Chapter 4

THE TWO-DIMENSIONAL WAVE EQUATION: VIBRATIONS OF MEMBRANES AND PLATES

4.1 VIBRATIONS OF A PLANE SURFACE

Consider transverse vibrations of two-dimensional systems, such as a drumhead or the diaphragm of a microphone. While analysis may seem more complicated because two spatial coordinates are needed to locate a point on the surface and a third to specify its displacement, the equation of motion (subject to the same simplifying assumptions invoked in the previous two chapters) will be merely the two-dimensional generalization of that for a string.

Generalization to two dimensions requires selecting a coordinate system. Choice of a coordinate system matching the boundary conditions (cartesian coordinates for a rectangular boundary and polar coordinates for a circular boundary) will greatly simplify obtaining and interpreting solutions. Unfortunately, the number of useful coordinate systems is strictly limited and, consequently, the number of easily solved membrane problems is similarly restricted.

4.2 THE WAVE EQUATION FOR A STRETCHED MEMBRANE

Assume a membrane is thin, is stretched uniformly in all directions, and vibrates transversely with small displacement amplitudes. Let ρ_S be the *surface density* (kg/m^2) of the membrane, and let $\mathcal{T}$ be the *membrane tension per unit length* (N/m); the material on opposite sides of a line segment of length dl will be pulled apart with a force $\mathcal{T}\, dl$.

In cartesian coordinates the transverse displacement of a point is expressed as $y(x, z, t)$. The force acting on a displaced surface element of area $dS = dx\,dz$ is the sum of the transverse forces acting on the edges parallel to the x and z axes. For the element shown in Fig. 4.2.1 the net vertical force arising from the pair of opposing tensions $\mathcal{T} dz$ is

$$\mathcal{T} dz\left[\left(\frac{\partial y}{\partial x}\right)_{x+dx} - \left(\frac{\partial y}{\partial x}\right)_{x}\right] = \mathcal{T}\frac{\partial^2 y}{\partial x^2}\, dx\, dz \qquad (4.2.1)$$

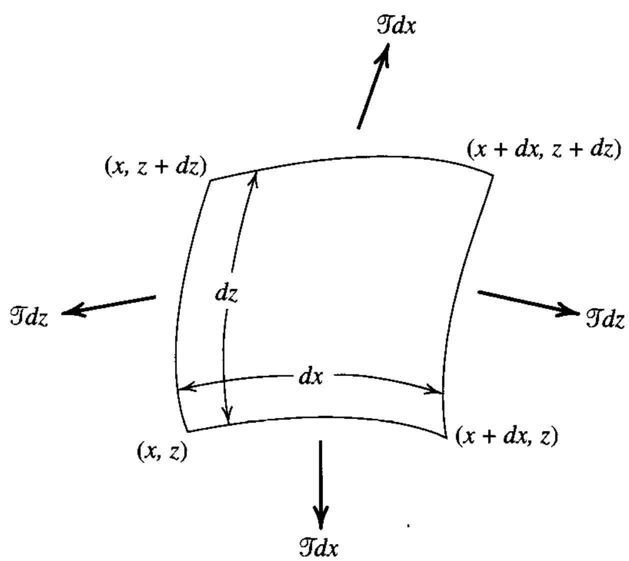

Figure 4.2.1 Elemental area of a membrane showing the forces acting when the membrane is displaced transversely.

and that from the pair of tensions $\mathcal{T}\,dx$ is $\mathcal{T}(\partial^2 y/\partial z^2)\,dx\,dz$. Equating the sum of these two to the product of the mass $\rho_S\,dx\,dz$ of the element and its acceleration $\partial^2 y/\partial t^2$ gives

$$\frac{\partial^2 y}{\partial x^2} + \frac{\partial^2 y}{\partial z^2} = \frac{1}{c^2}\frac{\partial^2 y}{\partial t^2} \tag{4.2.2}$$

with

$$\boxed{c^2 = \mathcal{T}/\rho_S} \tag{4.2.3}$$

Equation (4.2.2) may be expressed more generally in the form

$$\nabla^2 y = \frac{1}{c^2}\frac{\partial^2 y}{\partial t^2} \tag{4.2.4}$$

where ∇^2 is the *Laplacian operator* (in this case two-dimensional) and (4.2.4) is the *two-dimensional wave equation.*

The form of the Laplacian depends on the choice of the coordinate system. The Laplacian in two-dimensional cartesian coordinates,

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2} \tag{4.2.5}$$

is appropriate for rectangular membranes. For a circular membrane, polar coordinates (r, θ) are preferable and use of

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} \tag{4.2.6}$$

gives the appropriate wave equation,

$$\frac{\partial^2 y}{\partial r^2} + \frac{1}{r}\frac{\partial y}{\partial r} + \frac{1}{r^2}\frac{\partial^2 y}{\partial \theta^2} = \frac{1}{c^2}\frac{\partial^2 y}{\partial t^2} \tag{4.2.7}$$

Solutions to (4.2.4) will have all the properties of the waves studied previously, generalized to two dimensions. For calculating normal modes on membranes it is conventional to assume the solutions have the form

$$\mathbf{y} = \mathbf{\Psi} e^{j\omega t} \tag{4.2.8}$$

where $\mathbf{\Psi}$ is a function only of position. Substitution and identification of $k = \omega/c$ yields the *Helmholtz equation,*

$$\nabla^2\mathbf{\Psi} + k^2\mathbf{\Psi} = 0 \tag{4.2.9}$$

The solutions of (4.2.9) for a membrane with specified shape and boundary conditions are the normal modes of the problem.

4.3 FREE VIBRATIONS OF A RECTANGULAR MEMBRANE WITH FIXED RIM

If a stretched rectangular membrane is fixed at $x = 0, x = L_x, z = 0$, and $z = L_z$, the boundary conditions are

$$y(0,z,t) = y(L_x,z,t) = y(x,0,t) = y(x,L_z,t) = 0 \tag{4.3.1}$$

Assuming a solution

$$\mathbf{y}(x,z,t) = \mathbf{\Psi}(x,z)e^{j\omega t} \tag{4.3.2}$$

to (4.2.4) gives

$$\frac{\partial^2\mathbf{\Psi}}{\partial x^2} + \frac{\partial^2\mathbf{\Psi}}{\partial z^2} + k^2\mathbf{\Psi} = 0 \tag{4.3.3}$$

Now, apply the method of *separation of variables* by assuming that $\mathbf{\Psi}$ is the product of two functions, each dependent on only one of the dimensions,

$$\mathbf{\Psi}(x,z) = \mathbf{X}(x)\mathbf{Z}(z) \tag{4.3.4}$$

Substitution and division by $\mathbf{X}(x)\,\mathbf{Z}(z)$ gives

$$\frac{1}{\mathbf{X}}\frac{d^2\mathbf{X}}{dx^2} + \frac{1}{\mathbf{Z}}\frac{d^2\mathbf{Z}}{dz^2} + k^2 = 0 \tag{4.3.5}$$

Since the first term is a function only of x and the second only of z, both must be constants; otherwise the three terms cannot sum to zero for all x and z. This provides the pair of equations

$$\frac{d^2\mathbf{X}}{dx^2} + k_x^2\mathbf{X} = 0 \qquad \frac{d^2\mathbf{Z}}{dz^2} + k_z^2\mathbf{Z} = 0 \tag{4.3.6}$$

where the constants k_x and k_z are related by

$$k_x^2 + k_z^2 = k^2 \tag{4.3.7}$$

Solutions of (4.3.6) are sinusoids, so that

$$\mathbf{y}(x,z,t) = \mathbf{A}\sin(k_x x + \phi_x)\sin(k_z z + \phi_z)e^{j\omega t} \tag{4.3.8}$$

where k_x, k_z, ϕ_x, and ϕ_z are determined by the boundary conditions. The conditions $y(0,z,t) = 0$ and $y(x,0,t) = 0$ require $\phi_x = 0$ and $\phi_z = 0$, and the conditions $y(L_x, z, t) = 0$ and $y(x, L_z, t) = 0$ require the arguments k_xL_x and k_zL_z to be integral multiples of π. Thus, the standing waves on the membrane are given by

$$\begin{aligned} \mathbf{y}(x,z,t) &= \mathbf{A}\sin k_x x \sin k_z z\, e^{j\omega t} \\ k_x &= n\pi/L_x \qquad n = 1,2,3,\ldots \\ k_z &= m\pi/L_z \qquad m = 1,2,3,\ldots \end{aligned} \tag{4.3.9}$$

where $|\mathbf{A}|$ is the maximum displacement amplitude. These equations limit the wave numbers k_x and k_z to discrete sets of values, which in turn restrict the natural frequencies for the allowed modes to

$$f_{nm} = \omega_{nm}/2\pi = (c/2)[(n/L_x)^2 + (m/L_z)^2]^{1/2} \tag{4.3.10}$$

This is the two-dimensional extension of the comparable results for the freely vibrating fixed, fixed string. The fundamental frequency is obtained by substitution

