UE TSM - Les Séries de Volterra
TD

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Problem 1. System $\mathcal{F}$ is described with an input $u = v_j$ and an output $y = v_{j+1}$ and follows the equation (1):

$$\frac{1}{\omega}\frac{dv_j}{dt} + \tanh(v_j) = \tanh(v_{j-1}) \tag{1}$$

Question 1. Show the Volterra Kernel of $y = \tanh(u)$ in the time domain and Laplace domain:

$$\tanh(x) = \sum_{n=1}^{\infty} T_n x^n,$$

with $T_1 = 1, T_2 = 0, T_3 = -\frac{1}{3}, T_4 = 0, \cdots$

Solution. In time domain, this is a special case, static nonlinear system described by a polynomial in the input,

$$\begin{aligned} y(t) &= \sum_{i=1}^{\infty} g_n(t_1, \cdots, t_n) u(t - t_1) \cdots u(t - t_n) dt_1 \cdots dt_n \\ &= \sum_{n=1}^{\infty} T_n u^n(t) \end{aligned}$$

Therefore, the kernel

$$g_n(t_1, \cdots, t_n) = T_n \delta(t_1) \cdots \delta(t_n).$$

In Laplace domain,

$$\begin{aligned} G_n(s_1, \cdots, s_n) &= \int_{\mathbb{R}^n} g_n(t_1, \cdots, t_n) e^{-(s_1 t_1 + \cdots + s_n t_n)} dt_1 \cdots dt_n \\ &= T_n \int \delta(t_1) e^{-s_1 t_1} dt_1 \cdots \int \delta(t_n) e^{-s_n t_n} dt_n \\ &= T_n \end{aligned}$$

Therefore,

$$G_n(s_1, \cdots, s_n) = T_n$$

□

Question 2. Show a block-diagram of *Canceling system* of $\mathcal{F}$.

Solution. From equation (1), we could obtain the relation between input u and output y

$$\frac{1}{\omega}\dot{y} + \tanh(y) - \tanh(u) = 0.$$

Let $z = \frac{1}{\omega}\dot{y} + \tanh(y) - \tanh(u)$, the block diagram is shown as figure (1).

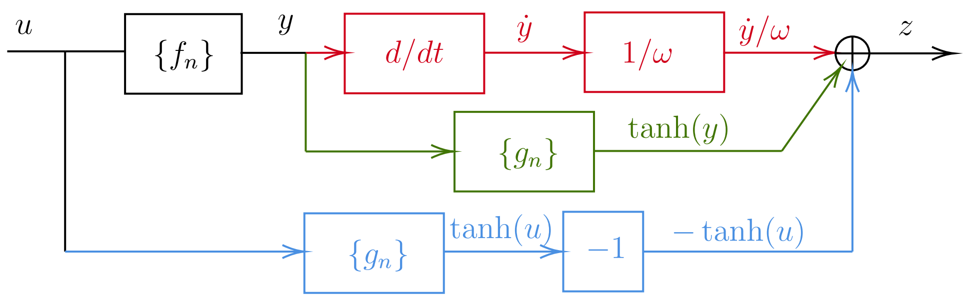

Figure 1: Canceling system schema of $\mathcal{F}$

□

Question 3. Show equations about $\{F_n\}$ of the diagram and calculate for $n = 1, 2, 3$.

Solution. The equivalent kernel of red part $R(s) = \frac{s}{\omega}$ cascade with $\{F_n\}$, so that the equivalent kernel of these three system is

$$H_{r,n}(s_1, \cdots, s_n) = F_n(s_1, \cdots, s_n)R(s_1 + \cdots + s_n)$$

The green part equivalent kernel $G_n(s_1, \cdots, s_n) = T_n$ cascade with $\{F_n\}$, so that the equivalent kernel of these two system is

$$H_{g,n}(s_1, \cdots, s_n) = \sum_{p=1}^{n} \sum_{i_1+\cdots+i_p=n} F_{i_1}(s_1, \cdots, s_{i_1}) \cdots F_{i_p}(s_{i_1+\cdots+i_{p-1}+1}, \cdots, s_n)$$
$$G_p(s_1 + \cdots + s_{i_1}, \cdots, s_{i_1+\cdots+i_{p-1}+1} + \cdots + s_n)$$

The blue part equivalent kernel is $H_{b,n}(s_1, \cdots, s_n) = -G_n(s_1, \cdots, s_n)$.
Therefore,

$$H_{r,n}(s_1, \cdots, s_n) + H_{g,n}(s_1, \cdots, s_n) + H_{b,n}(s_1, \cdots, s_n) = 0$$

