Chapter 7

RADIATION AND RECEPTION OF ACOUSTIC WAVES

7.1 RADIATION FROM A PULSATING SPHERE

The acoustic source simplest to analyze is a pulsating sphere—a sphere whose radius varies sinusoidally with time. While pulsating spheres are of little practical importance, their analysis is useful for they serve as the prototype for an important class of sources referred to as *simple sources*.

In a medium that is infinite, homogeneous, and isotropic, a pulsating sphere will produce an outgoing spherical wave

$$\mathbf{p}(r,t) = (\mathbf{A}/r)e^{j(\omega t - kr)} \tag{7.1.1}$$

where $\mathbf{A}$ is determined by an appropriate boundary condition.

Consider a sphere of average radius a, vibrating radially with complex speed $U_0 \exp(j\omega t)$, where the displacement of the surface is much less than the radius, $U_0/\omega \ll a$. The acoustic pressure of the fluid in contact with the sphere is given by (7.1.1) evaluated at $r = a$. (This is consistent with the small-amplitude approximation of linear acoustics.) The radial component of the velocity of the fluid in contact with the sphere is found using the specific acoustic impedance for the spherical wave (5.11.10) also evaluated at $r = a$,

$$\mathbf{z}(a) = \rho_0 c \cos\theta_a \, e^{j\theta_a} \tag{7.1.2}$$

where $\cot\theta_a = ka$. The pressure at the surface of the source is then

$$\mathbf{p}(a,t) = \rho_0 c U_0 \cos\theta_a \, e^{j(\omega t - ka + \theta_a)} \tag{7.1.3}$$

Comparing (7.1.3) with (7.1.1) gives

$$\mathbf{A} = \rho_0 c U_0 a \cos\theta_a \, e^{j(ka + \theta_a)} \tag{7.1.4}$$

so the pressure at any distance $r > a$ is

$$\mathbf{p}(r,t) = \rho_0 c U_0 (a/r) \cos\theta_a \, e^{j[\omega t - k(r-a) + \theta_a]} \tag{7.1.5}$$

The acoustic intensity, found from (5.11.20), is

$$I = \tfrac{1}{2}\rho_0 c U_0^2 (a/r)^2 \cos^2\theta_a \tag{7.1.6}$$

If the radius of the source is small compared to a wavelength, $\theta_a \to \pi/2$ and the specific acoustic impedance near the surface of the sphere is strongly reactive. (This reactance is a symptom of the strong radial divergence of the acoustic wave near a small source and represents the storage and release of energy because successive layers of the fluid must stretch and shrink circumferentially, altering the outward displacement. This inertial effect manifests itself in the mass-like reactance of the specific acoustic impedance.) In this long wavelength limit the pressure

$$\mathbf{p}(r,t) = j\rho_0 c U_0 (a/r) ka\, e^{j(\omega t - kr)} \qquad ka \ll 1 \tag{7.1.7}$$

is nearly $\pi/2$ out of phase with the particle speed (pressure and particle speed are not *exactly* $\pi/2$ out of phase, since that would lead to a vanishing intensity), and the acoustic intensity is

$$I = \tfrac{1}{2}\rho_0 c U_0^2 (a/r)^2 (ka)^2 \qquad ka \ll 1 \tag{7.1.8}$$

For constant U_0 this intensity is proportional to the square of the frequency and depends on the fourth power of the radius of the source. Thus, we see that sources small with respect to a wavelength are inherently poor radiators of acoustic energy.

In the next section, it will be shown that all *simple sources*, no matter what their shapes, will produce the same acoustic field as a pulsating sphere provided the wavelength is greater than the dimensions of the source and the sources have the same *volume velocity.*

7.2 ACOUSTIC RECIPROCITY AND THE SIMPLE SOURCE

Acoustic reciprocity is a powerful concept that can be used to obtain some very general results. Let us begin by deriving one of the more commonly encountered statements of acoustic reciprocity.

Consider a space occupied by two sources, as suggested by Fig. 7.2.1. By changing which source is active and which passive, it is possible to set up different sound fields. Choose two situations having the same frequency and denote them as 1 and 2. Establish a volume V of space that does not itself contain any sources but bounds them. Let the surface of this volume be S. The volume V and the surface S remain the same for both situations. Let the velocity potential be $\mathbf{\Phi}_1$ for situation 1 and $\mathbf{\Phi}_2$ for situation 2. Green's theorem (see Appendix A8) gives the general relation

$$\int_S (\mathbf{\Phi}_1 \nabla \mathbf{\Phi}_2 - \mathbf{\Phi}_2 \nabla \mathbf{\Phi}_1) \cdot \hat{n}\, dS = \int_V (\mathbf{\Phi}_1 \nabla^2 \mathbf{\Phi}_2 - \mathbf{\Phi}_2 \nabla^2 \mathbf{\Phi}_1)\, dV \tag{7.2.1}$$

where $\hat{n}$ is the unit outward normal to S. Since the volume does not include any sources, and since both velocity potentials are for excitations of the same

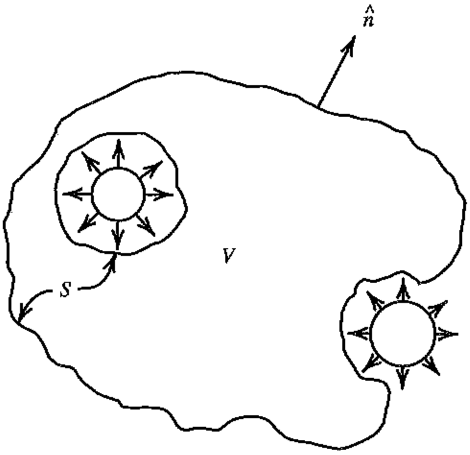

Figure 7.2.1 Geometry used in deriving the theorem of acoustical reciprocity.

frequency, the wave equation yields

$$\begin{aligned} \nabla^2 \mathbf{\Phi}_1 &= -k^2 \mathbf{\Phi}_1 \\ \nabla^2 \mathbf{\Phi}_2 &= -k^2 \mathbf{\Phi}_2 \end{aligned} \tag{7.2.2}$$

so that the right side of (7.2.1) vanishes identically throughout V. Furthermore, recall that the pressure is $\mathbf{p} = -j\omega\rho_0\mathbf{\Phi}$ and the particle velocity for irrotational motion is $\vec{\mathbf{u}} = \nabla\mathbf{\Phi}$. Substitution of these expressions into the left side of (7.2.1) gives

$$\int_S (\mathbf{p}_1\vec{\mathbf{u}}_2 \cdot \hat{n} - \mathbf{p}_2\vec{\mathbf{u}}_1 \cdot \hat{n})\, dS = 0 \tag{7.2.3}$$

This is one form of the *principle of acoustic reciprocity.* This principle states that, for example, if the locations of a small source and a small receiver are interchanged in an unchanging environment, the received signal will remain the same.