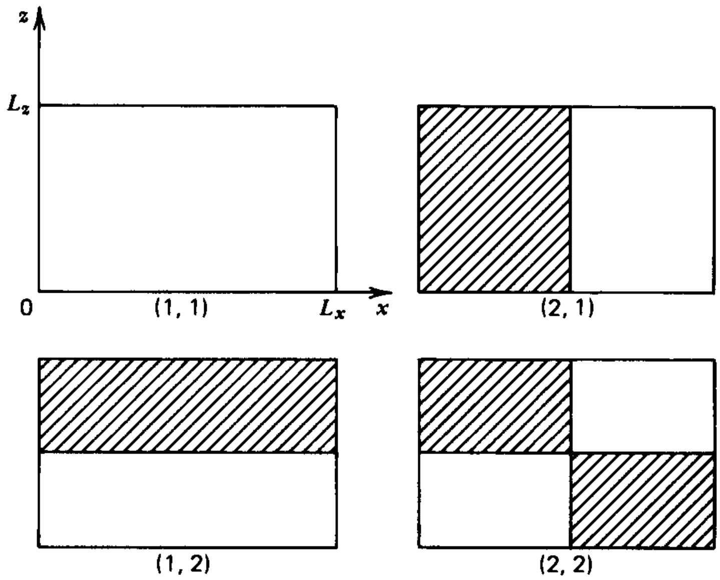

Figure 4.3.1 Schematic representation of four typical normal modes of a rectangular membrane with fixed rim. The modes are designated by the pair of integers (n, m). The hatched areas denote sections of the membrane that vibrate 180° out of phase with the unhatched areas. These areas are separated by nodal lines.

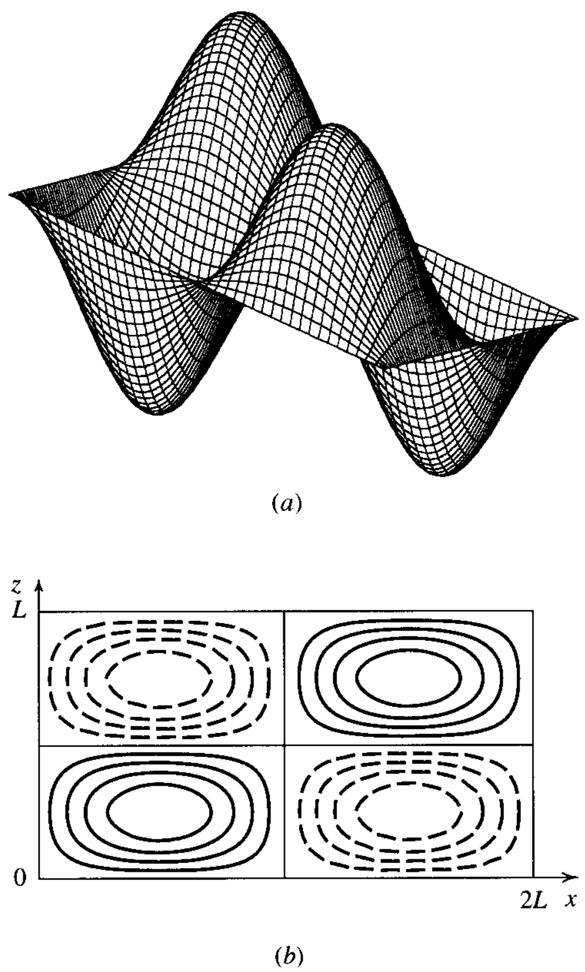

Figure 4.3.2 The displacement of a rectangular membrane with $L_x/L_z = 2$ vibrating in a (2, 2) mode. (*a*) Isometric view. (*b*) Contours of equal displacement. The regions denoted by contours shown with solid lines vibrate 180° out of phase from those shown with dashed lines.

of $n = 1$ and $m = 1$ into (4.3.10). Overtones having $n = m$ will be harmonics of the fundamental, while those for which $n \neq m$ may not be. Figure 4.3.1 illustrates a few modes for a rectangular membrane. The normal modes are labelled by the ordered pair (n, m). Figure 4.3.2 shows the displacement of a (2, 2) mode of a rectangular membrane with fixed rim. Since the nodal lines have zero displacement, it is possible to insert rigid supports along any of them without affecting the wave pattern for the particular frequency involved.

4.4 FREE VIBRATIONS OF A CIRCULAR MEMBRANE WITH FIXED RIM

For a circular membrane fixed at $r = a$, the Helmholtz equation in cylindrical coordinates

$$\frac{\partial^2 \Psi}{\partial r^2} + \frac{1}{r}\frac{\partial \Psi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \Psi}{\partial \theta^2} + k^2 \Psi = 0 \tag{4.4.1}$$

can be solved by assuming that $\boldsymbol{\Psi}(r, \theta)$ is the product of two terms, each a function of only one spatial variable,

$$\boldsymbol{\Psi} = \mathbf{R}(r)\boldsymbol{\Theta}(\theta) \tag{4.4.2}$$

subject to the boundary condition

$$\mathbf{R}(a) = 0 \tag{4.4.3}$$

In addition, $\boldsymbol{\Theta}$ must be a smooth and continuous function of θ. Substitution into (4.2.9) gives

$$\boldsymbol{\Theta}\frac{d^2\mathbf{R}}{dr^2} + \frac{\boldsymbol{\Theta}}{r}\frac{d\mathbf{R}}{dr} + \frac{\mathbf{R}}{r^2}\frac{d^2\boldsymbol{\Theta}}{d\theta^2} + k^2\mathbf{R}\boldsymbol{\Theta} = 0 \tag{4.4.4}$$

where $k = \omega/c$. Multiplying this equation by $r^2/\boldsymbol{\Theta}\mathbf{R}$ and moving those terms containing r to one side of the equality sign and those containing θ to the other side results in

$$\frac{r^2}{\mathbf{R}}\left(\frac{d^2\mathbf{R}}{dr^2} + \frac{1}{r}\frac{d\mathbf{R}}{dr}\right) + k^2r^2 = -\frac{1}{\boldsymbol{\Theta}}\frac{d^2\boldsymbol{\Theta}}{d\theta^2} \tag{4.4.5}$$

The left side of this equation, a function of r alone, cannot equal the right side, a function of θ alone, unless both functions equal the same constant. If we let this constant be m^2, then the right side becomes

$$\frac{d^2\boldsymbol{\Theta}}{d\theta^2} = -m^2\boldsymbol{\Theta} \tag{4.4.6}$$

which has harmonic solutions

$$\boldsymbol{\Theta}(\theta) = \cos(m\theta + \gamma_m) \tag{4.4.7}$$

where the γ_m are determined by the (spatial factor in the) initial conditions. Since $\boldsymbol{\Theta}$ must be smooth and single-valued, each m must be an integer. With m fixed in value, (4.4.5) is *Bessel's equation,*

$$\frac{d^2\mathbf{R}}{dr^2} + \frac{1}{r}\frac{d\mathbf{R}}{dr} + \left(k^2 - \frac{m^2}{r^2}\right)\mathbf{R} = 0 \tag{4.4.8}$$

Solutions to this equation are the *Bessel functions* of *order m* of the *first kind* $J_m(kr)$ and *second kind* $Y_m(kr)$,

$$\mathbf{R}(r) = \mathbf{A}J_m(kr) + \mathbf{B}Y_m(kr) \tag{4.4.9}$$

Some properties of Bessel functions are summarized in Appendixes A4 and A5. They are oscillatory functions of kr whose amplitudes diminish roughly as $1/\sqrt{kr}$. The $Y_m(kr)$ become unbounded in the limit $kr \to 0$.

While (4.4.9) is the general solution of (4.4.8), a membrane that extends across the origin must have finite displacement at $r = 0$. This requires $\mathbf{B} = 0$ so that

$$\mathbf{R}(r) = \mathbf{A}J_m(kr) \tag{4.4.10}$$

[If, however, the membrane were stretched between inner and outer rims, so that it did not span the origin, then both terms in (4.4.9) would have to be used to satisfy the two boundary conditions.]

Application of the boundary condition $\mathbf{R}(a) = 0$ requires $J_m(ka) = 0$. If the values of the argument of J_m that cause it to equal zero are denoted by j_{mn}, then k assumes the discrete values $k_{mn} = j_{mn}/a$. (See the Appendixes for values of, and formulas for, the arguments j_{mn}.)

