Chapter 3

VIBRATIONS OF BARS

3.1 LONGITUDINAL VIBRATIONS OF A BAR

Another important wave motion is the propagation of *longitudinal (compressional) waves*, often encountered in solid bars (and, at low frequencies, in gas-filled tubes and ducts with rigid walls). As a longitudinal disturbance moves along a bar, the displacements of particles of the bar are essentially parallel to its axis. When the lateral dimensions of the bar are small compared with its length, each cross-sectional plane of the bar may be considered to move as a unit. (Actually the bar shrinks somewhat laterally as it expands longitudinally, but for thin bars this lateral motion may be neglected.)

A number of acoustic devices utilize longitudinal vibrations in bars. Frequency standards used for producing sounds of definite pitches can be constructed from rods of various lengths. When longitudinal vibrations are excited in such rods, the frequency of vibration is observed to be inversely proportional to the length of the rod (if all are of the same composition). Longitudinal vibrations in nickel tubes are often used to drive the vibrating diaphragm of a sonar transducer. Piezoelectric crystals may be cut so that the frequency of longitudinal vibration in a selected direction in the crystal is used either to control the frequency of an oscillatory electric current or to drive an electroacoustic transducer.

Studying longitudinal vibrations of bars also aids in understanding acoustic waves. The mathematical expressions for the transmission of acoustic plane waves through fluid media are very similar to those for the transmission of compressional waves along a bar. If the fluid is confined to a rigid pipe, there is also a close analogy between the boundary conditions.

3.2 LONGITUDINAL STRAIN

Consider a bar of length L and uniform cross-sectional area S subjected to longitudinal forces. The application of these forces will produce a longitudinal displacement ξ of each of the particles in the bar, and for thin bars this displacement

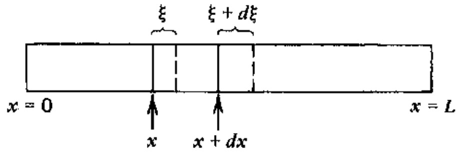

Figure 3.2.1 Longitudinal strain $d\xi/dx$ of an element of length dx in a bar.

will be essentially the same at all points in any particular cross section. Thus, ξ is assumed to be a function only of distance x along the bar and time t,

$$\xi = \xi(x, t) \tag{3.2.1}$$

Let the coordinates of the left and right ends of the bar be $x = 0$ and $x = L$, and consider a short segment dx of the unstrained bar lying between x and $x + dx$. Assume that the forces cause the plane originally located at x to move a distance ξ to the right, and the plane originally located at $x + dx$ to move a distance $\xi + d\xi$ to the right (Fig. 3.2.1). The convention adopted in this book is that a *positive* value of ξ signifies a displacement to the *right* (and a *negative* value to the *left*).

At any time t for small dx the displacement at $x + dx$ can be represented by the first two terms of a Taylor's series expansion of ξ about x,

$$\xi + d\xi = \xi + (\partial\xi/\partial x)\, dx \tag{3.2.2}$$

Since the left end of the segment has been displaced a distance ξ and the right end a distance $\xi + d\xi$, the increase in length of the segment is given by

$$(\xi + d\xi) - \xi = d\xi = (\partial\xi/\partial x)\, dx \tag{3.2.3}$$

The *strain* in the segment is defined as the ratio of its change in length $d\xi$ to its original length dx, or

$$\boxed{\text{strain} = \partial\xi/\partial x} \tag{3.2.4}$$

3.3 LONGITUDINAL WAVE EQUATION

Whenever a bar is strained, elastic forces are produced. These forces act across each cross-sectional plane in the bar and hold the bar together. Let $f = f(x, t)$ represent these longitudinal forces, where the convention is adopted of choosing a *positive* value of f to represent a force of *compression*, as indicated in Fig. 3.3.1, and a *negative*

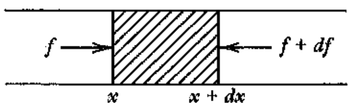

Figure 3.3.1 Compressional forces on an element of length dx in a bar.

value to represent a force of *tension*. This choice of sign is the opposite of that conventionally taken by many material scientists but has the distinct advantage for us of making the compression of a solid by a positive increment of force analogous to the compression of a fluid by a positive increment in pressure.

The stress in the bar of cross-sectional area S is defined as

$$\boxed{\text{stress} = f/S} \tag{3.3.1}$$

For most materials, if the strain is small the stress is proportional to it. This relationship is known as Hooke's law,

$$f/S = -Y(d\xi/dx) \tag{3.3.2}$$

where Y, the *Young's modulus* or *modulus of elasticity*, is a characteristic property of the material. Since a positive stress results in a negative strain, the minus sign in (3.3.2) ensures a positive value for Y. Values of Y for a number of common solids are given in Appendix A10. Rewriting (3.3.2), we obtain

$$f = -SY(d\xi/dx) \tag{3.3.3}$$

as an expression for the internal longitudinal forces in the bar.

If f represents the internal force at x, then $f + (\partial f/\partial x)\,dx$ represents the force at $x + dx$, and the net force to the right is

$$df = f - [f + (\partial f/\partial x)\,dx] = -(\partial f/\partial x)\,dx \tag{3.3.4}$$

Use of (3.3.3) yields

$$df = SY(\partial^2\xi/\partial x^2)\,dx \tag{3.3.5}$$

The mass of the segment dx is $\rho S\,dx$, where ρ is the *density* (mass per unit volume) of the bar. Therefore, the equation of motion of the segment is

$$(\rho S\,dx)(\partial^2\xi/\partial t^2) = SY(\partial^2\xi/\partial x^2)\,dx \tag{3.3.6}$$

or

$$\boxed{\frac{\partial^2\xi}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2\xi}{\partial t^2}} \tag{3.3.7}$$

A comparison of (3.3.7) with the corresponding (2.3.6) for the transverse motion of a string shows that they have identical form, with longitudinal displacement ξ replacing the transverse displacement y and the phase speed c now given by

$$\boxed{c^2 = Y/\rho} \tag{3.3.8}$$

Thus, the general solution has the same form as that for the transverse wave equation,

$$\xi(x, t) = \xi_1(ct - x) + \xi_2(ct + x) \tag{3.3.9}$$

The complex harmonic solution of (3.3.7) is

$$\boldsymbol{\xi}(x, t) = \mathbf{A}e^{j(\omega t - kx)} + \mathbf{B}e^{j(\omega t + kx)} \tag{3.3.10}$$

where $\mathbf{A}$ and $\mathbf{B}$ are complex amplitude constants and $k = \omega/c$ is the wave number.

Since Young's modulus Y is measured under conditions allowing the strained rod to alter its transverse dimensions, (3.3.8) gives the phase speed only when the solid is a thin bar. When the transverse dimensions of the solid are large compared to a wavelength, a combination of the *bulk modulus* $\mathcal{B}$ and the *shear modulus* $\mathcal{G}$ must be used in place of the Young's modulus to calculate the phase speed. (See Appendix A11.)

3.4 SIMPLE BOUNDARY CONDITIONS

Let the bar be rigidly fixed at both ends, so that $\boldsymbol{\xi} = 0$ at $x = 0$ and at $x = L$ for all times t. (The analysis that follows will be seen to be identical with that of Section 2.10 for a rigidly supported vibrating string.)

Application of $\boldsymbol{\xi} = 0$ at $x = 0$ gives $\mathbf{A} + \mathbf{B} = 0$, so that $\mathbf{B} = -\mathbf{A}$ and (3.3.10) becomes

$$\boldsymbol{\xi}(x, t) = \mathbf{A}e^{j\omega t}(e^{-jkx} - e^{jkx}) = -2j\mathbf{A}e^{j\omega t}\sin kx \tag{3.4.1}$$

The condition $\boldsymbol{\xi} = 0$ at $x = L$ gives $\sin kL = 0$, which requires

$$k_n L = n\pi \qquad n = 1, 2, 3, \ldots \tag{3.4.2}$$

(the same as for a fixed, fixed string). The angular frequencies of the natural modes of vibration are

$$\omega_n = n\pi c/L \quad \text{or} \quad f_n = (n/2)(c/L) \tag{3.4.3}$$

[identical with (2.10.5)]. The complex displacement $\boldsymbol{\xi}_n$ corresponding to the nth mode of vibration is

$$\boldsymbol{\xi}_n(x, t) = -2j\mathbf{A}_n e^{j\omega_n t}\sin k_n x \tag{3.4.4}$$

and the real part is

$$\xi_n(x, t) = (A_n \cos \omega_n t + B_n \sin \omega_n t)\sin k_n x \tag{3.4.5}$$

where the real amplitude constants A_n and B_n are defined by $2\mathbf{A}_n = B_n + jA_n$. The complete solution is the sum of all separate harmonic solutions,

$$\xi(x, t) = \sum_{n=1}^{\infty}(A_n \cos \omega_n t + B_n \sin \omega_n t)\sin k_n x \tag{3.4.6}$$

If the initial conditions of displacement and speed of the bar are known, Fourier's theorem can be used, as in Section 2.10, to evaluate A_n and B_n.

Since a solid bar is very rigid, it is difficult to provide supports of greater rigidity, and the assumed boundary condition is difficult to realize in practice. By contrast, a free end may be achieved readily by placing the bar on soft supports.

When a bar is free to move at an end, there can be no internal elastic force at the end, and hence $f = 0$ at this point. Since $f = -SY(\partial\xi/\partial x)$, this condition is equivalent to

$$\frac{\partial \xi}{\partial x} = 0 \tag{3.4.7}$$

at a free end.

Consider a free, free bar. The condition $\partial\xi/\partial x = 0$ applied to (3.3.10) at $x = 0$ gives

$$-\mathbf{A} + \mathbf{B} = 0 \quad \text{or} \quad \mathbf{B} = \mathbf{A} \tag{3.4.8}$$

so that

$$\boldsymbol{\xi}(x, t) = \mathbf{A}e^{j\omega t}(e^{-jkx} + e^{jkx}) = 2\mathbf{A}e^{j\omega t}\cos kx \tag{3.4.9}$$

Application of $\partial\xi/\partial x = 0$ at $x = L$ gives $\sin kL = 0$ or

$$\omega_n = n\pi c/L \qquad n = 1, 2, 3, \ldots \tag{3.4.10}$$

The natural frequencies of a free, free bar are identical with those of (3.4.3) for a fixed, fixed bar of the same shape and composition. The complex displacement of the nth mode of vibration is

$$\boldsymbol{\xi}_n(x, t) = 2\mathbf{A}_n e^{j\omega_n t}\cos k_n x \tag{3.4.11}$$

and the real displacement is

$$\xi_n(x, t) = (A_n \cos \omega_n t + B_n \sin \omega_n t)\cos k_n x \tag{3.4.12}$$

where now $2\mathbf{A}_n = A_n - jB_n$. In contrast with the fixed, fixed bar, which has nodes at either end, the free, free bar has antinodes at either end as shown by a $\cos k_n x$ term in the above equation, instead of a $\sin k_n x$ as in (3.4.4). A comparison of the nodal patterns for these two types of support is given in Fig. 3.4.1. It should be observed that whenever an *antinode* occurs at the center of the bar the vibrations are *symmetric* with respect to the center: when a segment of the bar left of center is displaced to the left, the corresponding segment to the right of center is also displaced the same distance to the left. Similarly, whenever there is a *node* at the center, the vibrations are *antisymmetric*.

A bar may be rigidly clamped at a point without interfering with any mode of vibration that has a node at this point. However, a mode not having a node at this point will be suppressed. It is impossible to find a position for clamping a free, free bar that will not eliminate some of the normal modes.

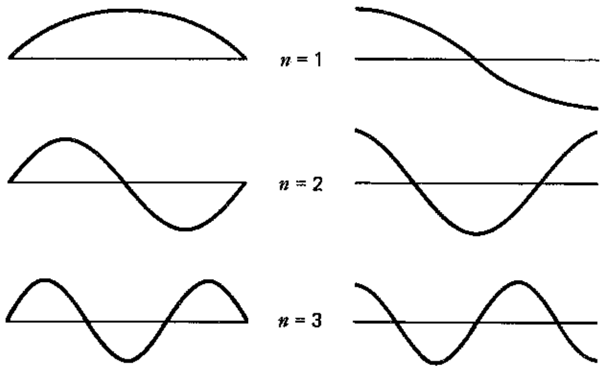

Figure 3.4.1 The lowest three longitudinal standing waves in fixed, fixed and free, free bars.

