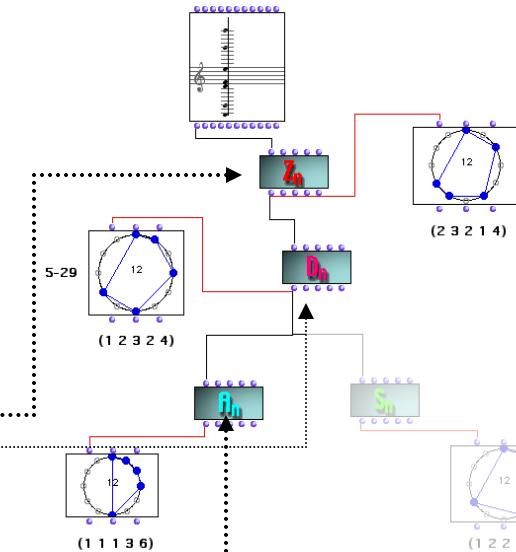
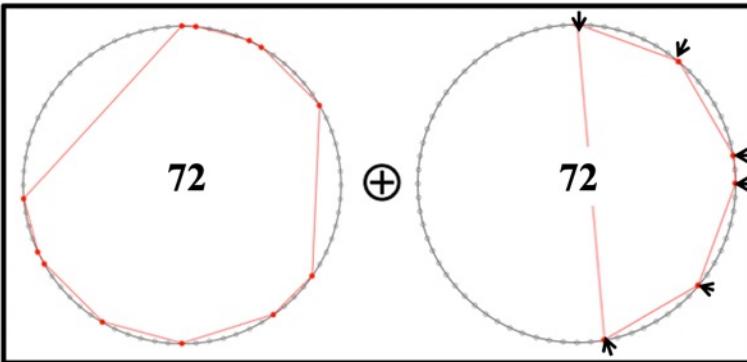
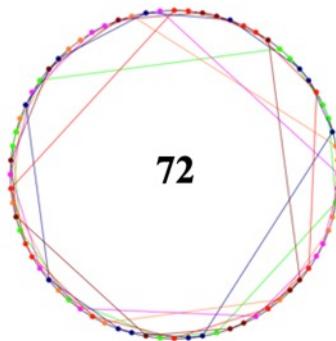
Anatol Vieru

A musical score for four voices, each represented by a treble clef staff. The music consists of vertical dashes of varying lengths, indicating rhythmic values. Arrows point from the top of each staff to specific notes, highlighting a repeating pattern of note heads across all four voices. The score is set against a grid of horizontal dotted lines.

The melody of a Vuza Canon



Towards the complete classification of Vuza Canons



R

(1 3 3 6 11 4 9 6 5 1 3 20)
 (20 3 1 5 6 9 4 11 6 3 3 1)
 (1 4 1 19 4 1 6 6 7 4 13 6)
 (6 13 4 7 6 6 1 4 19 1 4 1)
 (1 5 15 4 5 6 6 3 4 17 3 3)
 (3 3 17 4 3 6 6 5 4 15 5 1)

S

(8 8 2 8 8 38)
 (16 2 14 2 16 22)
 (14 8 10 8 14 18)

(1 3 3 6 11 4 9 6 5 1 3 20)
 (1 4 1 19 4 1 6 6 7 4 13 6)
 (1 5 15 4 5 6 6 3 4 17 3 3)

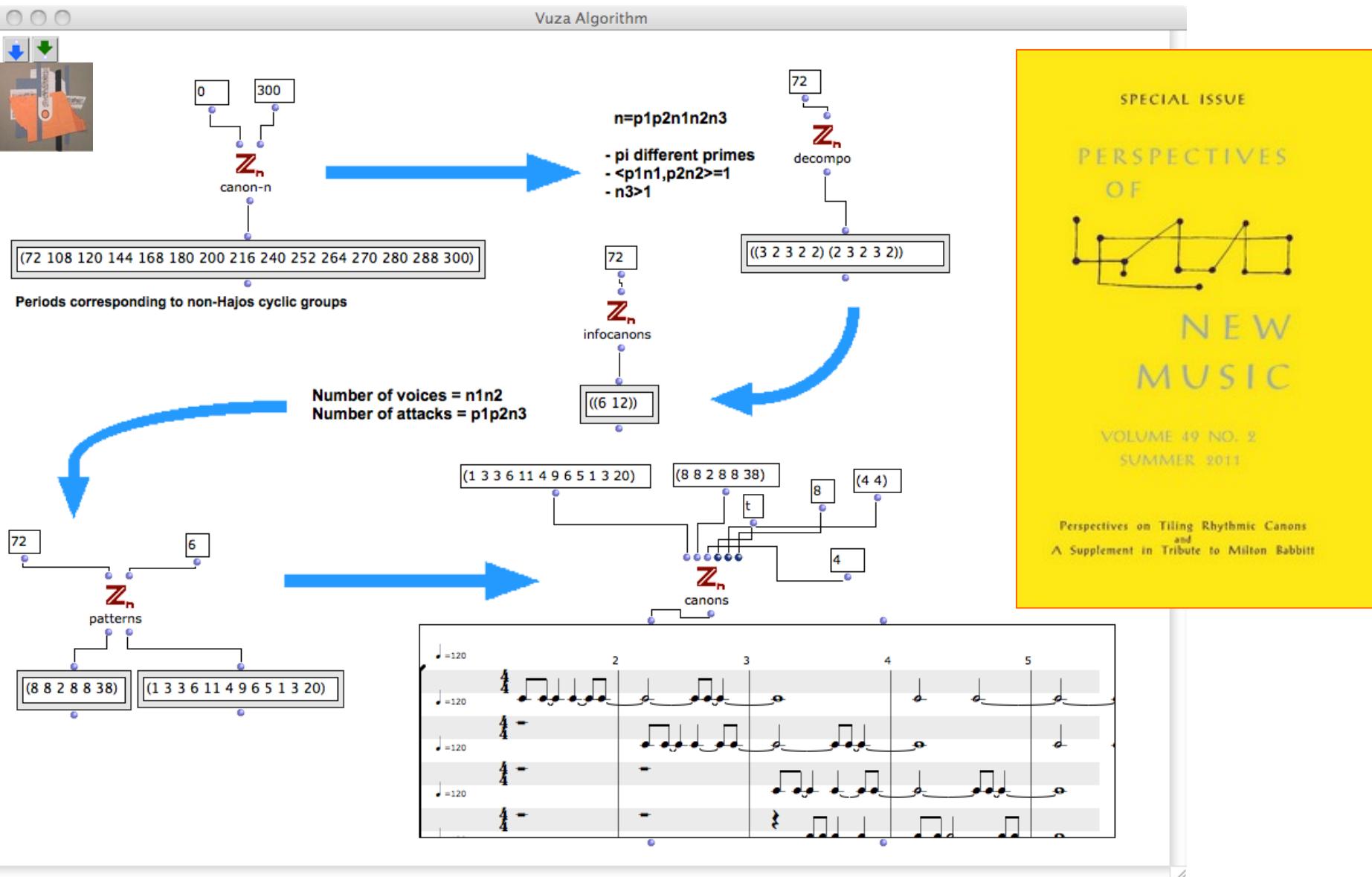
(8 8 2 8 8 38)
 (16 2 14 2 16 22)
 (14 8 10 8 14 18)

(1 3 3 6 11 4 9 6 5 1 3 20)
 (1 4 1 19 4 1 6 6 7 4 13 6)

(8 8 2 8 8 38)

There are only two « types » of Vuza Canons
of period 72 (up to an affine transformation)

Exploring Vuza Canons in OpenMusic



- Agon C. et M. Andreatta (2011), “Modelling and Implementing Tiling Rhythmic Canons in OpenMusic Visual Programming Language”, *Perspectives of New Music*, Special Issue, vol. 1-2, n° 49, p. 66-91.

Some compositional applications of the Vuza Canons model

voix I
voix II
voix III
voix IV
voix V
voix VI
mes. : 158 167 172 173 174 175 184

voix I
voix II
voix III
voix IV
voix V
voix VI
mes. : 187 188 189 190 192 193 194 195 197 199 200 203 204 205 206 208 209 210 211 212 213

voix I
voix II
voix III
voix IV
voix V
voix VI
mes. : 213 215 217 218 219 220 221 222 223 224

(superposition voix V, VI et I) | (superposition voix IV, I et III)

a/=: montée vers accord puis "mise en pulsation"
 b/=: mise en pulsation superposé à un gliss. descendant
 c/=: montée vers accord (tête de a/=:)
 d/=: mise en pulsation" en diminuendo (fin de a/=:)
 [e*f*]: accord mis en "cross rythm" (durée double)
 g/V: gliss. descendant puis ascendant
 [h+i+]: accord mis en "cross rythm" (durée double)
 JV: gliss. ascendant puis descendant avec accent
 kc: "son à l'envers"
 l": deux impacts courts et piano



F. Lévy

Coincidences (1999)



M. Lanza

La bataille de caresme et de charnage (2012)

(pour violoncelle et accompagnement)

A piece based on Monk (2007) ("Well You Need'n't")



G. Bloch

474 (poco accel.) Poco più mosso (a = 80 col.) AD



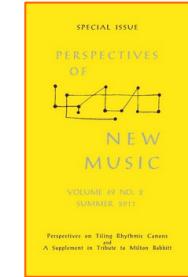
D. Ghisi

La notte poco prima della foresta (2009)