The solutions are

$$\begin{aligned} \mathbf{y}_{mn}(r,\theta,t) &= \mathbf{A}_{mn} J_m(k_{mn}r)\cos(m\theta + \gamma_{mn})e^{j\omega_{mn}t} \\ k_{mn}a &= j_{mn} \end{aligned} \tag{4.4.11}$$

and the natural frequencies are

$$f_{mn} = j_{mn}c/2\pi a \tag{4.4.12}$$

Recall that the physical motion of the (m,n)th solution is the real part of (4.4.11),

$$y_{mn}(r,\theta,t) = A_{mn} J_m(k_{mn}r)\cos(m\theta + \gamma_{mn})\cos(\omega_{mn}t + \phi_{mn}) \tag{4.4.13}$$

where $\mathbf{A}_{mn} = A_{mn}\exp(j\phi_{mn})$. The azimuthal phase angles γ_{mn} depend on the location of the initial excitation of the membrane.

Figure 4.4.1 illustrates some simpler modes of vibration for a circular membrane fixed at the rim. The integer m determines the number of *radial nodal lines* and the

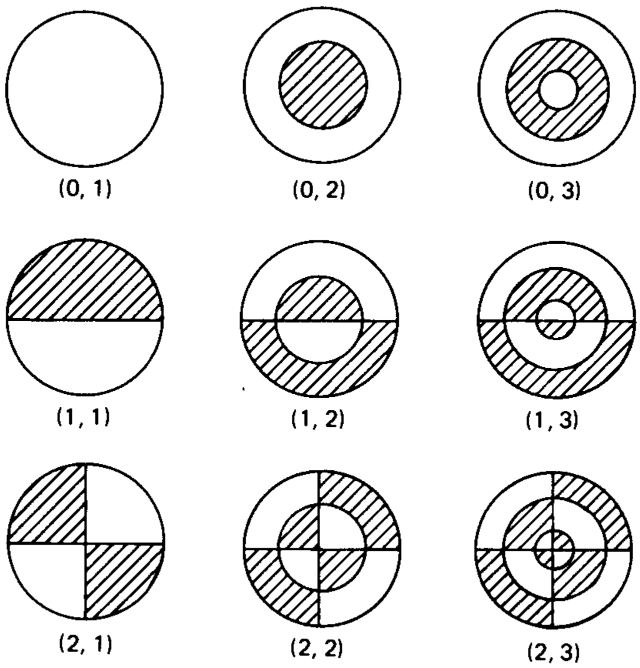

Figure 4.4.1 Normal modes of a circular membrane with fixed rim. The modes are designated by the pair of integers (m,n). The hatched areas denote sections of the membrane that vibrate 180° out of phase with the unhatched areas. These areas are separated by nodal lines. The frequency of the modes increases down each column. (See Table 4.4.1.)

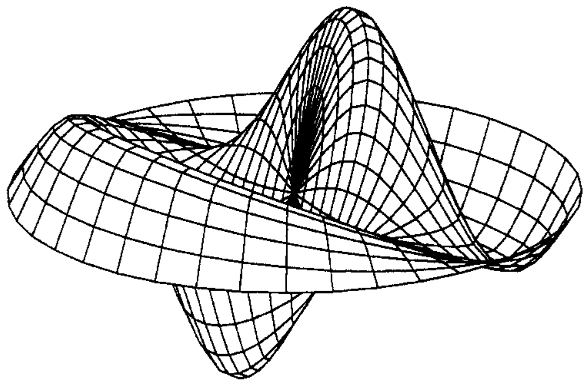

Figure 4.4.2 Isometric view of the displacement of a circular membrane with fixed rim vibrating in a (1, 2) mode.

Table 4.4.1 Normal-mode frequencies of a circular membrane

$f_{01} = 1.0f_{01}$	$f_{11} = 1.593f_{01}$	$f_{21} = 2.135f_{01}$
$f_{02} = 2.295f_{01}$	$f_{12} = 2.917f_{01}$	$f_{22} = 3.500f_{01}$
$f_{03} = 3.598f_{01}$	$f_{13} = 4.230f_{01}$	$f_{23} = 4.832f_{01}$

second integer n determines the number of *nodal circles*. It should be noted that $n = 1$ is the minimum allowed value of n and corresponds to a mode of vibration in which the (only) nodal circle occurs at the fixed boundary of the membrane. Figure 4.4.2 shows the displacement of a circular membrane vibrating in a (1, 2) mode.

For each m there exists a sequence of modes of increasing frequency. Table 4.4.1 lists a few of these frequencies f_{mn} expressed in terms of the fundamental frequency f_{01}. Note that none of the overtones are harmonics of the fundamental.

4.5 SYMMETRIC VIBRATIONS OF A CIRCULAR MEMBRANE WITH FIXED RIM

For many situations described by a circular membrane fixed at the rim, modes having circular symmetry are of greatest importance. Let us, therefore, confine our attention to those solutions that are independent of θ. Because $m = 0$ for these modes, we will suppress this subscript and retain only n,

$$\mathbf{y}_n(r,t) \equiv \mathbf{y}_{0n}(r,\theta,t) = \mathbf{A}_n J_0(k_n r)e^{j\omega_n t} \tag{4.5.1}$$

The natural frequencies are found from (4.4.12),

$$f_n/f_1 = j_{0n}/j_{01} \tag{4.5.2}$$

and the lowest three are given by the first column in Table 4.4.1. For all symmetric modes other than the fundamental, inner nodal circles will occur at radial distances for which $J_0(k_n r)$ vanishes.

The real part of $\mathbf{y}_n$ gives the displacement of the membrane in its nth symmetric mode, and the summation over all n gives the total displacement of the membrane when it is vibrating with circular symmetry,

$$y(r,t) = \sum_{n=1}^{\infty} A_n J_0(k_n r)\cos(\omega_n t + \phi_n) \tag{4.5.3}$$

where $A_n = |\mathbf{A}_n|$ is the displacement amplitude of the nth mode at $r = 0$.

Figure 4.4.1 shows that when the central part of the membrane is displaced up, the adjacent ring is displaced down, and vice versa. Consequently, a membrane vibrating at natural frequencies other than its fundamental produces little net displacement of the surrounding air. (For this reason, the vibrating head of a kettledrum has lower efficiencies of sound production for its overtone frequencies than for its fundamental.) One parameter for ranking the efficiency of each normal mode in producing sound is the average displacement amplitude of the mode. From (4.5.3), the average displacement amplitude $\langle\Psi_n\rangle_S$ of the nth symmetric normal mode is

$$\begin{aligned}\langle\Psi_n\rangle_S &= \frac{1}{\pi a^2}\int_S A_n J_0(k_n r)\,dS = \frac{1}{\pi a^2}\int_0^a A_n J_0(k_n r) 2\pi r\,dr \\ &= (2A_n/k_n a)J_1(k_n a)\end{aligned} \tag{4.5.4}$$

where we have used the relationship $zJ_0(z) = d[zJ_1(z)]/dz$ from Appendix A4. [Note that for all modes other than the symmetric ones, the angular dependence $\cos(m\theta + \gamma_m)$ guarantees that the average displacement is zero.]

In many situations involving sources of sound with dimensions smaller than the radiated wavelength, the radiated pressure field depends primarily on the *amount* of air displaced, and not on the exact shape of the moving surface. A measure of the amount of air displaced is the *volume displacement amplitude,* defined as the surface area of the vibrating surface multiplied by the average displacement amplitude of that surface.

When vibrating in its lowest mode, the circular membrane (with fixed rim) has $k_1 a = 2.405$ and, from (4.5.4), the average displacement amplitude is

$$\langle\Psi_1\rangle_S = (2A_1/2.405)J_1(2.405) = 0.432A_1 \tag{4.5.5}$$

where A_1 is the displacement amplitude at the center. A simple piston of the same surface area and a displacement amplitude of $0.432A_1$ will have the same volume displacement amplitude $0.432(\pi a^2)A_1$ as the membrane. If the membrane is vibrating in the mode of its first overtone, $\langle\Psi_2\rangle_S = -0.123A_2$. (The negative sign indicates that the average displacement is opposed to the displacement at the center.) If fundamental and first overtone have the same displacement amplitude at the center of the membrane, the fundamental would be about 3.5 times as effective as the first overtone in displacing air.

*4.6 THE DAMPED, FREELY VIBRATING MEMBRANE

Damping forces, such as those arising within the membrane from internal friction and external forces associated with the radiation of sound, cause the amplitude of each freely vibrating mode to decrease exponentially. As in Chapters 2 and 3, we will use a phenomenological approach. A generic loss term proportional to, and oppositely directed from, the velocity of the vibrating element is introduced into the wave equation. For convenience, let the proportionality constant be 2β so (4.2.4) becomes

$$\frac{\partial^2 y}{\partial t^2} + 2\beta\frac{\partial y}{\partial t} - c^2\nabla^2 y = 0 \tag{4.6.1}$$

For calculational simplicity, assume oscillatory behavior and generalize y to be complex,

$$\mathbf{y} = \mathbf{\Psi} e^{j\boldsymbol{\omega} t} \tag{4.6.2}$$

Since there are no applied driving forces, $\boldsymbol{\omega}$ must be complex if damping is to occur. Substituting (4.6.2) into (4.6.1) and dividing out $\exp(j\boldsymbol{\omega} t)$ results in the Helmholtz equation

$$\nabla^2\mathbf{\Psi} + \mathbf{k}^2\mathbf{\Psi} = 0 \tag{4.6.3}$$

with the complex separation constant $\mathbf{k}^2$ given by

$$\mathbf{k}^2 = (\boldsymbol{\omega}/c)^2 - j2(\beta/c)(\boldsymbol{\omega}/c) \tag{4.6.4}$$

In this case $\mathbf{k}$ must be real, since for membranes fixed at their edges the arguments of the normal modes must be real. Solution of (4.6.4) for $\boldsymbol{\omega}$ is straightforward:

$$\begin{aligned}\boldsymbol{\omega} &= \omega_d + j\beta \\ \omega_d &= (\omega^2 - \beta^2)^{1/2} \\ \omega &= kc\end{aligned} \tag{4.6.5}$$

where ω is the natural angular frequency of the undamped case, ω_d the natural angular frequency of the damped case, and β the temporal absorption coefficient.