Next consider a free, fixed bar. Application of $\partial\xi/\partial x = 0$ at $x = 0$ to (3.3.10) gives (3.4.9), and application of $\xi = 0$ at $x = L$ yields $\cos kL = 0$. This requires

$$k_nL = (2n-1)\pi/2 \qquad n = 1, 2, 3, \ldots \tag{3.4.13}$$

and the natural frequencies are

$$f_n = [(2n-1)/4](c/L) \tag{3.4.14}$$

The frequency of the fundamental is half that of a similar free, free bar, and only the odd-numbered harmonics are present; the frequency of the first overtone of a free, fixed bar is three times that of its fundamental. Because of the absence of even harmonics, the quality of the sound produced by a vibrating free, fixed bar differs markedly from that produced by a free, free bar.

3.5 THE FREE, MASS-LOADED BAR

In many practical applications, a vibrating bar is neither rigidly fixed nor completely free to move at its ends. Instead, it may be loaded with some kind of mechanical impedance, most commonly of the mass-controlled type.

To analyze this type of constraint, consider a bar that is free at $x = 0$ and is loaded with a concentrated mass m at $x = L$. (Ideally, this mass should be a point mass; otherwise it will not move as a unit but will have waves propagated through it.) The boundary condition $\partial\xi/\partial x = 0$ at $x = 0$ applied to (3.3.10) leads again to

$$\xi(x, t) = 2\mathbf{A}e^{j\omega t}\cos kx \tag{3.5.1}$$

The boundary condition at $x = L$ is obtained by the following argument. Since a positive value for f was chosen to indicate compression of the bar, the reaction to such a force will accelerate the mass attached to the right end of the bar toward the right. Since the mass is attached to the bar, the end of the bar and the mass must experience the same acceleration. Thus, the boundary condition must be

$$\mathbf{f}_L = m\left(\frac{\partial^2 \xi}{\partial t^2}\right)_{x=L} \tag{3.5.2}$$

or, with the help of (3.3.3),

$$-SY\left(\frac{\partial \xi}{\partial x}\right)_{x=L} = m\left(\frac{\partial^2 \xi}{\partial t^2}\right)_{x=L} \tag{3.5.3}$$

Applying the above boundary conditions to $\boldsymbol{\xi}$ gives $kSY \sin kL = -m\omega^2 \cos kL$, or

$$\tan kL = -m\omega c/SY \tag{3.5.4}$$

There is no explicit solution of this transcendental equation. For very small mass loading, however, $m \approx 0$ so that $\tan kL \approx 0$ or $kL \approx n\pi$, which is the condition for the natural frequencies of a free, free bar. Such a result is obviously to be expected, since for very light loadings the bar is essentially free at both ends. Similarly, for heavy mass loadings the mass acts very much like a rigid support, and the natural frequencies approach those of a free, fixed bar.

It should be noted that in practice "fixing" the end of a bar amounts to loading it with a large mass, the mass of the support. For light bars a heavy support will act essentially as an infinite mass, and hence like a rigid restraint, but for heavy bars it may be very difficult, if not impossible, to approximate the fixed condition.

The general case of mass loading can be solved most readily by graphical or numerical means. It will be convenient to replace Y by ρc^2 and to let $m_b = \rho SL$ represent the mass of the bar. Then (3.5.4) becomes

$$\tan kL = -(m/m_b)kL \tag{3.5.5}$$

This transcendental equation is identical with (2.9.23), developed for the forced, mass-loaded string, except that m_b (the mass of the bar) replaces m_s (the mass of the string). Analysis proceeds exactly as before. If we choose the special case $m_b = m$, then the allowed values of kL solving (3.5.5) are $kL = 2.03, 4.91, 7.98, \ldots$. The nodes of the vibrations must occur where

$$\cos kx = 0 \tag{3.5.6}$$

and the fundamental mode, for which $kL = 2.03$, yields a node at

$$2.03\,x/L = \pi/2 \quad \text{or} \quad x = 0.774\,L \tag{3.5.7}$$

In contrast with the free, free bar, the node is no longer at the center but has shifted toward the loading mass, as suggested by Fig. 3.5.1. The bar could be supported at this nodal position without interfering with the fundamental mode of vibration.

Clearly, as the value of m changes from $m \ll m_b$ to $m \gg m_b$, the position of the node of the fundamental shifts from $x \approx L/2$ to $x \approx L$. Thus, the larger the mass

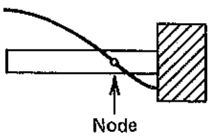

Figure 3.5.1 Fundamental mode of longitudinal vibration of a free, mass-loaded bar.

attached to a free, mass-loaded bar, the more the nodes of each normal mode of vibration are shifted toward the mass-loaded end.

Note that the overtones of the free, mass-loaded bar are not harmonics. The presence of nonharmonic overtones is sometimes advantageous in practical applications. As an illustration, consider a mass-loaded nickel tube that is intended to generate a pure tone and is driven magnetostrictively by alternating currents in a coil mounted on the tube. Harmonic frequency components other than the desired fundamental will be present in the output unless the current produced by the oscillator–amplifier unit driving the tube is well filtered. However, since the overtones of the mass-loaded tube are not harmonics of the fundamental, they will not be resonant at the harmonics of the driving current, and hence will be weakly excited, if at all.

*3.6 THE FREELY VIBRATING BAR: GENERAL BOUNDARY CONDITIONS

For a freely vibrating bar with arbitrary loading on each end, the normal modes of vibration can be determined in terms of the mechanical impedance at each end of the bar. If the mechanical impedance of the support at $x = 0$ is $\mathbf{Z}_{m0}$, the force acting on this support due to the bar is

$$\mathbf{f}_0 = -\mathbf{Z}_{m0}\mathbf{u}(0, t) \tag{3.6.1}$$

where the minus sign arises because a positive compressive force in the bar leads to an acceleration of the support to the left. On the other hand, a positive compressive force at the end $x = L$ leads to an acceleration of the adjacent support to the right so that the force acting on this support is

$$\mathbf{f}_L = +\mathbf{Z}_m\mathbf{u}(L, t) \tag{3.6.2}$$

where $\mathbf{Z}_m$ is the mechanical impedance of the support at the right end of the bar.

These equations can be expressed in terms of the particle displacement by using (3.3.3) to replace the compressive force and writing $\mathbf{u} = \partial\xi/\partial t$,

$$\begin{aligned}\left(\frac{\partial \xi}{\partial x}\right)_{x=0} &= \frac{\mathbf{Z}_{m0}}{\rho_L c^2}\left(\frac{\partial \xi}{\partial t}\right)_{x=0}\\ \left(\frac{\partial \xi}{\partial x}\right)_{x=L} &= -\frac{\mathbf{Z}_{m}}{\rho_L c^2}\left(\frac{\partial \xi}{\partial t}\right)_{x=L}\end{aligned} \tag{3.6.3}$$

where $\rho_L = \rho S$ is the density per unit length of the bar.

The choice of a trial solution satisfying the lossless wave equation for the bar and the boundary conditions (3.6.3) depends on the natures of the impedances $\mathbf{Z}_{m0}$ and $\mathbf{Z}_m$. If these loads are purely reactive, there can be no loss of acoustic energy so that there is no temporal or spatial damping. An appropriate trial solution would then be (3.3.10). We may go one step further and notice that, since there are no losses, the wave traveling to the right must possess the same energy as that going to the left. The amplitudes must therefore be equal, $|\mathbf{A}| = |\mathbf{B}|$. Application of the boundary conditions (3.6.3) then amounts to determining the phase angles of these complex amplitudes.

On the other hand, if either or both of $\mathbf{Z}_{m0}$ and $\mathbf{Z}_m$ have resistive components, a more general trial solution must be assumed. As was noted in the earlier discussion, in Section 2.11(b) on the freely vibrating string terminated by a resistive support, the presence of resistance requires that there be temporal damping. This means that the temporal behavior of the vibrating bar must be described by a complex angular frequency $\boldsymbol{\omega} = \omega + j\beta$ whose real part is the angular frequency of vibration and whose imaginary part is the temporal absorption coefficient β. Since there are no internal losses in the bar, the wave equation is still (3.3.7). Thus, we postulate

$$\boldsymbol{\xi}(x, t) = (\mathbf{A}e^{-j\mathbf{k}x} + \mathbf{B}e^{j\mathbf{k}x})e^{j\boldsymbol{\omega} t} \tag{3.6.4}$$

where $\mathbf{k}$ is determined by $\boldsymbol{\omega} = c\mathbf{k}$. Application of the boundary conditions (3.6.3) to the generalized trial solution (3.6.4) yields the pair of equations

$$\begin{aligned} \mathbf{A} - \mathbf{B} &= -(\mathbf{Z}_{m0}/\rho_L c)(\mathbf{A} + \mathbf{B}) \\ \mathbf{A}e^{-j\mathbf{k}L} - \mathbf{B}e^{j\mathbf{k}L} &= (\mathbf{Z}_m/\rho_L c)(\mathbf{A}e^{-j\mathbf{k}L} + \mathbf{B}e^{j\mathbf{k}L}) \end{aligned} \tag{3.6.5}$$

The first equation is solved for $\mathbf{B}$ in terms of $\mathbf{A}$ and this is substituted into the second equation. The results are

$$\begin{aligned} \mathbf{B} &= \frac{1 + (\mathbf{Z}_{m0}/\rho_L c)}{1 - (\mathbf{Z}_{m0}/\rho_L c)}\mathbf{A} \\ \tan \mathbf{k}L &= j\frac{(\mathbf{Z}_{m0}/\rho_L c) + (\mathbf{Z}_m/\rho_L c)}{1 + (\mathbf{Z}_{m0}/\rho_L c)(\mathbf{Z}_m/\rho_L c)} \end{aligned} \tag{3.6.6}$$

Given the impedances $\mathbf{Z}_{m0}$ and $\mathbf{Z}_m$, the properties of the vibration have been obtained, although explicit solution is not in general easy. Any resistive component in $\mathbf{Z}_{m0}$ or $\mathbf{Z}_m$ causes the argument of the tangent to be complex, introducing calculational difficulties in solving this transcendental equation.

*3.7 FORCED VIBRATIONS OF A BAR: RESONANCE AND ANTIRESONANCE REVISITED

In discussing the behavior of a forced, loaded string (Section 2.9), we defined resonance to occur when the speed amplitude was as large as possible and antiresonance to occur when the speed amplitude was as small as possible. It was seen there that resonance corresponded to the vanishing of the input mechanical reactance and that antiresonance corresponded, for purely *reactive* loads, to the reactance becoming infinite. We will now investigate these concepts in more detail and show that they must be modified for loads with a nonzero resistive component. We will find that at resonance the speed amplitude is maximized and at antiresonance it is minimized, but both resonance and antiresonance correspond to the vanishing of the input reactance.

Assume that a bar of length L is driven at $x = 0$ with a force $\mathbf{f}_0 = F_0 \exp(j\omega t)$ and is terminated at $x = L$ by a support possessing a mechanical impedance $\mathbf{Z}_m$. We assume the trial solution (3.3.10). The boundary condition at the forced end is (3.3.3):

$$F_0 e^{j\omega t} = -\rho_L c^2 \left(\frac{\partial \xi}{\partial x}\right)_{x=0} \tag{3.7.1}$$

where $\rho_L = \rho S$ and $Y = \rho c^2$. At the loaded end, the boundary condition is $\mathbf{f}_L = \mathbf{Z}_m \mathbf{u}(L, t)$:

$$\left(\frac{\partial \xi}{\partial x}\right)_{x=L} = -\frac{\mathbf{Z}_m}{\rho_L c^2}\left(\frac{\partial \xi}{\partial t}\right)_{x=L} \tag{3.7.2}$$

Either direct application of these boundary conditions to (3.6.4) or argument by analogy will determine **A** and **B** and the input mechanical impedance.

Let us argue by analogy. Direct comparison of (3.7.1) and (3.7.2) with (2.9.16) and (2.9.17) reveals that the boundary conditions are identical if we substitute $\mathbf{Z}_m$ for $j\omega m$. Since the trial solution remains unchanged, the same substitution into (2.9.22) gives a generalized form of the input impedance,

$$\mathbf{Z}_{m0} = \rho_L c \frac{(\mathbf{Z}_m/\rho_L c) + j \tan kL}{1 + (\mathbf{Z}_m/\rho_L c) j \tan kL} \tag{3.7.3}$$

If we define a scaled mechanical impedance by

$$\mathbf{Z}_m/\rho_L c = R/\rho_L c + jX/\rho_L c = r + jx \tag{3.7.4}$$

then (3.7.3) can be rewritten as

$$\frac{\mathbf{Z}_{m0}}{\rho_L c} = \frac{r + j(x + \tan kL)}{(1 - x \tan kL) + jr \tan kL} \tag{3.7.5}$$

(Here and to the end of this section, x represents the scaled mechanical reactance.)

