Travaux Dirigés d'Electromagnétisme - Corrigés

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Chapter 1

Vector Analysis

1.1 Vector analysis

In order to simplify the notations the partial derivatie f with respect to a varaible x will be denoted :

$$\frac{\partial f}{\partial x} = \partial_x f$$

Divergence of the product between a scalar field and a vecor field

We have :

$$f = f(x, y, z)$$

and

$$\mathbf{a} \left| \begin{array}{l} a_x(x, y, z) \\ a_y(x, y, z) \\ a_z(x, y, z) \end{array} \right.$$

If we develop $\text{div}(f\mathbf{a})$ we obtain :

$$\text{div}(f\mathbf{a}) = \partial_x(fa_x(x, y, z)) + \partial_y(fa_y(x, y, z)) + \partial_z(fa_z(x, y, z))$$

Next we just have to expand each term of the derivation :

$$\begin{array}{rcl} \text{div}(f\mathbf{a}) & = & \partial_x f a_x + f\partial_x \quad a_x \\ & + & \partial_y f \quad a_y + f\partial_y a_y \\ & + & \partial_z f \quad a_z + f\partial_z a_z \end{array}$$

And finally we obtain :

$$\text{div}(f\mathbf{a}) = \text{grad}(f) \cdot \mathbf{a} + f\text{div}(\mathbf{a})$$

Divergence of a curl product between two vector fields

We note :

$$\mathbf{c} = \mathbf{a} \wedge \mathbf{b}$$

We have :

$$\mathbf{c} \left| \begin{array}{l} a_y b_z - a_z b_y \\ a_z b_x - a_x b_z \\ a_x b_y - a_y b_x \end{array} \right.$$

Newt we apply the div operator and we group different terms of the expansion :

$$\begin{array}{rcl} \text{div}(\mathbf{c}) & = & a_x\{\partial_z b_y - \partial_y b_z\} + b_x\{\partial_y a_z - \partial_z a_y\} \\ & + & a_y\{\partial_x b_z - \partial_z b_x\} + b_y\{\partial_z a_x - \partial_x a_z\} \\ & + & a_z\{\partial_y b_x - \partial_x b_x\} + b_z\{\partial_x a_y - \partial_y a_x\} \end{array}$$

By identifiaction we have :

$$\text{div}(\mathbf{a} \wedge \mathbf{b}) = \mathbf{b} \cdot \text{rot}(\mathbf{a}) - \mathbf{a} \cdot \text{rot}(\mathbf{b})$$

Divergence of a curl:

On a

$$\mathrm{rot}\vec{a} = \left| \begin{array}{l} \partial_y a_z - \partial_z a_y \\ \partial_z a_x - \partial_x a_z \\ \partial_x a_y - \partial_y a_x \end{array} \right.$$

D'oÃź

$$\begin{aligned} \mathrm{div}(\mathrm{rot}\vec{a}) &= \partial_x \left(\partial_y a_z - \partial_z a_y\right) + \partial_y \left(\partial_z a_x - \partial_x a_z\right) + \partial_z \left(\partial_x a_y - \partial_y a_x\right) \\ &= \partial^2_{xy} a_z - \partial^2_{xz} a_y + \partial^2_{yz} a_z - \partial^2_{yx} a_z + \partial^2_{zx} a_y - \partial^2_{zy} a_x \\ &= 0 \end{aligned}$$

Curl of a gradient :

We have

$$\mathrm{grad}f = \left| \begin{array}{l} \partial_x f \\ \partial_y f \\ \partial_z f \end{array} \right.$$

So

$$\mathrm{rotgrad}f = \left| \begin{array}{l} \partial^2_{yz} f - \partial^2_{zy} f = 0 \\ \partial^2_{zx} f - \partial^2_{xz} f = 0 \\ \partial^2_{xy} f - \partial^2_{yx} f = 0 \end{array} \right.$$

Curl of a scalar function multiplied by a vector field :

We are going to do the calculations along the x component

$$\begin{aligned} \mathrm{rot}(f\mathbf{a})|_x &= \partial_y (f a_z) - \partial_z (f a_y) \\ &= \underbrace{-a_y \partial_z f + a_z \partial_y f}_{\mathrm{grad}f \wedge \mathbf{a}|_x} + \underbrace{-f \partial_z a_y + f \partial_y a_z}_{f\mathrm{rot}\mathbf{a}|_x} \end{aligned}$$

Curl-Curl of a vector :

$$\mathrm{rot}(\mathrm{rot}\mathbf{a}) \left| \begin{array}{l} \partial_y \left(\partial_x a_y - \partial_y a_x\right) - \partial_z \left(\partial_z a_x - \partial_x a_z\right) \\ \partial_z \left(\partial_y a_z - \partial_z a_y\right) - \partial_x \left(\partial_x a_y - \partial_y a_x\right) \\ \partial_x \left(\partial_z a_x - \partial_x a_z\right) - \partial_y \left(\partial_y a_z - \partial_z a_y\right) \end{array} \right.$$