If the membrane is excited into motion and allowed to come naturally to rest, the resulting motion of the surface is a superposition of the excited normal modes, each with its own decay coefficient β and damped natural angular frequency ω_d:

$$\mathbf{y} = \sum_m\sum_n \mathbf{\Psi}_{mn} e^{-\beta_{mn}t} e^{j(\omega_d)_{mn}t} \tag{4.6.6}$$

Each normal mode $\mathbf{\Psi}_{mn}$ has a complex amplitude $\mathbf{A}_{mn}$ whose magnitude A_{mn} and phase angle ϕ_{mn} are determined by the initial conditions at $t = 0$. The decay coefficients are usually functions of frequency. Losses associated with the flexing of the membrane tend to increase with increasing frequency as the nodal pattern becomes more segmented. On the other hand, losses to the surrounding medium by the radiation of sound become smaller with more complicated modal patterns. (This reflects the observation that the volume displacement amplitudes are smaller for higher modes and zero for unsymmetric modes.) These two effects tend to offset each other, but as a general rule, higher modes damp out faster than do lower ones.

*4.7 THE KETTLEDRUM

Damping and inertial forces are two of the motion-induced forces that may act on the surface of a membrane. Another arises from the changes in pressure within a closed space behind the head of a drum or the diaphragm of a condenser microphone as the volume of the entrapped gas is altered by the motion of the membrane.

For example, the kettledrum has its head stretched tightly over the open end of a hemispherical cavity of volume V. As the head vibrates, the air in the cavity may be alternately compressed and expanded. If the phase speed of transverse waves on the membrane is considerably less than the speed of sound in air, the pressure resulting from any compression and expansion of the enclosed air is nearly uniform within the entire volume and thus depends only on the average instantaneous displacement $\langle y \rangle_S$. The incremental change in volume of the enclosed air is $dV = \pi a^2 \langle y \rangle_S$, where a is the radius of the drumhead. If the equilibrium volume inside the vessel is V_0 and the equilibrium pressure is $\mathcal{P}_0$, then for adiabatic changes in volume the new pressures $\mathcal{P}$ and volumes V are related by

$$\mathcal{P} V^\gamma = \mathcal{P}_0 V_0^\gamma \tag{4.7.1}$$

where γ is the ratio of the heat capacity of the entrapped air at constant pressure to that at constant volume (see Appendix A9). Differentiation shows that the excess pressure $d\mathcal{P}$ inside the kettle will be

$$d\mathcal{P} \approx -(\gamma \mathcal{P}_0 / V_0)\, dV = -\gamma(\mathcal{P}_0 / V_0)\pi a^2 \langle y \rangle_S \tag{4.7.2}$$

This generates an additional force $d\mathcal{P} r\, dr\, d\theta$ on each incremental area $r\, dr\, d\theta$ of the membrane. From the discussion of the previous section, the normal modes affected by this force must be just the symmetric ones. While these are relatively unimportant for the musical properties of the kettledrum, the effect of this induced force has interest in other applications, and so we shall pursue the analysis further. Including this force in the discussion of Section 4.2 and writing $\mathbf{y}$ as (4.6.2) with $\mathbf{\Psi}$ real leads to

$$\nabla^2 \Psi + k^2 \Psi = (\gamma \mathcal{P}_0 \pi a^2 / \rho_S c^2 V_0) \langle \Psi \rangle_S \tag{4.7.3}$$

for each symmetric normal mode Ψ. The subscripts 0 and n have been suppressed for economy of expression. Because it is proportional to displacement, the right side is a spring-like term; the allowed wave numbers k will, therefore, be increased. The homogeneous solutions to (4.7.3) will still be Bessel functions, but they may not have zeros at the rim. The boundary condition requires the presence of a particular solution, which in this case is a constant. Adding this to the homogeneous solution and satisfying the boundary condition gives

$$\Psi = A\,[J_0(kr) - J_0(ka)] \tag{4.7.4}$$

as a solution for each symmetric normal mode. The right side of (4.7.3) can now be evaluated with the help of

$$\begin{aligned} \pi a^2 \langle \Psi \rangle_S &= \int_0^a \Psi 2\pi r\, dr = 2\pi A[(r/k)J_1(kr) - (r^2/2)J_0(ka)]\Big|_0^a \\ &= \pi a^2 A\left[2J_1(ka)/ka - J_0(ka)\right] = \pi a^2 A\, J_2(ka) \end{aligned} \tag{4.7.5}$$

Substitution shows that (4.7.4) is a solution of (4.7.3) if

$$\begin{aligned} J_0(ka) &= -B\, J_2(ka)/(ka)^2 \\ B &= \pi a^4 \gamma \mathcal{P}_0 / \mathcal{T} V_0 \end{aligned} \tag{4.7.6}$$

Solving (4.7.6) for ka determines the natural frequencies. The nondimensional parameter B measures the relative importance of the restoring force of the air in the vessel to the tension in the membrane.

Since the frequencies of only the modes Ψ_{0n} are affected by the pressure fluctuations within the vessel, the area πa^2 of the drumhead and the volume V_0 of the vessel are parameters that can be varied to alter the natural frequency distribution of the kettledrum. Variation of B affects the relative values of the f_{0n} frequencies. Altering a and V_0 such that a^4/V_0 remains constant will vary the nonsymmetric overtones f_{mn} ($m \neq 0$) with respect to the symmetric ones.

If damping is now considered, consistent with (4.6.5) each standing wave will have its angular frequency shifted from the value ω_{mn} for undamped motion to that with damping $(\omega_d)_{mn}$, and each standing wave will decay with its own decay constant β_{mn}. The form of each standing wave will be given by (4.6.6), with the symmetric Ψ_{0n} modes given by (4.7.4).

This development has not taken into consideration any inertance effects of the medium on the membrane. As the membrane vibrates, it radiates acoustic energy but also accelerates the surrounding medium locally, as if it were storing and recovering energy from the mass of the adjacent medium. This inertance is quite important in affecting the natural frequencies of the excited modes. In practice, the significant normal modes of the kettledrum are the lowest four or five of the asymmetric $(m, 1)$ family (beginning with $m = 1$). The inertance contributes an additional effective mass to the membrane, thereby lowering the frequency of the normal mode. The effect is greater for the lower modes, decreasing as the segmentations of the normal mode patterns increase. The natural frequencies are lowered with the lowest ones being most affected. The result brings the relative values close to 2:3:4:5 and this accounts for the distinctive timbre and clear pitch associated with the kettledrum. A quantitative treatment of inertance goes beyond our present purpose, but will be considered further starting in Chapter 7.

*4.8 FORCED VIBRATION OF A MEMBRANE

Introduction of a forcing function into the equation of motion is similarly straightforward. The units of each term in (4.6.1) are those of acceleration, so the forcing function must have the same. A suitable combination of terms is pressure divided by surface density. This gives the generalization of (4.6.1) that includes an external driving agent,

$$\frac{\partial^2 y}{\partial t^2} + 2\beta \frac{\partial y}{\partial t} - c^2 \nabla^2 y = \frac{P}{\rho_S} f(t) \tag{4.8.1}$$

where $f(t)$ is a dimensionless function of time. The pressure P can be a constant or any appropriate function of space, including a delta function. The function of time can be oscillatory, a delta function, or whatever is necessary to represent the temporal behavior of the applied force. For example, if both P and $f(t)$ were delta functions, this would approximate the stroke of a drumstick at a specific point on the membrane.