To obtain information about simple sources, let us develop a more restrictive but simpler form of (7.2.3). Assume that some portion of S is removed a great distance from the enclosed source. In any real case there is always some absorption of sound by the medium so the intensity at this surface will decrease faster than $1/r^2$. Since the area of the surface increases as r^2, the product of intensity and area vanishes in the limit $r \to \infty$. In addition, if each of the remaining portions of S is either (1) perfectly rigid so that $\vec{\mathbf{u}} \cdot \hat{n} = 0$, (2) pressure release so that $\mathbf{p} = 0$, or (3) normally reacting so that $\mathbf{p}/(\vec{\mathbf{u}} \cdot \hat{n}) = \mathbf{z}_n$, then the surface integrals over these surfaces must vanish. Under these conditions, (7.2.3) reduces to an integral over only those portions of S that correspond to sources active in situations 1 or 2:

$$\int_{sources} (\mathbf{p}_1\vec{\mathbf{u}}_2 \cdot \hat{n} - \mathbf{p}_2\vec{\mathbf{u}}_1 \cdot \hat{n})\, dS = 0 \tag{7.2.4}$$

This simple result will now be applied to develop some important general properties of sources that are small compared to a wavelength.

Consider a region of space in which there are two irregularly shaped sources, as shown in Fig. 7.2.2. Let source A be active and source B be perfectly rigid in

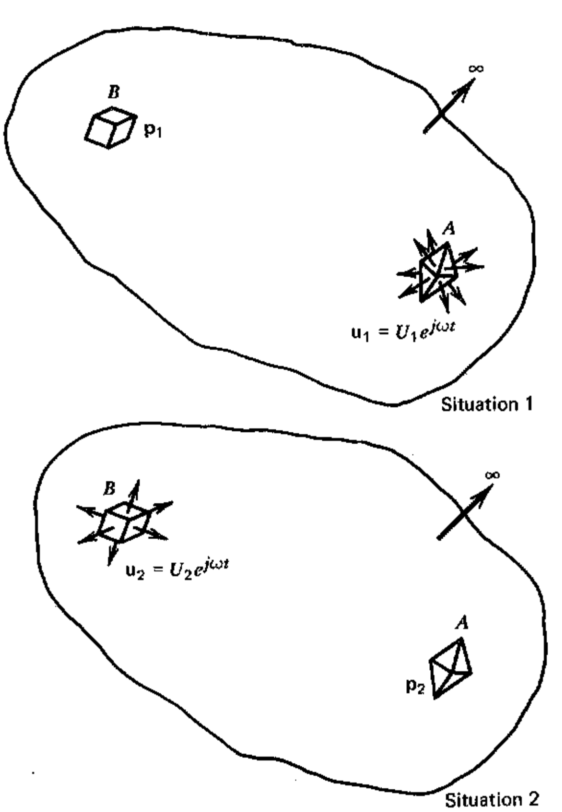

Figure 7.2.2 Reciprocity theorem applied to simple sources.

situation 1, and vice versa in situation 2. If we define $\mathbf{p}_1$ as the pressure at B when source A is active with $\vec{\mathbf{u}}_1$ the velocity of its radiating element, and $\mathbf{p}_2$ as the pressure at A when source B is active with $\vec{\mathbf{u}}_2$ the velocity of its radiating element, application of (7.2.4) yields

$$\int_{S_A} \mathbf{p}_2 \vec{\mathbf{u}}_1 \cdot \hat{n}\, dS = \int_{S_B} \mathbf{p}_1 \vec{\mathbf{u}}_2 \cdot \hat{n}\, dS \tag{7.2.5}$$

If the sources are small with respect to a wavelength and several wavelengths apart, then the pressure is uniform over each source so that

$$\frac{1}{\mathbf{p}_1} \int_{S_A} \vec{\mathbf{u}}_1 \cdot \hat{n}\, dS = \frac{1}{\mathbf{p}_2} \int_{S_B} \vec{\mathbf{u}}_2 \cdot \hat{n}\, dS \tag{7.2.6}$$

Assume that the moving elements of a source have complex vector displacements

$$\vec{\boldsymbol{\xi}} = \vec{\Xi} e^{j(\omega t + \phi)} \tag{7.2.7}$$

where $\vec{\Xi}$ gives the magnitude and direction of the displacement and ϕ the temporal phase of each element. If $\hat{n}$ is the unit outward normal to each element dS of the surface, the source will displace a volume of the surrounding medium

$$\mathbf{V} = \int_S \vec{\Xi} e^{j(\omega t + \phi)} \cdot \hat{n}\, dS = V e^{j(\omega t + \theta)} \tag{7.2.8}$$

where $\mathbf{V}$ is the *complex volume displacement*, V the generalization of the volume displacement amplitude discussed in Section 4.5, and θ the accumulated phase over the surface of the element. The time derivative $\partial\mathbf{V}/\partial t$, the *complex volume velocity*, defines the *complex source strength* $\mathbf{Q}$

$$\mathbf{Q}e^{j\omega t} = \frac{\partial \mathbf{V}}{\partial t} = \int_S \vec{\mathbf{u}} \cdot \hat{n}\, dS \tag{7.2.9}$$

where $\vec{\mathbf{u}} = \partial\vec{\boldsymbol{\xi}}/\partial t$ is the complex velocity distribution of the source surface. The complex source strength of the pulsating sphere has only a real part,

$$\mathbf{Q} = Q = 4\pi a^2 U_0 \tag{7.2.10}$$

Substitution of (7.2.9) and $\mathbf{p} = \mathbf{P}(r)\exp[j(\omega t - kr)]$ into (7.2.6) gives

$$\mathbf{Q}_1/\mathbf{P}_1(r) = \mathbf{Q}_2/\mathbf{P}_2(r) \tag{7.2.11}$$

which shows that the ratio of the source strength to the pressure amplitude at distance r from the source is the same for all simple sources (at the same frequency) in the same surroundings. This allows us to calculate the pressure field of any irregular simple source since it must be identical with the pressure field produced by a small pulsating sphere of the same source strength. If the simple sources are in free space, (7.1.7) and (7.2.10) show that the ratio of (7.2.11) is

$$\mathbf{Q}/\mathbf{P}(r) = -j\,2\lambda r/\rho_0 c \tag{7.2.12}$$

This is the *free field reciprocity factor.*

Rewriting (7.1.7) with the help of (7.2.10) results in

$$\mathbf{p}(r,t) = \tfrac{1}{2} j\rho_0 c(Q/\lambda r)e^{j(\omega t - kr)} \tag{7.2.13}$$

which, from the above, must be true for all simple sources. The pressure amplitude is

$$P = \tfrac{1}{2}\rho_0 c Q/\lambda r \qquad \text{(simple source)} \tag{7.2.14}$$

and the intensity is

$$I = \tfrac{1}{8}\rho_0 c(Q/\lambda r)^2 \tag{7.2.15}$$

Integration of the intensity over a sphere centered at the source gives the power radiated,

$$\Pi = \tfrac{1}{2}\pi\rho_0 c(Q/\lambda)^2 \tag{7.2.16}$$

Another case of practical interest is that of a simple source mounted on or very close to a rigid plane boundary. If the dimensions of the boundary are much greater than a wavelength of sound, the boundary can be considered a plane of infinite extent. This kind of boundary is termed a *baffle.* As shown in Section