It is left as an exercise to show that for $r = 0$ the input impedance is purely reactive and vanishes for frequencies such that $\tan kL = -x$ and becomes infinite when $\tan kL = 1/x$. Since the driving force amplitude is assumed constant, the vanishing of the input impedance $\mathbf{Z}_{m0} = \mathbf{f}_0/\mathbf{u}(0, t)$ means that the speed amplitude at the point of application of the force is infinite, the condition for mechanical *resonance* ($\tan kL = -x$). On the other hand, when the input impedance becomes infinite, the speed amplitude at the driver goes to zero, the condition for mechanical *antiresonance* ($\tan kL = 1/x$).

When the load resistance is not zero, the input impedance (3.7.5) has vanishing reactance if the phases of numerator and denominator are the same. This provides the condition

$$\frac{x + \tan kL}{r} = \frac{r \tan kL}{1 - x \tan kL} \tag{3.7.6}$$

which can be rewritten as a quadratic

$$x \tan^2 kL + (r^2 + x^2 - 1)\tan kL - x = 0 \tag{3.7.7}$$

Under the condition $|r^2 + x^2 - 1| \gg 2|x|$ and with the help of $\sqrt{1+\varepsilon} \approx 1 + \varepsilon/2$ for small ε, the roots are approximately

$$\tan kL \approx \begin{cases} x/(r^2 + x^2 - 1) \\ -(r^2 + x^2 - 1)/x \end{cases} \tag{3.7.8}$$

Assume that the mechanical support at the right end of the bar has small losses so that $r \ll 1$. Then x must be either large ($x \gg 1$) or small ($x \ll 1$). In either case, the pair of roots simplifies to

$$\tan kL \approx \begin{cases} -x \\ 1/x \end{cases} \tag{3.7.9}$$

Substitution of these roots into (3.7.5) gives (1) for $\tan kL \approx -x$

$$\mathbf{Z}_{m0}/\rho_L c \sim r \tag{3.7.10}$$

regardless of whether $x \gg 1$ or $x \ll 1$, and (2) for $\tan kL \approx 1/x$

$$\mathbf{Z}_{m0}/\rho_L c \sim 1/r \tag{3.7.11}$$

whether x is large or small. (These results are approximate, but simple. Higher accuracy would require much more mathematical manipulation and would result in expressions more complicated than warranted for our discussion.) For small load resistance and both large and small load reactance, the first root $\tan kL \approx -x$ corresponds to resonance, since the input impedance is real and small so that the velocity at the driving point has large amplitude. The second root $\tan kL \approx 1/x$ corresponds to antiresonance since the input impedance is real and large so that the velocity amplitude at the driving point is small. Both resonance and antiresonance frequencies occur when the input mechanical reactance vanishes. The input resistance is small at resonance and large at antiresonance.

These observations are consistent with the standing wave having large amplitude at resonance and small amplitude at antiresonance. For example, for the forced, nearly free bar there must be an antinode close to the end at L so that the maximum particle speed within the bar is nearly $U_L = |\mathbf{u}(L, t)|$. The power transmitted from the bar into the load at L is approximated by

$$\Pi \approx \tfrac{1}{2}U_L^2 R \tag{3.7.12}$$

while the power sent into the bar both at resonance and at antiresonance is

$$\Pi = \tfrac{1}{2}F_0^2/R_0 \tag{3.7.13}$$

where R_0 is the input mechanical resistance found from (3.7.5). Since the bar itself is assumed to be lossless, these powers must be equal and we can solve for the approximate antinodal speed amplitude, $U_L \approx F_0/\sqrt{R_0 R}$. Substitution of the appropriate values of R_0 gives $U_L \approx F_0/R$ at resonance and $U_L \approx F_0/\rho_L c$ at antiresonance, so that

$$\frac{U_L(antiresonance)}{U_L(resonance)} \approx \frac{R}{\rho_L c} \tag{3.7.14}$$

Since we have assumed $R \ll \rho_L c$, it is clear that the standing wave has much greater amplitude at resonance than at antiresonance.

Examination of the case $r \gg 1$ leads to analogous results and the same conclusions about resonance and antiresonance.

*3.8 TRANSVERSE VIBRATIONS OF A BAR

A bar is capable of vibrating transversely as well as longitudinally, and the internal coupling between strains makes it difficult to produce one motion without the other. For example, if

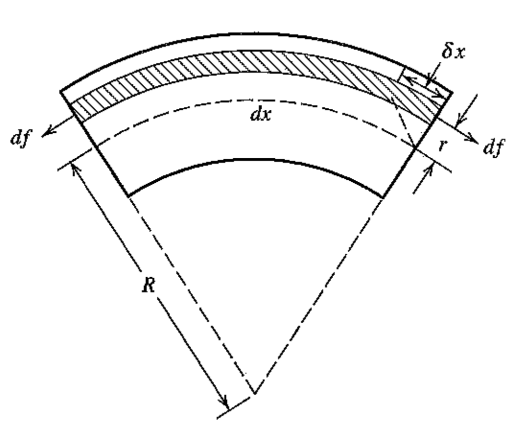

Figure 3.8.1 Bending strains and stresses set up by the transverse displacements at the ends of an element of a bar of length dx with radius of curvature R.

a long thin bar is supported at its center and set into vibration by a hammer blow directed along the axis of the bar, any slight eccentricity of the blow results in predominantly transverse vibrations rather than the desired longitudinal vibrations.

Consider a straight bar of length L, having a uniform cross section S with bilateral symmetry. Let the x coordinate measure positions along the bar, and the y coordinate the transverse displacements of the bar from its normal configuration. When the bar is bent as indicated in Fig. 3.8.1, the lower part is compressed and the upper part is stretched. Somewhere between the top and the bottom of the bar there will be a *neutral axis* whose length remains unchanged. (If the cross section of the bar is symmetric about a horizontal plane, this neutral axis will coincide with the central axis of the bar.)

Now consider a segment of the bar of length dx, and assume that the bending of the bar is measured by the radius of curvature R of the neutral axis. Let $\delta x = (\partial\xi/\partial x)\, dx$ be the increment of length, due to bending, of a filament of the bar located at a distance r from the neutral axis. Then the longitudinal force df is given by

$$df = -Y\, dS\, (\delta x/dx) = -Y\, dS\, (\partial\xi/\partial x) \tag{3.8.1}$$

where dS is the cross-sectional area of the filament. The value of δx for the particular filament considered in Fig. 3.8.1 is positive, so that df is a tension, and consequently negative. For filaments below the neutral axis δx is negative, giving a positive force of compression.

Now from the geometry $(dx + \delta x)/(R + r) = dx/R$ and hence $\delta x/dx = r/R$. Substitution into (3.8.1) yields

$$df = -(Y/R) r\, dS \tag{3.8.2}$$

The total longitudinal force $f = \int df$ is zero, negative forces above the neutral axis being canceled by positive forces below it. However, a bending moment M is present in the bar,

$$M = \int r\, df = -\frac{Y}{R}\int r^2\, dS \tag{3.8.3}$$

If we define a constant κ by

$$\kappa^2 = \frac{1}{S}\int r^2\, dS \tag{3.8.4}$$

then

$$M = -YS\kappa^2/R \tag{3.8.5}$$

The constant κ can be thought of as the *radius of gyration* of the cross-sectional area S, by analogy with the definition of the radius of gyration of a solid. The value of κ for a bar of rectangular cross section and thickness t (measured in the y direction) is $\kappa = t/\sqrt{12}$. For a circular rod of radius a, $\kappa = a/2$.

The radius of curvature R is not in general a constant but is rather a function of position along the neutral axis. If the displacements y of the bar are limited to small values, $\partial y/\partial x \ll 1$, then we may use the approximate relation

$$R = \frac{[1 + (\partial y/\partial x)^2]^{3/2}}{\partial^2 y/\partial x^2} \approx \frac{1}{\partial^2 y/\partial x^2} \tag{3.8.6}$$

Substitution of (3.8.6) into (3.8.5) yields

$$M = -YS\kappa^2(\partial^2 y/\partial x^2) \tag{3.8.7}$$

In the situation illustrated in Fig. 3.8.1, the curvature makes $\partial^2 y/\partial x^2$ negative, and the bending moment M is consequently positive. It is apparent that to obtain the curvature illustrated, the torque applied to the left end of the segment dx must act in a counterclockwise or positive angular direction, so that (3.8.7) gives the torque acting on the left end of the segment both as to magnitude and as to direction. Similarly, the torque acting on the right end of the segment must be clockwise, with the result that it is negative and is therefore represented both in direction and in magnitude by $-M$.

*3.9 TRANSVERSE WAVE EQUATION

The effect of distorting the bar is to produce not only bending moments but also shear forces. Consider an upward shear force F_y acting on the left end of the segment dx as positive (Fig. 3.9.1). Then the associated shear force acting on the right end of the segment must be downward, and is consequently negative. When a bent bar is in a condition of static equilibrium, the torques and shear forces acting on any segment must produce no net turning moment. Taking moments about the left end of the segment of Fig. 3.9.1, we have

$$M(x) - M(x + dx) = F_y(x + dx)\,dx \tag{3.9.1}$$

For segments of small length dx, $M(x + dx)$ and $F_y(x + dx)$ can be expanded in Taylor's expansions about x, and this yields

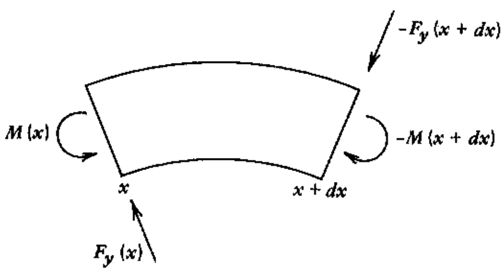

Figure 3.9.1 Bending moments and shear forces set up by the transverse displacements at the ends of an element of a bar of length dx.

$$F_y = -(\partial M/\partial x) = YS\kappa^2(\partial^3 y/\partial x^3) \tag{3.9.2}$$

where second-order terms in dx have been dropped.

This relation between the shear force F_y and the bending moment M has been derived for a condition of static equilibrium. For transverse vibrations of a bar the equilibrium is dynamic, rather than static, and the right side of (3.9.1) must equal the rate of increase of angular momentum of the segment. However, if the displacement and slope of the bar are limited to small values, the variations in angular momentum may be neglected, and (3.9.2) serves as an adequate approximation for the relation between F_y and y.

The net upward force dF_y acting on the segment dx is then given by

$$dF_y = F_y(x) - F_y(x + dx) = -\left(\frac{\partial F_y}{\partial x}\right)dx = -YS\kappa^2\left(\frac{\partial^4 y}{\partial x^4}\right)dx \tag{3.9.3}$$

By Newton's second law, this force will give the mass ($\rho S\,dx$) of the segment an upward acceleration $\partial^2 y/\partial t^2$ so that the equation of motion is

$$\frac{\partial^2 y}{\partial t^2} = -(\kappa c)^2\frac{\partial^4 y}{\partial x^4} \tag{3.9.4}$$

where $c^2 = Y/\rho$. One significant difference between this differential equation and the simpler equation for the transverse waves on a string is the presence of a fourth partial derivative with respect to x, rather than a second partial. As a result, direct substitution shows that functions of the form $f(ct - x)$ are not solutions of (3.9.4). Transverse waves do not travel along the bar with a constant speed c and unchanging shape.

Assume that (3.9.4) may be solved by separation of variables, and write the complex transverse displacement as

$$\mathbf{y}(x, t) = \mathbf{\Psi}(x)e^{j\omega t} \tag{3.9.5}$$

Upon substitution, the exponential function of time cancels, leaving a new *total* differential equation involving $\mathbf{\Psi}$,

$$\boxed{\begin{aligned}\frac{d^4\mathbf{\Psi}}{dx^4} &= \left(\frac{\omega}{v}\right)^4\mathbf{\Psi}\\ v^2 &= \omega(\kappa c)\end{aligned}} \tag{3.9.6}$$

where v has the dimensions of a speed. If we substitute a trial function $\mathbf{\Psi} = \exp(\gamma x)$ into (3.9.6), it is valid for $\gamma = \pm(\omega/v)$ and $\pm j(\omega/v)$. Thus, if we define a quantity g by

$$g = \omega/v \tag{3.9.7}$$

then a complete monofrequency solution can be written as

$$\begin{aligned}\mathbf{\Psi}(x) &= \mathbf{A}e^{gx} + \mathbf{B}e^{-gx} + \mathbf{C}e^{jgx} + \mathbf{D}e^{-jgx}\\ \mathbf{y}(x, t) &= (\mathbf{A}e^{gx} + \mathbf{B}e^{-gx})e^{j\omega t} + \mathbf{C}e^{j(\omega t + gx)} + \mathbf{D}e^{j(\omega t - gx)}\end{aligned} \tag{3.9.8}$$

where $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, and $\mathbf{D}$ are arbitrary constants and g is both a *wave number* and also a *spatial attenuation coefficient*. Note that g is proportional to $\sqrt{\omega}$. The solution represents *flexural* disturbances of two kinds: (1) two traveling waves each propagating with a *phase speed* v proportional to $\sqrt{\omega}$ and (2) two standing oscillations that are spatially damped, each with a spatial attenuation coefficient g depending on $\sqrt{\omega}$. Waves of different frequencies travel with different phase speeds, an effect known as *dispersion*. The high-frequency components outrun the low-frequency components, altering the shape of the wave. This is analogous to the transmission of light through glass, wherein the different component frequencies of a light beam travel with different speeds. A vibrating bar is a *dispersive medium* for transverse waves.