When $n = 1$,

$$\begin{cases} H_{r,1}(s_1) = F_1(s_1)\frac{s_1}{\omega} \\ H_{g,1}(s_1) = F_1(s_1)G_1(s_1) = F_1(s_1) \\ H_{b,1}(s_1) = -1 \end{cases}$$
$$F_1(s) = \frac{1}{1 + s_1/\omega}$$

When $n = 2$,

$$\begin{cases} H_{r,2}(s_1, s_2) = F_2(s_1, s_2)\frac{s_1+s_2}{\omega} \\ H_{g,2}(s_1, s_2) = F_2(s_1, s_2)G_1(s_1 + s_2) + F_1(s_1)F_1(s_2)G_2(s_1, s_2) = F_2(s_1, s_2) \\ H_{b,2}(s_1, s_2) = 0 \end{cases}$$

$$F_2(s_1, s_2) = 0$$

When $n = 3$,

$$\begin{cases} H_{r,3}(s_1, s_2, s_3) = F_3(s_1, s_2, s_3)\frac{s_1+s_2+s_3}{\omega} \\ H_{g,3}(s_1, s_2, s_3) = F_3(s_1, s_2, s_3)G_1(s_1 + s_2 + s_3) \\ \qquad +F_2(s_1, s_2)F_1(s_3)G_2(s_1 + s_2, s_3) + F_1(s_1)F_2(s_2, s_3)G_2(s_1, s_2 + s_3) \\ \qquad +F_1(s_1)F_1(s_2)F_1(s_3)G_3(s_1, s_2, s_3) \\ \qquad = F_3(s_1, s_2, s_3) - \frac{1}{3}F_1(s_1)F_1(s_2)F_1(s_3) \\ H_{b,3}(s_1, s_2, s_3) = 1/3 \end{cases}$$

$$F_3(s_1, s_2, s_3) = \frac{\frac{1}{3}(F_1(s_1)F_1(s_2)F_1(S_3) - 1)}{1 + \frac{s_1+s_2+s_3}{\omega}}$$

□

Question 4. Show a diagram of implementation for 3-order Volterra.

Solution. We could express H_1, H_2, H_3 as

$$\begin{aligned} H_1(s_1) &= A_{1,1}(s_1) \\ H_2(s_2, s_2) &= A_{2,1}(s_1)B_{2,1}(s_2)C_{2,1}(s_1 + s_2) \\ H_3(s_1, s_2, s_3) &= A_{3,1}(s_1)B_{3,1}(s_2)C_{3,1}(s_3)D_{3,1}(s_1 + s_2 + s_3) \end{aligned}$$

So that the diagram is shown as figure (2).

□

Problem 2. System $\mathcal{H}$ is described with an input $u = v_0$ and an output $y = v_4$, which cascades 4 $\mathcal{F}$-systems.

Question 5. Show the kernel of $\mathcal{H}$, H_n, for $n = 1, 2, 3, 4$.