Looking at the x component we have :

$$\begin{aligned} \mathrm{rot}(\mathrm{rot}\mathbf{a})|_x &= \partial_y \left(\partial_x a_y - \partial_y a_x\right) - \partial_z \left(\partial_z a_x - \partial_x a_z\right) \\ &= -\partial^2_{y^2} a_x - \partial^2_{z^2} a_x + \partial^2_{yx} a_y + \partial^2_{zx} a_z \\ &= -\partial^2_{y^2} a_x - \partial^2_{z^2} a_x + \partial^2_{yx} a_y + \partial^2_{zx} a_z - \partial^2_{x^2} a_x + \partial^2_{x^2} a_x \\ &= \underbrace{-\partial^2_{x^2} a_x - \partial^2_{y^2} a_x - \partial^2_{z^2} a_x}_{\triangle \vec{a}|_x} + \underbrace{+\partial^2_{x^2} a_x + \partial^2_{yx} a_y + \partial^2_{zx} a_z}_{\mathrm{grad}(\mathrm{div}\vec{a})|_x} \end{aligned}$$

1.2 Gradient of a scalar function

We have

$$\begin{aligned} R &= \| \mathbf{R} \| = \| \mathbf{MM}' \| = \| \mathbf{r}' - \mathbf{r} \| \\ &= \sqrt{(x'-x)^2 + (y'-y)^2 + (z'-z)^2} \end{aligned}$$

Direct differentiation gives :

$$\begin{aligned} \mathrm{grad}(R)_M &= -\frac{\mathbf{MM}'}{\| \mathbf{MM}' \|} \\ \mathrm{grad}(R)_{M'} &= \frac{\mathbf{MM}'}{\| \mathbf{MM}' \|} \end{aligned}$$

and we have :

$$\mathrm{grad}(R)_M = -\mathrm{grad}(R)_{M'}$$

1.3 Evaluation of line integrals

We have to evaluate

$$\mathbf{F}(x, y, z) = 5y\mathbf{a}_x + x^2\mathbf{a}_z$$

along the following path

$$\begin{cases} 2x = y \\ 9x = z^2 \end{cases}$$

. We are going ot determine the differential path along Γ. By differentiaing the path we have :

$$\begin{aligned} 2dx &= dy \\ 9dx &= 2zdz \end{aligned}$$

We can express $\mathbf{dl}$ with respect to dx :

$$\mathbf{dl} = dx\mathbf{a}_x + 2dx\mathbf{a}_y + \frac{9dx}{2z}\mathbf{a}_z$$

So the line integral can be expressed as :

$$\begin{aligned} \int_{(0,0,0)}^{(1,2,3)} \mathbf{F} \cdot \mathbf{dl} &= \int_{(0,0,0)}^{(1,2,3)} \left(5y + \frac{9x^2}{2z}\right) dx \\ &= \int_0^1 10x\, dx + \int_0^3 \left(\frac{z^2}{9}\right)^2 dz \\ &= 5.6 \end{aligned}$$

1.4 Field created by a distribution of charges

Depending if M is inside or outside the sphere of radius a the total quantity of charges is equal to :

$$\begin{cases} Q_{int} = \frac{4\pi}{3}r^3 & r < a \\ Q_{int} = \frac{4\pi}{3}a^3 & r > a \end{cases} \tag{1.1}$$

The distribution of charges is spherical so we can deduce that :

$$\mathbf{E}(r, \theta, \phi) = \mathbf{E}(r)$$

Due to symmetry properties of the distribution we have :

$$\mathbf{E}(r) = E(r)\mathbf{e}_r$$

So in order to apply the Gauss theorem we take a spherical surface centered in O. We have :

$$\iint_{\mathcal{S}} \mathbf{E} \cdot \mathbf{n}\, ds = \frac{Q_{int}}{\varepsilon_0}$$

We deduce the following relations :

$$\begin{cases} 4\pi r^2 E(r) = \frac{4\pi\rho}{3\varepsilon_0}r^3 & r < a \\ 4\pi r^2 E(r) = \frac{4\pi\rho}{3\varepsilon_0}a^3 & r \geq a \end{cases} \tag{1.2}$$

We deduce the value of the electric field :

$$\mathbf{E}(r) = \frac{\rho}{3\varepsilon_0}r\mathbf{e}_r \qquad r < a$$

And

$$\mathbf{E}(r) = \frac{\rho a^3}{3\varepsilon_0 r^2}\mathbf{e}_r \qquad r \geq a$$

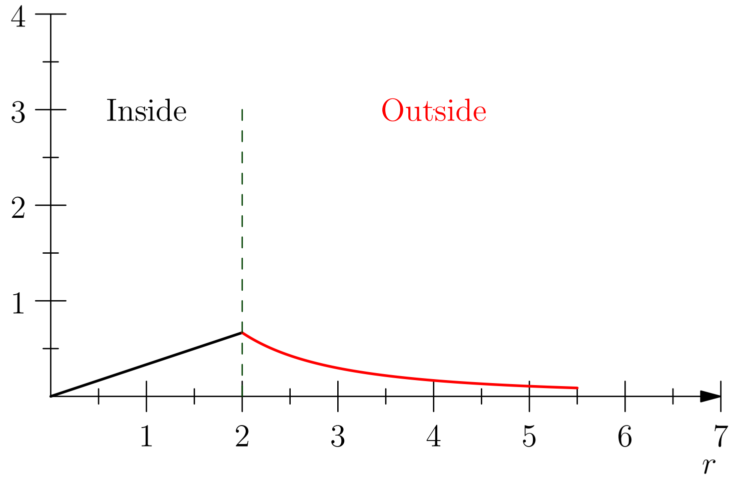

Figure 1.1: Evolution of the electric potential of a sphere with a radius $a = 2$ and $\frac{\rho}{\varepsilon_0} = 1$