The actual solution of (3.9.4) is the real part of (3.9.8). It may conveniently be expressed using hyperbolic and trigonometric identities (see Appendix A3),

$$y(x,t) = [A\cosh gx + B\sinh gx + C\cos gx + D\sin gx]\cos(\omega t + \phi) \tag{3.9.9}$$

where A, B, C, and D are new real constants. Although these constants are related to the complex constants $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, and $\mathbf{D}$, the relationships are unimportant, since in practice A, B, C, and D are evaluated directly through the application of initial and boundary conditions.

*3.10 BOUNDARY CONDITIONS

Since (3.9.9) contains twice as many arbitrary constants as the corresponding equation for the transverse vibrations of a string, the determination of these constants requires twice as many boundary conditions. This need is fulfilled by the existence of pairs of boundary conditions at the ends of the bar. The particular forms of these conditions depend on the nature of the support and include the following.

(a) Clamped End

If an end of the bar is rigidly clamped, both the displacement and the slope must be zero at that end for all times t. The boundary conditions are therefore

$$y = 0 \quad \text{and} \quad \frac{\partial y}{\partial x} = 0 \tag{3.10.1}$$

(b) Free End

At a free end there can be neither an externally applied torque nor a shearing force, and hence both M and F_y are zero at the end. However, the displacement and slope are not constrained, except that their values must be small. Then from (3.8.7) and (3.9.2), the boundary conditions are

$$\frac{\partial^2 y}{\partial x^2} = 0 \quad \text{and} \quad \frac{\partial^3 y}{\partial x^3} = 0 \tag{3.10.2}$$

(c) Simply Supported End

A simply supported end is obtained by constraining that end of the bar between a pair of knife edges mounted perpendicular to the plane of the transverse motion and centered on the neutral axis of the bar (or a pair of needle points, similarly placed on the neutral axis) so that both the transverse displacement and the torque are zero with no constraint on the slope.

$$y = 0 \quad \text{and} \quad \frac{\partial^2 y}{\partial x^2} = 0 \tag{3.10.3}$$

*3.11 BAR CLAMPED AT ONE END

Assume that a bar of length L is rigidly clamped at $x = 0$ and is free at $x = L$. Then applying the two conditions of (3.10.1) at $x = 0$ to the general solution of (3.9.9) we obtain $A + C = 0$ and $B + D = 0$ so that the general solution reduces to

$$y(x, t) = [A(\cosh gx - \cos gx) + B(\sinh gx - \sin gx)] \cos(\omega t + \phi) \tag{3.11.1}$$

A further application of the two conditions of (3.10.2) at $x = L$ gives

$$\begin{aligned} A(\cosh gL + \cos gL) &= -B(\sinh gL + \sin gL) \\ A(\sinh gL - \sin gL) &= -B(\cosh gL + \cos gL) \end{aligned} \tag{3.11.2}$$

While it is impossible for both of these equations to be true for all frequencies, at certain frequencies they become equivalent. To determine these allowed frequencies, divide one equation by the other, thus canceling out the constants A and B. Then cross-multiply and simplify by using the identities $\cos^2 \theta + \sin^2 \theta = 1$ and $\cosh^2 \theta - \sinh^2 \theta = 1$. This gives

$$\cosh gL \cos gL = -1 \tag{3.11.3}$$

It is easy to obtain the allowed values of gL by numerical techniques, particularly since the hyperbolic cosine grows as $\exp(gL)$ so that the cosine must approach zero very closely for arguments greater than about π. Solutions are found to be

$$gL = \omega L/v = (1.194, 2.988, 5, 7, \ldots)\pi/2 \tag{3.11.4}$$

Substituting $v = \sqrt{\omega \kappa c}$ into (3.11.4) and squaring both sides, we have for the natural frequencies of a transversely vibrating clamped, free bar

$$f = (1.194^2, 2.988^2, 5^2, 7^2, \ldots)\pi\kappa c/8L^2 \tag{3.11.5}$$

The application of boundary conditions limits the natural modes to a discrete set, just as it does for a vibrating string. However, in contrast with the string, the overtone frequencies are not harmonics of the fundamental. As shown in Table 3.11.1, the first overtone has a frequency more than six times that of the fundamental. If a bar is struck so that the amplitudes of vibration of its overtones are appreciable, the sound produced has a metallic quality. However, the overtones are rapidly damped so that the initial sound mellows into a nearly pure tone at the fundamental. Vibrating reeds in music boxes give good examples of this behavior. Note that the fundamental frequency can be adjusted by varying either the thickness or the length; doubling the length lowers the frequency by a factor of four.

The distribution of nodal points along the bar is more complex than in the examples previously considered, for the nodes are not evenly placed at intervals of $\lambda/2$ but are irregularly spaced. See Fig. 3.11.1. There are *three types* of nodal points where $y = 0$: (1) the node where the bar is clamped, characterized by the additional condition $\partial y/\partial x = 0$; (2) the so-called *true* nodes, lying nearly $\lambda/2$ apart and very close to the points of inflection where $\partial^2 y/\partial x^2 \approx 0$; and (3) the node adjacent to the free end of the bar. (A point of inflection does not lie near this last nodal position but instead is shifted out to the end.) It is also to be noted that the vibrational amplitudes at the various antinodal positions are not the same and that the antinode at the free end has the greatest amplitude of motion.

Table 3.11.1 gives the nodal positions for transverse vibrations of a bar 100 cm in length (clamped at $x = 0$ and free at $x = 100$ cm), the ratios of the frequencies and phase speeds of the overtones to those for the fundamental, and the wavelengths $\lambda = v/f$ for each natural frequency. The increase in phase speed with frequency is quite apparent. As discussed

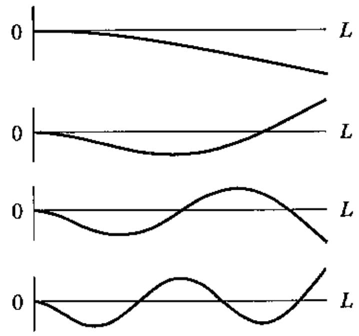

Figure 3.11.1 The four lowest modes of transverse vibration of a clamped, free bar. Note the boundary conditions at each end and the different classes of the nodes.

Table 3.11.1 Transverse vibration characteristics of a clamped, free bar with $L = 100$ cm

Frequency	*Phase Speed*	*Wavelength (cm)*	*Nodal Positions (cm from clamped end)*
f_1	v_1	335.0	0
$6.267f_1$	$2.50v_1$	133.4	0, 78.3
$17.55f_1$	$4.18v_1$	80.0	0, 50.4, 86.8
$34.39f_1$	$5.87v_1$	57.2	0, 35.8, 64.4, 90.6

earlier, the wavelengths are not in general equal to twice the distance between adjacent nodes. However, the nodal spacing between true nodes for the third overtone is $\lambda/2$ within the accuracy of the data [$64.4 - 35.8 = 28.6 = (57.2)/2$ cm].

*3.12 BAR FREE AT BOTH ENDS

Another important kind of transverse vibration is that of a free, free bar. The boundary conditions are satisfied at $x = 0$ if $A - C = 0$ and $B - D = 0$. Application of the same conditions at $x = L$ and the same kind of trigonometric and hyperbolic reductions yields the transcendental equation

$$\cosh gL \cos gL = 1 \tag{3.12.1}$$

As before, numerical solution is relatively simple, and we obtain the natural frequencies for the transversely vibrating free, free bar,

$$f = (3.011^2, 5^2, 7^2, 9^2, \ldots)\pi\kappa c/8L^2 \tag{3.12.2}$$

Again the overtones are not harmonics of the fundamental.

Table 3.12.1 gives information concerning the frequencies, phase speeds, and nodal positions of a free, free bar 100 cm long. An inspection of Fig. 3.12.1 shows that the modes

Table 3.12.1 Transverse vibration characteristics of a free, free bar with $L = 100$ cm

Frequency	*Phase Speed*	*Wavelength (cm)*	*Nodal Positions (cm from end)*
f_1	v_1	133.0	22.4, 77.6
$2.756f_1$	$1.66v_1$	80.0	13.2, 50.0, 86.8
$5.404f_1$	$2.32v_1$	57.2	9.4, 35.6, 64.4, 90.6
$8.933f_1$	$2.99v_1$	44.5	7.3, 27.7, 50.0, 72.3, 92.7

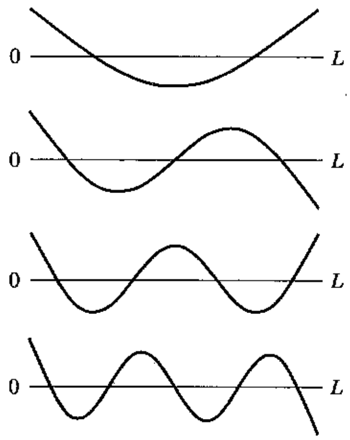

Figure 3.12.1 The four lowest modes of transverse vibration of a free, free bar. Note the boundary conditions at each end and the different classes of the nodes.

of vibration corresponding to the fundamental and all *even* overtones (corresponding to $f_1, f_3, f_5, \ldots$ in the figure) are symmetric about the center. There is a *true antinode* at the center where $\partial y/\partial x = 0$. In contrast, the *odd* overtones ($f_2, f_4, f_6, \ldots$) correspond to antisymmetric modes with respect to the center. In all modes, the nodes are symmetrically distributed about the center. The bar may be supported on a knife edge, or held by a knife-edge clamp, at any nodal point without interfering with the mode of vibration having a node at this point. A knife-edge clamp (or needle-point support) is required, since it must merely restrict the displacement to zero and must not restrict the changes in slope that occur at a node.

Each bar of a xylophone is supported at points corresponding to the nodes of its fundamental. Since the nodes of the accompanying overtones will not in general be located at these same two points, the overtones will rapidly be damped out, leaving the fundamental. This is one of a number of factors that contribute to the mellow sound of a xylophone or marimba.

The free, free bar may be used qualitatively to describe a tuning fork. This is basically a ßU-shaped bar with a stem attached to the center. The bend and the mass-loading of the stem reduce the separation of the two nodes present in the fundamental mode. Compare Fig. 3.12.2 with Fig. 3.12.1. As above, when a tuning fork is struck the overtones rapidly dampen, leaving the fundamental frequency. The stem, at an antinode, vibrates and couples the motion to any surface it touches. The radiation efficiency is enhanced if the surface has large area, or forms a side of a resonator box tuned to the fundamental.

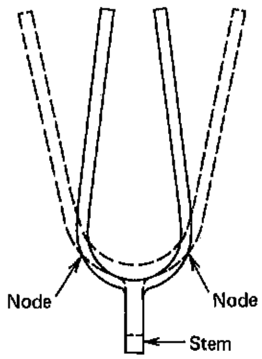

Figure 3.12.2 Vibration of a tuning fork.

If a bar is rigidly clamped at both ends, the boundary conditions $y = 0$ and $\partial y/\partial x = 0$ at the ends $x = 0$ and $x = L$ lead to the same set of natural frequencies as for a free, free bar. However, as is to be expected, the locations of the nodes are different.

*3.13 TORSIONAL WAVES ON A BAR

A bar is capable of vibrating torsionally as well as longitudinally and transversely. For example, if a long, thin bar (or a fiber used as an activator for a torsional-pendulum clock) is fixed at one end and the other end is twisted about the long axis of the bar, the restoring torque will increase as the angle of twist is increased. If the twisted end is then released, a torsional wave will travel down the bar.