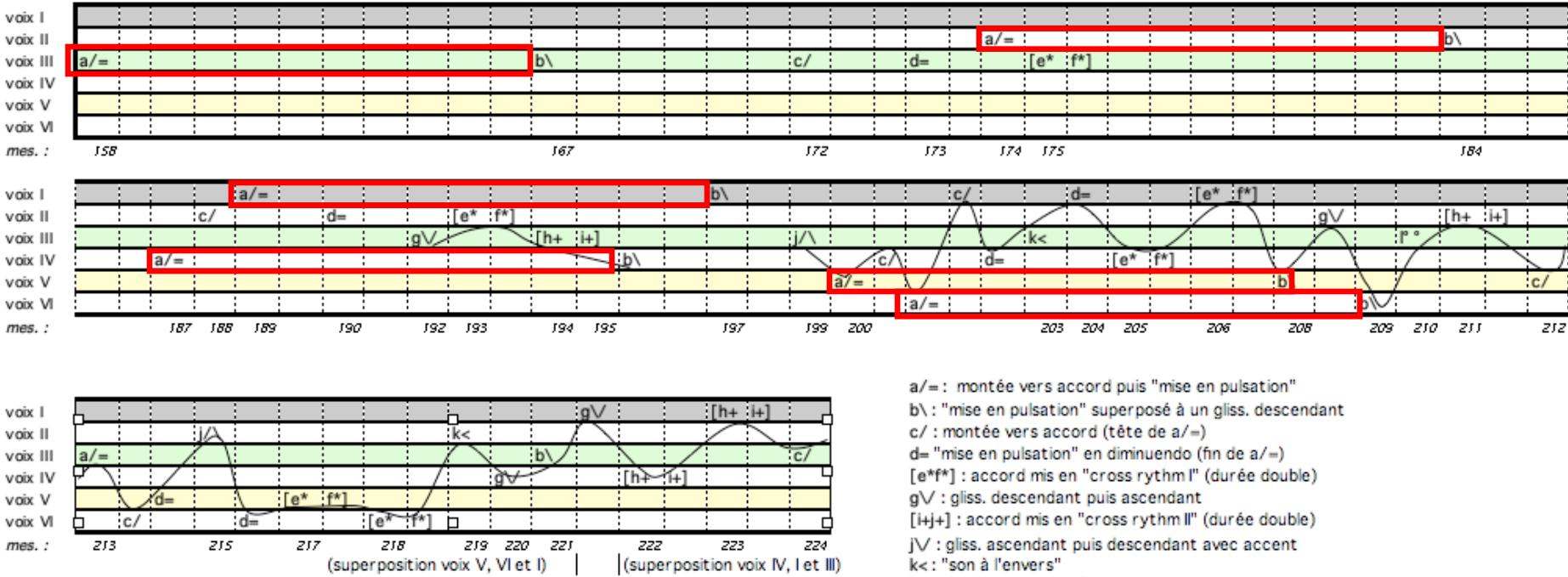
(opéra de chambre pour acteur, mezzo-soprano, baryton, ensemble et électronique)

Fabien Lévy

Morphological Tiling Canons



- *Coïncidences* (pour 33 musiciens, 1999-2007)



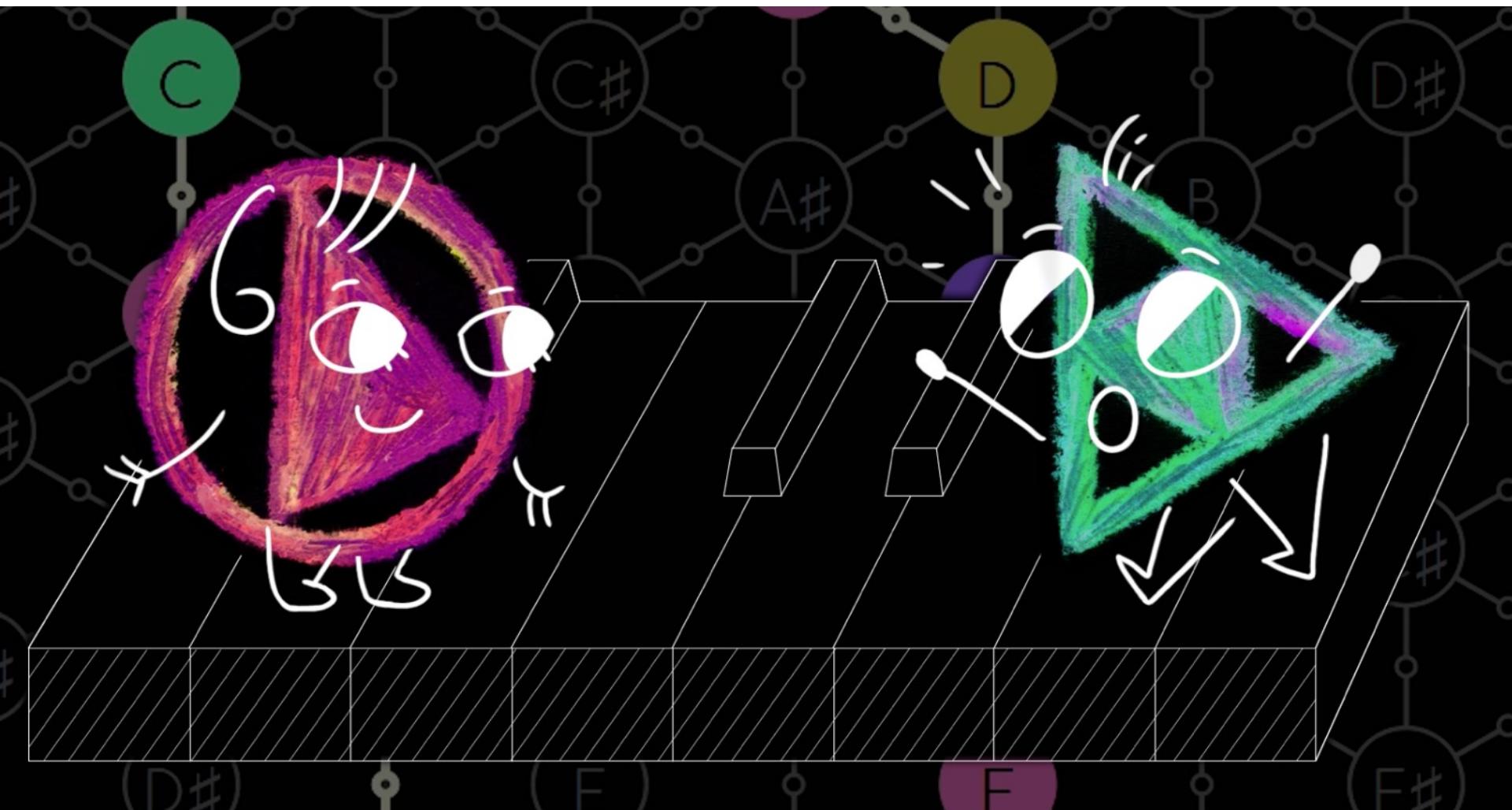
Coïncidences - Fabien Levy : déroulement du canon (mes. 158 à 226)
 (chaque impact fait 3 temps)



Tokyo Symphony Orchestra, Dir.: Kazuyoshi Akiyama, 05/09/2007, Suntory Hall, Tokyo, Japon