Expression of the electrostatic potential

In this case we know $\mathbf{E}$ and we want to determine $V(r)$, we are going to use the relation :

$$\mathbf{E} = -\mathrm{grad}(V)$$

and since the electric field is radial and depending only of r we have :

$$E(r) = -\frac{\partial V}{\partial r}$$

We have by direct integration :

$$\begin{aligned} V(r) &= -\frac{\rho r^2}{6\varepsilon_0} + C_1 \qquad r < a \\ V(r) &= \frac{\rho a^3}{3\varepsilon_0 r} + C_2 \qquad r \geq a \end{aligned}$$

The value of C_1 and C_2 is determinded by the continuity of the potenial at the surface of the sphere and also by the fact that :

$$\lim_{r \to \infty} V(r) = 0$$

From this relation we deduce $C_2 = 0$ and :

$$-\frac{\rho a^2}{6\varepsilon_0} + C_1 = \frac{\rho a^3}{3\varepsilon_0 a}$$

So

$$C_1 = \frac{\rho a^2}{2\varepsilon_0}$$

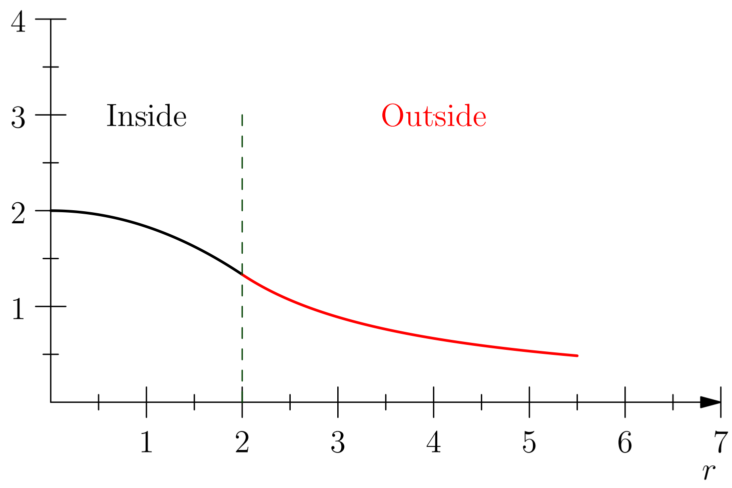

Figure 1.2: Evolution of the electric potential of a sphere with a radius $a = 2$ and $\frac{\rho}{\varepsilon_0} = 1$

Case of a spherical shell

The same kind of treatment can be performed in that case and we obtain in that case, for the total quantity of charges :

$$\begin{cases} Q_{int} = 0 & r < a \\ Q_{int} = 4\pi\sigma a^2 & r > a \end{cases} \tag{1.3}$$

With the help of the Gauss theorem we have :

$$\mathbf{E}(r) = \mathbf{0} \qquad r < a$$

And

$$\mathbf{E}(r) = \frac{\sigma a^2}{3\varepsilon_0 r^2}\mathbf{e}_r \qquad r \geq a$$

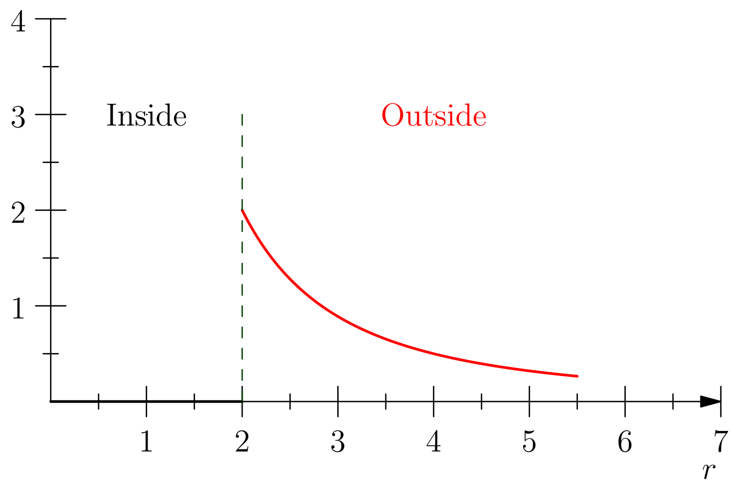

Figure 1.3: Evolution of the electric field produced by a spherical shell with a radius $a = 2$ and $\frac{\sigma}{\varepsilon_0} = 1$