6.8, the pressure field in the half-space occupied by the source will be twice that generated by the source (with the same source strength) in free space,

$$P = \rho_0 c Q/\lambda r \qquad \text{(baffled simple source)} \tag{7.2.17}$$

The intensity is increased by a factor of four,

$$I = \tfrac{1}{2}\rho_0 c(Q/\lambda r)^2 \tag{7.2.18}$$

and integration of the intensity over a hemisphere (there is no acoustic penetration of the space behind the baffle) gives twice the radiated power,

$$\Pi = \pi\rho_0 c(Q/\lambda)^2 \tag{7.2.19}$$

A doubling of the power output of the source may seem surprising but results from the fact that the source has the same source strength in both cases: the source face is moving with the same velocity in both cases, but in the baffled case it is working into twice the force and therefore must expend twice the power to maintain its own motion in the presence of the doubled pressure.

7.3 THE CONTINUOUS LINE SOURCE

As an example of a distribution of point sources used to describe an extended source, consider a long, thin cylindrical source of length L and radius a. This configuration, suggested in Fig. 7.3.1, is termed a *continuous line source.* Let the surface vibrate radially with speed $U_0 \exp(j\omega t)$. Consider the source to be made up of a large number of cylinders of length dx. Each of these elements can be considered an unbaffled simple source of strength $dQ = U_0\, 2\pi a\, dx$. Each generates the increment of pressure given by (7.2.13) with r replaced by the distance r' from

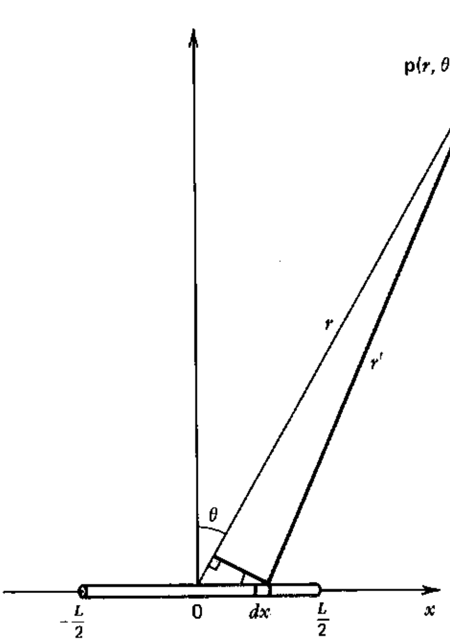

Figure 7.3.1 The far field acoustic field at (r, θ) of a continuous line source of length L and radius a is found by summing the contributions of simple sources of length dx and radius a.

the element to the field point at (r,θ). The total pressure is found by integrating $d\mathbf{p}$ over the length of the source,

$$\mathbf{p}(r,\theta,t) = \frac{j}{2}\rho_0 c U_0 ka \int_{-L/2}^{L/2} \frac{1}{r'} e^{j(\omega t - kr')} dx \tag{7.3.1}$$

The acoustic field close to the source is complicated, but a simple expression can be obtained in the *far field approximation.* Under the assumption $r \gg L$, the denominator of the integrand can be replaced by its approximate value r, which amounts to making very small errors in the amplitudes of the acoustic fields at (r,θ) generated by each of the simple sources. In the exponent, however, this simplification cannot always be made because the relative phases of the elements will be very strong functions of angle when kL approaches or exceeds unity. Then the more accurate approximation $r' \approx r - x\sin\theta$ must be used, and the integral takes the form

$$\mathbf{p}(r,\theta,t) = \frac{j}{2}\rho_0 c U_0 \frac{ka}{r} e^{j(\omega t - kr)} \int_{-L/2}^{L/2} e^{jkx\sin\theta} dx \tag{7.3.2}$$

Evaluation is immediate,

$$\mathbf{p}(r,\theta,t) = \frac{j}{2}\rho_0 c U_0 \frac{a}{r} kL \left(\frac{\sin\left(\frac{1}{2}kL\sin\theta\right)}{\frac{1}{2}kL\sin\theta} \right) e^{j(\omega t - kr)} \tag{7.3.3}$$

The acoustic pressure amplitude in the far field can be written

$$\boxed{P(r,\theta) = P_{ax}(r)H(\theta)} \tag{7.3.4}$$

where

$$H(\theta) = \left| \frac{\sin v}{v} \right| \qquad v = \tfrac{1}{2}kL\sin\theta \tag{7.3.5}$$

is the *directional factor* and

$$P_{ax}(r) = \tfrac{1}{2}\rho_0 c U_0 (a/r) kL \tag{7.3.6}$$

is the amplitude of the *far field axial pressure.*

Separating the far field pressure amplitude into one factor that depends only on angle and has maximum value of unity on the *acoustic axis* and another that depends only on the distance from the source is common practice in describing the sound fields of complicated sources. Note that in the far field the axial pressure is proportional to $1/r$, as for a simple source. This is a feature common to all acoustic sources.

The behavior of $(\sin v)/v$ is shown in Fig. 7.3.2. This function is known as the *sinc function* or the *zeroth order spherical Bessel function of the first kind.* The corresponding *beam pattern* $b(\theta) = 20\log H(\theta)$ is plotted in Fig. 7.3.3 for the case $kL = 24$. There are *nodal surfaces* (cones in the present case) at angles where $H(\theta) = 0$, for which $\frac{1}{2}kL\sin\theta_n = \pm n\pi$, with $n = 1, 2, 3, \ldots$. These nodal surfaces are separated by *lobes*

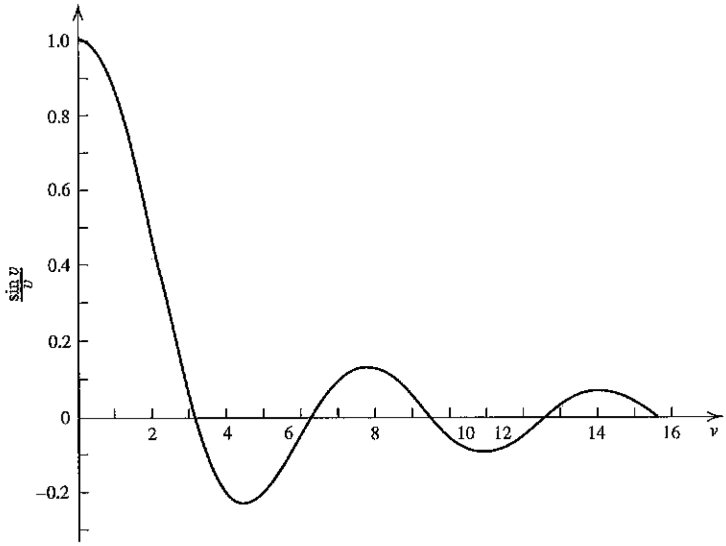

Figure 7.3.2 Functional behavior of $(\sin v)/v$.