Additional slides

« Musique et mathématiques »: a pedagogical film

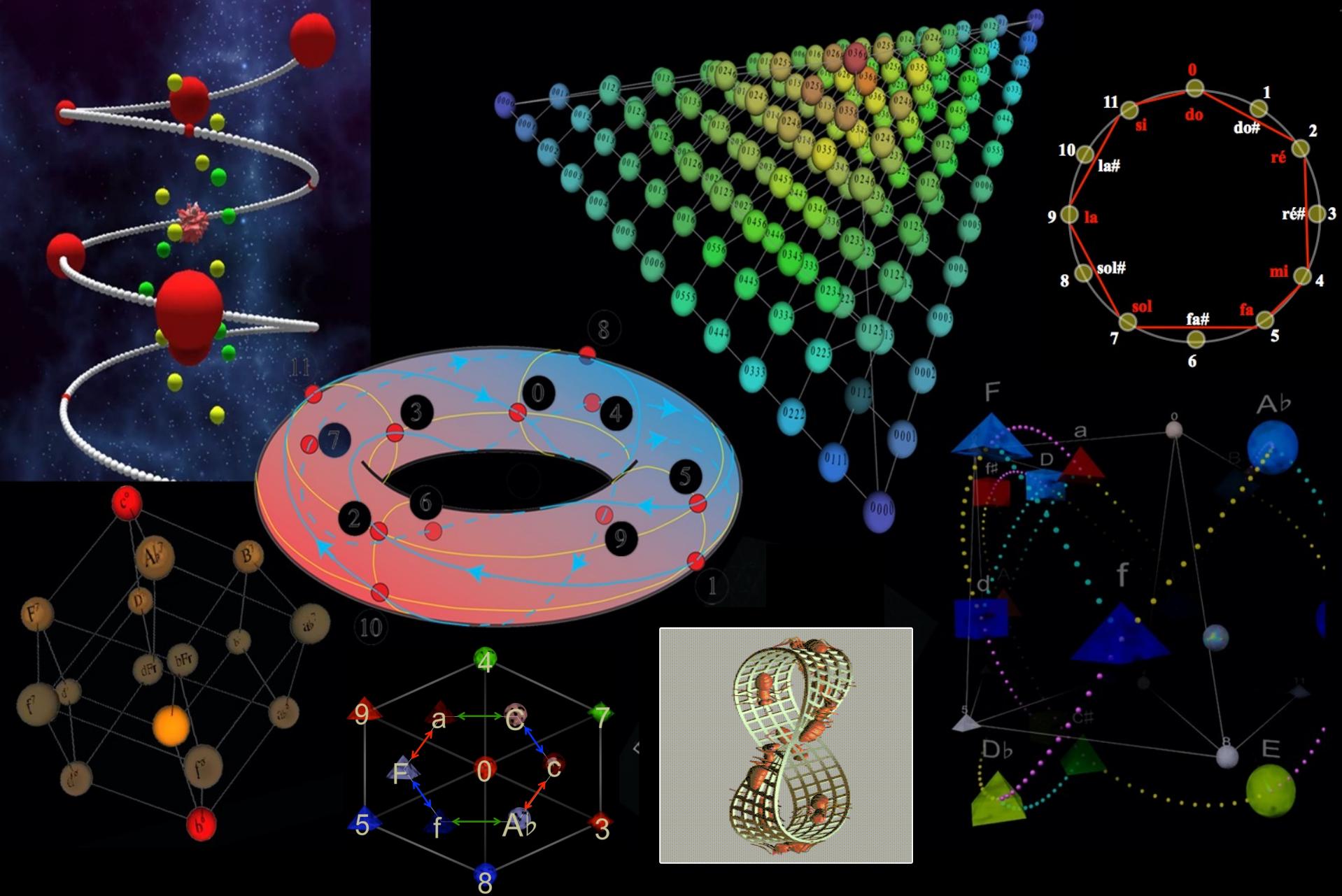


→ www.morenoandreatta.com

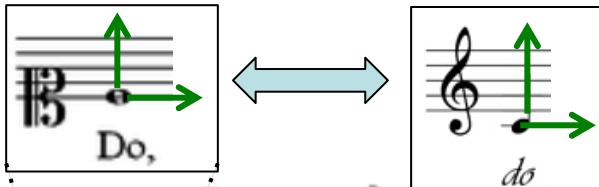


AuDiMATH
AUTOUR DE LA DIFFUSION
DES MATHÉMATIQUES

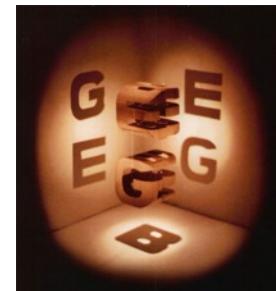
The galaxy of geometrical models at the service of music



Bach's enigmatic canons and geometry



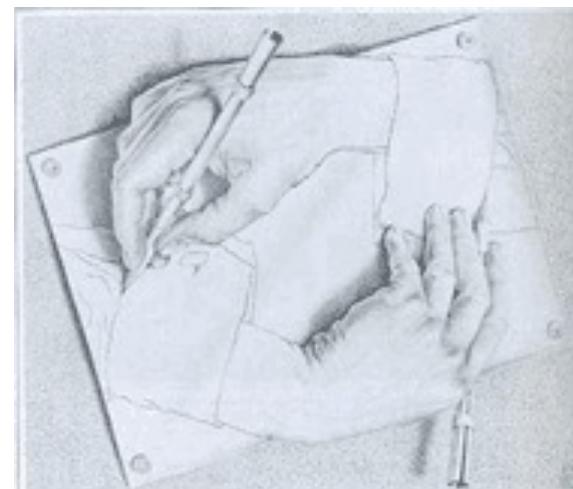
Canones diversi
super thema regium



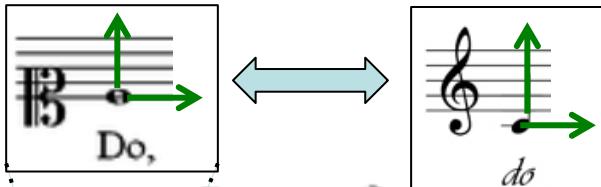
Canon a 2

1.

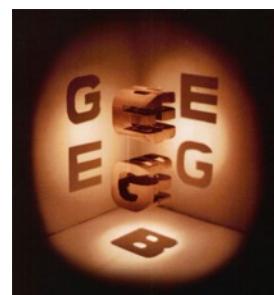
Musical score for 'Canon a 2'. The score consists of two staves. The top staff is in common time, has a key signature of one flat (B-flat), and starts with a bass clef (B). The bottom staff is also in common time and has a key signature of one flat (B-flat), starting with a bass clef (B). The music features various note heads and stems, with some notes connected by beams. A blue arrow points from the bass clef on the first staff to the bass clef on the second staff, indicating a transposition or a canon entry. Another blue arrow points from the end of the second staff back to the bass clef on the first staff, suggesting a return or continuation of the theme.



Bach's enigmatic canons and geometry



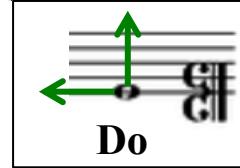
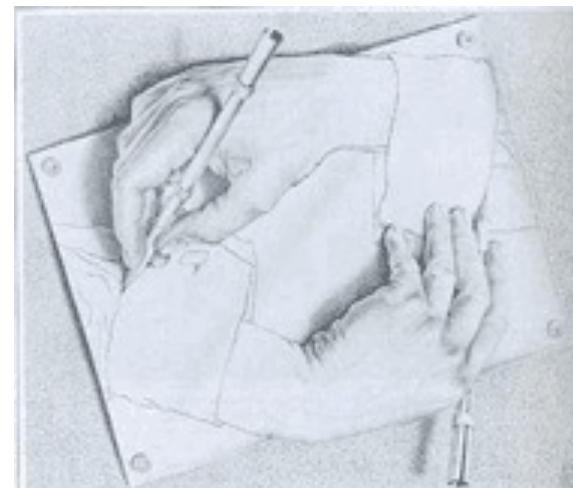
Canones diversi
super thema regium

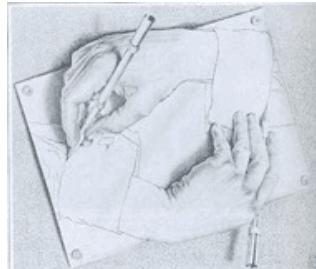


Canon a 2

1.

The musical score consists of two staves. The first staff begins with a bass clef (B) and a 'c' (common time). The second staff begins with a treble clef (G) and an 'e' (common time). A blue arrow points from the bass clef in the first staff to the treble clef in the second staff. The music is composed of eighth and sixteenth note patterns.





My end is my beginning (but twisted!)

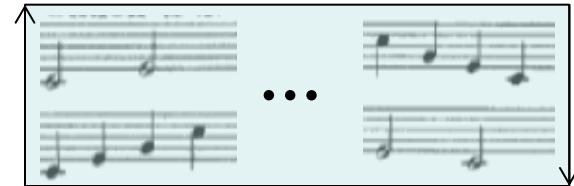
Canones diversi

super thema regium

Canon a 2

4.

B-flat major, common time

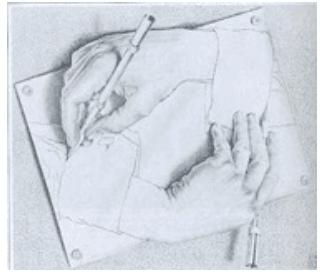


Canones diversi
super thema regium

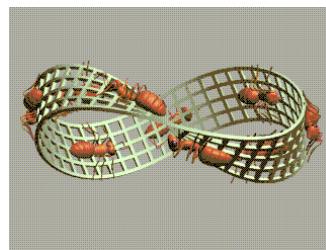
Canon a 2

4.

B-flat major, common time



My end is my beginning (but twisted!)



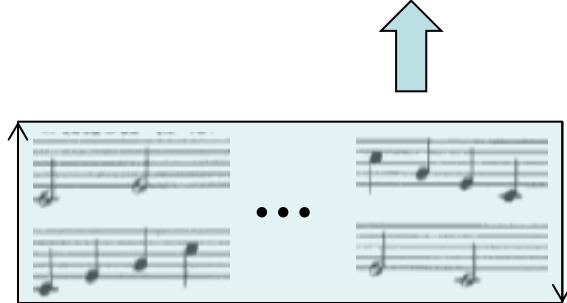
Canones diversi

super thema regium

4.

Canon a 2

Musical score for Canon a 2, showing three staves of music in E-flat major, 2/4 time. The score consists of three staves of music, each with a different rhythm pattern. The first staff starts with a quarter note, the second with an eighth note, and the third with a sixteenth note. The music is composed of eighth notes throughout.



Canones diversi
super thema regium

4.

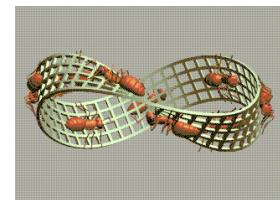
Canon a 2

Musical score for Canon a 2, showing three staves of music in E-flat major, 2/4 time. The score consists of three staves of music, each with a different rhythm pattern. The first staff starts with a quarter note, the second with an eighth note, and the third with a sixteenth note. The music is composed of eighth notes throughout.



