Devoir Maison UE FpA - Acoustique
Partie Vibrations

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September 14, 2024

Question 1. Show

$$H(\omega) = \frac{V(\omega)}{F(\omega)} = \frac{j\omega/K}{1 + j\frac{\omega}{\omega_0 Q} - \left(\frac{\omega}{\omega_0}\right)^2}$$

and ω_0, Q.

Solution. Consider $x = L$, the equation of motion for {M} can be written as

$$M\ddot{y}(t) + R\dot{y}(t) + Ky(t) = f(t). \tag{1}$$

Taking the Fourier Transform of (1), we obtain

$$\begin{aligned} -\omega^2 MY(\omega) + j\omega RY(\omega) + KY(\omega) &= F(\omega) \\ \left(-\omega^2 M + j\omega R + K\right) Y(\omega) &= F(\omega). \end{aligned}$$

Thus, the displacement $Y(\omega)$ is

$$Y(\omega) = \frac{F(\omega)}{-\omega^2 M + j\omega R + K}. \tag{2}$$

The mobility $H(\omega)$ is defined as $H(\omega) = \frac{V(\omega)}{F(\omega)}$, and $V(\omega) = j\omega Y(\omega)$. Substituting $Y(\omega)$ by $V(\omega)$ in (2)

$$\begin{aligned} \frac{V(\omega)}{j\omega} &= \frac{F(\omega)}{-\omega^2 M + j\omega R + K} \\ H(\omega) = \frac{V(\omega)}{F(\omega)} &= \frac{j\omega}{-\omega^2 M + j\omega R + K} \\ &= \frac{j\omega/K}{1 + j\frac{\omega}{K/R} - \left(\frac{\omega}{\sqrt{K/M}}\right)^2} \\ &= \frac{j\omega/K}{1 + j\frac{\omega}{\omega_0 Q} - \left(\frac{\omega}{\omega_0}\right)^2} \end{aligned} \tag{3}$$

where

$$\omega_0 = \sqrt{\frac{K}{M}}$$
$$Q = \frac{\omega_0 M}{R}.$$

□

Question 2. Show $H(\omega) = G(\omega) + jB(\omega)$. Define G and B. Calculate G and B when $\omega = \omega_0$.

Solution. Consider the denominator of (3)

$$1 + j\frac{\omega}{\omega_0 Q} - \left(\frac{\omega}{\omega_0}\right)^2.$$

It can be rewrite as

$$\left[1 - \left(\frac{\omega}{\omega_0}\right)^2\right] + j\frac{\omega}{\omega_0 Q}.$$

This complex number can be expressed as $\alpha + j\beta$, where

$$\alpha = 1 - \left(\frac{\omega}{\omega_0}\right)^2$$
$$\beta = \frac{\omega}{\omega_0 Q}.$$

In order to simplify (3), multiply both the numerator and denominator by the conjugate $\alpha - j\beta$, then (3) becomes

$$H(\omega) = \frac{j\omega/K \cdot \left[\left(1 - \left(\frac{\omega}{\omega_0}\right)^2\right) - j\frac{\omega}{\omega_0 Q}\right]}{\left[1 - \left(\frac{\omega}{\omega_0}\right)^2\right]^2 + \left(\frac{\omega}{\omega_0 Q}\right)^2}$$
$$= \frac{j\omega/K \cdot \left[1 - \left(\frac{\omega}{\omega_0}\right)^2\right] + \frac{\omega^2}{K\omega_0 Q}}{\left[1 - \left(\frac{\omega}{\omega_0}\right)^2\right]^2 + \left(\frac{\omega}{\omega_0 Q}\right)^2}.$$

Thus,

$$G(\omega) = \frac{\frac{\omega^2}{K\omega_0 Q}}{\left[1 - \left(\frac{\omega}{\omega_0}\right)^2\right]^2 + \left(\frac{\omega}{\omega_0 Q}\right)^2}$$
$$B(\omega) = \frac{\omega/K \cdot \left[1 - \left(\frac{\omega}{\omega_0}\right)^2\right]}{\left[1 - \left(\frac{\omega}{\omega_0}\right)^2\right]^2 + \left(\frac{\omega}{\omega_0 Q}\right)^2}$$

When $\omega = \omega_0$,

$$G(\omega_0) = 1/R, B(\omega_0) = 0.$$

$\square$

Question 3. Physical meaning of ω_0 and Q. How to estimate Q?

Solution. ω_0 is the natural frequency at which the system oscillates when it is not subjected to external forces. Q is the quality factor, which characterizes the damping in a system, and it has relations with the time constant τ.