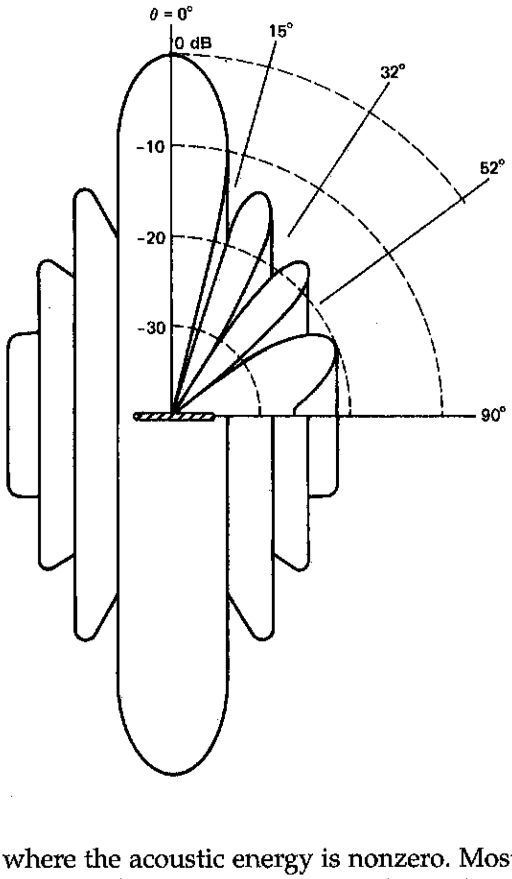

Figure 7.3.3 Beam pattern $b(\theta)$ for a continuous line source of length L radiating sound of wave number k with $kL = 24$.

where the acoustic energy is nonzero. Most of the acoustic energy is projected in the *major lobe,* contained within the angles given by $n = 1$ and centered on a plane perpendicular to the line source. The amplitudes of the *minor lobes* are less than unity and tend to decrease away from this plane. Clearly, the larger the value of kL the more narrowly directed will be the major lobe and the greater the number of minor lobes.

Note that the pressure, when expressed in terms of the source strength $Q = U_0\, 2\pi aL$, becomes

$$\mathbf{p}(r,\theta,t) = \frac{j}{2}\rho_0 c \frac{Q}{\lambda r}\frac{\sin v}{v} e^{j(\omega t - kr)} \tag{7.3.7}$$

Comparison with (7.2.13) shows that the pressure field is the product of that generated by a simple source of source strength Q and a directional factor $\sin v/v$.

7.4 RADIATION FROM A PLANE CIRCULAR PISTON

An acoustic source of practical interest is the plane circular piston, which is the model for a number of sources, including loudspeakers, open-ended organ pipes, and ventilation ducts. Consider a piston of radius a mounted on a flat rigid baffle of infinite extent. Let the radiating surface of the piston move uniformly with speed $U_0 \exp(j\omega t)$ normal to the baffle. The geometry and coordinates are sketched in Fig. 7.4.1.

The pressure at any field point can be obtained by dividing the surface of the piston into infinitesimal elements, each of which acts as a baffled simple source of strength $dQ = U_0\, dS$. Since the pressure generated by one of these sources is given by (7.2.17), the total pressure is

$$\mathbf{p}(r,\theta,t) = j\rho_0 c \frac{U_0}{\lambda} \int_S \frac{1}{r'} e^{j(\omega t - kr')}\, dS \tag{7.4.1}$$

where the surface integral is taken over the region $\sigma \le a$. While this integral is difficult to solve for a general field point, closed-form solutions are possible for two regions: (a) along a line perpendicular to the face of the piston and passing through its center (the acoustic axis), and (b) at sufficiently large distances, in the *far field.*

(a) Axial Response

The field along the acoustic axis (the z axis) is relatively simple to calculate. With reference to Fig. 7.4.1, we have

$$\mathbf{p}(r,0,t) = j\rho_0 c \frac{U_0}{\lambda} e^{j\omega t} \int_0^a \frac{\exp\left(-jk\sqrt{r^2+\sigma^2}\right)}{\sqrt{r^2+\sigma^2}} 2\pi\sigma\, d\sigma \tag{7.4.2}$$

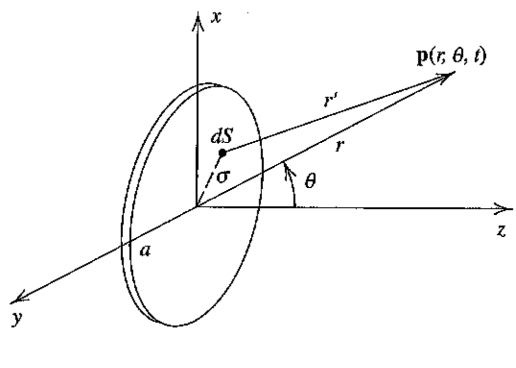

Figure 7.4.1 Geometry used in deriving the acoustic field of a baffled circular plane piston of radius a radiating sound of wave number k.

The integrand is a perfect differential,

$$\frac{\sigma \exp\left(-jk\sqrt{r^2+\sigma^2}\right)}{\sqrt{r^2+\sigma^2}} = -\frac{d}{d\sigma}\left(\frac{\exp\left(-jk\sqrt{r^2+\sigma^2}\right)}{jk}\right) \tag{7.4.3}$$

so the complex acoustic pressure is

$$\mathbf{p}(r,0,t) = \rho_0 c\, U_0 \left\{1 - \exp\left[-jk\left(\sqrt{r^2+a^2} - r\right)\right]\right\} e^{j(\omega t - kr)} \tag{7.4.4}$$

The pressure amplitude on the axis of the piston is the magnitude of the above expression,

$$P(r,0) = 2\rho_0 c U_0 \left|\sin\left\{\tfrac{1}{2}kr\left[\sqrt{1+(a/r)^2} - 1\right]\right\}\right| \tag{7.4.5}$$

For $r/a \gg 1$, the square root can be simplified to

$$\sqrt{1+(a/r)^2} \approx 1 + \tfrac{1}{2}(a/r)^2 \tag{7.4.6}$$

If also $r/a > ka/2$, the pressure amplitude on the axis has asymptotic form

$$P_{ax}(r) = \tfrac{1}{2}\rho_0 c U_0 (a/r) ka \tag{7.4.7}$$

which reveals the expected spherical divergence at sufficiently large distances. (The inequality $r/a > ka/2$ can be rewritten as $r > \pi a^2/\lambda$. In general, the quantity S/λ, where S is the moving area of the source, is called the *Rayleigh length*.)