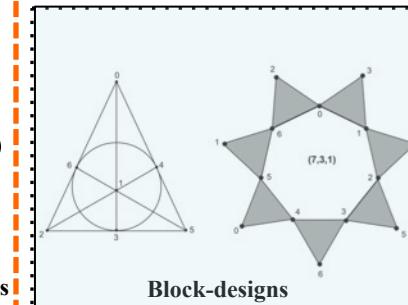
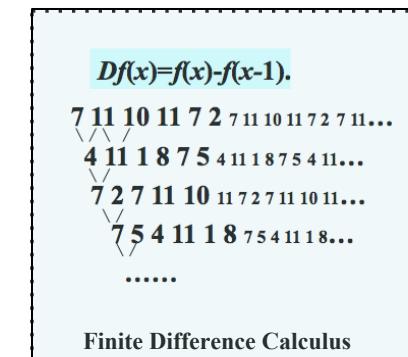
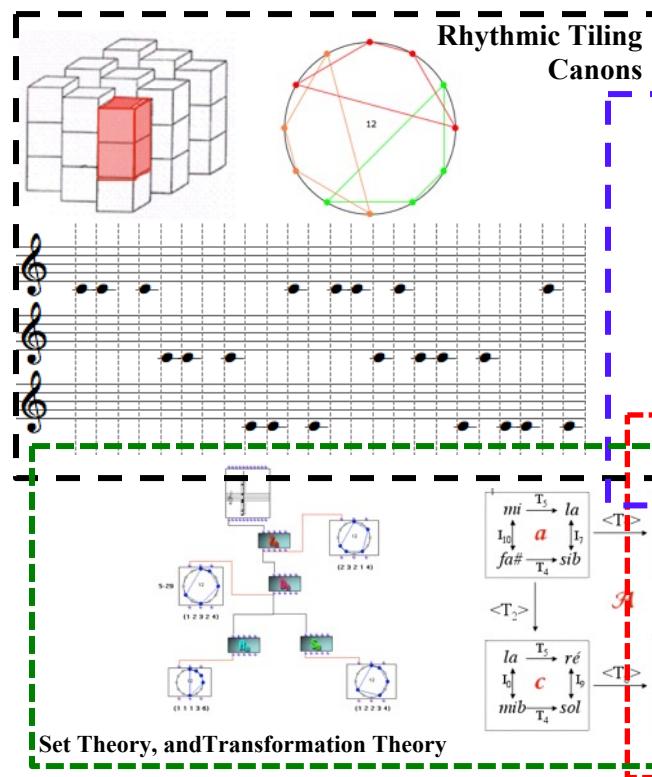
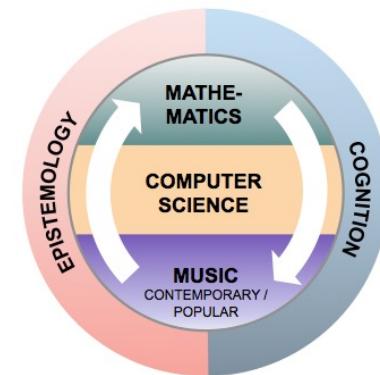
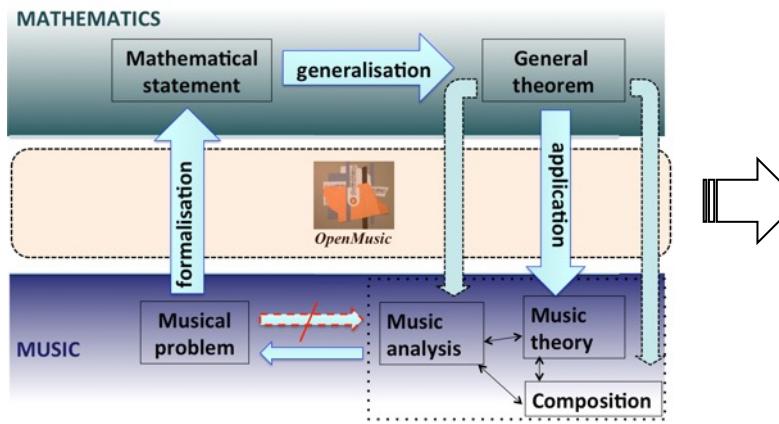
<http://www.josleys.com/Canon/Canon.html>

[min. 1'14"]



Some musically-driven mathematical problems

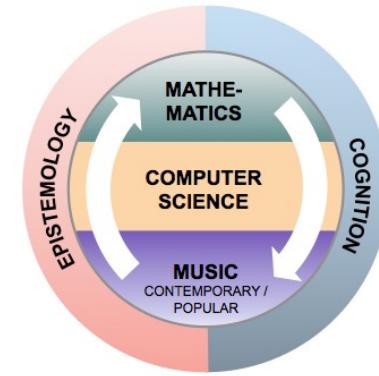
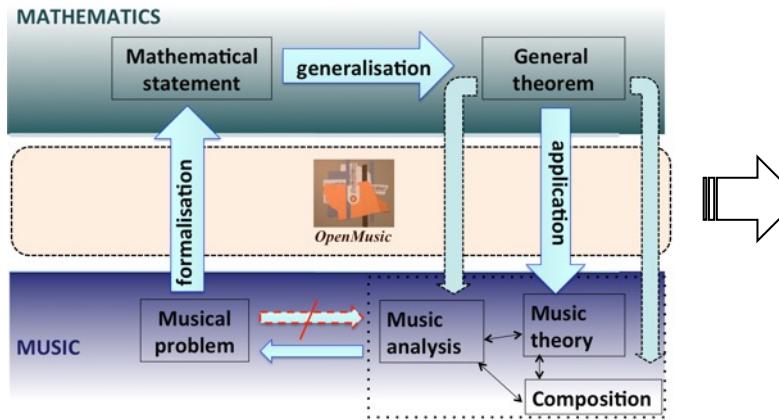
- Tiling Rhythmic Canons
- Z relation and homometry
- Transformational Theory
- Music Analysis, SC and FCA
- Diatonic Theory and ME-Sets
- Periodic sequences and FDC
- Block-designs in composition



Some musically-driven mathematical problems

M. Andreatta : *Mathematica est exercitium musicae*, Habilitation Thesis, IRMA University of Strasbourg, 2010

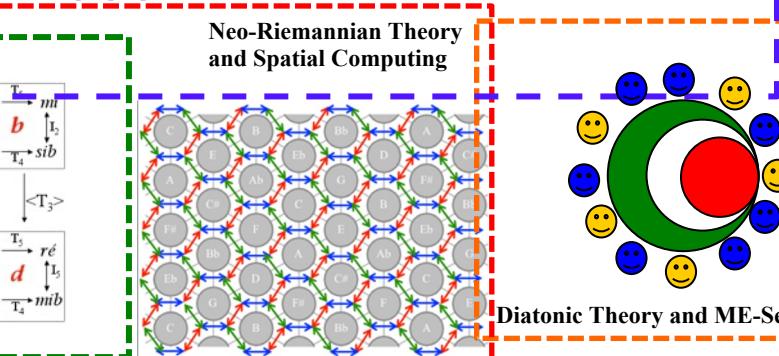
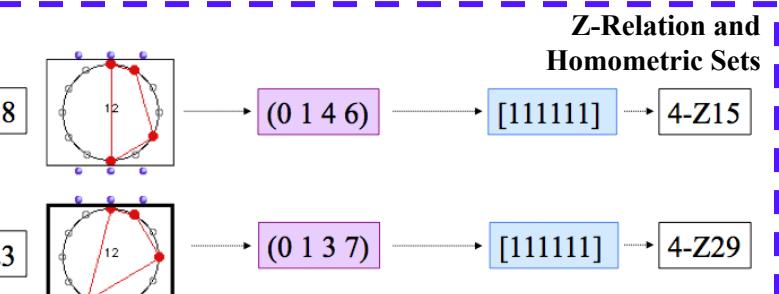
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Rhythmic Tiling Canons

Set Theory, and Transformation Theory

Set Theory, and Transformation Theory



Finite Difference Calculus

$$Df(x) = f(x) - f(x-1).$$

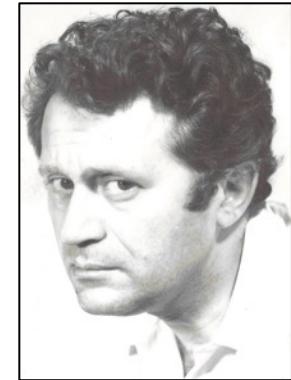
7 11 10 11 7 2 7 11 10 11 7 2 7 11...
 4 11 1 8 7 5 4 11 18 7 5 4 11...
 7 2 7 11 10 11 7 2 7 11 10 11...
 7 5 4 11 1 8 7 5 4 11 18...

