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Playing Billiards in the Concert Hall: The Mathematical Foundations of Geometrical Room Acoustics

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ABSTRACT

Billiard theory is emerging as a fundamental model for understanding the behaviour of complex physical systems. Yet it reduces to the analysis of one ball moving at constant velocity along a straight trajectory that experiences specular reflections on the billiard boundaries. Hence, the analogy between billiard trajectories and sound rays. This analogy makes it possible to derive ergodic properties for sound rays. Among these, the concept of mixing singles itself out as the key to a proper understanding of diffusion. Furthermore combining billiard theory with energy conservation allows one to estimate the average number of rays reaching a receiver at any time, thus casting some light on the rôle of hidden image sources. Billiard theory also casts some new light on the reverberation process; this is illustrated by means of a simple stochastical computer model.

1 INTRODUCTION

In room acoustics, where the temporal structure of a hall response plays such an important rôle, most computer simulations by means of all-numerical models make use of geometrical acoustics, either as ray tracing or as its dual technique, mirror images. Initially, these techniques were taken over from geometrical optics[1] where ray tracing is now widely used to design microwaves antenna, or in computer graphics. But optical wavelengths are

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much shorter than the objects they reflect on, while acoustical wavelengths have the same order of magnitude. These considerations lead therefore to the need for a careful analysis of the foundations of geometrical acoustics, in order to find out the underlying assumptions, and determine the range of validity of this approach.

Being of a theoretical nature, this analysis is clearly beyond the scope of the present paper, and has been published elsewhere.[2] Here, only the outline of the principles on which the theory is built are given, and some practical consequences of the theory are indicated.

2 WAVES, WAVE-FRONTS, AND RAYS

When dealing with waves, the notion of *wave-front* naturally emerges from the wave equation (partial differential equation). Hadamard[3] and Courant and Hilbert[4] showed that wave-fronts are equivalent to discontinuities propagating in a fluid, and that such discontinuities can only take place on well-defined surfaces, called *characteristic surfaces* because their equations can be derived from the wave equation, and more generally from the differential equations governing the fluid, in a characteristic way.

Now, the equation of the wave-front is also a differential equation and as such admits characteristics. Since it is a first-order differential equation, the characteristics are no longer surfaces, but curves. Hadamard[3] and Courant and Hilbert[4] have also shown that the wave-fronts travel along their characteristic curves, which therefore correspond to sound *rays*. It is then straightforward to calculate the speed at which the wave-front travels on the ray: this speed is equal to the velocity of sound. In a homogenous, isotropic medium at rest, the velocity of sound is constant, therefore wave-fronts are normal to rays and travel at constant speed along them.

Just as important for practical purposes is the variation of the wave amplitude along the ray,[3,4] that is, the *transport equation*.[5,6] As for geometrical optics,[1] application of energy conservation[7] to a tube of rays (Fig. 1) shows that it reduces to the *conservation of sound energy flux through the cross-section of the tube*: this result expresses *the intensity law of geometrical acoustics*.

What happens if the cross-section of the tube vanishes, as is the case when it reaches a caustic surface? The intensity law of geometrical acoustics breaks down, but not ray theory. In fact, the intensity must be expanded in a proper asymptotic manner[6,8,9] that introduces a phase shift behind the caustics, the magnitude of which depends on the type of caustics.[7–9] In a room, such caustic surfaces can effectively be created by reflections on non-planar walls.

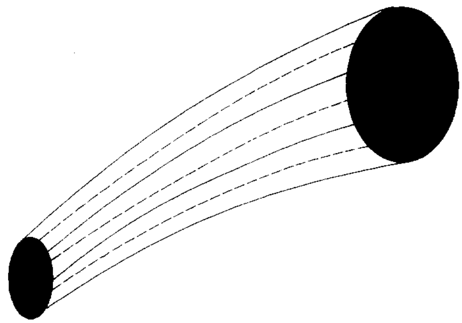

Fig. 1. A tube of rays.