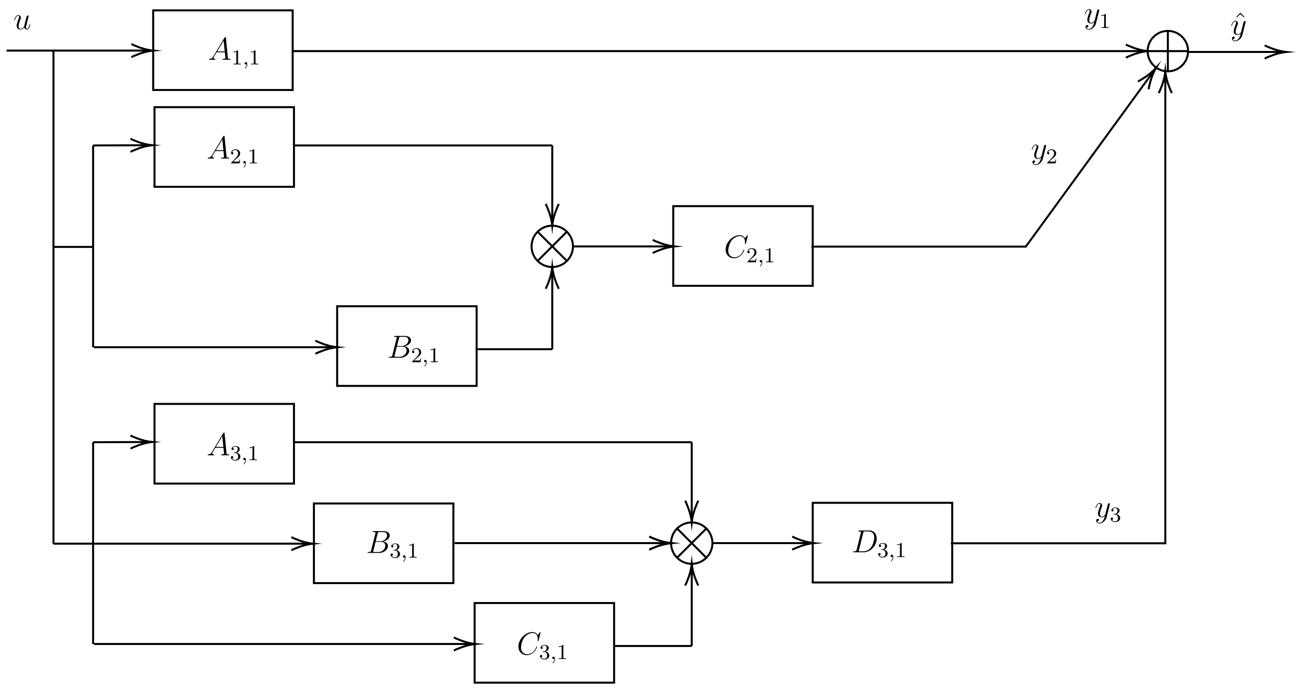

Figure 2: Diagram of Implementation for 3-order Volterra

Solution. First, consider system $\mathcal{G}$ is the cascade of two systems $\mathcal{F}$,

$$\begin{aligned}
G_1(s_1) &= [F_1(s_1)]^2 \\
G_2(s_1, s_2) &= F_2(s_1, s_2)[F_1(s_1 + s_2) + F_1(s_1)F_1(s_2)] \\
G_3(s_1, s_2, s_3) &= F_3(s_1, s_2, s_3)[F_1(s_1 + s_2 + s_3) + F_1(s_1)F_1(s_2)F_1(s_3)] \\
&\quad + F_2(s_1, s_2)F_1(s_3)F_2(s_1 + s_2, s_3) + F_1(s_1)F_2(s_2, s_3)F_2(s_1, s_2 + s_3) \\
G_4(s_1, s_2, s_3, s_4) &= F_4(s_1, s_2, s_3, s_4)[F_1(s_1 + s_2 + s_3 + s_4) + F_1(s_1)F_1(s_2)F_1(s_3)F_1(s_4)] \\
&\quad + F_3(s_1, s_2, s_3)F_1(s_4)F_2(s_1 + s_2 + s_3, s_4) \\
&\quad + F_2(s_1, s_2)F_2(s_3, s_4)F_2(s_1 + s_2, s_3 + s_4) \\
&\quad + F_1(s_1)F_3(s_2, s_3, s_4)F_2(s_1, s_2 + s_3 + s_4) \\
&\quad + F_2(s_1, s_2)F_1(s_3)F_1(s_4)F_3(s_1 + s_2, s_3, s_4) \\
&\quad + F_1(s_1)F_2(s_2, s_3)F_1(s_4)F_3(s_1, s_2 + s_3, s_4) \\
&\quad + F_1(s_1)F_1(s_2)F_2(s_3, s_4)F_3(s_1, s_2, s_4 + s_4)
\end{aligned}$$

And then consider system $\mathcal{H}$ is the cascade of two systems $\mathcal{G}$,