UNIVERSITÀ DEGLI STUDI DI PADOVA
 Dipartimento di Matematica
 Corso di Laurea Triennale in Matematica
 Tesi di Laurea
 ON SOME ALGEBRAIC ASPECTS OF ANATOL VIERU PERIODIC SEQUENCES
 APPLIED TO MUSIC CANONS
 Relatore: Dott.ssa LUISA FIOROT MCCXXI
 Lavoratore: NICOLA ANCIELLOTTI
 Matricola: 062907
 ANNO ACCADEMICO 2014-2015
 4 dicembre 2015

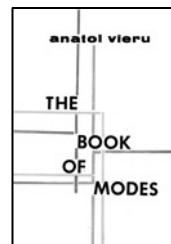
Periodic sequences and finite difference calculus

$$Df(x) = f(x) - f(x-1)$$

$$\begin{aligned} f &= 7 \ 11 \ 10 \ 11 \ 7 \ 2 \ 7 \ 11 \ 10 \ 11 \ 7 \ 2 \ 7 \ 11 \dots \\ Df &= 4 \ 11 \ 1 \ 8 \ 7 \ 5 \ 4 \ 11 \ 1 \ 8 \ 7 \ 5 \ 4 \ 11 \dots \\ D^2 f &= 7 \ 2 \ 7 \ 11 \ 10 \ 11 \ 7 \ 2 \ 7 \ 11 \ 10 \ 11 \dots \\ D^3 f &= 7 \ 5 \ 4 \ 11 \ 1 \ 8 \ 7 \ 5 \ 4 \ 11 \ 1 \ 8 \dots \\ D^k f &= \dots \dots \end{aligned}$$



Anatol Vieru



dolcissimo

mf *mp* *pp* *pt* *mp* *p* *mf* *mp* *pp* *pt* *pp*

V	0	3	8	7	11	0	11	10	6	9	0	9	1	2	9	8	4	3	6
VIII	0	0	0	0	3	3	7	2	0	0	0	6	3	3	3	4	8	0	0
IV	3	3	4	4	1	11	11	8	3	3	9	4	1	7	11	8	11	3	9
IX	0	0	0	0	0	3	6	[1]	3	3	3	3	9	0	3	6	[10]	6	6
IV	0	10	3	9	10	0	9	7	0	6	7	9	6	4	9	3	4	6	3

Anatol Vieru: *Zone d'oubli* pour alto (1973)

Reducible and reproducible sequences

$$\begin{aligned} f &= 11 \begin{smallmatrix} 6 \\ \backslash \diagup \\ 7 \end{smallmatrix} 7 \ 2 \ 3 \ 10 \ 11 \ 6 \dots \\ Df &= 7 \begin{smallmatrix} 1 \\ \backslash \diagup \\ 7 \end{smallmatrix} 1 \ 7 \ 1 \ 7 \ 1 \ 7 \ 1 \dots \\ D^2f &= 6 \begin{smallmatrix} 6 \\ \backslash \diagup \\ 6 \end{smallmatrix} 6 \ 6 \ 6 \dots \\ D^4f &= 0 \ 0 \ 0 \end{aligned}$$

Reducible sequences:
 $\exists k \geq 1$ such that
 $D^k f = 0$

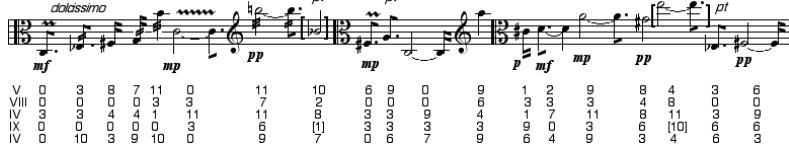
$$\begin{aligned} f &= 7 \begin{smallmatrix} 11 \\ \backslash \diagup \\ 10 \end{smallmatrix} 11 \ 7 \ 2 \ 7 \ 11 \dots \\ Df &= 4 \begin{smallmatrix} 11 \\ \backslash \diagup \\ 1 \end{smallmatrix} 8 \ 7 \ 5 \ 4 \ 11 \ 1 \dots \\ D^2f &= 7 \begin{smallmatrix} 2 \\ \backslash \diagup \\ 7 \end{smallmatrix} 11 \ 10 \ 11 \ 7 \ 2 \ 7 \dots \\ D^3f &= 7 \begin{smallmatrix} 5 \\ \backslash \diagup \\ 4 \end{smallmatrix} 11 \ 1 \ 8 \ 7 \ 5 \ 4 \ 11 \ 1 \ 8 \dots \\ D^4f &= 10 \ 11 \ 7 \ 2 \ 7 \ 11 \ 10 \ 11 \dots \\ D^5f &= 1 \ 8 \ 7 \ 5 \ 4 \ 11 \ 1 \ 8 \dots \\ D^6f &= 7 \ 11 \ 10 \ 11 \ 7 \ 2 \ 7 \ 11 \dots \end{aligned}$$

Reproducible sequences:
 $\exists k \geq 1$ such that
 $D^k f = f$

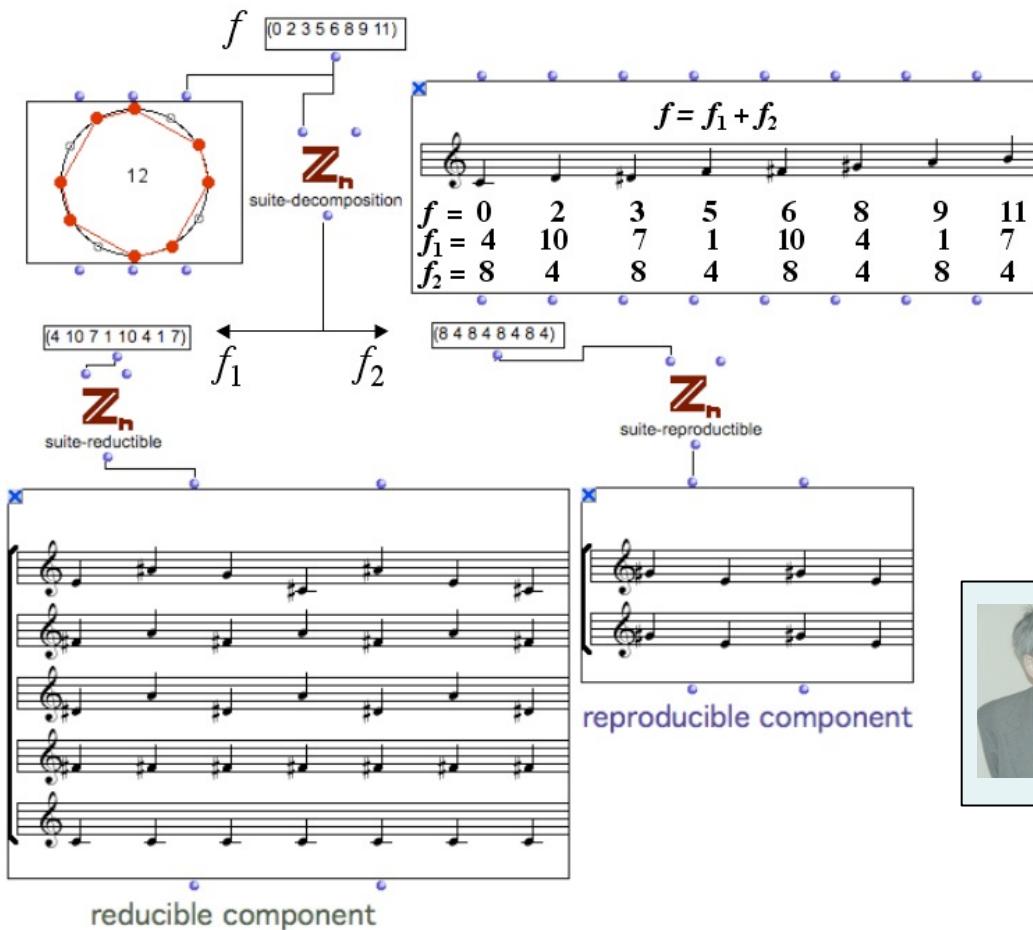
A decomposition property of any periodic sequence

$$Df(x) = f(x) - f(x-1).$$

7 11 10 11 7 2 7 11 10 11 7 2 7 11...
 4 11 1 8 7 5 4 11 1 8 7 5 4 11...
 7 2 7 11 10 11 7 2 7 11 10 11...
 7 5 4 11 1 8 7 5 4 11 1 8...



Anatol Vieru: *Zone d'oubli* for viola (1973)



Reducible sequences:
 $\exists k \geq 1$ such that $D^k f = 0$

Reproducible sequences:
 $\exists k \geq 1$ such that $D^k f = f$

• Decomposition theorem
 (Vuza & Andreatta, *Tatra M.*, 2001)
 Every periodic sequence f
 can be decomposed in a
 unique way as a sum $f_1 + f_2$
 of a reducible sequence f_1 and
 a reproducible sequence f_2

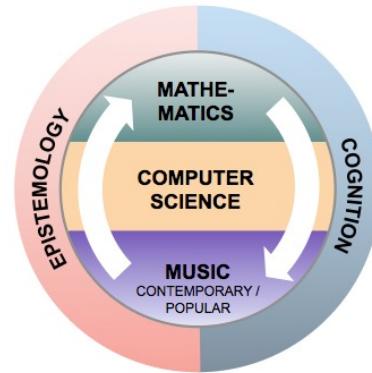
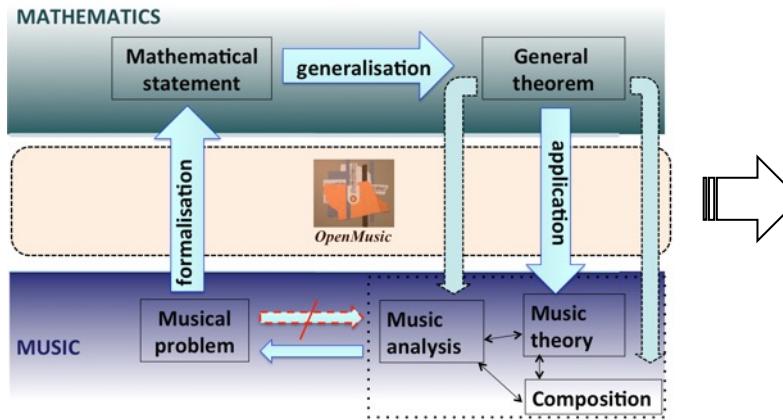


D. Vuza, M. Andreatta (2001), « On some properties of periodic sequences in Anatol Vieru's modal theory », Tatra Mountains Mathematical Publications, Vol. 23, p. 1-15

Some musically-driven mathematical problems

M. Andreatta : *Mathematica est exercitium musicae*, Habilitation Thesis, IRMA University of Strasbourg, 2010

- Tiling Rhythmic Canons
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Rhythmic Tiling Canons

Diagram illustrating rhythmic tiling canons, showing a 3D stack of cubes and a circle divided into 12 segments.