For simplicity of discussion, let the bar have a circular cross section of radius a. Isolate an element of the bar of length dx (Fig. 3.13.1a). Break this element into a series of concentric

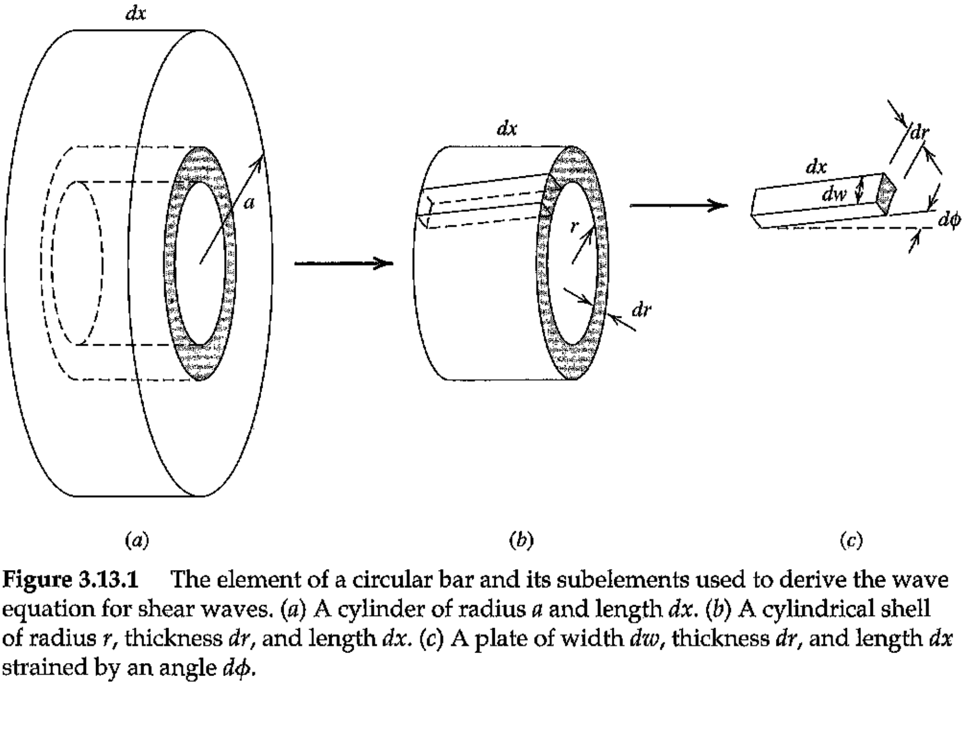

Figure 3.13.1 The element of a circular bar and its subelements used to derive the wave equation for shear waves. (a) A cylinder of radius a and length dx. (b) A cylindrical shell of radius r, thickness dr, and length dx. (c) A plate of width dw, thickness dr, and length dx strained by an angle $d\phi$.

hollow tubes of radius r and thickness dr (Fig. 3.13.1*b*), and further divide each hollow cylinder into side-by-side rectangular plates of length dx, thickness dr, and (curved) width dw (Fig. 3.13.1*c*). When the tube is twisted away from equilibrium through a small angle $d\phi$, this rectangle is distorted by an angle $r(d\phi/dx)$ which is the *shearing strain*. The *shearing stress* required to produce that shearing strain is proportional to it (Hooke's law) and the constant of proportionality is the *shear modulus* (the *modulus of rigidity*) $\mathscr{G}$ (see Appendix A11),

$$\text{stress} = \mathscr{G} r(d\phi/dx) \tag{3.13.1}$$

This is the torsional equivalent to (3.3.2). The force df required to produce this distortion is the shearing stress multiplied by the area over which that stress acts,

$$df = \mathscr{G}(dw\,dr)\,r(d\phi/dx) \tag{3.13.2}$$

The torque dM required to produce the strain in the hollow tube of height dx is found by multiplying df by its moment arm r and integrating around the circumference of the tube. Because of the circumferential symmetry, $\int dw = 2\pi r$ so the torque on a tube is

$$d\tau = \mathscr{G}(2\pi r^3)(d\phi/dx)\,dr \tag{3.13.3}$$

The total torque τ twisting this end of the elemental solid cylinder is found by integrating over the concentric tubes from $r = 0$ to $r = a$,

$$\tau = \mathscr{G}\tfrac{1}{2}\pi a^4(d\phi/dx) \tag{3.13.4}$$

The net torque on the elemental cylinder of height dx is the difference of the torques on each end, which by a Taylor's expansion is $\tau(x+dx) - \tau(x) = (d\tau/dx)\,dx = \mathscr{G}(\pi a^4/2)(\partial^2\phi/\partial x^2)\,dx$. This net torque is equal to the moment of inertia of the cylinder $(a^2/2)\,dm$, where $dm = \rho\pi a^2\,dx$, times its angular acceleration $\partial^2\phi/\partial t^2$. This gives the familiar one-dimensional wave equation

$$\frac{\partial^2\phi}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2} \tag{3.13.5}$$

with the phase speed c obtained from

$$\boxed{c^2 = \mathscr{G}/\rho} \tag{3.13.6}$$

All the techniques for finding the solutions for waves on strings and longitudinal waves in bars apply here. Examples of boundary conditions applicable to torsional oscillations include (1) fixed end, $\phi = 0$; (2) free end, $\tau = 0$ so that $\partial\phi/\partial x = 0$; and (3) mass-loaded end, $\partial\phi/\partial x = (\partial\phi^2/\partial x^2)I$, where I is the moment of inertia of the load about the axis of the bar.

Problems

3.4.1. A bar of length L is rigidly fixed at $x = 0$ and free to move at $x = L$. (*a*) Show that only odd integral harmonic overtones are allowed. (*b*) Determine the fundamental frequency of the bar, if it is composed of steel and has a length of 0.5 m. (*c*) If a static force F is applied to the free end of the bar so as to displace this end by h, show that, when the bar vibrates longitudinally subsequent to the sudden release of this force, the amplitudes of the various harmonic vibrations are given by $A_n = [8h/(n\pi)^2]\sin(n\pi/2)$. (*d*) Determine these amplitudes for the above steel bar, if the force is 5000 N and the cross-sectional area of the bar is 0.00005 m^2.

3.4.2. Verify whether or not the normal modes of a longitudinally vibrating bar form an orthogonal set if the boundary conditions are (*a*) fixed, fixed; (*b*) free, free; (*c*) fixed, free.

3.5.1. A steel bar of 0.0001 m^2 cross-sectional area and 0.25 m length is free to move at $x = 0$ and is loaded with 0.15 kg at $x = 0.25$ m. (*a*) Compute the fundamental frequency of longitudinal vibrations of the above mass-loaded bar. (*b*) Determine the position at which the bar may be clamped to cause the least interference with its fundamental mode of vibration. (*c*) When this bar is vibrating in its fundamental mode, what is the ratio of the displacement amplitude of the free end to that of the mass-loaded end? (*d*) What is the frequency of the first overtone of this bar?

3.5.2. A 2 kg mass is hanging on a steel wire of 0.00001 m^2 cross-sectional area and 1.0 m length. (*a*) Compute the fundamental frequency of vertical oscillation of the mass by considering it to be a simple oscillator. (*b*) Compute the fundamental frequency of vertical oscillation of the mass by considering the system to be that of a longitudinally vibrating bar fixed at one end and mass-loaded at the other. (*c*) Show that for $kL < 0.2$, the equation derived in (*b*) reduces to (1.2.4).

3.5.3 Are the normal modes orthogonal for (*a*) a fixed, mass-loaded bar, (*b*) a free, mass-loaded bar?

3.6.1. A thin bar of length L and mass M is fixed at one end and free at the other. What mass m must be attached to the free end to decrease the fundamental frequency of longitudinal vibration by 25% from its fixed, free value?

3.6.2. A steel bar of 0.2 m length and 0.04 kg mass is loaded at one end with 0.027 kg and at the other end with 0.054 kg. (*a*) Calculate the fundamental frequency of longitudinal vibration of this system. (*b*) Calculate the position of the node in the bar. (*c*) Calculate the ratio of the displacement amplitudes at the two ends of the bar.

3.6.3. Assuming very small losses, find a condition relating the mechanical impedances of the supports at the ends of a bar of length L and longitudinal wave speed c if the bar is to have an integral number of wavelengths between its ends when it is vibrating longitudinally.

3.6.4. Determine an expression giving the fundamental frequency of longitudinal vibrations of a fixed, free bar of length L and mass m, if the reaction of the fixture corresponds to a mechanical reactance of $-js/\omega$ (stiffness).

3.6.5. A bar of length L has circular cross-sectional area S. The material of the bar has a linear density ρ_L and a Young's modulus Y. The bar is terminated at $x = 0$ with a mass m and at $x = L$ by a longitudinal spring of spring constant s. (*a*) In terms of L, S, ρ_L, s, and Y, find the transcendental equation for kL that must be solved to find the normal modes for longitudinal motion. (*b*) The bar is aluminum with length 1 m and radius 1 cm. The mass m is 0.848 kg and the spring has $s = 2.23 \times 10^7$ N/m. Find the lowest eigenfrequency. (*c*) Find the nodal locations.

3.7.1. Show that for $r = 0$ the input impedance $\mathbf{Z}_{m0}$ is purely reactive and has magnitude zero when $\tan kL = -x$ and becomes infinitely large when $\tan kL = 1/x$.

3.7.2. Assume that the load mechanical impedance is $\mathbf{Z}_m/\rho_L c = 1 + jx$. (*a*) If the load reactance is large ($x \gg 1$) in the frequency range of interest, show that the equations determining resonance and antiresonance are $\tan kL = -x$ and $\tan kL = 1/x$, respectively. (*b*) If in the frequency range of interest the load reactance is small ($x \ll 1$) show that $\tan kL \approx \pm 1$ and the input impedance becomes $\mathbf{Z}_{m0} \approx \rho_L c$.

3.7.3. A long thin bar of length L is driven by a longitudinal force $F \cos \omega t$ at $x = 0$ and is free at $x = L$. (*a*) Derive the equation that gives the amplitude of the standing waves set up in the bar. (*b*) What is the input mechanical impedance? (*c*) What is the input mechanical impedance of a similar bar of infinite length? (*d*) If the material of the bar is aluminum, the length is 1.0 m, the cross-sectional area is 0.0001 m^2, and the amplitude of the driving force is 10 N, plot the amplitude of the driven end of the bar of part (*a*) as a function of frequency over the range from 200 to 2000 Hz.

3.7.4C. For the longitudinal vibrations of a bar driven at one end by a force of constant amplitude F and loaded at the other with an impedance $R + j(\omega m - s/\omega)$, plot the input power for frequencies covering the first three resonances for (*a*) three values of R keeping m and s constant, (*b*) three values of m keeping R and s constant, and (*c*) three values of s keeping R and m constant.

3.7.5. A thin bar of length L with longitudinal phase speed c is driven at $x = 0$ by a force with adjustable frequency. (*a*) Find the frequencies for which the driving force experiences an input mechanical impedance equal to the mechanical impedance of the support at $x = L$. (*b*) At the frequencies of (*a*), compare the amplitudes of the velocities of the two ends of the bar.

3.7.6. Show that for $r \gg 1$ the conclusions pertaining to resonance and antiresonance following (3.7.11) are also reached.

3.8.1. Find the radius of gyration for a bar of circular cross section of radius a.

3.8.2. Calculate the radius of gyration for a bar of rectangular cross section with thickness t and width w for bending in the direction of (*a*) the thickness, (*b*) the width, and (*c*) parallel to a diagonal.

3.9.1. Show by direct substitution that (3.9.9) is a solution of (3.9.4).

3.9.2. Show that $v = \sqrt{\omega \kappa c}$ has the dimensions of a speed. For what frequency will the transverse vibrations of an aluminum rod of 0.01 m diameter have the same phase speed as that of longitudinal vibrations in the rod?

3.10.1. An aluminum bar of 100 cm length with circular cross section of 1 cm diameter is simply supported at both ends. For transverse vibrations, (*a*) show that the normal modes are the same as for the fixed, fixed string. (*b*) Find the frequencies of the normal modes. (*c*) Are the overtones harmonics as they are for the fixed, fixed string?

3.11.1. For a bar of rectangular cross section with $w = 2t$, one end clamped and the other free, find the ratio of the fundamental frequencies of free vibration for bending in the direction of the thickness to that for bending in the direction of the width.

3.11.2. For a bar of length 100 cm clamped at both ends, find, in terms of the fundamental frequency and phase speed, the frequencies, phase speeds, wavelengths, and nodal positions of the first three normal modes of transverse vibration.

3.11.3C. For a clamped, free bar vibrating in its third transverse mode, (*a*) plot the displacement amplitude and first three derivatives of the displacement as a function of length. (*b*) Use these calculated values to show that the boundary conditions are satisfied at both ends and discuss the nature of each node.

3.11.4C. If the same bar used in creating Table 3.11.1 is clamped at both ends, (*a*) create a table similar to Table 3.11.1, and (*b*) plot the shape of the first four normal modes.

3.11.5. An aluminum bar 100 cm long with a 1.0 cm radius is clamped at one end and free at the other. (*a*) Find the frequency of the lowest mode of transverse vibration. (*b*) If the free end has a displacement amplitude of 5.0 cm, determine all the constants in the equation for the transverse displacement of the bar. (*c*) Plot the displacement amplitude of the bar.

3.12.1. A steel rod of 0.005 m radius has a length of 0.5 m. (*a*) What is its fundamental frequency of free, free transverse vibrations? (*b*) If the displacement amplitude at the center of the rod is 2 cm when vibrating in its fundamental mode, what is the displacement amplitude at the ends?