$$
\begin{aligned}
H_1(s_1) &= [G_1(s_1)]^2 \\
H_2(s_1, s_2) &= G_2(s_1, s_2)[G_1(s_1+s_2) + G_1(s_1)G_1(s_2)] \\
H_3(s_1, s_2, s_3) &= G_3(s_1, s_2, s_3)[G_1(s_1+s_2+s_3) + G_1(s_1)G_1(s_2)G_1(s_3)] \\
&\quad + G_2(s_1, s_2)G_1(s_3)G_2(s_1+s_2, s_3) + G_1(s_1)G_2(s_2, s_3)G_2(s_1, s_2+s_3) \\
H_4(s_1, s_2, s_3, s_4) &= G_4(s_1, s_2, s_3, s_4)[G_1(s_1+s_2+s_3+s_4) + G_1(s_1)G_1(s_2)G_1(s_3)G_1(s_4)] \\
&\quad + G_3(s_1, s_2, s_3)G_1(s_4)G_2(s_1+s_2+s_3, s_4) \\
&\quad + G_2(s_1, s_2)G_2(s_3, s_4)G_2(s_1+s_2, s_3+s_4) \\
&\quad + G_1(s_1)G_3(s_2, s_3, s_4)G_2(s_1, s_2+s_3+s_4) \\
&\quad + G_2(s_1, s_2)G_1(s_3)G_1(s_4)G_3(s_1+s_2, s_3, s_4) \\
&\quad + G_1(s_1)G_2(s_2, s_3)G_1(s_4)G_3(s_1, s_2+s_3, s_4) \\
&\quad + G_1(s_1)G_1(s_2)G_2(s_3, s_4)G_3(s_1, s_2, s_4+s_4)
\end{aligned}
$$

□

Question 6. Show a diagram of implementation for 3-order Volterra of system $\mathcal{F}$.

Solution. For $F_1(s) = \frac{1}{1+s_1/\omega}$, it is a first order of low pass filter.

For $F_3(s_1, s_2, s_3) = \frac{\frac{1}{3}(F_1(s_1)F_1(s_2)F_1(S_3)-1)}{1+\frac{s_1+s_2+s_3}{\omega}}$, it can be re-written by

$$
F_3(s_1, s_2, s_3) = A_1(s_1)B_1(s_2)C_1(s_3)D_1(s_1+s_2+s_3) - D_1(s_1+s_2+s_3)
$$

$$
\begin{cases}
A_1(s_1) = F_1(s_1) \\
B_1(s_2) = F_1(s_2) \\
C_1(s_3) = F_1(s_3) \\
D_1(s_1+s_2+s_3) = \frac{1}{3}F_1(s_1+s_2+s_3)
\end{cases}
$$

Therefore, the diagram of implementation is shown as figure (3).

□

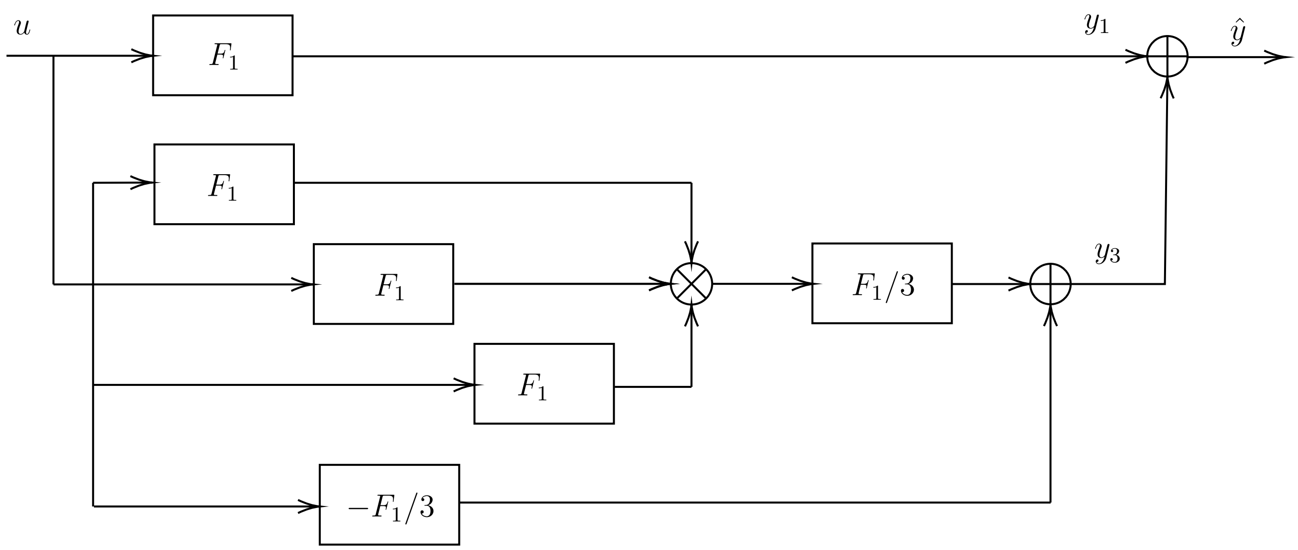

Figure 3: Diagram of Implementation of $\mathcal{F}$