Chapter 10

PIPES, RESONATORS, AND FILTERS

10.1 INTRODUCTION

The behavior of sound in a rigid-walled pipe depends strongly on the properties of the driver, the length of the pipe, the behavior of its cross section as a function of distance, the presence of any perforations of its wall, and the boundary conditions describing any termination. If the wavelength of the sound is sufficiently large, the wave motion can be considered to be well approximated by a collimated plane wave; this affords great simplification. Applications include measuring the absorptive and reflective properties of materials, predicting the behavior of wind instruments (brasses, woodwinds, organ pipes, etc.), and determining the design of ventilation ducts.

Segments of pipes having all dimensions sufficiently small compared to the relevant wavelengths can be considered as *lumped acoustic elements* whose behaviors resemble those of simple oscillators. These lumped elements find application as convenient models for more complicated systems at low frequencies, allowing straightforward design of the noise transmission characteristics of pipes, ducts, mufflers, and so forth, without materially affecting any required steady flow of fluid through the system.

10.2 RESONANCE IN PIPES

Assume that the fluid in a pipe of cross-sectional area S and length L is driven by a piston at $x = 0$ and that the pipe is terminated at $x = L$ in a mechanical impedance $\mathbf{Z}_{mL}$. If the piston vibrates at frequencies for which only plane waves propagate, the wave in the pipe will be of the form

$$\mathbf{p} = \mathbf{A}e^{j[\omega t + k(L-x)]} + \mathbf{B}e^{j[\omega t - k(L-x)]} \tag{10.2.1}$$

where $\mathbf{A}$ and $\mathbf{B}$ are determined by the boundary conditions at $x = 0$ and $x = L$.

The continuities of force and particle speed require that the mechanical impedance of the wave at $x = L$ equals the mechanical impedance $\mathbf{Z}_{mL}$ of the termination. Since the force of the fluid on the termination is $\mathbf{p}(L,t)S$ and the particle speed is $\mathbf{u}(L,t) = -(1/\rho_0)\int(\partial\mathbf{p}/\partial x)\,dt$,

$$\mathbf{Z}_{mL} = \rho_0 c S \frac{\mathbf{A}+\mathbf{B}}{\mathbf{A}-\mathbf{B}} \tag{10.2.2}$$

The input mechanical impedance $\mathbf{Z}_{m0}$ at $x = 0$ is

$$\mathbf{Z}_{m0} = \rho_0 c S \frac{\mathbf{A}e^{jkL}+\mathbf{B}e^{-jkL}}{\mathbf{A}e^{jkL}-\mathbf{B}e^{-jkL}} \tag{10.2.3}$$

Combining these equations to eliminate $\mathbf{A}$ and $\mathbf{B}$, we obtain

$$\frac{\mathbf{Z}_{m0}}{\rho_0 c S} = \frac{(\mathbf{Z}_{mL}/\rho_0 c S) + j\tan kL}{1 + j(\mathbf{Z}_{mL}/\rho_0 c S)\tan kL} \tag{10.2.4}$$

which is identical to (3.7.3) with the replacement of $\rho_L c$ with $\rho_0 c S$, the *characteristic mechanical impedance* of the fluid. The substitution

$$\mathbf{Z}_{mL}/\rho_0 c S = r + jx \tag{10.2.5}$$

leads directly to (3.7.5). Recalling the discussion following that equation, the frequencies of resonance and antiresonance are determined by the vanishing of the input mechanical reactance, which requires

$$x\tan^2 kL + (r^2 + x^2 - 1)\tan kL - x = 0 \tag{10.2.6}$$

The solution identified with *small* input resistance denotes *resonance*, and that identified with *large* input resistance denotes *antiresonance*. (In the limiting case $r = 0$, there is only one solution, corresponding to resonance.)