Notice that wavelengths are extrinsic to the concept of characteristics, but not to the concept of caustics. As a consequence, the present approach is restricted to the limit of high frequencies. And the fact that caustics are neglected in the remainder of this paper precludes any attempt to evaluate the lower frequency limit of validity.

3 RAYS AND BILLIARDS

In a room, air can be considered as an isotropic, homogenous medium at rest, therefore rays are straight lines on which wave-fronts propagate at constant speed. The movement of a wave-front along a ray can thus be assimilated to the movement of a point mass in the absence of gravity: it carries on unperturbed, until it reaches an obstacle, i.e. the walls of the room. On the walls, some reflection occurs, and the analogy of the point mass can be carried out to postulate specular reflection: this amounts to very hard walls.

The uniform movement of a point mass in a bounded space without gravity, but with specular reflections on the walls, has received attention from mathematicians: they call it a *billiard*,[10–13] by analogy to the well-known game. Indeed, most of the mathematical work on billiards has been

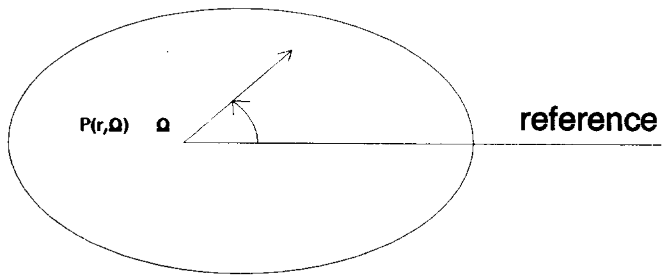

Fig. 2. The phase space.

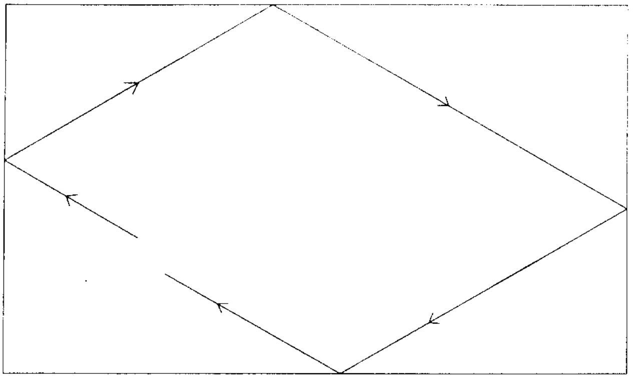

Fig. 3. The rectangular billiard.

restricted to two-dimensional billiards, but most properties can be generalised to three dimensions.

Definition:[13] A billiard is defined by three elements:

(1) a *phase-space*, describing the position r and the direction Ω taken by the mass at any time (Fig. 2);
(2) rules for computing the *trajectories*: uniform rectilinear motion of the mass, and specular reflections on the boundaries;
(3) a *measure* on the phase space, invariant along the trajectories

$$d\mu = (1/4\pi V)\, dr\, d\Omega$$

where V is the volume of the billiard. It measures the probability of the mass to take any position and direction in the phase space at any time.

The best-known billiards are the rectangular billiards: it can be shown that each trajectory reaches nearly all positions, but only takes four different directions in two dimensions (Fig. 3), that is, eight in three dimensions. But other shapes have also been investigated, such as polygonal or smooth convex billiards, and even concave billiards.

4 SOME STATISTICAL PROPERTIES OF BILLIARDS

For room acoustical applications, the statistical properties of billiards are preponderant. For two-dimensional billiards, the billiard classes govern the properties, but little is known for three-dimensional billiards: the existence of any property must be checked for each particular shape.