Z-Relation and Homometric Sets

Diagram showing two examples of Z-relation and homometric sets. For number 18, a circle of 12 points leads to the set (0 1 4 6) which corresponds to the sequence [111111] and the transformation 4-Z15. For number 23, it leads to (0 1 3 7) corresponding to [111111] and 4-Z29.

Set Theory, and Transformation Theory

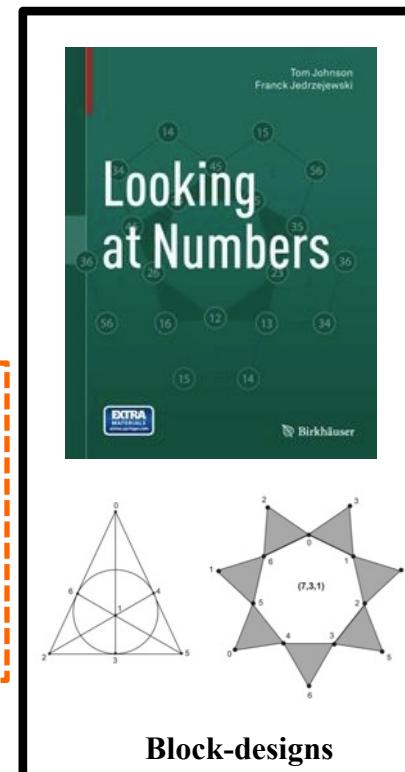
Diagram showing musical transformations between notes like mi, la, etc., with labels a, b, c, d.

Neo-Riemannian Theory and Spatial Computing

Diagram showing a complex network of nodes (A, B, C, etc.) connected by arrows, representing spatial computing in Neo-Riemannian theory.

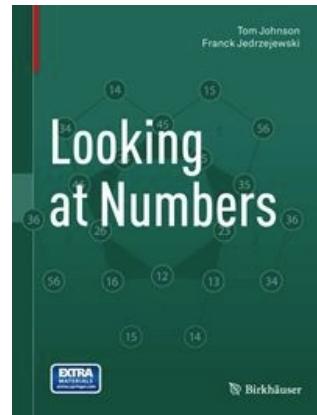
Diatonic Theory and ME-Sets

Diagram showing a diatonic circle with various notes and a central red circle surrounded by smiley faces.

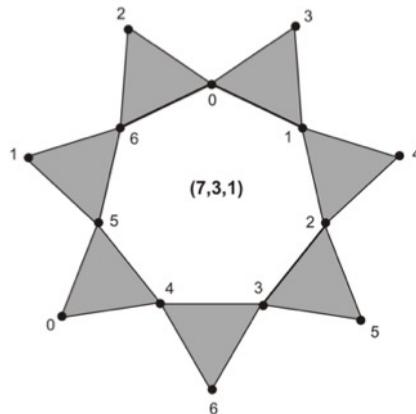
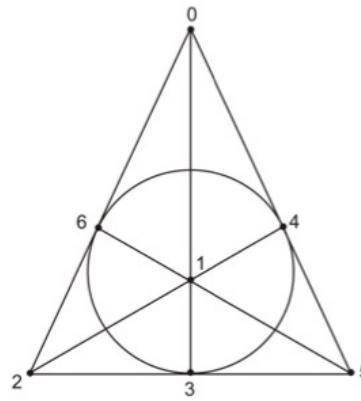


Block-designs

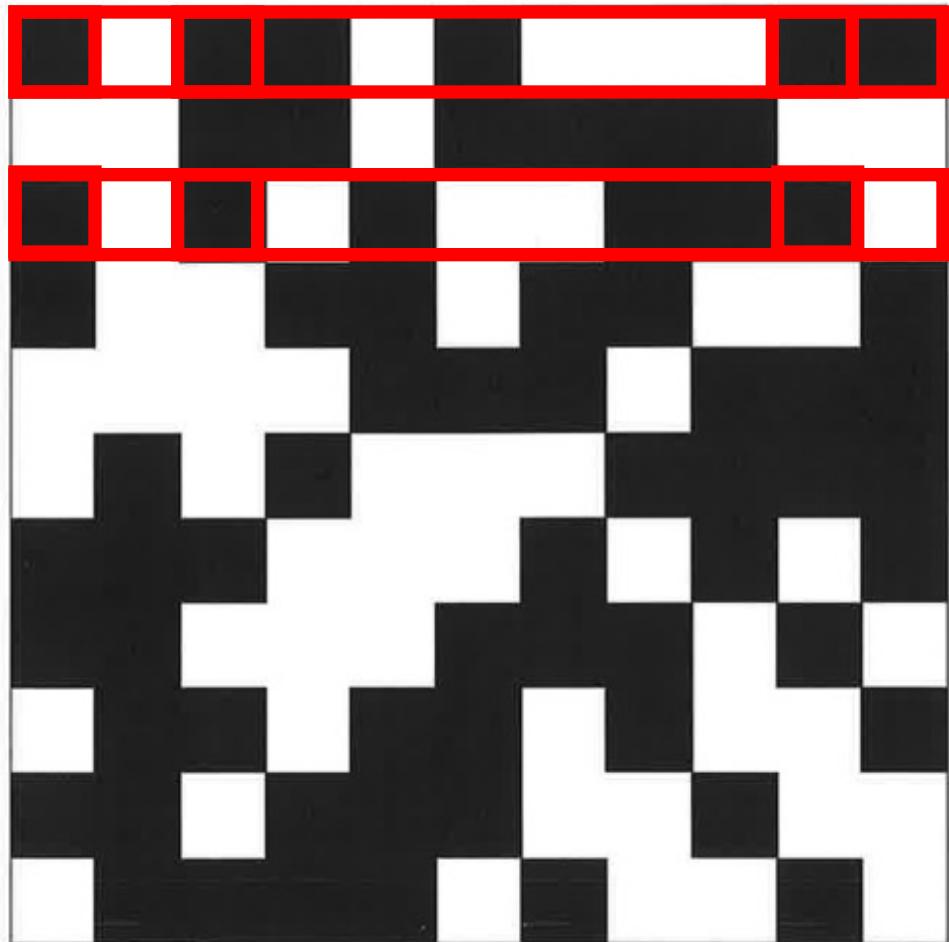
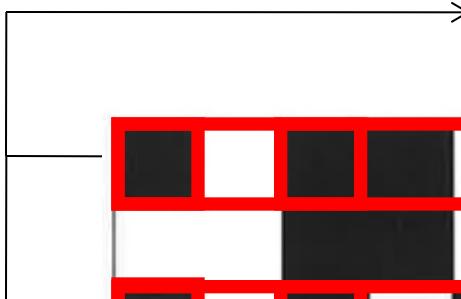
Block-Designs in Johnson's music



Tom Johnson



Fano plane = (7,3,1)



Block-design (11,6,3)