By integration of the electric field with respect to r we obtain for the potential :

$$\begin{aligned} V(r) &= \frac{\sigma a}{\varepsilon_0} \qquad r < a \\ V(r) &= \frac{\sigma a^2}{\varepsilon_0 r} \qquad r \geq a \end{aligned}$$

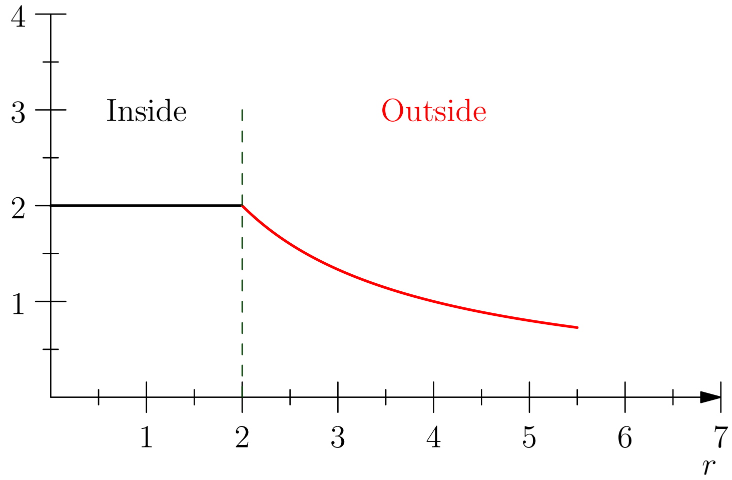

Figure 1.4: Evolution of the electric potential produced by a spherical shell with a radius $a = 2$ and $\frac{\sigma}{\varepsilon_0} = 1$

The electric field is indeed discontinuous at the crossing of the sphere and it is normal and can be related to the discontinuity of the normal component of the electric field in presence of a surfacic density of charges found from the boundary conditions deduced from the Maxwell's equations.

Chapter 2

Calculation of electrostatic fields and potenitals

2.1 Field created by two charges

The field created by two charges is the sum of both field created by each charge separately :

$$\mathbf{E}(M) = \mathbf{E}_1(M) + \mathbf{E}_2(M)$$

We deduce :

$$\mathbf{E}(M) = \frac{12\pi\varepsilon_0}{4\pi\varepsilon_0}\frac{\mathbf{P_1M}}{\| \mathbf{P_1M} \|^3} + \frac{-4\pi\varepsilon_0}{4\pi\varepsilon_0}\frac{\mathbf{P_2M}}{\| \mathbf{P_2M} \|^3}$$

with

$$\begin{aligned} \mathbf{P_1M} &= 2\mathbf{e}_x + \mathbf{e}_z \\ \mathbf{P_2M} &= -\mathbf{e}_x + \mathbf{e}_z \end{aligned}$$

we have :

$$\mathbf{E}\left|\begin{array}{l} \frac{6}{5\sqrt{5}} + \frac{1}{2\sqrt{2}} \\ 0 \\ \frac{3}{5\sqrt{5}} - \frac{1}{2\sqrt{2}} \end{array}\right.$$

The same kind of treatment can be done for the electrostatis field. We have :

$$V(M) = \frac{12\pi\varepsilon_0}{4\pi\varepsilon_0}\frac{1}{\| \mathbf{P_1M} \|} + \frac{-4\pi\varepsilon_0}{4\pi\varepsilon_0}\frac{1}{\| \mathbf{P_2M} \|}$$

which yields :

$$V(M) = \frac{3}{5\sqrt{5}} - \frac{1}{\sqrt{2}}$$

2.1.1 Field created by a thin loop

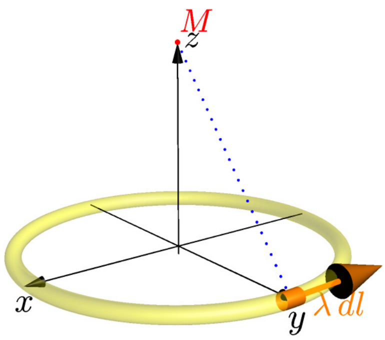

Figure 2.1: Configuration of the loop

Due to symmetries of the problem each piece of the wire ant its symmetric with resect to the z axis will give a field with a non null component only along z. We have

$$dE_z = \frac{\lambda dl}{4\pi\varepsilon_0}\frac{1}{R^2}\cos(\alpha)$$

with $\cos(\alpha) = \frac{z}{R}$. Next we have to integrate over the wire and we obtain :

$$E_z = \int_0^{2\pi} \frac{\lambda r}{4\pi\varepsilon_0}\frac{z\,d\theta}{\sqrt[3]{r^2+z^2}}$$

Finally :

$$\mathbf{E}(0,0,z) = \frac{\lambda r}{2\varepsilon_0}\frac{z}{\sqrt[3]{r^2+z^2}}\mathbf{e}_z$$

Field created by a disk

The probkem is also xysymmetric here, the only thing which changes is the expression of the quantituy of charges and we have :

$$dE_z = \frac{\lambda ds}{4\pi\varepsilon_0}\frac{1}{l^2}\cos(\alpha)$$

with

$$\begin{aligned} l^2 &= r^2+z^2 \\ ds &= r dr d\theta \\ \cos(\alpha) &= \frac{z}{l} \end{aligned}$$