Here, we concentrate on applied oscillatory forces. Let $\mathbf{f}(t) = \exp(j\omega t)$ and assume that the steady-state solution for y has the form

$$\mathrm{y} = \mathbf{\Psi} e^{j\omega t} \tag{4.8.2}$$

with the angular frequency ω real. (In the case of forced motion, where there is a steady-state solution, ω cannot have an imaginary component.) Substitution into (4.8.1) and cancellation of the exponentials gives

$$(-\omega^2 + j2\beta\omega - c^2\nabla^2)\mathbf{\Psi} = P/\rho_S \tag{4.8.3}$$

The solution of (4.8.3) consists of the sum of the solution to the homogeneous equation and a solution to the particular equation. The homogeneous equation can be written as

$$\begin{aligned} \nabla^2\boldsymbol{\Psi} + \mathbf{k}^2\boldsymbol{\Psi} &= 0 \\ \mathbf{k} &= k - j\alpha \\ k &= (\omega/c)[1 + (\beta/\omega)^2]^{1/2} \\ \alpha/k &= (\beta/\omega)/[1 + (\beta/\omega)^2] \approx \beta/\omega \end{aligned} \tag{4.8.4}$$

The top equation is the familiar Helmholtz equation, but with complex $\mathbf{k}$ rather than real k. This means that whatever functions solve the Helmholtz equation for lossless conditions are still solutions, but with k replaced with $\mathbf{k}$. For the cases we have studied (rectangular and circular membranes), the functions now have complex arguments and cannot satisfy the boundary condition of a fixed rim without the help of the particular solution.

For the case of uniform pressure P distributed over the circular membrane with fixed rim at $r = a$, the azimuthal symmetry of the problem restricts the homogeneous solution $\boldsymbol{\Psi}_h$ to the zeroth order Bessel function $J_0(kr)$. The appropriate particular solution $\boldsymbol{\Psi}_p$ to (4.8.3) is a constant,

$$\boldsymbol{\Psi}_p = -(P/\rho_S)/(\mathbf{k}c)^2 \tag{4.8.5}$$

Adding this to the homogeneous solution $\boldsymbol{\Psi}_h$ and requiring that the sum vanish when evaluated at the rim results in the desired solution,

$$\boldsymbol{\Psi} = (P/\mathscr{T}\mathbf{k}^2)\left[J_0(\mathbf{k}r)/J_0(\mathbf{k}a) - 1\right] \tag{4.8.6}$$

The tension $\mathscr{T}$ has replaced $\rho_S c^2$. The values for $\mathbf{k} = k - j\alpha$ are determined from (4.8.4). Inspection of (4.8.6) shows that the amplitude of the displacement is directly proportional to that of the driving force and inversely proportional to the tension $\mathscr{T}$. The dependence on frequency of the amplitude of vibration at any location is given by the relatively complicated expression within the square bracket. When the driving frequency matches any natural frequency [found from $J_0(ka) = 0$], then $J_0(\mathbf{k}a)$ has very small magnitude and $|\boldsymbol{\Psi}|$ may be very large, depending on the damping.

*4.9 THE DIAPHRAGM OF A CONDENSER MICROPHONE

An important case of a driven membrane is the circular diaphragm of a condenser microphone. The incident sound wave, acting on a tightly stretched metallic membrane placed above a metal plate, produces a nearly uniform driving force. As the membrane is displaced, the electrical capacitance between the membrane and the adjacent metal plate is changed. This generates an output voltage that is (for small motion) a linear function of the averaged displacement amplitude of the membrane,

$$\langle\boldsymbol{\Psi}\rangle_S = \frac{1}{\pi a^2}\frac{P}{\mathscr{T}}\frac{1}{\mathbf{k}^2}\int_0^a\left(\frac{J_0(\mathbf{k}r)}{J_0(\mathbf{k}a)} - 1\right)2\pi r\,dr = \frac{Pa^2}{\mathscr{T}}\frac{1}{(\mathbf{k}a)^2}\frac{J_2(\mathbf{k}a)}{J_0(\mathbf{k}a)} \tag{4.9.1}$$

If the frequency is below the region of the lowest resonance, $\mathbf{k}$ can be replaced with the wave number k and use of the small-argument approximations for the Bessel functions gives

$$\langle\boldsymbol{\Psi}\rangle_S \approx \tfrac{1}{8}(Pa^2/\mathscr{T})[1 + (ka)^2/6] \tag{4.9.2}$$

Thus, $\langle\boldsymbol{\Psi}\rangle_S$ is nearly constant for $ka < 1$, or for frequencies

$$f < c/2\pi a = (\mathscr{T}/\rho_S)^{1/2}/2\pi a \tag{4.9.3}$$

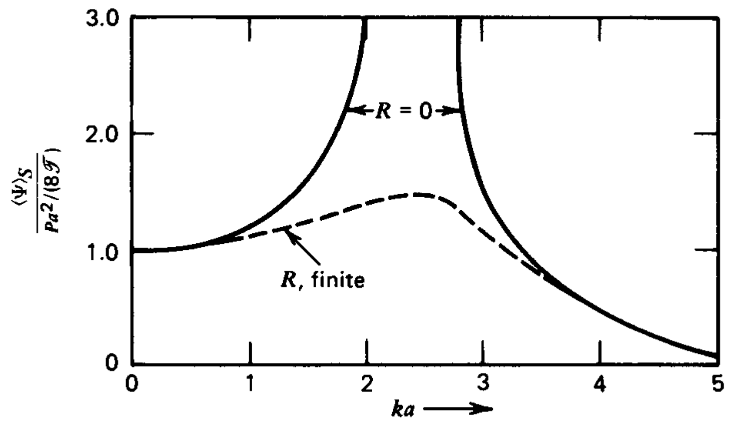

Figure 4.9.1 Average displacement $\langle\psi\rangle_S$ of a driven circular membrane with and without resistance.

Below this frequency limit, $\langle\Psi\rangle_S$ resembles the displacement amplitude of a stiffness-controlled harmonic oscillator. This upper frequency limit may be increased by increasing the tension or by decreasing either the radius or the surface density of the membrane. However, an increase in $\mathcal{T}$ or a decrease in a reduces $\langle\Psi\rangle_S$ and thus the voltage output of the microphone.

With sufficient damping, the response at the first resonance, $k_1a = 2.405$, is reduced considerably and the region of fairly uniform response can be extended up to, and somewhat beyond, the first resonance. In the immediate vicinity of resonance, the term $J_0(\mathbf{k}a)$ in the denominator of (4.9.1) can be expanded in a Taylor's series about k_1a,

$$J_0(\mathbf{k}a) = -J_1(k_1a)(\mathbf{k}a - k_1a) + \cdots \tag{4.9.4}$$

Writing $\mathbf{k} = k(1 - j\beta/\omega)$ from (4.8.4), substituting the quality factor $Q = \omega_1/2\beta$ from (1.10.7) for the resonance at ω_1, and restricting ω to values close to ω_1 casts (4.9.1) into the form

$$|\langle\Psi\rangle_S| = \frac{2Pa^2}{\mathcal{T}}\frac{1}{(ka)^3}\frac{J_2(j_{01})}{J_1(j_{01})}\frac{1}{[(\omega/\omega_1 - \omega_1/\omega)^2 + 1/Q^2]^{1/2}} \tag{4.9.5}$$

This displays the same behavior in the vicinity of its resonance as does a damped harmonic oscillator. The resonance peak and bandwidth are controlled by the quality factor in the same way.

Response curves showing the normalized average displacement amplitudes of a driven membrane with and without losses are given in Fig. 4.9.1. Note that (4.9.1) indicates minimal response at the frequencies for which $J_2(ka) = 0$. From (4.8.6), at frequencies for which $ka > 3.83$, a nodal circle appears within the rim of the membrane. With increasing frequency the radius of this circle decreases. The displacements within this circle are out of phase with those between it and the rim. As the nodal circle continues to shrink there is increasing cancellation leading to nearly zero response when $ka \approx 5.136$.

*4.10 NORMAL MODES OF MEMBRANES

Orthogonality was developed in Section 2.13 for a set of one-dimensional normal modes describing the vibration of a string for certain simple boundary conditions. It is appropriate here to extend that discussion to a treatment of normal modes on a two-dimensional surface. For the free vibration of each membrane we have studied, a set of normal modes Ψ that satisfy the Helmholtz equation and the boundary conditions has been obtained. For each normal mode Ψ_{mn} of the set, there is an associated separation constant k_{mn}^2 determined by the boundary conditions. If the Helmholtz equation for each of two normal modes Ψ_{mn} and $\Psi_{m'n'}$ is multiplied by the other of the pair and one equation subtracted from the other, the result is

$$\Psi_{m'n'}\nabla^2\Psi_{mn} - \Psi_{mn}\nabla^2\Psi_{m'n'} + (k_{mn}^2 - k_{m'n'}^2)\Psi_{mn}\Psi_{m'n'} = 0 \tag{4.10.1}$$

Integration over the surface S of the membrane gives

$$(k_{mn}^2 - k_{m'n'}^2)\int_S \Psi_{mn}\Psi_{m'n'}\,dS = \int_S \nabla\cdot(\Psi_{mn}\nabla\Psi_{m'n'} - \Psi_{m'n'}\nabla\Psi_{mn})\,dS \tag{4.10.2}$$

where use has been made of the identity in Appendix A8. Application of Gauss's theorem in two dimensions now gives the desired result,

$$(k_{mn}^2 - k_{m'n'}^2)\int_S \Psi_{mn}\Psi_{m'n'}\,dS = \int_{rim} [\Psi_{mn}(\hat{n}\cdot\nabla\Psi_{m'n'}) - \Psi_{m'n'}(\hat{n}\cdot\nabla\Psi_{mn})]\,dl \tag{4.10.3}$$

where the line integral is over the perimeter (rim) of the surface S and $\hat{n}$ is the unit normal in the plane of the membrane directed outward at each point on the rim. This equation is the generalization of the right side of (2.13.7) to a two-dimensional situation. There are two cases of interest to us here: (1) for a free rim, the gradient of Ψ at the rim is at right angles to the normal; and (2) for a fixed rim, Ψ is zero on the rim. In both cases, the right side vanishes and the normal modes form an orthogonal set.