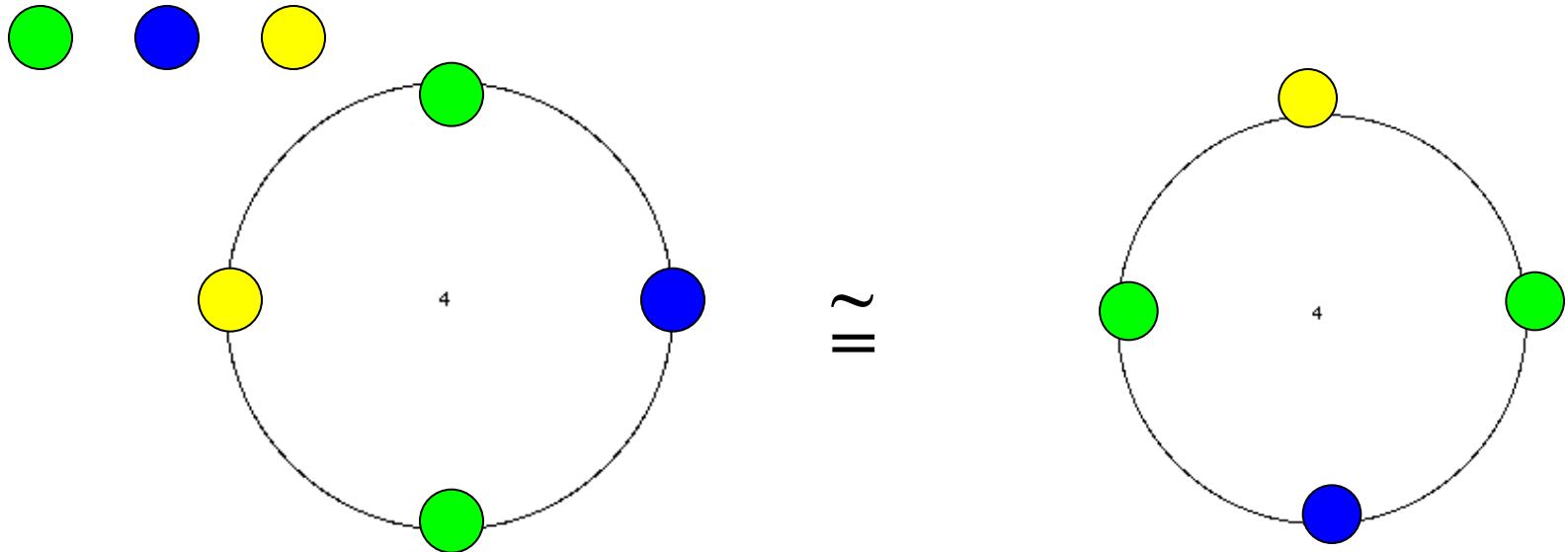
➔ Jedrzejewski, F., M. Andreatta, T. Johnson (2009), « Musical experiences with Block Designs », *Proceedings of the Conference MCM 2009*, vol. 38, New Haven, p. 154-165

Enumeration of chord classes (modulo a group action)

Burnside Lemma

$$n = \frac{1}{|G|} \sum_{g \in G} |X^g|.$$

$$X^g = \{x \in X : gx = x\}$$



?



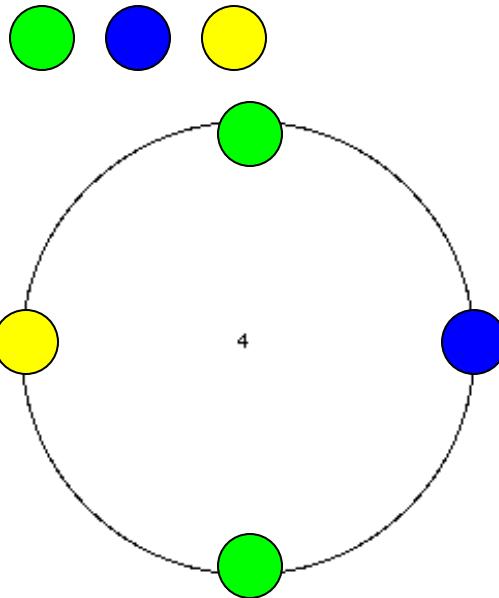
How many possible configurations could you find?

Enumeration of chord classes (modulo a group action)

Burnside Lemma

$$n = \frac{1}{|G|} \sum_{g \in G} |X^g|.$$

$$X^g = \{x \in X : gx = x\}$$



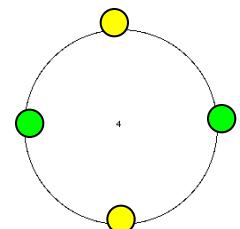
Possible configurations = $3^4 = 81$

T_0 fixes all configurations $\Rightarrow |X^{T_0}| = 81$

T_1 fixes all monochromatic configurations $\Rightarrow |X^{T_1}| = 3$

T_3 idem

T_2 fixes all «double-diameter» configurations $\Rightarrow |X^{T_2}| = 3^2 = 9$



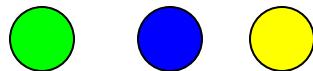
$$\rightarrow n = 1/4 (81 + 3 + 3 + 9) = 24$$

Enumeration of chord classes (modulo a group action)

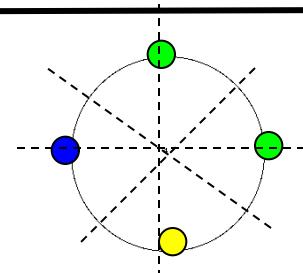
Burnside Lemma

$$n = \frac{1}{|G|} \sum_{g \in G} |X^g|.$$

$$X^g = \{x \in X : gx = x\}$$



Action de \mathbf{Z}_4



Transformation	Action	Cycle representation	No. of cycles	Fixed configs.	Cycle type	Cycle index
T_0	$0 \rightarrow 0, 1 \rightarrow 1, 2 \rightarrow 2, 3 \rightarrow 3$	$(0)(1)(2)(3)$	4	$3^4 = 81$	1^4	t_1^4
T_1	$0 \rightarrow 1 \rightarrow 2 \rightarrow 3 \rightarrow 0$	$(0 \ 1 \ 2 \ 3)$	1	$3^1 = 3$	4^1	t_4^1
T_2	$0 \rightarrow 2 \rightarrow 0, 1 \rightarrow 3 \rightarrow 1$	$(0 \ 2)(1 \ 3)$	2	$3^2 = 9$	2^2	t_2^2
T_3	$0 \rightarrow 3 \rightarrow 2 \rightarrow 1 \rightarrow 0$	$(0 \ 3 \ 2 \ 1)$	1	$3^1 = 3$	4^1	t_4^1

Julian Hook, « Why are there 29 Tetrachords? A Tutorial on Combinatorics and Enumeration in Music Theory », MTO, 13(4), 2007

$$n = 1/4 (81+3+3+9) = 24$$

Enumeration of chord classes (modulo a group action)

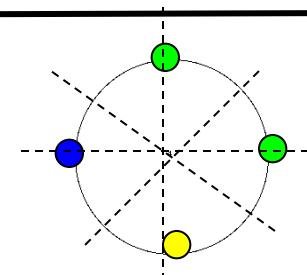
Burnside Lemma

$$n = \frac{1}{|G|} \sum_{g \in G} |X^g|.$$

$$X^g = \{x \in X : gx = x\}$$



Action of D_4



<i>Transformation</i>	<i>Action</i>	<i>Cycle representation</i>	<i>No. of cycles</i>	<i>Fixed configs.</i>	<i>Cycle type</i>	<i>Cycle index</i>
T_0	$0 \rightarrow 0, 1 \rightarrow 1, 2 \rightarrow 2, 3 \rightarrow 3$	$(0)(1)(2)(3)$	4	$3^4 = 81$	1^4	t_1^4
T_1	$0 \rightarrow 1 \rightarrow 2 \rightarrow 3 \rightarrow 0$	$(0 \ 1 \ 2 \ 3)$	1	$3^1 = 3$	4^1	t_4^1
T_2	$0 \rightarrow 2 \rightarrow 0, 1 \rightarrow 3 \rightarrow 1$	$(0 \ 2)(1 \ 3)$	2	$3^2 = 9$	2^2	t_2^2
T_3	$0 \rightarrow 3 \rightarrow 2 \rightarrow 1 \rightarrow 0$	$(0 \ 3 \ 2 \ 1)$	1	$3^1 = 3$	4^1	t_4^1
I	$0 \rightarrow 0, 1 \rightarrow 3 \rightarrow 1, 2 \rightarrow 2$	$(0)(1 \ 3)(2)$	3	$3^3 = 27$	$1^2 2^1$	$t_1^2 t_2^1$
$T_1 I$	$0 \rightarrow 1 \rightarrow 0, 2 \rightarrow 3 \rightarrow 2$	$(0 \ 1)(2 \ 3)$	2	$3^2 = 9$	2^2	t_2^2
$T_2 I$	$0 \rightarrow 2 \rightarrow 0, 1 \rightarrow 1, 3 \rightarrow 3$	$(0 \ 2)(1)(3)$	3	$3^3 = 27$	$1^2 2^1$	$t_1^2 t_2^1$
$T_3 I$	$0 \rightarrow 3 \rightarrow 0, 1 \rightarrow 2 \rightarrow 1$	$(0 \ 3)(1 \ 2)$	2	$3^2 = 9$	2^2	t_2^2

Julian Hook, « Why are there 29 Tetrachords? A Tutorial on Combinatorics and Enumeration in Music Theory », MTO, 13(4), 2007

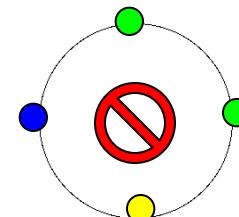
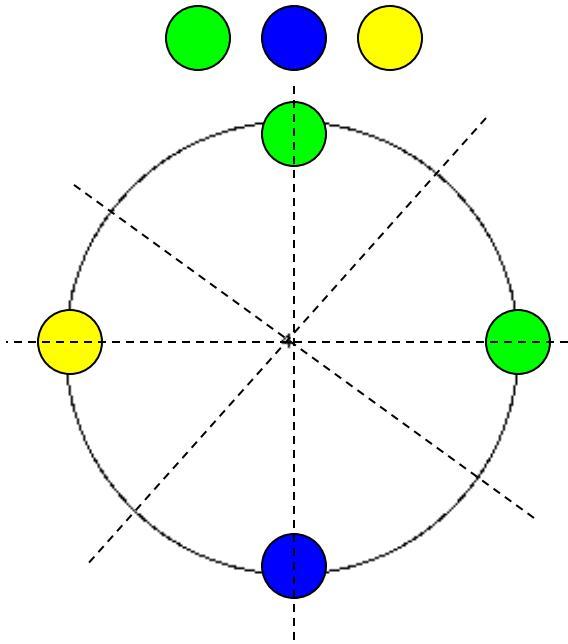
$$n = 1/8 (81+3+3+9+27+9+27+9) = 168/8=21$$