So :

$$\begin{aligned} E_z(0,0,z) &= \int_0^R\int_0^{2\pi}\frac{\sigma z}{4\pi\varepsilon_0}\frac{r\,dr\,d\theta}{\sqrt[3]{r^2+z^2}} \\ &= \frac{\sigma z}{4\varepsilon_0}\int_0^R \frac{2r\,dr}{\sqrt[3]{r^2+z^2}} \\ &= \frac{\sigma z}{4\varepsilon_0}\left[\frac{-2}{2\sqrt{r^2+z^2}}\right]_0^R \end{aligned}$$

Finally we have :

$$\mathbf{E}(0,0,z) = \frac{\sigma z}{2\varepsilon_0}\left[\frac{1}{|z|} - \frac{1}{\sqrt{R^2+z^2}}\right]\mathbf{e}_z$$

2.1.2 Electric dipole

The elctric potential generated by the two charges can be written as :

$$V(r) = \frac{1}{4\pi\varepsilon_0}\left[\frac{-q}{\parallel \mathbf{P_1M} \parallel} + \frac{q}{\parallel \mathbf{P_2M} \parallel}\right]$$

$$V(r) = \frac{1}{4\pi\varepsilon_0}\left[\frac{-q}{\sqrt{\frac{l^2}{4}+r^2+lr\cos(\theta)}} + \frac{q}{\sqrt{\frac{l^2}{4}+r^2-lr\cos(\theta)}}\right]$$

In the limit where $r >> l$ we can do a Taylor expansion of the previous terms :

$$\begin{aligned} \frac{1}{\sqrt{\frac{l^2}{4}+r^2-lr\cos(\theta)}} &\approx \frac{1}{r}\left(1+\frac{l}{2r}\cos(\theta)\right) \\ \frac{1}{\sqrt{\frac{l^2}{4}+r^2-lr\cos(\theta)}} &\approx \frac{1}{r}\left(1+\frac{l}{2r}\cos(\theta)\right) \end{aligned}$$

We obtain for V :

$$V(r) = \frac{ql\cos(\theta)}{4\pi\varepsilon_0 r^2}$$

We can remark here that the electric potential varies as $\frac{1}{r^2}$ instead of a classical variation of $\frac{1}{r}$. For the derivation of the electric field we use the gradient formula in cylindrical coordiante system and we have :

$$\mathrm{grad}(f)\left|\begin{array}{l}\partial_r f\\ \frac{1}{r}\partial_\theta f\end{array}\right.$$

we obtain :

$$\mathbf{E}\left|\begin{array}{l}\partial_r f\\ \frac{1}{r}\partial_\theta f\end{array}\right.$$

And we finally have :

$$\mathbf{E}\left|\begin{array}{l}\frac{2ql\cos(\theta)}{4\pi\varepsilon_0 r^3}\\ \frac{ql\sin(\theta)}{4\pi\varepsilon_0 r^3}\end{array}\right.$$

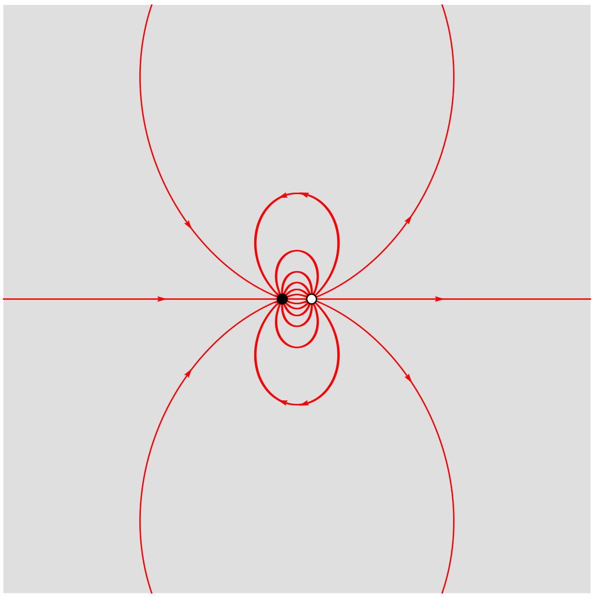

Figure 2.2: sketch of the field lines

2.1.3 Image Theory

The electrostatic potential is such that :

$$\Delta V = \frac{q}{\varepsilon_0}\delta(0,0,a)$$

and $V = 0$ at $z = 0$. Equivalent distribution of charges : The problem that we have at hand is symetrical along the z axis. The distribution of charges must be such that the potential created by the charge q in the plane $z = 0$ is cancelled out by other charges.
The unique way to do this is to place a charge $-q$ at $(0,0,-a)$.
The equivalent distribution of charges is then the following : The potential given by this distribution is then :

$$V(x,y,z) = \frac{q}{4\pi\varepsilon_0}\left[\frac{1}{\sqrt{(x^2+y^2+(z-a)^2}} - \frac{1}{\sqrt{(x^2+y^2+(z+a)^2}}\right]$$