Let the pipe be driven at $x = 0$ and *closed* at $x = L$ by a rigid cap. To obtain the condition of resonance most simply, let $|\mathbf{Z}_{mL}/\rho_0 c S| \to \infty$ in (10.2.4). This yields

$$\mathbf{Z}_{m0}/\rho_0 c S = -j\cot kL \tag{10.2.7}$$

The reactance is zero and resonance occurs when $\cot kL = 0$,

$$k_n L = (2n-1)\pi/2 \qquad n = 1,2,3,\ldots \tag{10.2.8}$$

This is identical to (2.9.9) for the forced, fixed string. The resonance frequencies are the odd harmonics of the fundamental. The driven, closed pipe has a pressure antinode at $x = L$ and a pressure node at $x = 0$. Note that this requires that the driver presents a vanishing mechanical impedance to the pipe. The implication of this, and the effects of the mechanical properties of the driver on the behavior of the driver–pipe system, will be discussed in Section 10.6.

Now, consider a pipe driven at $x = 0$ and *open-ended* at $x = L$. On first examination, it might be thought that this will lead to $\mathbf{Z}_{mL} = 0$ for which

$\mathbf{Z}_{m0}/\rho_0 cS = j\tan kL$ with resonances occurring at $f_n = (n/2)c/L$ for $n = 1, 2, 3, \ldots$. However, this is *not* the case, most elementary physics textbooks notwithstanding. The condition at $x = L$ is not $\mathbf{Z}_{mL} = 0$ since the open end of the pipe radiates sound into the surrounding medium. The appropriate value for $\mathbf{Z}_{mL}$ is therefore

$$\mathbf{Z}_{mL} = \mathbf{Z}_r \tag{10.2.9}$$

where $\mathbf{Z}_r$ is the radiation impedance of the open end of the pipe.

For example, assume that the open end of a circular pipe of radius a is surrounded by a *flange* large with respect to the wavelength of the sound. Consistent with the assumption that the wavelength is large compared to the transverse dimensions of the pipe ($\lambda \gg a$), the opening resembles a baffled piston in the low-frequency limit. We have, then, from (7.5.11) *et seq.*

$$\mathbf{Z}_{mL}/\rho_0 cS = \tfrac{1}{2}(ka)^2 + j(8/3\pi)ka \qquad \text{(flanged)} \tag{10.2.10}$$

where both $r = (ka)^2/2$ and $x = 8ka/3\pi$ are much less than unity. Solution of (10.2.6) under these conditions gives $\tan kL = -x$ for the resonance frequencies. Since $x \ll 1$, this yields

$$\tan(n\pi - k_n L) = (8/3\pi)ka \approx \tan(8ka/3\pi) \qquad n = 1, 2, 3, \ldots \tag{10.2.11}$$

Therefore,

$$n\pi = k_n L + (8/3\pi)k_n a \tag{10.2.12}$$

and the resonance frequencies are

$$f_n = \frac{n}{2}\frac{c}{L + (8/3\pi)a} \tag{10.2.13}$$

These resonance frequencies are all harmonics of the fundamental, and the *effective length* L_{eff} of such a pipe is not L but rather $L + 8a/3\pi$. This predicted *end correction* is in reasonable agreement with measured values of around $0.85a$.

For an *unflanged* open pipe, both experiments and theory indicate that the radiation impedance is approximately

$$\mathbf{Z}_{mL}/\rho_0 cS = \tfrac{1}{4}(ka)^2 + j\,0.6\,ka \qquad \text{(unflanged)} \tag{10.2.14}$$

so the effective length of an unflanged open pipe is $L_{eff} = L + 0.6a$.

In both cases, the end corrections are independent of frequency. The resonance frequencies of flanged and unflanged open pipes are harmonics of the fundamental (as long as $\lambda_n \gg a$). This result has been obtained only for pipes of constant cross section. The presence of *flare* in the pipe, as found in many wind instruments and some organ pipes, modifies these results. In particular, the resonance frequencies may no longer be harmonics of the fundamental. Designing the flare is very important in emphasizing or reducing certain of the harmonics present in the forcing function, thereby controlling the *timbre* of the sound radiated by the pipe.