Study of (7.4.5) reveals that the axial pressure exhibits strong interference effects, fluctuating between 0 and $2\rho_0 c U_0$ as r ranges between 0 and ∞. These extremes of pressure occur for values of r satisfying

$$\tfrac{1}{2}kr\left[\sqrt{1+(a/r)^2} - 1\right] = m\pi/2 \qquad m = 0, 1, 2, \ldots \tag{7.4.8}$$

Solution of the above for the values of r at the extrema yields

$$r_m/a = a/m\lambda - m\lambda/4a \tag{7.4.9}$$

Moving in toward the source from large r, one encounters the first local maximum in axial pressure at a distance r_1 given by

$$r_1/a = a/\lambda - \lambda/4a \tag{7.4.10}$$

For still smaller r, the pressure amplitude falls to a local minimum at r_2 given by

$$r_2/a = a/2\lambda - \lambda/2a \tag{7.4.11}$$

and then continues to fluctuate until the face of the piston is reached. A sketch of this behavior is shown in Fig. 7.4.2.

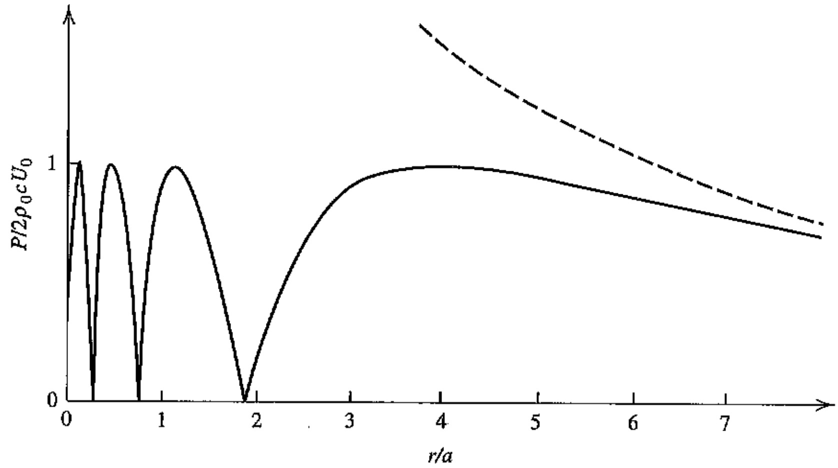

Figure 7.4.2 Axial pressure amplitude for a baffled circular plane piston of radius a radiating sound of wave number k with $ka = 8\pi$. Solid line is calculated from the exact theory. Dashed line is the far field approximation extrapolated into the near field. For this case, the far field approximation is accurate only for distances beyond about seven piston radii.

For $r > r_1$ the axial pressure decreases monotonically, approaching an asymptotic $1/r$ dependence. For $r < r_1$ the axial pressure displays strong interference effects, suggesting that the acoustic field close to the piston is complicated. The distance r_1 serves as a convenient demarcation between the complicated *near field* found close to the source and the simpler *far field* found at large distances from the source. The quantity r_1 has physical meaning only if the ratio a/λ is large enough that $r_1 > 0$. Indeed, if $a = \lambda/2$, then $r_1 = 0$ and there is no near field. At still lower frequencies the radiation from the piston approaches that of a simple source.

(b) Far Field

To aid in the evaluation of the far field, additional coordinates are introduced as indicated in Fig. 7.4.3. Let the x and y axes be oriented so the field point (r, θ) lies in the x-z plane. This allows the piston surface to be divided into an array of continuous line sources of differing lengths, each parallel to the y axis, so the field point is on the acoustic axis of each line source. The far field radiation pattern is found by imposing the restriction $r \gg a$ so the contribution to the field point from each of the line sources is simply its far field axial pressure. Since each line

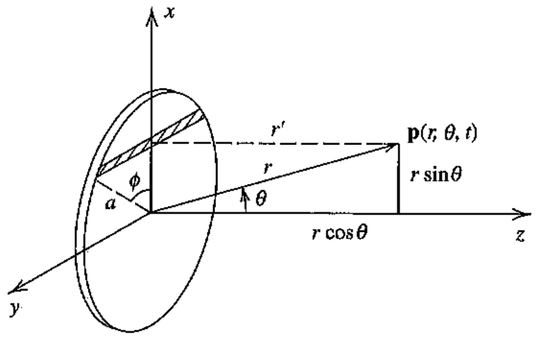

Figure 7.4.3 Geometry used in deriving the far field at (r, θ) of a baffled circular plane piston of radius a.

is of length $2a \sin\phi$ and width dx, the source strength from one such source is $dQ = 2U_0 a \sin\phi \, dx$ and the incremental pressure $d\mathbf{p}$ for this *baffled* source is, from (7.3.7),

$$d\mathbf{p} = j\rho_0 c \frac{U_0}{\pi r'} ka \sin\phi \, e^{j(\omega t - kr')} \, dx \tag{7.4.12}$$

For $r \gg a$, the value of r' is well approximated by

$$r' \approx r + \Delta r = r - a \sin\theta \cos\phi \tag{7.4.13}$$

and the acoustic pressure is

$$\mathbf{p}(r, \theta, t) = j\rho_0 c \frac{U_0}{\pi r'} ka \, e^{j(\omega t - kr')} \int_{-a}^{a} e^{jka \sin\theta \cos\phi} \sin\phi \, dx \tag{7.4.14}$$

where $r' \to r$ in the denominator, but $r' = r + \Delta r$ in the phase in accordance with the far field approximation. Using $x = a \cos\phi$, we can convert the integration from dx to $d\phi$:

$$\mathbf{p}(r, \theta, t) = j\rho_0 c \frac{U_0}{\pi} \frac{a}{r} ka \, e^{j(\omega t - kr)} \int_0^{\pi} e^{jka \sin\theta \cos\phi} \sin^2\phi \, d\phi \tag{7.4.15}$$