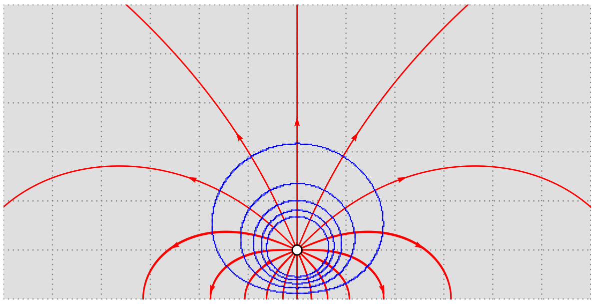

Figure 2.3: sketch of the field lines, and equipoltentials

Chapter 3

Tutorial 3 : Induction field

3.1 Induction field created by a straight wire

We have :

$$dB = \frac{\mu_0}{4\pi}\frac{Idx}{r^2}\sin(\phi)$$

with $\pi - \phi - \theta = \frac{\pi}{2}$ so :

$$dB = \frac{\mu_0}{4\pi}\frac{Idx}{r^2}\cos(\theta)$$

$$x = R\tan(\theta)$$

so

$$\frac{\partial x}{\partial \theta} = \frac{R}{\cos(\theta)^2} = \frac{r^2}{R}$$

And finally :

$$dB = \frac{\mu_0}{4\pi}\frac{I}{R}\cos(\theta)d\theta$$

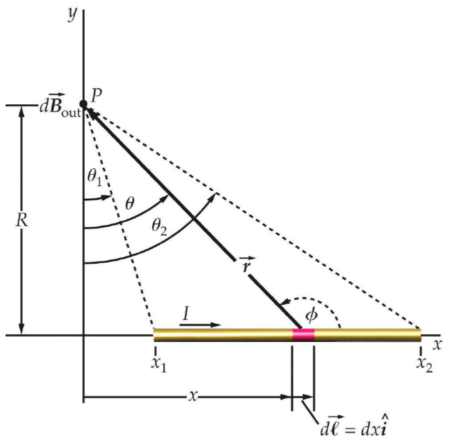

Figure 3.1: sketch of the proposed problem

After an integration over θ we obtain :

$$\begin{aligned} B &= \frac{\mu_0}{4\pi}\frac{I}{R}\int_{\theta_1}^{\theta^2}\cos(\theta)d\theta \\ &= \frac{\mu_0}{4\pi}\frac{I}{R}\left[\sin(\theta_2) - \sin(\theta_1)\right] \end{aligned}$$

3.2 Induction created by a wire loop

The application of the Biot and Savart law and symmetric properties of the configuration implies that only the z compenent of the induction field is non null if it is measured along the z axis. We have :

$$dB_z = \underbrace{\frac{\mu_0}{4\pi}\frac{Idl}{\sqrt{a^2+z^2}}}_{\text{Biot and Savart}}\underbrace{\frac{a}{\sqrt{a^2+z^2}}}_{\text{Projection over z}}$$

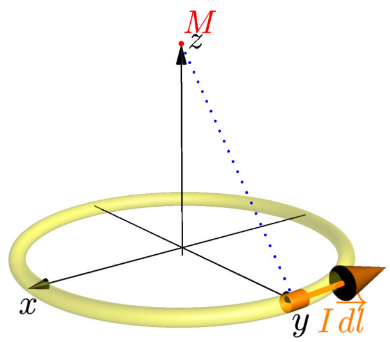

Figure 3.2: Configuration of the loop

Since in that case $dl = a d\theta$. The integration can be performed easily and we have :

$$\begin{aligned} \mathbf{B}(0,0,z) &= \frac{\mu_0 I a^2}{2\sqrt[3]{a^2+z^2}}\mathbf{e}_z \\ &= \frac{\mu_0 I}{2a}\sin^3(\theta)\mathbf{e}_z \end{aligned}$$

3.3 Induction generated by a solenoid

Let us assume that for a width dz of the solenoid the are dN current loops. We have :

$$dN = \frac{N}{L}dz$$

The induction field related to this dN current loops can be written as :

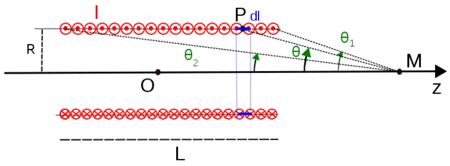

Figure 3.3: Solenoid of finite length

$$dB_z(0,0,z) = dN\frac{\mu_0 I}{2R}\sin^3(\theta)\mathbf{e}_z$$

In order to calculate the field created by the solenoid we have to integrate over the length of the solenoid. We have

$$\tan(\theta) = \frac{R}{z}$$

so $z = \frac{R}{\tan(\theta)}$ and by differentiation we obtain :

$$dx = \frac{-Rd\theta}{\sin^2(\theta)}$$

It yields :

$$dB_z(0,0,z) = \frac{-\mu_0 NI}{2L}\frac{\sin^3(\theta)}{\sin^2(\theta)}d\theta$$