3.12.2. Calculate A/B for (*a*) the clamped, free bar and (*b*) the free, free bar.

3.13.1. A 100 cm long aluminum bar has a diameter of 1.0 cm. (*a*) Find the torque required to give one end of the bar a static twist of 360° relative to the other end. (*b*) Find the phase speed of torsional waves on this bar. (*c*) If the bar is rigidly supported in the middle and free at the ends, find the lowest frequency at which it will support a torsional normal mode.

3.13.2. For an aluminum bar free at both ends, find the ratio of the lowest frequencies of a normal mode of longitudinal vibration to the lowest frequency of a normal mode of torsional vibration.

Chapter **4**

THE TWO-DIMENSIONAL WAVE EQUATION: VIBRATIONS OF MEMBRANES AND PLATES

4.1 VIBRATIONS OF A PLANE SURFACE

Consider transverse vibrations of two-dimensional systems, such as a drumhead or the diaphragm of a microphone. While analysis may seem more complicated because two spatial coordinates are needed to locate a point on the surface and a third to specify its displacement, the equation of motion (subject to the same simplifying assumptions invoked in the previous two chapters) will be merely the two-dimensional generalization of that for a string.

Generalization to two dimensions requires selecting a coordinate system. Choice of a coordinate system matching the boundary conditions (cartesian coordinates for a rectangular boundary and polar coordinates for a circular boundary) will greatly simplify obtaining and interpreting solutions. Unfortunately, the number of useful coordinate systems is strictly limited and, consequently, the number of easily solved membrane problems is similarly restricted.

4.2 THE WAVE EQUATION FOR A STRETCHED MEMBRANE

Assume a membrane is thin, is stretched uniformly in all directions, and vibrates transversely with small displacement amplitudes. Let ρ_S be the *surface density* (kg/m^2) of the membrane, and let $\mathcal{T}$ be the *membrane tension per unit length* (N/m); the material on opposite sides of a line segment of length dl will be pulled apart with a force $\mathcal{T}\,dl$.

In cartesian coordinates the transverse displacement of a point is expressed as $y(x,z,t)$. The force acting on a displaced surface element of area $dS = dx\,dz$ is the sum of the transverse forces acting on the edges parallel to the x and z axes. For the element shown in Fig. 4.2.1 the net vertical force arising from the pair of opposing tensions $\mathcal{T}\,dz$ is

$$\mathcal{T}\,dz\left[\left(\frac{\partial y}{\partial x}\right)_{x+dx}-\left(\frac{\partial y}{\partial x}\right)_{x}\right]=\mathcal{T}\frac{\partial^2 y}{\partial x^2}\,dx\,dz \tag{4.2.1}$$

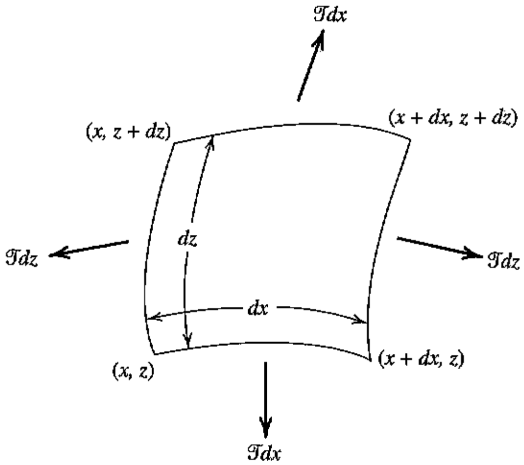

Figure 4.2.1 Elemental area of a membrane showing the forces acting when the membrane is displaced transversely.

and that from the pair of tensions $\mathcal{T}\,dx$ is $\mathcal{T}(\partial^2 y/\partial z^2)\,dx\,dz$. Equating the sum of these two to the product of the mass $\rho_S\,dx\,dz$ of the element and its acceleration $\partial^2 y/\partial t^2$ gives

$$\frac{\partial^2 y}{\partial x^2} + \frac{\partial^2 y}{\partial z^2} = \frac{1}{c^2}\frac{\partial^2 y}{\partial t^2} \tag{4.2.2}$$

with

$$\boxed{c^2 = \mathcal{T}/\rho_S} \tag{4.2.3}$$

Equation (4.2.2) may be expressed more generally in the form

$$\nabla^2 y = \frac{1}{c^2}\frac{\partial^2 y}{\partial t^2} \tag{4.2.4}$$

where ∇^2 is the *Laplacian operator* (in this case two-dimensional) and (4.2.4) is the *two-dimensional wave equation.*

The form of the Laplacian depends on the choice of the coordinate system. The Laplacian in two-dimensional cartesian coordinates,

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2} \tag{4.2.5}$$

is appropriate for rectangular membranes. For a circular membrane, polar coordinates (r, θ) are preferable and use of

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} \tag{4.2.6}$$

gives the appropriate wave equation,

$$\frac{\partial^2 y}{\partial r^2} + \frac{1}{r}\frac{\partial y}{\partial r} + \frac{1}{r^2}\frac{\partial^2 y}{\partial \theta^2} = \frac{1}{c^2}\frac{\partial^2 y}{\partial t^2} \tag{4.2.7}$$

Solutions to (4.2.4) will have all the properties of the waves studied previously, generalized to two dimensions. For calculating normal modes on membranes it is conventional to assume the solutions have the form

$$\mathbf{y} = \mathbf{\Psi} e^{j\omega t} \tag{4.2.8}$$

where $\mathbf{\Psi}$ is a function only of position. Substitution and identification of $k = \omega/c$ yields the *Helmholtz equation,*

$$\nabla^2 \mathbf{\Psi} + k^2 \mathbf{\Psi} = 0 \tag{4.2.9}$$

The solutions of (4.2.9) for a membrane with specified shape and boundary conditions are the normal modes of the problem.

4.3 FREE VIBRATIONS OF A RECTANGULAR MEMBRANE WITH FIXED RIM

If a stretched rectangular membrane is fixed at $x = 0, x = L_x, z = 0$, and $z = L_z$, the boundary conditions are

$$y(0, z, t) = y(L_x, z, t) = y(x, 0, t) = y(x, L_z, t) = 0 \tag{4.3.1}$$

Assuming a solution

$$\mathbf{y}(x, z, t) = \mathbf{\Psi}(x, z) e^{j\omega t} \tag{4.3.2}$$

to (4.2.4) gives

$$\frac{\partial^2 \mathbf{\Psi}}{\partial x^2} + \frac{\partial^2 \mathbf{\Psi}}{\partial z^2} + k^2 \mathbf{\Psi} = 0 \tag{4.3.3}$$

Now, apply the method of *separation of variables* by assuming that $\mathbf{\Psi}$ is the product of two functions, each dependent on only one of the dimensions,

$$\mathbf{\Psi}(x, z) = \mathbf{X}(x)\mathbf{Z}(z) \tag{4.3.4}$$

Substitution and division by $\mathbf{X}(x)\ \mathbf{Z}(z)$ gives

$$\frac{1}{\mathbf{X}}\frac{d^2\mathbf{X}}{dx^2} + \frac{1}{\mathbf{Z}}\frac{d^2\mathbf{Z}}{dz^2} + k^2 = 0 \tag{4.3.5}$$

Since the first term is a function only of x and the second only of z, both must be constants; otherwise the three terms cannot sum to zero for all x and z. This provides the pair of equations

$$\frac{d^2\mathbf{X}}{dx^2} + k_x^2\mathbf{X} = 0 \qquad \frac{d^2\mathbf{Z}}{dz^2} + k_z^2\mathbf{Z} = 0 \tag{4.3.6}$$

where the constants k_x and k_z are related by

$$k_x^2 + k_z^2 = k^2 \tag{4.3.7}$$

Solutions of (4.3.6) are sinusoids, so that

$$\mathbf{y}(x,z,t) = \mathbf{A}\sin(k_x x + \phi_x)\sin(k_z z + \phi_z)e^{j\omega t} \tag{4.3.8}$$

where k_x, k_z, ϕ_x, and ϕ_z are determined by the boundary conditions. The conditions $y(0,z,t) = 0$ and $y(x,0,t) = 0$ require $\phi_x = 0$ and $\phi_z = 0$, and the conditions $y(L_x,z,t) = 0$ and $y(x,L_z,t) = 0$ require the arguments k_xL_x and k_zL_z to be integral multiples of π. Thus, the standing waves on the membrane are given by

$$\begin{aligned} \mathbf{y}(x,z,t) &= \mathbf{A}\sin k_x x \sin k_z z\, e^{j\omega t} \\ k_x &= n\pi/L_x \qquad n = 1,2,3,\ldots \\ k_z &= m\pi/L_z \qquad m = 1,2,3,\ldots \end{aligned} \tag{4.3.9}$$

where $|\mathbf{A}|$ is the maximum displacement amplitude. These equations limit the wave numbers k_x and k_z to discrete sets of values, which in turn restrict the natural frequencies for the allowed modes to

$$f_{nm} = \omega_{nm}/2\pi = (c/2)[(n/L_x)^2 + (m/L_z)^2]^{1/2} \tag{4.3.10}$$

This is the two-dimensional extension of the comparable results for the freely vibrating fixed, fixed string. The fundamental frequency is obtained by substitution

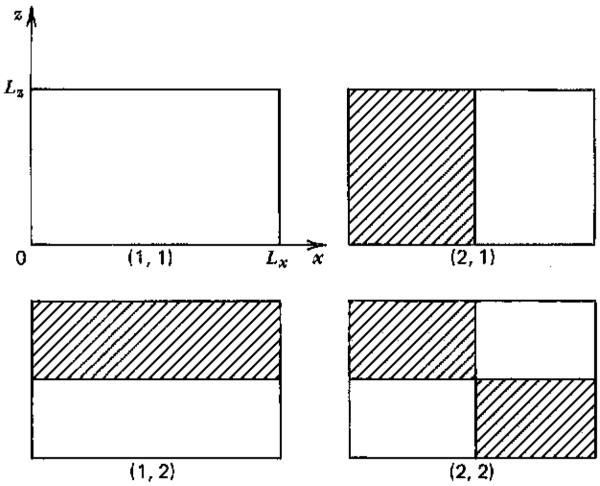

Figure 4.3.1 Schematic representation of four typical normal modes of a rectangular membrane with fixed rim. The modes are designated by the pair of integers (n, m). The hatched areas denote sections of the membrane that vibrate 180° out of phase with the unhatched areas. These areas are separated by nodal lines.

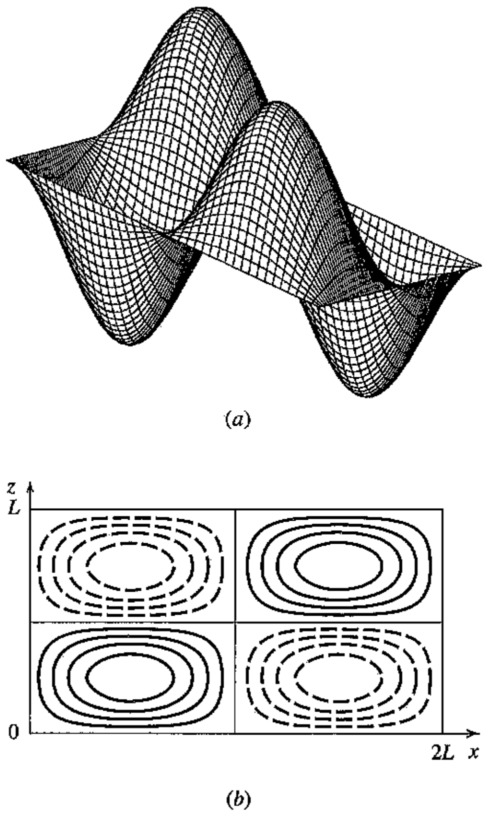

Figure 4.3.2 The displacement of a rectangular membrane with $L_x/L_z = 2$ vibrating in a (2, 2) mode. (*a*) Isometric view. (*b*) Contours of equal displacement. The regions denoted by contours shown with solid lines vibrate 180° out of phase from those shown with dashed lines.

of $n = 1$ and $m = 1$ into (4.3.10). Overtones having $n = m$ will be harmonics of the fundamental, while those for which $n \neq m$ may not be. Figure 4.3.1 illustrates a few modes for a rectangular membrane. The normal modes are labelled by the ordered pair (n, m). Figure 4.3.2 shows the displacement of a (2, 2) mode of a rectangular membrane with fixed rim. Since the nodal lines have zero displacement, it is possible to insert rigid supports along any of them without affecting the wave pattern for the particular frequency involved.