By symmetry, the imaginary part of the integral vanishes. The real part is tabulated in terms of a Bessel function,

$$\int_0^{\pi} \cos(z \cos\phi) \sin^2\phi \, d\phi = \pi \frac{J_1(z)}{z} \tag{7.4.16}$$

so that

$$\mathbf{p}(r, \theta, t) = \frac{j}{2} \rho_0 c U_0 \frac{a}{r} ka \left[\frac{2J_1(ka \sin\theta)}{ka \sin\theta} \right] e^{j(\omega t - kr)} \tag{7.4.17}$$

All the angular dependence is in the bracketed term. Since this term goes to unity as θ goes to 0, we can make the identifications

$$\begin{aligned} |\mathbf{p}(r, \theta)| &= P_{ax}(r) H(\theta) \\ H(\theta) &= \left| \frac{2J_1(v)}{v} \right| \qquad v = ka \sin\theta \end{aligned} \tag{7.4.18}$$

Note that the axial pressure amplitude is identical with the asymptotic expression (7.4.7). A plot of $2J_1(v)/v$ is given in Fig. 7.4.4, and numerical values are in Appendix A6. It is well worth comparing and contrasting Figs. 7.3.2 and 7.4.4.

The angular dependence of $H(\theta)$ reveals that there are pressure nodes at angles θ_m given by

$$ka \sin\theta_m = j_{1m} \qquad m = 1, 2, 3, \ldots \tag{7.4.19}$$

where j_{1m} designates the values of the argument of J_1 that reduce this Bessel function to zero, $J_1(j_{1m}) = 0$. (See Appendix A5.) Note that the form of $H(\theta)$ yields

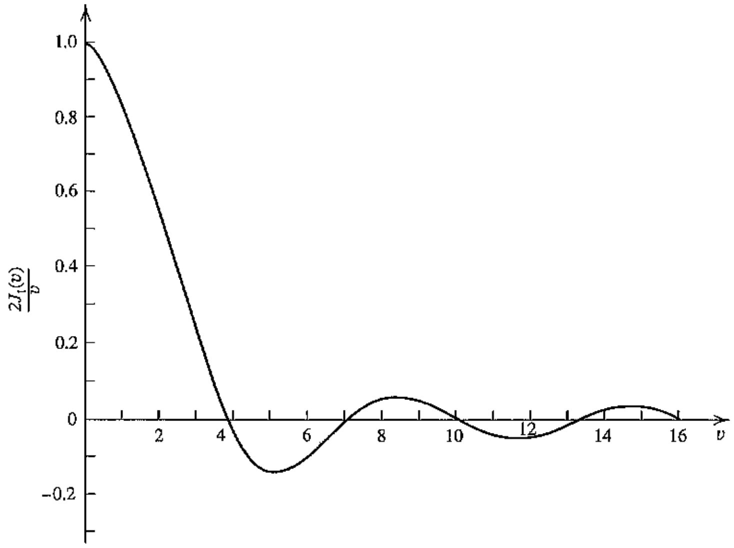

Figure 7.4.4 Functional behavior of $2J_1(v)/v$.

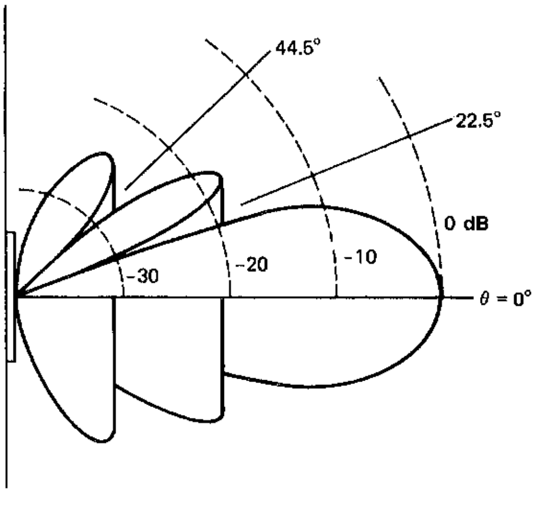

Figure 7.4.5 Beam pattern $b(\theta)$ for a circular plane piston of radius a radiating sound with $ka = 10$.

a maximum along $\theta = 0$. The angles θ_m define conical nodal surfaces with vertices at $r = 0$. Between these surfaces lie pressure lobes, as suggested in Fig. 7.4.5. The relative strengths and angular locations of the acoustic pressure maxima in the lobes are given by the relative maxima of $H(\theta)$. Thus, for constant r, if the intensity level on the axis is set at 0 dB, then the level of the maximum of the first side lobe is about -17.5 dB.

For wavelengths much smaller than the radius of the piston ($ka \gg 1$) the radiation pattern has many side lobes and the angular width of the major lobe is small. If the wavelength is sufficiently large ($ka < 3.83$) only the major lobe will be present. For $ka \ll 1$, the directional factor is nearly unity for all angles, so that the piston becomes a baffled simple source with source strength $Q = \pi a^2 U_0$.

The radiation patterns produced by a piston-type loudspeaker differ to some extent from these idealized patterns for reasons including the following: (1) The area of the baffle in which the speaker is mounted is finite. At low frequencies the wavelength of the sound may be the same as, or greater than, the linear dimensions of the baffle and the assumption that each element of the piston radiates with hemispherical divergence will be in error. (2) If the loudspeaker cabinet is not closed, the radiation from the back of the speaker may propagate into the region in front of the speaker, resulting in a radiation pattern approximating an acoustic doublet rather than a piston in an infinite baffle. (3) The material of a loudspeaker cone is not perfectly rigid. Driving the speaker at its center establishes velocity amplitudes higher near the center of the cone than near its rim at low frequencies, and at high frequencies the cone may vibrate in standing waves. Under these circumstances, U_0 may become a complex function $\mathbf{U}_0$ of the radial distance σ and angle ϕ. By a suitable choice of the relation between U_0 and σ, a wide variety of radiation patterns can be obtained. Altering radiation patterns with the help of a flexible radiating surface is an important consideration in loudspeaker design. Even in small rooms, loudspeakers that project higher frequencies into narrow major lobes often sound "sharp" or "edgy" to listeners on the acoustic axis and "dull" to listeners off the axis. Broadening the major lobes for higher frequencies helps to counteract this beaming of sound. In small rooms, avoiding high-frequency absorption at the walls is another aid in scattering high-frequency energy. When public address systems are used outdoors or in large auditoriums, scattering is negligible and uniform distribution of higher frequencies must be obtained by employing multidirectional clusters of speakers or groups of speakers aimed in different directions.