The first statistical property, *ergodicity*, considers only one trajectory:

along almost any trajectory of an ergodic billiard, the mass spends equal time in the vicinity of each position and direction.[14] As a consequence, (phase) space average is equal to time average along almost any trajectory. However, not all billiards are ergodic: Fig. 3 plainly shows that a rectangular billiard is never ergodic (only four directions along each trajectory). This counter-example stresses the necesssity of considering directions as well as positions, since in a rectangular billiard, the mass spends equal time in the vicinity of all positions.[10] Mathematicians also strongly conjecture that not all strictly convex three-dimensional billiards are ergodic either, to the extent that whole subsets of the billiard are never reached by the mass![10]

The next statistical property, *mixing*, involves two different observation times along a group of initially adjacent trajectories, and can therefore be considered as a correlation property: if sufficient time separates the two observations (strictly speaking an infinitely long time), a set of masses, following initially adjacent trajectories, separates completely and spreads over the whole billiard.[14] The masses can thereafter be considered as statistically independent: no one knows any longer the position nor the direction of any of them. Mixing billiards constitute a subset of ergodic billiards.

Billiards can also be characterised by their *entropy*: directly linked to ergodicity, this quantity can be shown to be related to the logarithmic rate of divergence of adjacent trajectories:[15] the Lyapunov characteristic number. The practical importance of entropy comes from the fact that all mixing billiards have a strictly positive entropy.[11,15] Thus, it is sufficient to evaluate the divergence of trajectories to check whether a billiard is mixing or not. This approach has been used to prove that all two-dimensional concave billiards are mixing.[10]

More precisely, the rate of divergence of trajectories is calculated by measuring at regular time intervals the distance (in phase space) between two initially adjacent trajectories. In practice, difficulties arise due to the finite dimensions of the billiard, but can be overcome by choosing a new adjacent trajectory at the end of each time interval, while cumulating all the distances (see Ref. 2, and especially Ref. 15, for more details).

5 THE NUMBER OF ARRIVALS

In the rest of this paper, special trajectories issued from a fixed sound source and reaching a fixed receiver position after some given lapse of time are considered. Such trajectories will be called *arrivals*. These arrivals are directly linked to the different reflection paths, and the number of arrivals shorter than a given length plays an important rôle in room acoustics.

Intuitively, this number must compensate the geometrical energy decrease along each path.

In the case of a mixing room with perfectly reflecting boundaries (no losses on the boundaries), this assumption can be proved by means of energy conservation. Mixing ensures that the energy is equidistributed in the room after sufficient time, i.e. after mixing time. Since adjacent trajectories in a mixing room diverge at an exponential rate, intensity reduces exponentially along any trajectory (*cf.* section 2). Therefore, to conserve energy, the number of arrivals must compensate for the divergence. Hence, *the number of arrivals is exponentially increasing with time in any mixing room.* In the case of a rectangular room, which is not mixing, energy conservation cannot be used, but direct computation shows the number of simultaneous arrivals of length l to be proportional to l^2,[16] and the energy decrease along each of them to be also proportional to l^2. Notice that in this case, trajectories diverge at a sub-exponential rate, that is, the logarithmic rate of divergence is equal to zero, as is expected for a non-ergodic billiard.

This quantitative difference between a rectangular and a mixing room leads to a fundamental *qualitative* difference in terms of image sources. Since each arrival corresponds to one image source, there are exponentially many image sources contributing to the sound field in a mixing room: it is therefore blatantly erroneous to extrapolate the rectangular room to the mixing room without first considering the mixing time. The question arises therefore of locating all these image sources. In fact, this location is meaningless, since mixing ensures that they are evenly distributed. A simple way of achieving a distribution of image sources equivalent to the correct one consists in keeping and exploiting the so-called *hidden image sources*, neglected in most approaches. Extrapolating from the rectangular room where all image sources are visible, only visible sources are considered in computations. In fact, all the other sources, i.e. the hidden sources, are so numerous that their contribution is likely to overpower the contribution from visible sources. More precisely, the mixing time can be considered to be the lapse of time after which visible sources can be neglected, since trajectories issued from visible sources remain correlated while trajectories issued from hidden sources are uncorrelated by virtue of mixing (*cf.* section 4). Therefore, it is erroneous to neglect hidden image sources in computer models.

Another way to increase the number of image sources is by the introduction of non-specular reflections in random directions. Despite the fact that billiard theory excludes all randomness, random reflections can accurately simulate a billiard with very corrugated walls since the probability that one ray hits the same position on the walls twice with the same incidence (that is, of choosing a periodic trajectory) is negligible.