Enumeration of chord classes (modulo a group action)

Burnside Lemma

$$n = \frac{1}{|G|} \sum_{g \in G} |X^g|.$$

$$X^g = \{x \in X : gx = x\}$$



Action of \mathbf{D}_4

$$T_0 = \text{id}$$

$$T_1 = \text{rot } 90^\circ$$

$$T_2 = \text{rot } 180^\circ$$

$$T_3 = \text{rot } 270^\circ$$

$T_0 I = \text{inversion}$

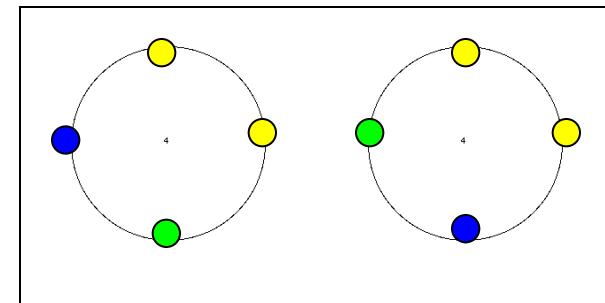
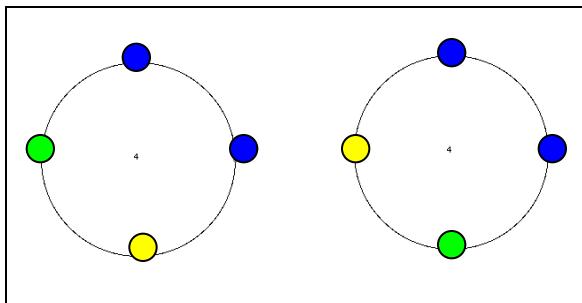
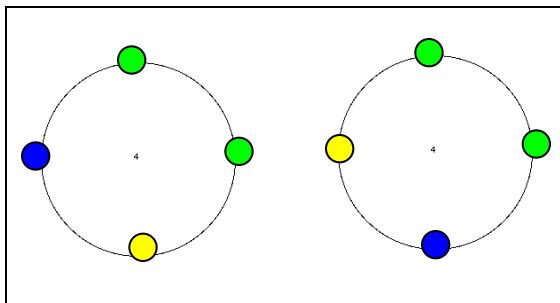
$T_1 I = \text{inv.}$

$T_2 I = \text{inv.}$

$T_3 I = \text{inv.}$



$$21=24-3$$



Enumeration of transposition chord classes



<i>Transformation</i>	<i>Cycle representation</i>	<i>No. of cycles</i>	<i>Fixed configs.</i>	<i>Cycle type</i>	<i>Cycle index</i>
T_0	(0)(1)(2)(3)(4)(5)(6)(7)(8)(9)(A)(B)	12	$2^{12} = 4096$	1^{12}	t_1^{12}
T_1	(0 1 2 3 4 5 6 7 8 9 A B)	1	$2^1 = 2$	12^1	t_{12}^{-1}
T_2	(0 2 4 6 8 A)(1 3 5 7 9 B)	2	$2^2 = 4$	6^2	t_6^2
T_3	(0 3 6 9)(1 4 7 A)(2 5 8 B)	3	$2^3 = 8$	4^3	t_4^3
T_4	(0 4 8)(1 5 9)(2 6 A)(3 7 B)	4	$2^4 = 16$	3^4	t_3^4
T_5	(0 5 A 3 8 1 6 B 4 9 2 7)	1	$2^1 = 2$	12^1	t_{12}^{-1}
T_6	(0 6)(1 7)(2 8)(3 9)(4 A)(5 B)	6	$2^6 = 64$	2^6	t_2^6
T_7	(0 7 2 9 4 B 6 1 8 3 A 5)	1	$2^1 = 2$	12^1	t_{12}^{-1}
T_8	(0 8 4)(1 9 5)(2 A 6)(3 B 7)	4	$2^4 = 16$	3^4	t_3^4
T_9	(0 9 6 3)(1 A 7 4)(2 B 8 5)	3	$2^3 = 8$	4^3	t_4^3
T_{10}	(0 A 8 6 4 2)(1 B 9 7 5 3)	2	$2^2 = 4$	6^2	t_6^2
T_{11}	(0 B A 9 8 7 6 5 4 3 2 1)	1	$2^1 = 2$	12^1	t_{12}^{-1}

Burnside Lemma

$$n = \frac{1}{|G|} \sum_{g \in G} |X^g|.$$

$$X^g = \{x \in X : gx = x\}$$

Action of \mathbf{Z}_{12}

(Hook, MTO)



chords = $1/12[4096+2+4+8+16+2+64+2+16+8+4+2]=4224/12=352$

Enumeration of pitch-class sets



<i>Transformation</i>	<i>Cycle representation</i>	<i>No. of cycles</i>	<i>Fixed configs.</i>	<i>Cycle type</i>	<i>Cycle index</i>
T_0	(0)(1)(2)(3)(4)(5)(6)(7)(8)(9)(A)(B)	12	$2^{12} = 4096$	1^{12}	t_1^{12}
T_1	(0 1 2 3 4 5 6 7 8 9 A B)	1	$2^1 = 2$	12^1	t_{12}^{-1}
T_2	(0 2 4 6 8 A)(1 3 5 7 9 B)	2	$2^2 = 4$	6^2	t_6^2
T_3	(0 3 6 9)(1 4 7 A)(2 5 8 B)	3	$2^3 = 8$	4^3	t_4^3
T_4	(0 4 8)(1 5 9)(2 6 A)(3 7 B)	4	$2^4 = 16$	3^4	t_3^4
T_5	(0 5 A 3 8 1 6 B 4 9 2 7)	1	$2^1 = 2$	12^1	t_{12}^{-1}
T_6	(0 6)(1 7)(2 8)(3 9)(4 A)(5 B)	6	$2^6 = 64$	2^6	t_2^6
T_7	(0 7 2 9 4 B 6 1 8 3 A 5)	1	$2^1 = 2$	12^1	t_{12}^{-1}
T_8	(0 8 4)(1 9 5)(2 A 6)(3 B 7)	4	$2^4 = 16$	3^4	t_3^4
T_9	(0 9 6 3)(1 A 7 4)(2 B 8 5)	3	$2^3 = 8$	4^3	t_4^3
T_{10}	(0 A 8 6 4 2)(1 B 9 7 5 3)	2	$2^2 = 4$	6^2	t_6^2
T_{11}	(0 B A 9 8 7 6 5 4 3 2 1)	1	$2^1 = 2$	12^1	t_{12}^{-1}
I	(0)(1 B)(2 A)(3 9)(4 8)(5 7)(6)	7	$2^7 = 128$	$1^2 2^5$	$t_1^2 t_2^5$
$T_1 I$	(0 1)(2 B)(3 A)(4 9)(5 8)(6 7)	6	$2^6 = 64$	2^6	t_2^6
$T_2 I$	(0 2)(1)(3 B)(4 A)(5 9)(6 8)(7)	7	$2^7 = 128$	$1^2 2^5$	$t_1^2 t_2^5$
$T_3 I$	(0 3)(1 2)(4 B)(5 A)(6 9)(7 8)	6	$2^6 = 64$	2^6	t_2^6
$T_4 I$	(0 4)(1 3)(2)(5 B)(6 A)(7 9)(8)	7	$2^7 = 128$	$1^2 2^5$	$t_1^2 t_2^5$
$T_5 I$	(0 5)(1 4)(2 3)(6 B)(7 A)(8 9)	6	$2^6 = 64$	2^6	t_2^6
$T_6 I$	(0 6)(1 5)(2 4)(3)(7 B)(8 A)(9)	7	$2^7 = 128$	$1^2 2^5$	$t_1^2 t_2^5$
$T_7 I$	(0 7)(1 6)(2 5)(3 4)(8 B)(9 A)	6	$2^6 = 64$	2^6	t_2^6
$T_8 I$	(0 8)(1 7)(2 6)(3 5)(4)(9 B)(A)	7	$2^7 = 128$	$1^2 2^5$	$t_1^2 t_2^5$
$T_9 I$	(0 9)(1 8)(2 7)(3 6)(4 5)(A B)	6	$2^6 = 64$	2^6	t_2^6
$T_{10} I$	(0 A)(1 9)(2 8)(3 7)(4 6)(5)(B)	7	$2^7 = 128$	$1^2 2^5$	$t_1^2 t_2^5$
$T_{11} I$	(0 B)(1 A)(2 9)(3 8)(4 7)(5 6)	6	$2^6 = 64$	2^6	t_2^6

Burnside Lemma

$$n = \frac{1}{|G|} \sum_{g \in G} |X^g|.$$

$$X^g = \{x \in X : gx = x\}$$

Action of \mathbf{D}_{12}

(Hook, MTO)

chords = $1/12[4096+2+4+8+16+2+64+2+16+8+4+2] = 4224/12 = 352$

chords = $1/24[4224+1152] = 224$