Méthode de la bande passante à -3dB (frequency domain)

Find the peak point of ω in the frequency response curve, which occurs at $\omega = \omega_0$. Then measure the frequencies ω_1 and ω_2, where the amplitude is half of the maximum amplitude and corresponds to a drop of -3dB. Then Q could be estimated by

$$Q = \frac{\omega_0}{\omega_2 - \omega_1}.$$

Méthode du décrément logarithmique (time domain)

The logarithmic decrement δ is defined as the logarithm of the ratio of two nearest peaks

$$\delta = \log \frac{x(t)}{x(t+T)}.$$

By using δ, we could easily obtain the damping ξ, which defined as

$$\xi = \frac{\xi}{\sqrt{4\pi^2 + \delta^2}}.$$

Thus, Q could be estimated by

$$Q = \frac{1}{2\xi}.$$

$\square$

Question 4. Show the equation of motion of transverse vibrations in the string is

$$\frac{\partial^2 y}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 y}{\partial t^2} = 0$$

and the propagation speed of transverse waves c.

Solution. Take a small segment of the string with length dx, as figure 1 shows below.

The tension T exists on both ends. At position x, the vertical component of T can be expressed as $T_y(x) = T\sin\theta(x) = T\tan\theta(x) \approx \left[\frac{\partial y(x,t)}{\partial x}\right]_{x=x}$ because of a small angle. Similarly, at position $x+dx$, $T_y(x+dx) = T\sin\theta(x+dx) \approx \left[\frac{\partial y(x,t)}{\partial x}\right]_{x=x+dx}$. Then use Taylor formula to expand $T_y(x)$ at position $x+dx$

$$T_y(x)|_{x=x+dx} = T_y(x) + dx\left[\frac{\partial T_y(x)}{\partial x}\right]_{x=x} + o(dx)^2.$$

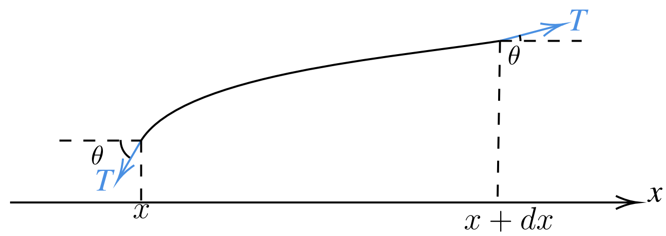

Figure 1: A small segment of the string

Therefore, in the vertical direction

$$\begin{aligned}
\rho_L \cdot S \cdot dx \cdot \ddot{y}(x,t) &= T_y(x+dx) - T_y(x) \\
&= \frac{\partial T_y(x)}{\partial x} \\
&= T \frac{\partial}{\partial x}\left(\frac{\partial y(x,t)}{\partial x}\right) \\
&= T \frac{\partial^2 y(x,t)}{\partial x^2} dx.
\end{aligned}$$

And finally the equation of motion could be expressed as

$$\begin{aligned}
\frac{\partial^2 y(x,t)}{\partial x^2} - \frac{\rho_L S}{T}\frac{\partial^2 y(x,t)}{\partial t^2} &= 0 \\
\frac{\partial^2 y}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 y}{\partial t^2} &= 0 \qquad (4)
\end{aligned}$$

where $c = \sqrt{\frac{T}{\rho_L S}}$.

□

Question 5. The boundary condition for $\Phi_i(x)$ at $x = 0$.

Solution. String is fixed at $x = 0$, so that $y(0,t) = 0\ \forall t$. Thus $\Phi_i(x)$ satisfy $\Phi_i(0) = 0$. □

Question 6. Show the transverse force at $x = L$ is $f(t) = -T\left(\frac{\partial y}{\partial x}\right)_{x=L}$.

Solution. At $x = L$, string is also fixed, so that $y(L,t) = 0\ \forall t$. The vertical component of tension T is $T_y(L)$. In the vertical direction:

$$\begin{aligned}
\rho_L \cdot S \cdot \ddot{y}(x,t) &= f(t) + T\left(\frac{\partial y}{\partial x}\right) \\
0 &= f(t) + T\left(\frac{\partial y}{\partial x}\right).
\end{aligned}$$

Thus,

$$f(t) = -T\left(\frac{\partial y}{\partial x}\right)_{x=L}. \qquad (5)$$

□

Question 7. For mode n, show the differential equation of the modal deformation and solve this equation by using boundary condition of $x = 0$.