One complication arises when there are two or more normal modes with the same natural frequency. When this happens, the separation constants are the same and the left side of (4.10.3) vanishes identically whether or not the degenerate modes are orthogonal. This means that if the membrane is excited into motion at this frequency and then left to vibrate freely, the shape of the resulting standing wave depends on the details of the excitation. This can result in standing waves of substantially different shapes depending on the relative phases and amplitudes of the degenerate normal modes. While these cases may need a little special attention, they present few problems. Mathematically, if the modes are not orthogonal, it is possible to choose two combinations of the pair that will be.

Solving for the motion of the membrane after excitation by some initial distribution of displacement and velocity proceeds just as for the string. The initial conditions at $t = 0$ are written as sums of the normal modes with unspecified amplitudes and phases. These equations are then multiplied by each of the normal modes and integrated over the surface, orthogonality applied, and the resulting reduced set of integrals evaluated to obtain the amplitudes and phases.

(a) The Rectangular Membrane with Fixed Rim

Since the rim is fixed, the normal modes for the rectangular membrane

$$\Psi_{nm}(x,z) = A_{nm}\sin k_{xn}x\sin k_{zm}z \tag{4.10.4}$$

form an orthogonal set. The separation constant in the Helmholtz equation (4.3.3) for each of these normal modes is

$$k_{nm}^2 = k_{xn}^2 + k_{zm}^2 \tag{4.10.5}$$

Thus,

$$\int_0^{L_z}\int_0^{L_x} \Psi_{nm}\Psi_{n'm'}\,dx\,dz = \frac{4A_{nm}^2}{L_xL_z}\delta_{n'n}\delta_{m'm} \tag{4.10.6}$$

If the membrane is struck at some position $\vec{r}_0$ with an impulse at $t = 0$, then the initial transverse speed of the membrane can be approximated by $v(\vec{r}, 0) = \mathscr{V}\delta(\vec{r} - \vec{r}_0)$. The two-dimensional delta function is given by

$$\int_{S_0} \delta(\vec{r} - \vec{r}_0)\, dS = \begin{cases} 1 & \vec{r}_0 \in S_0 \\ 0 & \vec{r}_0 \notin S_0 \end{cases} \tag{4.10.7}$$

where $\delta(\vec{r} - \vec{r}_0)$ has dimension m^{-2} and $\mathscr{V}$ has dimension $(\mathrm{m/s})\mathrm{m}^2$. If the point is at (x_0, z_0), then it can be seen (Problem 4.10.2) that $\delta(\vec{r} - \vec{r}_0)$ can be represented as a product of one-dimensional delta functions

$$\delta(\vec{r} - \vec{r}_0) = \delta(x - x_0)\,\delta(z - z_0) \tag{4.10.8}$$

The real standing wave associated with the (n, m) mode can be written as

$$y_{nm}(x, z, t) = A_{nm} \sin k_{xn}x \sin k_{zm}z \sin(\omega_{nm}t + \phi_{nm}) \tag{4.10.9}$$

At $t = 0$ the membrane is at rest, $y(x, z, 0) = 0$. This can be satisfied by choosing $\phi_{nm} = 0$ for all modes. With the ϕ's established, the particle speed when $t = 0$ provides that

$$\sum_{n,m} \omega_{nm} A_{nm} \sin k_{xn}x \sin k_{zm}z = \mathscr{V}\delta(x - x_0)\,\delta(z - z_0) \tag{4.10.10}$$

Since the normal modes are orthogonal, use of (4.10.6) gives the values of each A_{nm}, and we have

$$y(x, z, t) = \frac{4\mathscr{V}}{L_x L_z} \sum_{n,m} \frac{1}{\omega_{nm}} \sin k_{xn}x_0 \sin k_{zm}z_0 \sin k_{xn}x \sin k_{zm}z \sin \omega_{nm}t \tag{4.10.11}$$

Using the delta function introduces some convergence problems in this expression. Practically, time should be restricted to finite values, $t \geq t_0$, and the summations should be truncated at realistic values of the indices, $n < N$ and $m < M$, where t_0, N, and M are based on the true duration of the impact and the finite area of impact of the drumstick. Suitable decay can also be included as discussed earlier by introducing an exponential decay factor $\exp(-\beta_{nm}t)$ and shifting the natural angular frequency according to (4.6.5) for each normal mode.

(b) The Circular Membrane with Fixed Rim

Conceptually, analysis is exactly the same as for the rectangular membrane. The two-dimensional delta function must be expressed in terms of one-dimensional delta functions in the coordinates r and θ,

$$\delta(\vec{r} - \vec{r}_0) = \frac{1}{r}\delta(r - r_0)\,\delta(\theta - \theta_0) \tag{4.10.12}$$

(see Problem 4.10.6). We can orient the axes so that $\theta = 0$ corresponds to the azimuthal direction of the blow (which now requires $\theta_0 = 0$) and assert that the normal modes must be maximized in this direction. This requires that all $\gamma_{mn} = 0$ in (4.4.13). For a strike at $t = 0$ on a stationary membrane, we must have $\phi_{mn} = -\pi/2$. The individual standing waves can now be extracted,

$$y_{mn}(r, \theta, t) = A_{mn} J_m(k_{mn}r) \cos m\theta \sin \omega_{mn}t \tag{4.10.13}$$

Solution proceeds as before with the help of

$$\int_0^a \int_0^{2\pi} \left[J_m(k_{mn}r)\cos m\theta\right]^2 r\,dr\,d\theta = \begin{cases} \pi a^2 [J_m'(k_{mn}a)]^2 & m = 0 \\ \dfrac{\pi a^2}{2}[J_m'(k_{mn}a)]^2 & m > 0 \end{cases} \tag{4.10.14}$$

where $k_{mn}a = j_{mn}$. The resulting standing wave is

$$y(r,\theta,t) = \frac{\mathcal{V}}{\pi a^2} \sum_{m,n} \frac{\varepsilon_m}{\omega_{mn}} \frac{J_m(k_{mn}r_0)}{\left[J_m'(k_{mn}a)\right]^2} J_m(k_{mn}r)\cos m\theta \sin \omega_{mn}t \tag{4.10.15}$$

for a strike at the point $(r_0, 0)$ at time $t = 0$. The quantity ε_m is 1 for $m = 0$ and 2 for all other m. As before, times should be restricted to $t \geq t_0$ and the summation truncated at appropriate N and M. Notice that if the membrane is struck exactly at the center, then only terms with $m = 0$ contribute.

*4.11 VIBRATION OF THIN PLATES

There is an essential difference between the vibration of a membrane and of a thin plate. In a membrane, the restoring force arises entirely from the tension applied to the membrane, whereas in a thin plate the restoring force results from the stiffness of the diaphragm. This same difference exists between the transverse restoring forces in strings and bars. Analysis of the plate will be limited to the *symmetric vibrations of a uniform circular diaphragm*. A rigorous development of the equation of motion lies beyond our interests. The equation is

$$\frac{\partial^2 y}{\partial t^2} = -\frac{\kappa^2 Y}{\rho(1-\sigma^2)}\nabla^2(\nabla^2 y) \tag{4.11.1}$$

where ρ is the *volume density* of the material, σ the *Poisson's ratio*, Y the *Young's modulus*, and κ the *radius of gyration* given by $\kappa = d/\sqrt{12}$, where d is the plate thickness.

In partial explanation, since the restoring force acting on a plate depends on its elastic response to bending, the coefficient of the right term in (4.11.1) should resemble that for the transverse vibration of a bar (3.9.4), $\kappa^2 Y/\rho$. However, like a bar, a sheet curls transversely when it is bent lengthwise, but the lateral extent of the sheet hampers the curling. Thus, there should be a slight decrease in the resultant strain of the sheet for the impressed bending stress and therefore a slight increase in the effective stiffness of the sheet. Analysis provides the factor $1/(1-\sigma^2)$. Values of Poisson's ratio for various materials are given in Appendix A10. Note that $\sigma \sim 0.3$ for most materials.