4.4 FREE VIBRATIONS OF A CIRCULAR MEMBRANE WITH FIXED RIM

For a circular membrane fixed at $r = a$, the Helmholtz equation in cylindrical coordinates

$$\frac{\partial^2 \mathbf{\Psi}}{\partial r^2} + \frac{1}{r}\frac{\partial \mathbf{\Psi}}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \mathbf{\Psi}}{\partial \theta^2} + k^2 \mathbf{\Psi} = 0 \tag{4.4.1}$$

can be solved by assuming that $\mathbf{\Psi}(r,\theta)$ is the product of two terms, each a function of only one spatial variable,

$$\mathbf{\Psi} = \mathbf{R}(r)\mathbf{\Theta}(\theta) \tag{4.4.2}$$

subject to the boundary condition

$$\mathbf{R}(a) = 0 \tag{4.4.3}$$

In addition, $\mathbf{\Theta}$ must be a smooth and continuous function of θ. Substitution into (4.2.9) gives

$$\mathbf{\Theta}\frac{d^2\mathbf{R}}{dr^2} + \frac{\mathbf{\Theta}}{r}\frac{d\mathbf{R}}{dr} + \frac{\mathbf{R}}{r^2}\frac{d^2\mathbf{\Theta}}{d\theta^2} + k^2\mathbf{R}\mathbf{\Theta} = 0 \tag{4.4.4}$$

where $k = \omega/c$. Multiplying this equation by $r^2/\mathbf{\Theta}\mathbf{R}$ and moving those terms containing r to one side of the equality sign and those containing θ to the other side results in

$$\frac{r^2}{\mathbf{R}}\left(\frac{d^2\mathbf{R}}{dr^2} + \frac{1}{r}\frac{d\mathbf{R}}{dr}\right) + k^2r^2 = -\frac{1}{\mathbf{\Theta}}\frac{d^2\mathbf{\Theta}}{d\theta^2} \tag{4.4.5}$$

The left side of this equation, a function of r alone, cannot equal the right side, a function of θ alone, unless both functions equal the same constant. If we let this constant be m^2, then the right side becomes

$$\frac{d^2\mathbf{\Theta}}{d\theta^2} = -m^2\mathbf{\Theta} \tag{4.4.6}$$

which has harmonic solutions

$$\mathbf{\Theta}(\theta) = \cos(m\theta + \gamma_m) \tag{4.4.7}$$

where the γ_m are determined by the (spatial factor in the) initial conditions. Since $\mathbf{\Theta}$ must be smooth and single-valued, each m must be an integer. With m fixed in value, (4.4.5) is *Bessel's equation,*

$$\frac{d^2\mathbf{R}}{dr^2} + \frac{1}{r}\frac{d\mathbf{R}}{dr} + \left(k^2 - \frac{m^2}{r^2}\right)\mathbf{R} = 0 \tag{4.4.8}$$

Solutions to this equation are the *Bessel functions* of *order m* of the *first kind $J_m(kr)$* and *second kind $Y_m(kr)$,*

$$\mathbf{R}(r) = \mathbf{A}J_m(kr) + \mathbf{B}Y_m(kr) \tag{4.4.9}$$

Some properties of Bessel functions are summarized in Appendixes A4 and A5. They are oscillatory functions of kr whose amplitudes diminish roughly as $1/\sqrt{kr}$. The $Y_m(kr)$ become unbounded in the limit $kr \to 0$.

While (4.4.9) is the general solution of (4.4.8), a membrane that extends across the origin must have finite displacement at $r = 0$. This requires $\mathbf{B} = 0$ so that

$$\mathbf{R}(r) = \mathbf{A}J_m(kr) \tag{4.4.10}$$

[If, however, the membrane were stretched between inner and outer rims, so that it did not span the origin, then both terms in (4.4.9) would have to be used to satisfy the two boundary conditions.]

Application of the boundary condition $\mathbf{R}(a) = 0$ requires $J_m(ka) = 0$. If the values of the argument of J_m that cause it to equal zero are denoted by j_{mn}, then k assumes the discrete values $k_{mn} = j_{mn}/a$. (See the Appendixes for values of, and formulas for, the arguments j_{mn}.)

The solutions are

$$\begin{aligned} \mathbf{y}_{mn}(r,\theta,t) &= \mathbf{A}_{mn} J_m(k_{mn}r)\cos(m\theta + \gamma_{mn})e^{j\omega_{mn}t} \\ k_{mn}a &= j_{mn} \end{aligned} \tag{4.4.11}$$

and the natural frequencies are

$$f_{mn} = j_{mn}c/2\pi a \tag{4.4.12}$$

Recall that the physical motion of the (m,n)th solution is the real part of (4.4.11),

$$y_{mn}(r,\theta,t) = A_{mn}J_m(k_{mn}r)\cos(m\theta + \gamma_{mn})\cos(\omega_{mn}t + \phi_{mn}) \tag{4.4.13}$$

where $\mathbf{A}_{mn} = A_{mn}\exp(j\phi_{mn})$. The azimuthal phase angles γ_{mn} depend on the location of the initial excitation of the membrane.

Figure 4.4.1 illustrates some simpler modes of vibration for a circular membrane fixed at the rim. The integer m determines the number of *radial nodal lines* and the

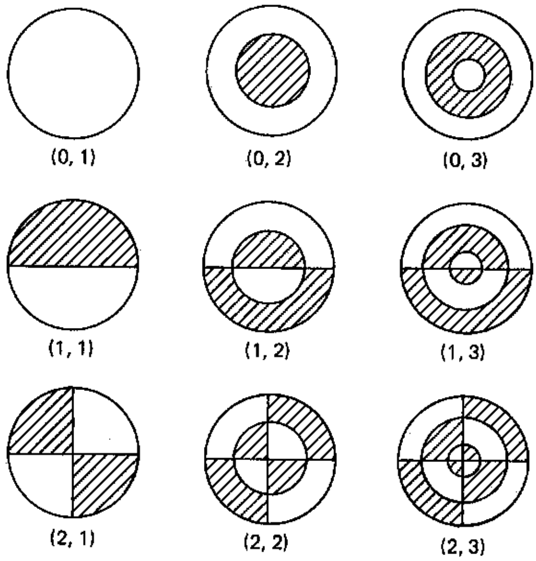

Figure 4.4.1 Normal modes of a circular membrane with fixed rim. The modes are designated by the pair of integers (m,n). The hatched areas denote sections of the membrane that vibrate 180° out of phase with the unhatched areas. These areas are separated by nodal lines. The frequency of the modes increases down each column. (See Table 4.4.1.)

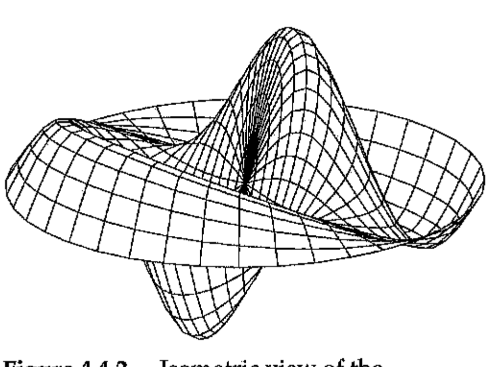

Figure 4.4.2 Isometric view of the displacement of a circular membrane with fixed rim vibrating in a (1, 2) mode.

Table 4.4.1 Normal-mode frequencies of a circular membrane

$f_{01} = 1.0f_{01}$	$f_{11} = 1.593f_{01}$	$f_{21} = 2.135f_{01}$
$f_{02} = 2.295f_{01}$	$f_{12} = 2.917f_{01}$	$f_{22} = 3.500f_{01}$
$f_{03} = 3.598f_{01}$	$f_{13} = 4.230f_{01}$	$f_{23} = 4.832f_{01}$

second integer n determines the number of *nodal circles*. It should be noted that $n = 1$ is the minimum allowed value of n and corresponds to a mode of vibration in which the (only) nodal circle occurs at the fixed boundary of the membrane. Figure 4.4.2 shows the displacement of a circular membrane vibrating in a (1, 2) mode.

For each m there exists a sequence of modes of increasing frequency. Table 4.4.1 lists a few of these frequencies f_{mn} expressed in terms of the fundamental frequency f_{01}. Note that none of the overtones are harmonics of the fundamental.

4.5 SYMMETRIC VIBRATIONS OF A CIRCULAR MEMBRANE WITH FIXED RIM

For many situations described by a circular membrane fixed at the rim, modes having circular symmetry are of greatest importance. Let us, therefore, confine our attention to those solutions that are independent of θ. Because $m = 0$ for these modes, we will suppress this subscript and retain only n,

$$\mathbf{y}_n(r,t) \equiv \mathbf{y}_{0n}(r,\theta,t) = \mathbf{A}_n J_0(k_n r)e^{j\omega_n t} \tag{4.5.1}$$

The natural frequencies are found from (4.4.12),

$$f_n/f_1 = j_{0n}/j_{01} \tag{4.5.2}$$

and the lowest three are given by the first column in Table 4.4.1. For all symmetric modes other than the fundamental, inner nodal circles will occur at radial distances for which $J_0(k_n r)$ vanishes.

The real part of $\mathbf{y}_n$ gives the displacement of the membrane in its nth symmetric mode, and the summation over all n gives the total displacement of the membrane when it is vibrating with circular symmetry,

$$y(r,t) = \sum_{n=1}^{\infty} A_n J_0(k_n r)\cos(\omega_n t + \phi_n) \tag{4.5.3}$$

where $A_n = |\mathbf{A}_n|$ is the displacement amplitude of the nth mode at $r = 0$.

Figure 4.4.1 shows that when the central part of the membrane is displaced up, the adjacent ring is displaced down, and vice versa. Consequently, a membrane vibrating at natural frequencies other than its fundamental produces little net displacement of the surrounding air. (For this reason, the vibrating head of a kettledrum has lower efficiencies of sound production for its overtone frequencies than for its fundamental.) One parameter for ranking the efficiency of each normal mode in producing sound is the average displacement amplitude of the mode. From (4.5.3), the average displacement amplitude $\langle\Psi_n\rangle_S$ of the nth symmetric normal mode is

$$\begin{aligned}\langle\Psi_n\rangle_S &= \frac{1}{\pi a^2}\int_S A_n J_0(k_n r)\,dS = \frac{1}{\pi a^2}\int_0^a A_n J_0(k_n r)2\pi r\,dr \\ &= (2A_n/k_n a)J_1(k_n a)\end{aligned} \tag{4.5.4}$$

where we have used the relationship $zJ_0(z) = d[zJ_1(z)]/dz$ from Appendix A4. [Note that for all modes other than the symmetric ones, the angular dependence $\cos(m\theta + \gamma_m)$ guarantees that the average displacement is zero.]

In many situations involving sources of sound with dimensions smaller than the radiated wavelength, the radiated pressure field depends primarily on the *amount* of air displaced, and not on the exact shape of the moving surface. A measure of the amount of air displaced is the *volume displacement amplitude,* defined as the surface area of the vibrating surface multiplied by the average displacement amplitude of that surface.

When vibrating in its lowest mode, the circular membrane (with fixed rim) has $k_1 a = 2.405$ and, from (4.5.4), the average displacement amplitude is

$$\langle\Psi_1\rangle_S = (2A_1/2.405)J_1(2.405) = 0.432A_1 \tag{4.5.5}$$

where A_1 is the displacement amplitude at the center. A simple piston of the same surface area and a displacement amplitude of $0.432A_1$ will have the same volume displacement amplitude $0.432(\pi a^2)A_1$ as the membrane. If the membrane is vibrating in the mode of its first overtone, $\langle\Psi_2\rangle_S = -0.123A_2$. (The negative sign indicates that the average displacement is opposed to the displacement at the center.) If fundamental and first overtone have the same displacement amplitude at the center of the membrane, the fundamental would be about 3.5 times as effective as the first overtone in displacing air.

*4.6 THE DAMPED, FREELY VIBRATING MEMBRANE

Damping forces, such as those arising within the membrane from internal friction and external forces associated with the radiation of sound, cause the amplitude of each freely vibrating mode to decrease exponentially. As in Chapters 2 and 3, we will use a phenomenological approach. A generic loss term proportional to, and oppositely directed from, the velocity of the vibrating element is introduced into the wave equation. For convenience, let the proportionality constant be 2β so (4.2.4) becomes

$$\frac{\partial^2 y}{\partial t^2} + 2\beta \frac{\partial y}{\partial t} - c^2 \nabla^2 y = 0 \tag{4.6.1}$$

For calculational simplicity, assume oscillatory behavior and generalize y to be complex,

$$\mathbf{y} = \mathbf{\Psi} e^{j\boldsymbol{\omega} t} \tag{4.6.2}$$

Since there are no applied driving forces, $\boldsymbol{\omega}$ must be complex if damping is to occur. Substituting (4.6.2) into (4.6.1) and dividing out $\exp(j\boldsymbol{\omega}t)$ results in the Helmholtz equation

$$\nabla^2 \mathbf{\Psi} + \mathbf{k}^2 \mathbf{\Psi} = 0 \tag{4.6.3}$$

with the complex separation constant $\mathbf{k}^2$ given by

$$\mathbf{k}^2 = (\boldsymbol{\omega}/c)^2 - j2(\beta/c)(\boldsymbol{\omega}/c) \tag{4.6.4}$$

In this case k must be real, since for membranes fixed at their edges the arguments of the normal modes must be real. Solution of (4.6.4) for $\boldsymbol{\omega}$ is straightforward:

$$\begin{aligned} \boldsymbol{\omega} &= \omega_d + j\beta \\ \omega_d &= (\omega^2 - \beta^2)^{1/2} \\ \omega &= kc \end{aligned} \tag{4.6.5}$$

where ω is the natural angular frequency of the undamped case, ω_d the natural angular frequency of the damped case, and β the temporal absorption coefficient.