Solution. For mode n,

$$y(x,t) = \Phi_n(x) q_n(t) \tag{6}$$

where $q_n(t) = e^{j\beta t}$. Then we could simplify (4):

$$\begin{aligned} e^{j\beta t}\frac{\partial^2 \Phi_n}{\partial x^2} + \frac{\beta^2}{c^2}\Phi_n e^{j\beta t} &= 0 \\ \frac{\partial^2 \Phi_n}{\partial x^2} + \frac{\beta^2}{c^2}\Phi_n &= 0. \end{aligned} \tag{7}$$

The solution for (7) can be expressed as

$$\Phi_n(x) = A\cos(\frac{\beta}{c}x) + B\sin(\frac{\beta}{c}x). \tag{8}$$

At $x = 0$, the boundary condition of $\Phi_i(x)$ is $\Phi_i(0) = 0$. By replacing (8) with this condition, we obtain $A = 0$. The modal deformation has a property called relation of orthogonality:

$$\int_0^L \sin(\frac{\beta_n}{c}x)\sin(\frac{\beta_m}{c}x) = \begin{cases} L/2 & n = m \\ 0 & n \neq m \end{cases}. \tag{9}$$

When $t = 0$, $y(x,0) = \Phi_n(x) = B\sin(\frac{\beta}{c}x)$, multiplied by $\sin(\frac{\beta}{c}x)$ and integrated from 0 to L:

$$\begin{aligned} \int_0^L y(x,0)\sin(\frac{\beta}{c}x)dx &= B \cdot L/2 \\ B &= \frac{2}{L}\int_0^L y(x,0)\sin(\frac{\beta}{c}x)dx. \end{aligned}$$

Thus,

$$\Phi_n(x) = \sin(\frac{\beta}{c}x)\frac{2}{L}\int_0^L y(x,0)\sin(\frac{\beta}{c}x)dx. \tag{10}$$

□

Question 8. Show

$$-T\left(\frac{\partial \Phi_n(x)}{\partial x}\right)_{x=L} H(\omega) = j\beta\Phi_n(L).$$

Solution. Consider again (6). Taking the Fourier Transform of $y(x,t)$, we obtain

$$Y(x,\omega) = \Phi_n(x)Q_n(\omega)$$

where $Q_n(\omega) = 2\pi\delta(\omega - \beta)$. From previous question, we replace $y(x,t)$ by (6) in (5):

$$f(t) = -T\left(\frac{\partial \Phi_n(x)}{\partial x}\right)_{x=L} \cdot q_n(t) \tag{11}$$

and take the Fourier Transform of (11)

$$F(\omega) = -T\left(\frac{\partial \Phi_n(x)}{\partial x}\right)_{x=L} \cdot Q_n(\omega). \tag{12}$$

For the speed at $x = L$,

$$v(t) = \left[\frac{\partial y(x,t)}{\partial t}\right]_{x=L} = \Phi_n(L)\cdot\frac{\partial q_n(t)}{\partial t} = \Phi_n(L)\cdot j\beta\cdot q_n(t) \tag{13}$$

and take the Fourier Transform of (13)

$$V(\omega) = j\beta\Phi_n(L)Q_n(\omega). \tag{14}$$

$H(\omega)$ is the ratio of (14) and (12)

$$H(\omega) = \frac{V(\omega)}{F(\omega)} = \frac{j\beta\Phi_n(L)Q_n(\omega)}{-T\left(\frac{\partial\Phi_n(x)}{\partial x}\right)_{x=L}\cdot Q_n(\omega)}.$$

Thus,

$$-T\left(\frac{\partial \Phi_n(x)}{\partial x}\right)_{x=L} H(\omega) = j\beta\Phi_n(L). \tag{15}$$

□

Question 9. Show

$$-j\frac{1}{H} = \frac{T}{\beta L} + 2\frac{\beta T}{L}\sum_{i=1}^{\infty}\frac{1}{\beta^2-\omega_i^2}.$$

Solution. The cotangent function could be also expressed as

$$\begin{aligned}\pi\cot(\pi z) &= \frac{1}{z} + 2z\sum_{n=1}^{\infty}\frac{1}{z^2-n^2}\\ \cot(z) &= \frac{1}{z} + 2z\sum_{n=1}^{\infty}\frac{1}{z^2-(n\pi)^2}.\end{aligned} \tag{16}$$

Replace z by $z = \frac{\beta L}{c}$ in (16), we obtain

$$\begin{aligned}\cot(\frac{\beta L}{c}) &= \frac{1}{\frac{\beta L}{c}} + 2\frac{\beta L}{c}\sum_{n=1}^{\infty}\frac{1}{\left(\frac{\beta L}{c}\right)^2-(\pi n)^2}\\ &= c\left(\frac{1}{\beta L} + 2\frac{\beta}{L}\sum_{n=1}^{\infty}\frac{1}{\beta^2-\left(\frac{\pi n c}{L}\right)^2}\right).\end{aligned} \tag{17}$$