Assume periodic vibration,

$$\mathbf{y} = \mathbf{\Psi} e^{j\omega t} \tag{4.11.2}$$

where, for circular symmetry, $\mathbf{\Psi}$ is a function only of r. Substitution into (4.11.1) yields

$$\begin{aligned} \nabla^2(\nabla^2\mathbf{\Psi}) - g^4\mathbf{\Psi} &= 0 \\ g^4 &= \omega^2\rho(1-\sigma^2)/\kappa^2 Y \end{aligned} \tag{4.11.3}$$

Now, direct substitution shows that (4.11.3) can be satisfied by

$$\nabla^2\mathbf{\Psi} \pm g^2\mathbf{\Psi} = 0 \tag{4.11.4}$$

In polar coordinates and with circular symmetry, (4.11.4) with the + sign is satisfied by $J_0(gr)$ and $Y_0(gr)$ and the solutions for the − sign are the Bessel functions of imaginary argument, $J_0(jgr) \equiv I_0(gr)$ and $Y_0(jgr)$. As before, the solutions involving Y_0 can be discarded since they have singularities at $r = 0$, so we have

$$\mathbf{\Psi} = \mathbf{A}J_0(gr) + \mathbf{B}I_0(gr) \tag{4.11.5}$$

Some properties and tables of values of the *modified Bessel functions of the first kind* I_m are given in Appendixes A4 and A5.

To evaluate the constants $\mathbf{A}$ and $\mathbf{B}$ we must know how the diaphragm is supported. The most common type of support is rigid clamping of the diaphragm around its circumference at $r = a$. This is equivalent to

$$\mathbf{\Psi} = 0 \quad \text{and} \quad \frac{\partial \mathbf{\Psi}}{\partial r} = 0 \qquad \text{at } r = a \tag{4.11.6}$$

These conditions give

$$\begin{aligned} \mathbf{A}J_0(ga) &= -\mathbf{B}I_0(ga) \\ \mathbf{A}J_1(ga) &= \mathbf{B}I_1(ga) \end{aligned} \tag{4.11.7}$$

and dividing one by the other gives the transcendental equation for the allowed values of g,

$$J_0(ga)/J_1(ga) = -I_0(ga)/I_1(ga) \tag{4.11.8}$$

Since both I_0 and I_1 are positive for all values of ga, solutions occur only when J_0 and J_l are of opposite sign. The tables of Bessel functions show that this equation is satisfied by $g_n a = 3.20,\ 6.30,\ 9.44,\ 12.57, \ldots \approx n\pi$ with $n = 1, 2, 3, \ldots$. The approximation improves with increasing n.

Solving (4.11.3) for the lowest natural frequency f_1 gives

$$f_1 = \frac{g_1^2}{2\pi a^2}\frac{d}{\sqrt{12}}\left(\frac{Y}{\rho(1-\sigma^2)}\right)^{1/2} = 0.47\frac{d}{a^2}\left(\frac{Y}{\rho(1-\sigma^2)}\right)^{1/2} \tag{4.11.9}$$

The frequencies of the other symmetric modes are not harmonics of the fundamental: $f_2/f_1 = (g_2/g_1)^2 = 3.88$, $f_3/f_1 = 8.70$, and so forth. The natural frequencies are spread much farther apart than those of the circular membrane.

The displacement of a thin circular plate vibrating in its fundamental mode is

$$y_1 = A_1\left[J_0(3.2r/a) + 0.0555 I_0(3.2r/a)\right]\cos(\omega_1 t + \phi_1) \tag{4.11.10}$$

where the ratio of coefficients is obtained from (4.11.7). Note that the amplitude at the center $|y_1(0)|$ is not A_1 but $1.0555A_1$. Comparing the displacement of the thin circular plate vibrating in its fundamental mode with that of a membrane vibrating at its fundamental shows that the relative displacement of the plate near its edge is much smaller than that of the membrane. Consequently, we should expect the ratio of its average amplitude to that at the center to be less than the same ratio for the membrane. The average displacement amplitude is $\langle \Psi_1 \rangle_S = 0.326A_1$, or

$$\langle \Psi_1 \rangle_S = 0.309|y_1(0)| \tag{4.11.11}$$

This is smaller by a factor of $(0.432/0.309) = 1.40$ than the averaged displacement for the circular fixed membrane (4.5.5) for the same amplitude at the center.

Treatments of loaded and driven plates are analogous to those for membranes, and the response curves for a uniform driving force are similar to those in Fig. 4.9.1, with large amplitudes at the fundamental resonance frequency unless there is considerable damping. Condenser microphones may be constructed with a thin circular plate instead of a stretched membrane for greater strength. However, the reduced sensitivity usually restricts such microphones to high-intensity applications where strength is necessary.

The most important utilization of the thin plate is in the diaphragms of ordinary telephone microphones and receivers. Although the responses of these devices are not uniform over a wide range of frequencies, they give adequate intelligibility and are simple and rugged. Another application is in sonar transducers used for producing sounds in water at frequencies below 1 kHz; sound is generated by the motion of relatively thin circular steel plates driven by alternations in the magnetic field of an adjacent electromagnet.

PROBLEMS

Except when otherwise noted, all membranes should be assumed fixed at their rims.

4.3.1. A square membrane of width a vibrates at its fundamental frequency with an amplitude A at its center. (*a*) Derive a general expression for its average displacement amplitude. (*b*) Derive a general expression for locating points on the membrane having an amplitude of $0.5A$. (*c*) Compute and plot a few points given by the equation derived in part (*b*). Do they form a circle?

4.3.2. A rectangular membrane has width a and length b. If $b = 2a$, compute the ratio of each of the first four overtone frequencies relative to the fundamental frequency.

4.3.3. A square membrane with sides of length L, uniform surface density ρ_S, and uniform tension $\mathcal{T}$ is fixed on three sides and free on the other. (*a*) Find the frequency of the fundamental mode. (*b*) Write a general expression for the natural frequencies and one for the normal modes. (*c*) Sketch the nodal patterns for the three normal modes with the lowest natural frequencies.

4.3.4. A square membrane of sides L and phase speed c is fixed on two of its opposed sides ($x = 0$ and $x = L$) and free on the other two ($z = 0$ and $z = L$). (*a*) Write the equation for the displacement of the membrane valid for all normal modes. (*b*) What are the frequencies for the five lowest modes? (*c*) Sketch the nodal patterns for these five modes.

4.4.1. Show that the total energy of a circular membrane when vibrating in its fundamental mode is given by $0.135(\pi a^2)\rho_S(\omega A_1)^2$, where a is its radius, ρ_S the area density, ω the angular frequency of vibration, and A_1 the amplitude at its center.

4.4.2. Although it may be hard to do physically, it is not hard to imagine a circular membrane with a free rim. (*a*) Write the general expression for the normal modes. (*b*) Sketch the nodal patterns for the three normal modes with the lowest natural frequencies. (*c*) Find the frequencies of these three normal modes in terms of the tension and surface density.

4.4.3. The maximum tensile stress that may be applied to aluminum is 2×10^8 Pa and to steel is 10^9 Pa. What is the maximum fundamental frequency (*a*) of a stretched aluminum membrane of 0.01 m radius and (*b*) of a steel membrane of equal radius? (For thin membranes these frequencies are independent of thickness.)

4.4.4. A circular membrane of 0.25 m radius has an area density of 1.0 kg/m^2 and is stretched to a tension of 25,000 N/m. (*a*) Compute the four lowest frequencies of free vibration. (*b*) For each of these frequencies locate any nodal circles.

4.5.1. A circular membrane of 1 cm radius and 0.2 kg/m^2 area density is stretched to a linear tension of 4000 N/m. When vibrating in its fundamental mode, the amplitude at the center is observed to be 0.01 cm. (*a*) What is its fundamental frequency? (*b*) What is the maximum volume of air displaced by the membrane?

4.5.2. At what fraction of the radius of a circular membrane does the nodal circle of the second symmetric mode occur?

4.5.3. A steel membrane of 0.02 m radius and 0.0001 m thickness is stretched to a tension of 20,000 N/m. (*a*) For circularly symmetric vibration, what is the frequency of the second overtone mode? (*b*) What are the radii of the two nodal circles when the membrane is vibrating at the above frequency? (*c*) When the membrane is vibrating at the above frequency, the displacement amplitude at the center is observed to be 0.0001 m. What is the average displacement amplitude?

4.5.4C. Plot the displacement as a function of radius and angle for the modes of a circular membrane shown in Fig. 4.4.1.

4.5.5C. Plot J_0 and J_1 for argument $0 < x < 10$ and compare to the small- and large-argument approximations. Comment on the range of x for which each approximation is good.

4.6.1. A circular membrane is acted on uniformly over its surface by a damping force per unit area of $-R(\partial y/\partial t)$. Introduce this term into (4.2.7) in a manner consistent with the dimensions, and solve the resulting equation to show that the amplitudes of the resulting free vibrations are damped exponentially as $\exp(-Rt/2\rho_S)$.