7.5 RADIATION IMPEDANCE

In Chapter 2 it was found useful to define the input mechanical impedance of a string as the force applied to the string divided by the resulting speed of the string at the point where the force is applied. If the force is not applied *directly* to the string, but to some device attached to the string, then it was shown in Problem 2.9.2 that the force applied to the device divided by the speed of the device was equal to the mechanical impedance of the device plus the input mechanical impedance of the string as seen by the device. Similarly, in the discussion of acoustic sources it will be useful to express the input mechanical impedance of the source in terms of the *mechanical impedance* of the source radiating into a vacuum and the *radiation impedance* of the acoustic wave propagated into the fluid.

Consider a transmitter whose active face (*diaphragm*) of area S moves with a normal velocity component $\mathbf{u}$ whose magnitude and phase may be a function of position. If $d\mathbf{f}_S$ is the normal component of force on an element dS of the active face, the radiation impedance is

$$\mathbf{Z}_r = \int \frac{d\mathbf{f}_S}{\mathbf{u}} \tag{7.5.1}$$

If the diaphragm has mass m, mechanical resistance R_m, and stiffness s and moves *uniformly* with a normal component of velocity $\mathbf{u}_0 = U_0 \exp(j\omega t) = j\omega\boldsymbol{\xi}_0$ under the externally applied force $\mathbf{f} = F\exp(j\omega t)$, Newton's law of motion yields

$$\mathbf{f} - \mathbf{f}_S - R_m\frac{d\boldsymbol{\xi}_0}{dt} - s\boldsymbol{\xi}_0 = m\frac{d^2\boldsymbol{\xi}_0}{dt^2} \tag{7.5.2}$$

where the force of the diaphragm on the fluid is $\mathbf{f}_S = \mathbf{Z}_r\mathbf{u}_0$. Recalling that $\mathbf{Z}_m = \mathbf{R}_m + j(\omega m - s/\omega)$ and solving for $\mathbf{u}_0$ gives

$$\mathbf{u}_0 = \mathbf{f}/(\mathbf{Z}_m + \mathbf{Z}_r) \tag{7.5.3}$$

Thus, in the presence of fluid loading, the applied force encounters the sum of the mechanical impedance of the source and the radiation impedance. The radiation impedance can be expressed as

$$\mathbf{Z}_r = Z_r e^{j\theta} = R_r + jX_r \tag{7.5.4}$$

where R_r is the *radiation resistance* and X_r is the *radiation reactance.*

A positive R_r will increase the total resistance, increasing the power dissipated by the source by an amount equal to the power radiated into the fluid,

$$\Pi = \frac{1}{T}\int_0^T \mathrm{Re}\{\mathbf{f}_S\}\mathrm{Re}\{\mathbf{u}_0\}\,dt \tag{7.5.5}$$

or

$$\Pi = \tfrac{1}{2}U_0^2 Z_r \cos\theta = \tfrac{1}{2}U_0^2 R_r \tag{7.5.6}$$

The radiation resistance can be found directly from the power radiated into the fluid. For example, use of (7.2.16) and (7.2.19) shows that for a simple source

$$R_r = \rho_0 c(kS)^2/4\pi \qquad \text{(simple source)} \tag{7.5.7}$$

$$R_r = \rho_0 c(kS)^2/2\pi \qquad \text{(baffled simple source)} \tag{7.5.8}$$

where in each case S is the surface area of the relevant source.

A positive X_r will manifest itself as a mass loading that decreases the resonance frequency ω_0 of the oscillator from $\sqrt{s/m}$ to $\sqrt{s/(m + m_r)}$, where $m_r = X_r/\omega$ is the *radiation mass.* The effect of the *radiation mass* can be slight for sources operating in light media such as air, but for a dense fluid like water the decrease in resonance frequency resulting from the presence of the medium may be quite marked.

(a) The Circular Piston

To calculate the radiation impedance of a baffled circular piston of radius a and normal complex velocity $\mathbf{u}_0 = U_0 \exp(j\omega t)$, consider an infinitesimal area dS of the surface of the piston (Fig. 7.5.1) and let $d\mathbf{p}$ be the incremental pressure that the motion of dS produces at some other element of area dS' of the piston. The total pressure $\mathbf{p}$ at dS' can be obtained by integrating (7.4.1) over the surface of the piston,

$$\mathbf{p} = j\rho_0 c\frac{U_0}{\lambda}\int_S \frac{1}{r}e^{j(\omega t - kr)}\,dS \tag{7.5.9}$$

where r is the distance between dS and dS'. The total force $\mathbf{f}_S$ on the piston from the pressure is the integral of $\mathbf{p}$ over dS', so that $\mathbf{f}_S = \int \mathbf{p}\,dS'$. The integrations over dS to get $\mathbf{p}$ and then over dS' to get $\mathbf{f}_S$ include both the force on dS' resulting from the motion of dS and vice versa. But from acoustic reciprocity, these two forces must be the same. Consequently, the result of the double integration is twice what

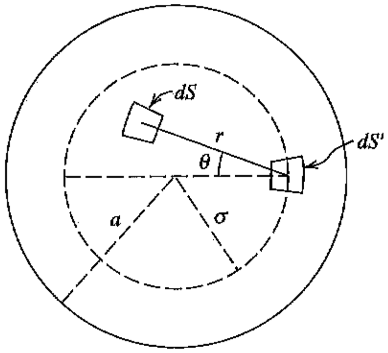

Figure 7.5.1 Surface elements dS and dS' used in obtaining the reaction force on a radiating plane circular piston.

would be obtained if the limits of integration were chosen to include the force between each pair of elements only once. This latter choice of limits leads to a considerable simplification of the problem. Refer to Fig. 7.5.1. With σ the radial distance from the center of the piston to dS', each pair of elements is used only once by integrating over the area of the piston within this circle of radius σ. The maximum distance from dS' to any point within the circle is $2\sigma\cos\theta$, so the entire area within the circle will be covered if we integrate r from 0 to $2\sigma\cos\theta$ and then integrate θ from $-\pi/2$ to $\pi/2$. The integration of dS' is now extended over the entire surface of the piston by setting $dS' = \sigma\, d\sigma\, d\psi$ and integrating ψ from 0 to 2π and then σ from 0 to a. After multiplying this by two, we have our desired expression,

$$\mathbf{f}_S = 2j\rho_0 c\frac{U_0}{\lambda}e^{j\omega t}\int_0^a\int_0^{2\pi}\int_{-\pi/2}^{\pi/2}\int_0^{2\sigma\cos\theta}\sigma e^{-jkr}dr\, d\theta\, d\psi\, d\sigma \tag{7.5.10}$$

The details of the integration are left to Problem 7.5.2. The result for the radiation impedance $\mathbf{Z}_r = \mathbf{f}_S/\mathbf{u}_0$ is

$$\mathbf{Z}_r = \rho_0 cS[R_1(2ka) + jX_1(2ka)] \tag{7.5.11}$$

where $S = \pi a^2$ is the area of the piston face. The *piston resistance function* R_1 and *piston reactance function* X_1 are given by

$$\begin{aligned} R_1(x) &= 1 - \frac{2J_1(x)}{x} = \frac{x^2}{2\cdot 4} - \frac{x^4}{2\cdot 4^2\cdot 6} + \frac{x^6}{2\cdot 4^2\cdot 6^2\cdot 8} - \cdots \\ X_1(x) &= \frac{2\mathbf{H}_1(x)}{x} = \frac{4}{\pi}\left(\frac{x}{3} - \frac{x^3}{3^2\cdot 5} + \frac{x^5}{3^2\cdot 5^2\cdot 7} - \cdots\right) \end{aligned} \tag{7.5.12}$$

with $\mathbf{H}_1(x)$ the *first order Struve function*, described in Appendix A4. Sketches of R_1 and X_1 are shown in Fig. 7.5.2 and numerically tabulated in Appendix A6.