Then multiply by $\frac{T}{c}$ for (17)

$$\frac{T}{c}\cot(\frac{\beta L}{c}) = \frac{T}{\beta L} + 2\frac{\beta T}{L}\sum_{n=1}^{\infty}\frac{1}{\beta^2-\left(\frac{\pi n c}{L}\right)^2} \tag{18}$$

which is exactly the right hand of the equation that we want to prove. Use (15) again, we could obtain

$$-j\frac{1}{H} = \frac{T\left(\frac{\partial \Phi_n(x)}{\partial x}\right)_{x=L}}{\beta \Phi_n(L)}$$

and replace $\Phi_n(x)$ by (10)

$$\begin{aligned} -j\frac{1}{H} &= \frac{TB\frac{\beta}{c}\cos\left(\frac{\beta}{c}L\right)}{\beta B \sin\left(\frac{\beta}{c}L\right)} \\ &= \frac{T}{c}\cot\left(\frac{\beta}{c}L\right). \end{aligned} \tag{19}$$

Combine (18) and (19)

$$\begin{aligned} -j\frac{1}{H} &= \frac{T}{\beta L} + 2\frac{\beta T}{L}\sum_{n=1}^{\infty}\frac{1}{\beta^2 - \left(\frac{\pi n c}{L}\right)^2} \\ -j\frac{1}{H} &= \frac{T}{\beta L} + 2\frac{\beta T}{L}\sum_{i=1}^{\infty}\frac{1}{\beta^2 - \omega_i^2} \end{aligned}$$

where $\omega_i = \frac{\pi i c}{L}$.

□

Question 10. For a pulsation β close to a natural pulsation ω_n of the mode n of the string fixed at its ends, what happens to this expression?

Solution. When $\beta \to \omega_n$, $\beta^2 - \omega_n^2 \to 0$, which means $\frac{1}{\beta^2-\omega_n^2}$ dominate the part of sum. Thus,

$$-j\frac{1}{H} \approx \frac{T}{\beta L} + 2\frac{\beta T}{L}\frac{1}{\beta^2 - \omega_n^2} \approx \frac{T}{\beta L} + 2\frac{\beta T}{L}\frac{1}{2\omega_n(\beta - \omega_n)} \approx \frac{T}{\beta L} + \frac{T}{L}\frac{1}{\beta - \omega_n}. \tag{20}$$

□

Question 11. Show β can be expressed as

$$\beta = \omega_n + \delta_n + j\alpha_n.$$

Solution. Replace $\beta - \omega_n$ by $\delta_n + j\alpha_n$ in (20)

$$\begin{aligned} -j\frac{1}{H} &= \frac{T}{\beta L} + \frac{T}{L}\frac{1}{\delta_n + j\alpha_n} \\ -j\frac{L/T}{H} &= \frac{1}{\omega_n} + \frac{1}{\delta_n + j\alpha_n} \\ -\frac{1}{\omega_n} - j\frac{L/T}{H} &= \frac{1}{\delta_n^2 + \alpha_n^2}(\delta_n - j\alpha_n) \end{aligned} \tag{21}$$

and we could obtain two equations from (21)

$$\begin{cases} -\frac{1}{\omega_n} & = \frac{\delta_n}{\delta_n^2+\alpha_n^2} \\ \frac{L/T}{H} & = \frac{\alpha_n}{\delta_n^2+\alpha_n^2} \end{cases}. \tag{22}$$

Solve (22), δ_n and α_n are

$$\begin{cases} \delta_n & = -\frac{H^2T^2\omega_n}{H^2T^2+L^2\omega_n^2} \\ \alpha_n & = \frac{HTL\omega_n^2}{H^2T^2+L^2\omega_n^2} \end{cases}$$

where $\omega_n = \frac{nc}{L}\pi$ and $c = \sqrt{\frac{T}{\rho_L S}}$.

□

Question 12. The physical meaning of δ_n and α_n. What will happen when $\omega_n = \omega_0$? The estimated value of δ_n and α_n under the cases of $\omega_n \ll \omega_0$ and $\omega_n \gg \omega_0$.

Solution. δ_n represents the shift of ω_i due to the system {masse-ressort-amortisseur} and α_n represents the shift of ω_i effected by damping.

When $\omega_n = \omega_0$, resonance occurs. The system experiences maximum energy transfer and α_n decides how much energy dissipated, which prevent the system oscillates indefinitely.

When $\omega_n \ll \omega_0$

$$\begin{cases} \delta_n & = -\omega_n \\ \alpha_n & = \frac{L}{HT}\omega_n^2 \end{cases}.$$

When $\omega_n \ll \omega_0$

$$\begin{cases} \delta_n & = -\frac{H^2T^2}{L^2\omega_n} \\ \alpha_n & = \frac{HT}{L} \end{cases}.$$

□

Question 13. How the modes affect the vibration of a guitar string?

Solution. When the frequency of the vibration of a guitar string is close to the resonance frequency (or its harmonic frequencies), the guitar will amplify the sound. The modes provide different harmonic frequencies to the guitar. □