If the membrane is excited into motion and allowed to come naturally to rest, the resulting motion of the surface is a superposition of the excited normal modes, each with its own decay coefficient β and damped natural angular frequency ω_d:

$$\mathbf{y} = \sum_m \sum_n \mathbf{\Psi}_{mn} e^{-\beta_{mn} t} e^{j(\omega_d)_{mn} t} \tag{4.6.6}$$

Each normal mode $\mathbf{\Psi}_{mn}$ has a complex amplitude $\mathbf{A}_{mn}$ whose magnitude A_{mn} and phase angle ϕ_{mn} are determined by the initial conditions at $t = 0$. The decay coefficients are usually functions of frequency. Losses associated with the flexing of the membrane tend to increase with increasing frequency as the nodal pattern becomes more segmented. On the other hand, losses to the surrounding medium by the radiation of sound become smaller with more complicated modal patterns. (This reflects the observation that the volume displacement amplitudes are smaller for higher modes and zero for unsymmetric modes.) These two effects tend to offset each other, but as a general rule, higher modes damp out faster than do lower ones.

*4.7 THE KETTLEDRUM

Damping and inertial forces are two of the motion-induced forces that may act on the surface of a membrane. Another arises from the changes in pressure within a closed space behind the head of a drum or the diaphragm of a condenser microphone as the volume of the entrapped gas is altered by the motion of the membrane.

For example, the kettledrum has its head stretched tightly over the open end of a hemispherical cavity of volume V. As the head vibrates, the air in the cavity may be alternately compressed and expanded. If the phase speed of transverse waves on the membrane is considerably less than the speed of sound in air, the pressure resulting from any compression and expansion of the enclosed air is nearly uniform within the entire volume and thus depends only on the average instantaneous displacement $\langle y \rangle_S$. The incremental change in volume of the enclosed air is $dV = \pi a^2 \langle y \rangle_S$, where a is the radius of the drumhead. If the equilibrium volume inside the vessel is V_0 and the equilibrium pressure is $\mathcal{P}_0$, then for adiabatic changes in volume the new pressures $\mathcal{P}$ and volumes V are related by

$$\mathcal{P} V^\gamma = \mathcal{P}_0 V_0^\gamma \tag{4.7.1}$$

where γ is the ratio of the heat capacity of the entrapped air at constant pressure to that at constant volume (see Appendix A9). Differentiation shows that the excess pressure $d\mathcal{P}$ inside the kettle will be

$$d\mathcal{P} \approx -(\gamma \mathcal{P}_0 / V_0)\, dV = -\gamma(\mathcal{P}_0/V_0)\pi a^2 \langle y \rangle_S \tag{4.7.2}$$

This generates an additional force $d\mathcal{P} r\, dr\, d\theta$ on each incremental area $r\, dr\, d\theta$ of the membrane. From the discussion of the previous section, the normal modes affected by this force must be just the symmetric ones. While these are relatively unimportant for the musical properties of the kettledrum, the effect of this induced force has interest in other applications, and so we shall pursue the analysis further. Including this force in the discussion of Section 4.2 and writing **y** as (4.6.2) with $\boldsymbol{\Psi}$ real leads to

$$\nabla^2 \Psi + k^2 \Psi = (\gamma \mathcal{P}_0 \pi a^2 / \rho_S c^2 V_0) \langle \Psi \rangle_S \tag{4.7.3}$$

for each symmetric normal mode Ψ. The subscripts 0 and n have been suppressed for economy of expression. Because it is proportional to displacement, the right side is a spring-like term; the allowed wave numbers k will, therefore, be increased. The homogeneous solutions to (4.7.3) will still be Bessel functions, but they may not have zeros at the rim. The boundary condition requires the presence of a particular solution, which in this case is a constant. Adding this to the homogeneous solution and satisfying the boundary condition gives

$$\Psi = A\,[J_0(kr) - J_0(ka)] \tag{4.7.4}$$

as a solution for each symmetric normal mode. The right side of (4.7.3) can now be evaluated with the help of

$$\begin{aligned} \pi a^2 \langle \Psi \rangle_S &= \int_0^a \Psi 2\pi r\, dr = 2\pi A[(r/k)J_1(kr) - (r^2/2)J_0(ka)]\Big|_0^a \\ &= \pi a^2 A\left[2J_1(ka)/ka - J_0(ka)\right] = \pi a^2 A\, J_2(ka) \end{aligned} \tag{4.7.5}$$

Substitution shows that (4.7.4) is a solution of (4.7.3) if

$$\begin{aligned} J_0(ka) &= -B\, J_2(ka)/(ka)^2 \\ B &= \pi a^4 \gamma \mathcal{P}_0 / \mathcal{T} V_0 \end{aligned} \tag{4.7.6}$$

Solving (4.7.6) for ka determines the natural frequencies. The nondimensional parameter B measures the relative importance of the restoring force of the air in the vessel to the tension in the membrane.

Since the frequencies of only the modes Ψ_{0n} are affected by the pressure fluctuations within the vessel, the area πa^2 of the drumhead and the volume V_0 of the vessel are parameters that can be varied to alter the natural frequency distribution of the kettledrum. Variation of B affects the relative values of the f_{0n} frequencies. Altering a and V_0 such that a^4/V_0 remains constant will vary the nonsymmetric overtones f_{mn} $(m \neq 0)$ with respect to the symmetric ones.

If damping is now considered, consistent with (4.6.5) each standing wave will have its angular frequency shifted from the value ω_{mn} for undamped motion to that with damping $(\omega_d)_{mn}$, and each standing wave will decay with its own decay constant β_{mn}. The form of each standing wave will be given by (4.6.6), with the symmetric Ψ_{0n} modes given by (4.7.4).

This development has not taken into consideration any inertance effects of the medium on the membrane. As the membrane vibrates, it radiates acoustic energy but also accelerates the surrounding medium locally, as if it were storing and recovering energy from the mass of the adjacent medium. This inertance is quite important in affecting the natural frequencies of the excited modes. In practice, the significant normal modes of the kettledrum are the lowest four or five of the asymmetric $(m, 1)$ family (beginning with $m = 1$). The inertance contributes an additional effective mass to the membrane, thereby lowering the frequency of the normal mode. The effect is greater for the lower modes, decreasing as the segmentations of the normal mode patterns increase. The natural frequencies are lowered with the lowest ones being most affected. The result brings the relative values close to 2:3:4:5 and this accounts for the distinctive timbre and clear pitch associated with the kettledrum. A quantitative treatment of inertance goes beyond our present purpose, but will be considered further starting in Chapter 7.

*4.8 FORCED VIBRATION OF A MEMBRANE

Introduction of a forcing function into the equation of motion is similarly straightforward. The units of each term in (4.6.1) are those of acceleration, so the forcing function must have the same. A suitable combination of terms is pressure divided by surface density. This gives the generalization of (4.6.1) that includes an external driving agent,

$$\frac{\partial^2 y}{\partial t^2} + 2\beta\frac{\partial y}{\partial t} - c^2\nabla^2 y = \frac{P}{\rho_S}f(t) \tag{4.8.1}$$

where $f(t)$ is a dimensionless function of time. The pressure P can be a constant or any appropriate function of space, including a delta function. The function of time can be oscillatory, a delta function, or whatever is necessary to represent the temporal behavior of the applied force. For example, if both P and $f(t)$ were delta functions, this would approximate the stroke of a drumstick at a specific point on the membrane.

Here, we concentrate on applied oscillatory forces. Let $\mathbf{f}(t) = \exp(j\omega t)$ and assume that the steady-state solution for $\mathbf{y}$ has the form

$$\mathbf{y} = \mathbf{\Psi} e^{j\omega t} \tag{4.8.2}$$

with the angular frequency ω real. (In the case of forced motion, where there is a steady-state solution, ω cannot have an imaginary component.) Substitution into (4.8.1) and cancellation of the exponentials gives

$$(-\omega^2 + j2\beta\omega - c^2\nabla^2)\mathbf{\Psi} = P/\rho_S \tag{4.8.3}$$

The solution of (4.8.3) consists of the sum of the solution to the homogeneous equation and a solution to the particular equation. The homogeneous equation can be written as

$$\begin{aligned} \nabla^2\mathbf{\Psi} + \mathbf{k}^2\mathbf{\Psi} &= 0 \\ \mathbf{k} &= k - j\alpha \\ k &= (\omega/c)[1 + (\beta/\omega)^2]^{1/2} \\ \alpha/k &= (\beta/\omega)/[1 + (\beta/\omega)^2] \approx \beta/\omega \end{aligned} \tag{4.8.4}$$

The top equation is the familiar Helmholtz equation, but with complex $\mathbf{k}$ rather than real k. This means that whatever functions solve the Helmholtz equation for lossless conditions are still solutions, but with k replaced with $\mathbf{k}$. For the cases we have studied (rectangular and circular membranes), the functions now have complex arguments and cannot satisfy the boundary condition of a fixed rim without the help of the particular solution.

For the case of uniform pressure P distributed over the circular membrane with fixed rim at $r = a$, the azimuthal symmetry of the problem restricts the homogeneous solution $\mathbf{\Psi}_h$ to the zeroth order Bessel function $J_0(kr)$. The appropriate particular solution $\mathbf{\Psi}_p$ to (4.8.3) is a constant,

$$\mathbf{\Psi}_p = -(P/\rho_S)/(\mathbf{k}c)^2 \tag{4.8.5}$$

Adding this to the homogeneous solution $\mathbf{\Psi}_h$ and requiring that the sum vanish when evaluated at the rim results in the desired solution,

$$\mathbf{\Psi} = (P/\mathcal{T}\mathbf{k}^2)\left[J_0(\mathbf{k}r)/J_0(\mathbf{k}a) - 1\right] \tag{4.8.6}$$

The tension $\mathcal{T}$ has replaced $\rho_S c^2$. The values for $\mathbf{k} = k - j\alpha$ are determined from (4.8.4). Inspection of (4.8.6) shows that the amplitude of the displacement is directly proportional to that of the driving force and inversely proportional to the tension $\mathcal{T}$. The dependence on frequency of the amplitude of vibration at any location is given by the relatively complicated expression within the square bracket. When the driving frequency matches any natural frequency [found from $J_0(ka) = 0$], then $J_0(\mathbf{k}a)$ has very small magnitude and $|\mathbf{\Psi}|$ may be very large, depending on the damping.

*4.9 THE DIAPHRAGM OF A CONDENSER MICROPHONE

An important case of a driven membrane is the circular diaphragm of a condenser microphone. The incident sound wave, acting on a tightly stretched metallic membrane placed above a metal plate, produces a nearly uniform driving force. As the membrane is displaced, the electrical capacitance between the membrane and the adjacent metal plate is changed. This generates an output voltage that is (for small motion) a linear function of the averaged displacement amplitude of the membrane,

$$\langle\mathbf{\Psi}\rangle_S = \frac{1}{\pi a^2}\frac{P}{\mathcal{T}}\frac{1}{\mathbf{k}^2}\int_0^a \left(\frac{J_0(\mathbf{k}r)}{J_0(\mathbf{k}a)} - 1\right)2\pi r\,dr = \frac{Pa^2}{\mathcal{T}}\frac{1}{(\mathbf{k}a)^2}\frac{J_2(\mathbf{k}a)}{J_0(\mathbf{k}a)} \tag{4.9.1}$$

If the frequency is below the region of the lowest resonance, $\mathbf{k}$ can be replaced with the wave number k and use of the small-argument approximations for the Bessel functions gives

$$\langle\mathbf{\Psi}\rangle_S \approx \tfrac{1}{8}(Pa^2/\mathcal{T})[1 + (ka)^2/6] \tag{4.9.2}$$

Thus, $\langle\mathbf{\Psi}\rangle_S$ is nearly constant for $ka < 1$, or for frequencies

$$f < c/2\pi a = (\mathcal{T}/\rho_S)^{1/2}/2\pi a \tag{4.9.3}$$