The only drawback of this method is that it relies on arbitrary coefficients that have no better justification that a good fit with reality.[17]

Energy considerations also lead to a rigorous definition of *diffusion* as a property of trajectories, and not of the sound field. Remembering that a diffuse sound field is characterised by *a constant instantaneous mean density* of acoustical energy throughout the room, and *a constant mean incident energy flux* within an infinitesimal solid angle around every direction, it is obvious that the sound field in a mixing room with perfectly reflecting boundaries is diffuse. Introducing absorption on the boundaries does modify the sound field, but there is no reason to assume it modifies the trajectories in the room: the room remains mixing. Hence, *diffusion must be redefined as equivalent to mixing.*

6 REVERBERATION

Reverberation takes place when absorption is introduced in a room: the sound field, built by superposition of all the arrivals simultaneously issued from the source and gradually reaching the receiver, decays at a rate proportional to the amount of absorption introduced in the room (Sabine's law).

Ergodicity allows one to consider only one trajectory instead of space averaging, since almost any trajectory runs through the entire phase space, leading to the equivalence of all trajectories. Mixing ensures that events further apart than the mixing time are statistically independent (*cf.* section 4). Hence, the distribution of the number of reflections undergone by one trajectory during the mixing time is independent of the actual choice of the trajectory, and therefore the whole process of reverberation is completely determined as soon as this distribution is known along one trajectory. In fact, this process gradually sets in, and is only fully established after mixing time, when even distribution of the sound energy is achieved.[2]

The results from billiard theory can be incorporated into a stochastic model for generating reverberation. Such a model makes use of the random generators: the first one generates the different arrivals reaching the receiver; and the second the number of reflections undergone by each arrival. When the number of 'simultaneous' arrivals reaches a fixed limit (about 10), statistical treatment takes over to speed up the process. So far, the model has assumed Poisson's distribution both for reflections and for arrivals. In the first distribution, the mean number of reflections is proportional to time, and inversely proportional to the mean free path ($4V/S$), which leads to Sabine's law of reverberation.[2,18] In the second distribution, the mean number of arrivals increases sub-exponentially in t^2, as in rectangular rooms. Since the model does not rely on the actual choice of the distribution laws, the latter can readily be replaced by more realistic ones.

Preliminary results were obtained for a room of 2000 m^3 with a reverberation time of 1 s and a mean absorption coefficient of 33%, amounting to a wall surface of 985 m^2 according to Sabine's reverberation formula: this is roughly equivalent to a room of dimensions 10 × 10 × 20 m. They confirm the low number of arrivals in rectangular rooms since the sound field become statistic (10 arrivals within 1 ms) after a very long time (150 ms on average). They also confirm that the late reverberant field can be considered as uniform throughout the room, as is assumed in Barron's revised theory.[19] In Fig. 4, the overall sound level, as a function of the distance between source and receiver, is compared to predictions according to Barron's model. The scatter of simulated data around Barron's predictions has a standard deviation of 0·6 dB, which represents 60% of the standard deviation measured by Barron in British halls.[19] Figure 5 displays the same comparison for Clarity: data scatter around predictions with a standard deviation of 0·85 dB, which again represents 60% of the experimental value.[19] Notice that random generation only creates a short range of source/receiver distances: the receiver was located at distances varying from one to five times the hall radius (2·54 m). Last but not least, the reverberation time was always equal to its target value of 1 s, confirming that