In the low-frequency limit ($ka \ll 1$) the radiation impedance can be approximated by the first terms of the power expansions. The radiation resistance becomes

$$R_r \approx \tfrac{1}{2}\rho_0 cS(ka)^2 \tag{7.5.13}$$

and the radiation reactance becomes

$$X_r \approx (8/3\pi)\rho_0 cSka \tag{7.5.14}$$

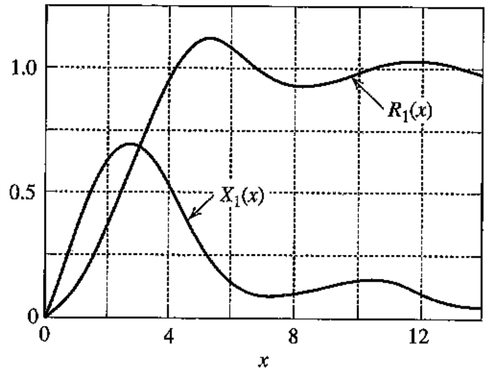

Figure 7.5.2 Radiation resistance and reactance for a plane circular piston of radius a radiating sound of wave number k ($x = 2ka$).

Note that, in the low-frequency limit, the radiation resistance for the piston is identical with that for a baffled simple source of the same surface area S. The low-frequency reactance is that of a mass

$$m_r = X_r/\omega = \rho_0 S(8a/3\pi) \tag{7.5.15}$$

Thus, the piston appears to be loaded with a cylindrical volume of fluid whose cross-sectional area S is that of the piston and whose effective height is $8a/3\pi \approx 0.85a$.

In the high-frequency limit $ka \gg 1$, we have $X_1(2ka) \to (2/\pi)/(ka)$ and $R_1(2ka) \to 1$, so that $Z_r \to R_r \approx S\rho_0 c$. This yields

$$\Pi \approx \tfrac{1}{2}\rho_0 c S U_0^2 \tag{7.5.16}$$

which is the same as the power that would be carried by a plane wave of particle speed amplitude U_0 in a fluid of characteristic impedance $\rho_0 c$ through a cross-sectional area S.

(b) The Pulsating Sphere

The radiation impedance of the pulsating sphere is easily found from (7.1.2) to be

$$\mathbf{Z}_r = \rho_0 c S \cos\theta_a \, e^{j\theta_a} \tag{7.5.17}$$

where $S = 4\pi a^2$ is the surface area of the sphere. For high frequencies ($ka \gg 1$), this reduces to a pure radiation resistance $\mathbf{Z}_r = R_r$, where

$$R_r = \rho_0 c S \tag{7.5.18}$$

For low frequencies ($ka \ll 1$), $\mathbf{Z}_r$ becomes

$$\mathbf{Z}_r \approx \rho_0 c S (ka)^2 + j\rho_0 c S ka \tag{7.5.19}$$

The radiation resistance is much less than the radiation reactance, and the radiation reactance is again like a mass,

$$m_r = X_r/\omega = 3\rho_0 V \tag{7.5.20}$$

where $V = 4\pi a^3/3$ is the volume of the sphere. In the low-frequency limit the radiation mass is three times the mass of the fluid displaced by the sphere.

7.6 FUNDAMENTAL PROPERTIES OF TRANSDUCERS

Several definitions are used to describe the more important aspects of the field without the necessity of displaying the entire radiation pattern.

(a) Directional Factor and Beam Pattern

We have shown that the far field radiation for each of two uncomplicated sources (continuous line and piston) can be expressed as a product of an axial pressure $P_{ax}(r)$ and a directional factor $H(\theta)$. For sources of lower symmetry, this same separation is possible, although the directional factor may depend on two angles, $H(\theta, \phi)$. The directional factor is always normalized so its maximum value is unity, as illustrated by (7.3.5) and (7.4.18). The directions for which $H = 1$ determine the acoustic axes. An acoustic "axis" may be a line, a plane, or a conical surface. The normalized far field pressure along any radial line designated by angles θ and ϕ is simply $H(\theta, \phi)/r$.

The variation of intensity level (or sound pressure level) with angle is the *beam pattern*

$$\begin{aligned} b(\theta, \phi) &= 10 \log[I(r, \theta, \phi)/I_{ax}(r)] = 20 \log[P(r, \theta, \phi)/P_{ax}(r)] \\ &= 20 \log H(\theta, \phi) \end{aligned} \tag{7.6.1}$$

(b) Beam Width

No single definition has been agreed upon for determining the angles that mark the effective extremities of the major lobe. Hence, the criterion must be clearly stated when beam widths are specified. The values of $I(r, \theta, \phi)/I_{ax}(r)$ used to delineate the effective width of a major lobe range from a maximum of 0.5 (down 3 dB or "half-power"), through 0.25 (down 6 dB or "quarter-power"), to a minimum of 0.1 (down 10 dB). As an illustration of the ambiguity that arises if the ratio of intensities is not specified, consider a piston that is radiating sound of wavelength $\lambda = a/4$. The calculated beam widths corresponding to the three ratios given above are 7.4° (down 3 dB), 10.1° (down 6 dB), and 12.9° (down 10 dB), whereas the beam width corresponding to the first null is 17.3°. Even when the outer limit of the major lobe is defined as being down 10 dB relative to the axial level, it is still some 7.5 dB higher than the maximum level of the first minor lobe.

(c) Source Level

A measure of the axial output of a source is the *source level SL*. Assume that the acoustic axis of the source has been determined and the pressure amplitude along this line is measured in the far field (where the pressure varies as $1/r$). The curve of $P_{ax}(r)$ versus $1/r$ can be extrapolated from large r to a position $r = 1$ m from the source to give

$$P_{ax}(1) = \lim_{r \downarrow 1} P_{ax}(r) \tag{7.6.2}$$