So

$$\begin{aligned} B_z(0,0,z) &= \frac{-\mu_0 NI}{2L}\int_{\theta_1}^{\theta_2}\sin(\theta)\,d\theta \\ &= \frac{\mu_0 NI}{2L}\left[\cos(\theta_2) - \cos(\theta_1)\right] \end{aligned}$$

3.4 Toroidal coil

Due to symetries the current is invariant along θ, so $\mathbf{B}$ is independant of θ.
Each plane containing two symetrical turn is a plane of symetry for the geometry so :

$$\mathbf{B} = B(r)\mathbf{e}_\theta$$

By using the Ampere law where the contour is a circle of radius r inside the tore we have :

$$\int_{\mathcal{C}} \mathbf{B}\cdot\mathbf{dl} = \mu_0 NI$$

So we deduce for a point inside the tore :

$$\mathbf{B} = \frac{\mu_0 NI}{2\pi r}\mathbf{e}_\theta$$

for a point outside the tore :

$$\mathbf{B} = \mathbf{0}$$

since the net current is null.

3.5 Field created by a magnetic dipole

As it has been done for the electric dipole we are going to evaluate first the vector potential when the observation is far from the sources and we will deduce afterwards $\mathbf{B}$ from $\mathbf{A}$

Determination of the vector potential A

By using the definition of the vector potential we have :

$$\mathbf{A} = \frac{\mu_0}{4\pi}\int_{\mathcal{C}} \frac{\mathbf{dl}}{\parallel \mathbf{PM} \parallel}$$

with :

$$\begin{aligned} \parallel \mathbf{PM} \parallel &= \parallel \mathbf{OM} - \mathbf{OP} \parallel \\ &= \sqrt{r^2 + R^2 - 2\mathbf{OM}\cdot\mathbf{OP}} \end{aligned}$$

Since $R << r$ we can do a power expansion of $\frac{1}{PM}$ in terms of $\frac{R}{r}$.

$$\frac{1}{PM} = \frac{1}{r}\left(1 + \frac{R^2}{r^2} - \frac{2\mathbf{OM}\cdot\mathbf{OP}}{r^2}\right)^{-\frac{1}{2}} \approx \frac{1}{r} + \frac{\mathbf{e}_r\cdot\mathbf{OP}}{r^2}$$

We deduce :

$$\mathbf{A} \approx \frac{\mu_0}{4\pi} I\left[\frac{1}{r}\int_{\mathcal{C}}\mathbf{dl} + \frac{1}{r^2}\int_{\mathcal{C}}(\mathbf{e}_r\cdot\mathbf{OP})\mathbf{dl}\right]$$

The first integral is null since it is evaluated along a closed loop. For the second term we project all the vectors in a cartesian coordinate system, For a loop in the (xOy) plane the quantity $(\mathbf{e}_r \cdot \mathbf{OP})\mathbf{dl}$ can be written as :

$$(\mathbf{e}_r\cdot\mathbf{OP})\mathbf{dl}\left|\begin{array}{l} (a\sin(\theta)\cos(\phi)\cos(\phi') + a\sin(\theta)\sin(\phi)\sin(\phi')) \times -a\sin(\phi')\,d\phi' \\ (a\sin(\theta)\cos(\phi)\cos(\phi') + a\sin(\theta)\sin(\phi)\sin(\phi')) \times a\cos(\phi')\,d\phi' \\ 0 \end{array}\right.$$

we obtain :

$$(\mathbf{e}_r\cdot\mathbf{OP})\mathbf{dl} \qquad a^2\sin(\theta)\left|\begin{array}{l} \left(-\cos(\phi)\cos(\phi')\sin(\phi') - \sin(\phi)\sin^2(\phi')\right)\,d\phi' \\ \left(\sin(\phi)\sin(\phi')\cos(\phi') + \cos(\phi)\cos^2(\phi')\right)\,d\phi' \\ 0 \end{array}\right.$$

By integerating fro 0 to 2π the first term in each component cancels out and we finally obtain :

$$\mathbf{A} \approx \frac{\mu_0 I}{4\pi}\frac{\pi a^2\sin(\theta)}{r^2}\mathbf{e}_\phi$$

Calculation of the induction field

By using the defintion of the curl in a spherical coordinate we deduce $\mathbf{B}$. We have :

$$\mathbf{B} = \mathrm{rot}(\mathbf{A})$$

which yields :

$$\mathbf{B} = \frac{1}{r\sin(\theta)}\partial_\theta(\sin(\theta)A_\phi)\mathbf{e}_r - \frac{1}{r}\partial_r(rA_\phi)\mathbf{e}_\theta$$

We obtain :

$$\mathbf{B}\left|\begin{array}{l} \frac{2\mu_0 I}{4\pi}\frac{\pi a^2\cos(\theta)}{r^3} \\ \frac{\mu_0 I}{4\pi}\frac{\pi a^2\sin(\theta)}{r^3} \\ 0 \end{array}\right.$$