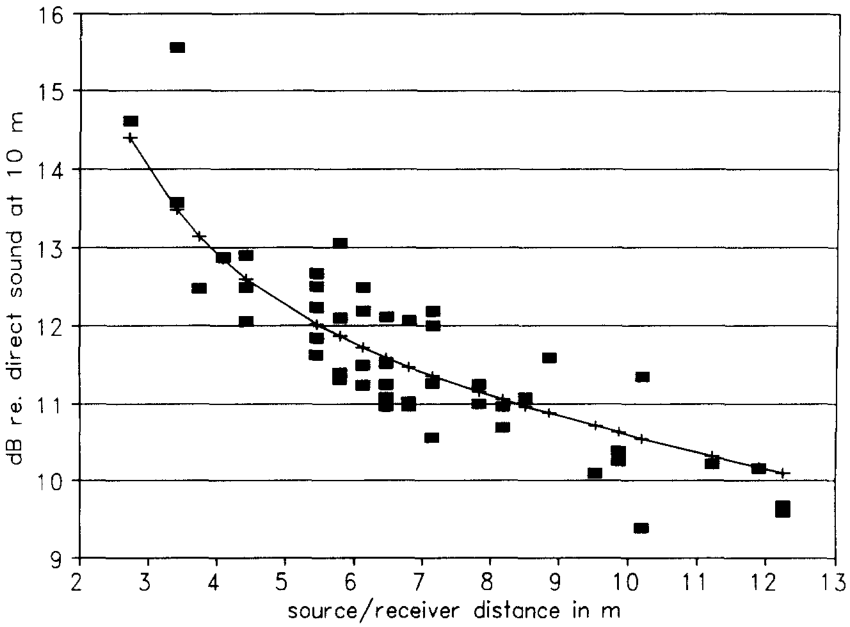

Fig. 4. Overall sound level as a function of source/receiver distance. (■) Simulated data, (+) Barron's revised theory.

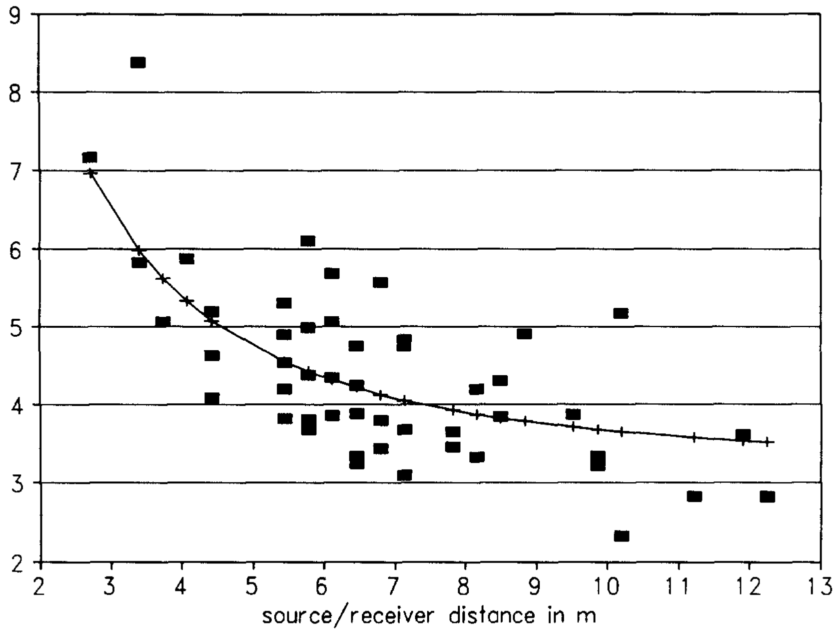

Fig. 5. Clarity as a function of source/receiver distance. (■) Simulated data, (+) Barron's revised theory.

Poisson's distribution for reflections leads to Sabine's reverberation formula.[2,18]

7 CONCLUSION

This short presentation of billiards in concert halls is meant to cast some light on the underlying assumptions of geometrical acoustics. It can be proved that the most interesting properties of room acoustics are statistical and can be deduced from following two neighbour trajectories throughout the room.

The Lyapunov characteristic number emerges as a fundamental quantity because of its relation to the mean number of arrivals. Proper estimation of this number could speed up algorithms by reducing the domain in which image sources, or rays, have to be calculated precisely, thus allowing one to optimise the number of rays or sources.

Available literature on billiards plainly shows the limitations of present numerical simulations: plane surfaces, that never exhibit mixing in two dimensions; visible image sources only, leading to unrealistic mixing times; no caustics, i.e. no phase shifts along the trajectories, and high frequency approximation only. Improvements must be carried out on along these lines in order to simulate concert halls accurately.

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