4.7.1. The circular membrane of a kettledrum has a radius of 0.25 m, an area density of 1.0 kg/m^2, and is stretched to a tension of 10,000 N/m. (*a*) What is its fundamental frequency without the kettle? (*b*) What is its fundamental frequency if the kettle is a hemispherical bowl of 0.25 m radius? Assume the kettle is filled with air at a pressure of 10^5 Pa, and the ratio of heat capacities is 1.4.

4.7.2. For the kettledrum, calculate the effect of B in (4.7.6) in changing the natural resonance frequencies associated with the lowest three symmetric normal modes. Calculate the values of ka for $B = 0, 1, 2, 5, 10$. Which frequency is the most changed?

4.7.3. (*a*) Find the values of ka for the lowest five members of the $(m, 1)$ family (beginning with $m = 1$) of the freely vibrating circular membrane. (Because these modes have no volume displacement amplitudes, they can represent those for a kettledrum.) (*b*) Assuming that f_{51} is not changed, calculate the fractional reduction in each of the lower frequencies to bring the series into the ratios 2:3:4:5:6. How uniform is the shifting of frequencies?

4.8.1. Find the resonance frequencies of a circular membrane with a free rim (but still under tension) driven by a uniform pressure $P \exp(j\omega t)$.

4.8.2. (*a*) Compute and plot the shape of the circular membrane when driven at one-half its fundamental frequency. (*b*) Similarly, compute and plot the shape of the membrane when driven at twice its fundamental frequency.

4.8.3. An undamped circular membrane of 0.02 m radius, 1.5 kg/m^2 area density, and 950 N/m tension is driven by a pressure of 6000 cos(ωt) Pa. (*a*) Compute and plot the amplitude of the displacement at the center as a function of frequency from 0 to 1 kHz. (*b*) Compute and plot the shape of the membrane when driven at 400 Hz. (*c*) Repeat part (*b*) for 1 kHz.

4.8.4C. For the forced vibration of a circular membrane, (*a*) plot the shape of the membrane for $ka = 1$ to $ka = 8$ in steps of 1.0. (*b*) Plot the displacement amplitude of the center for the same range of ka.

4.9.1. Perform the integration of (4.8.6) to obtain (4.9.1). *Hints:* Make a change of variable to $z = \mathbf{k}r$. Use the formula in the Appendix for the differentiation of $[zJ_1(z)]$ to determine the integral of $[zJ_0(z)]$. Relate J_1 to the appropriate combination of J_2 and J_0.

4.9.2. The diaphragm of a condenser microphone is a circular sheet of aluminum of 0.03 m diameter and 0.00002 m thickness. It may be stretched to a maximum tensile stress of 2×10^8 Pa. (*a*) What is the maximum tension (N/m)? (*b*) What will be its fundamental frequency when stretched to this tension? (*c*) What will be the displacement amplitude at its center when acted on by a sound wave of 500 Hz having a pressure amplitude of 2.0 Pa? (*d*) What will be the average displacement amplitude under these conditions?

4.9.3. If the volume of air trapped behind the diaphragm of the condenser microphone of Problem 4.9.2 is 3×10^{-7} m^3, by what percentage will its fundamental frequency be raised? Assume $\mathcal{P}_0 = 10^5$ Pa and $\gamma = 1.4$.

4.9.4. (*a*) Obtain the Taylor's expansion (4.9.4) with the help of the Appendix. (*b*) Show that (4.9.1) can be approximated by (4.9.5) for angular frequencies close to the lowest resonance. *Hint:* Show that $[(\omega/\omega_1) - (\omega_1/\omega)] \approx 2\Delta\omega/\omega_1$ for $\Delta\omega/\omega_1 \ll 1$ and use these relationships to simplify the angular frequency terms. (*c*) Compare the square root with that in Problem 1.10.1 to show that near resonance the average diaphragm displacement behaves like the displacement of a damped oscillator with the same resonance and damping.

4.9.5C. (*a*) Plot the average displacement of the driven circular membrane as a function of $\log(ka)$ for $0.01 < ka < 10$ using the exact solution and the low-frequency approximation. (*b*) Find the ratio of the frequency for which the low-frequency approximation is within 10% of the exact value to the frequency for which $J_0(ka) = 0$.

4.10.1. Show by direct integration of (4.10.4) over the surface area that the normal modes of the rectangular membrane form an orthogonal set. Find the values of $\mathbf{A}_{nm}$ that would make them an orthonormal set.

4.10.2. Show by direct application of (4.10.7) that $\delta(\vec{r} - \vec{r}_0) = \delta(x - x_0)\delta(z - z_0)$ is an appropriate representation, where $\vec{r}_0$ is directed from $(0, 0)$ to (x_0, z_0).

4.10.3. A rectangular membrane has dimensions such that the (3, 1) and (1, 2) modes are degenerate. (*a*) What is the ratio of lengths L_z/L_x? (*b*) If the membrane is set into motion at the degenerate frequency f_{31} at the point $(L_x/2, L_z/2)$, which of the pair is excited? (*c*) Repeat (*b*) for locations of $(L_x/2, L_z/3)$, $(L_x/3, L_z/2)$, and $(L_x/3, L_z/3)$. (*d*) Find three other degenerate pairs of frequencies as multiples of f_{31}.

4.10.4. Verify (4.10.6) for $\mathbf{y}_{nm}(x, z, t) = \sin(n\pi x/L_x)\sin(m\pi z/L_z)\exp(j\omega_{nm}t)$.

4.10.5. Verify (4.10.11) by writing the displacement as a summation of standing waves and applying the initial condition that the membrane is at rest and given an initial impulse so that its transverse speed is described by $\partial y/\partial t = \mathcal{V}\delta(x - x_0)\delta(z - z_0)$ at $t = 0$, where $0 < x_0 < L_x$ and $0 < z_0 < L_z$.

4.10.6. Show by direct application of (4.10.7) that $\delta(\vec{r} - \vec{r}_0) = (1/r)\delta(r - r_0)\delta(\theta - \theta_0)$ is an appropriate representation of the two-dimensional delta function as a product of one-dimensional delta functions in polar coordinates, where the vector $\vec{r}_0$ has magnitude r_0 and polar angle θ_0. *Hint:* Let $\delta(\vec{r} - \vec{r}_0)$ be written as $f(r)g(\theta)$, integrate over an elemental area $dS = r\,dr\,d\theta$, separate the integrals into a product of one on r and the other on θ, and note the form of the integrands.

4.10.7. Fill in the mathematical steps to verify the steps from (4.10.13) to (4.10.15) for the fixed-rim circular membrane struck at a point a distance r_0 from its center.

4.10.8. Obtain the normal mode expansion for a circular membrane fixed at its rim and struck at its center.

4.10.9C. Design a program to show plots of the displacement of a rectangular membrane spanning $(0, L_x)$ and $(0, L_z)$ at various times after it has been struck at a point (x_0, z_0).

4.10.10C. Design a program to show plots at various times of the displacement of a circular membrane of radius a after it has been struck at a point $(r_0, 0)$, where $r_0 < a$.

4.11.1. The diaphragm of a telephone receiver consists of a circular sheet of steel 4 cm in diameter and 0.02 cm thick. (*a*) If it is rigidly clamped at its rim, what is its fundamental frequency of vibration? What will be the effect on this frequency (*b*) of doubling the thickness of the diaphragm and (*c*) of doubling the diameter?

4.11.2. To what tension would the diaphragm of Problem 4.11.1 need to be stretched if its fundamental frequency, considered as resulting from the restoring forces of tension alone, were to equal that resulting from stiffness forces alone?

4.11.3. (*a*) Determine the ratio of the constants $\mathbf{B}_2/\mathbf{A}_2$ for a thin circular plate clamped at its rim and vibrating in its first overtone mode. (*b*) Express the resulting motion by an equation analogous to (4.11.10). (*c*) Plot the shape function of the diaphragm. (*d*) What is the ratio of the radius of the nodal circle to the radius of the plate?

4.11.4. The vibrating circular steel plate of an electromagnetic sonar transducer of radius 0.1 m and thickness 0.005 m is clamped at its rim. What is its fundamental frequency of vibration?

4.11.5. For a circular plate of thickness d, (*a*) show that the surface radius of gyration is $\kappa = d/\sqrt{12}$. (*b*) If the thickness of the plate is doubled, what happens to the frequencies of the normal modes?

4.11.6. (*a*) By direct integration obtain (4.11.11). (*b*) Show that the average displacement amplitude is $0.309A$, where A is the displacement amplitude at the center.

4.11.7. Find the frequencies of the symmetric normal modes for a circular plate fixed at both center and rim.

4.11.8C. Plot the modified Bessel functions of the first three orders for arguments $0 < x < 6$.

4.11.9C. Plot the shape of a thin circular plate clamped at the rim when it is vibrating in each of its first three symmetric normal modes.