Chapter 4

Electromotive force

4.1 Faraday disc

The current is prodcuced by the electromotive field $\mathbf{E}_m = \mathbf{v} \times \mathbf{B}$. The electromotive voltage between O and A is equal to :

$$e = \int_A^O \mathbf{E}_m \cdot \mathbf{dl} = \int_A^O (\mathbf{v} \times \mathbf{B}) \cdot \mathbf{dl}$$

with $\mathbf{v} = \omega r \mathbf{e}_\theta$, $\mathbf{B} = B_0 \mathbf{e}_z$ and $\mathbf{dl} = dr\mathbf{e}_r$, so we obtain :

$$\begin{aligned} e &= \int_A^O \omega B_0 r \mathbf{e}_r \cdot dr \mathbf{e}_r \\ &= -\frac{B_0 w a^2}{2} \end{aligned}$$

the current is then equal to :

$$i = -\frac{B_0 w a^2}{2R}$$

Potential at the terminals of a square loop

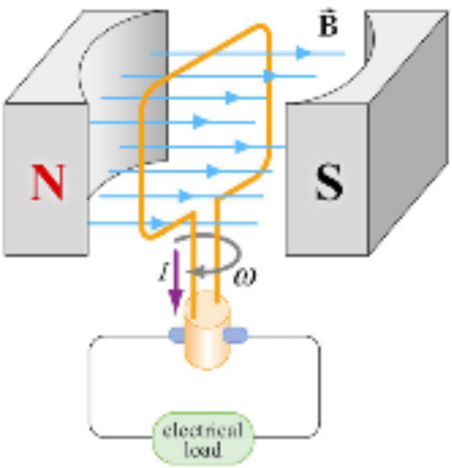

Figure 4.1: setup

We have

$$e = \frac{-d\phi}{dt}$$

where

$$\phi = \int_{\text{squareloop}} \mathbf{B} \cdot \mathbf{n}\, ds$$

and

$$\mathbf{n}\left|\begin{array}{l}\sin(\omega t)\\0\\\cos(\omega t)\end{array}\right.$$

so

$$\phi = B\pi a^2\cos(\omega t)$$

and

$$e = \omega B\pi a^2\sin(\omega t)$$

Self induction of a solenoid

We have :

$$\mathbf{B} = B_0\cos(\omega t)\mathbf{e}_z$$

we deduce that the flux is equal to :

$$\begin{aligned} e &= -\frac{d\phi}{dt}\\ &= \omega N\pi a^2 B_0\sin(\omega t)\end{aligned}$$

So we deuce the value of the current circulating into the solenoid :

$$i(t) = \frac{\omega N\pi a^2 B_0\sin(\omega t)}{R}$$

and

$$B_1 = \mu_0 n i(t)$$

$\mathbf{B}_2(t)$ is the field repsonsible of the current $i(t)$ flowing in the solenoid. So we have :

$$\mathbf{B}_2(t) = \mu_0\frac{N}{L}i(t)\mathbf{e}_z$$

Faraday law gives us :

$$\begin{aligned} e(t) &= -N\pi a^2\frac{dB}{dt}\\ &= -N\pi a^2\left(-B_0\omega\sin(\omega t) + \frac{dB_2}{dt}\right)\end{aligned}$$

Ohm law in the circuit gives, $i(t) = \frac{e(t)}{R}$ and $B_2(t) = \mu_0\frac{N}{L}i(t)$, so we can deduce the ODE :

$$\frac{dB_2}{dt} + \frac{LR}{\mu_0 N^2\pi a^2}B_2(t) = B_0\omega\sin(\omega t)$$

We set :

$$\frac{1}{\tau} = \frac{LR}{\mu_0 N^2\pi a^2}$$

We are going to look for a solution of the following type :

$$B_2(t) = \alpha\cos(\omega t) + \beta\sin(\omega t)$$

where α and β must be determined. They must satisfy the system :

$$\left\{\begin{array}{c} -\omega\alpha + \frac{\beta}{\tau} = \omega B_0\\ \frac{\alpha}{\tau} + \omega\beta = 0\end{array}\right.$$

We finally obtain :

$$B_2(t) = B_0\frac{1}{\omega\tau + \frac{1}{\omega\tau}}\left[-\omega\tau\cos(\omega t) + \sin(\omega t)\right]$$

we can see that that at high frequency ($\omega\tau >> 1$) we have :

$$B_2(t) \approx -B_0(t)$$

so self induction is very high and can cancel the total field in the solenoid.

4.2 Airplane over the North pole

We have

$$U = |\int_B^T \mathbf{E}_m \cdot \mathbf{dl}|$$

with :

$$\begin{aligned} \mathbf{E}_m &= \mathbf{v} \times \mathbf{B}_T \\ \mathbf{B}_T &= -B_T \mathbf{e}_z \end{aligned}$$

$$\mathbf{v} = v\mathbf{e}_y$$

So we deduce :

$$U = vB_tL \approx 1.2V$$

Chapter 5

Heating by induction - Strucutre of wave propagation

5.1 Heating with eddy currents

The expression of the induction field is the following :

$$\mathbf{B} = \mu_0 \frac{N}{L} I \cos(\omega t)\mathbf{e}_z$$

Induction phenomena and heating effect in the rod are due to the circulation of induced current. Every plane containing (Oz) is an antisymetrical plane for the problem so $\mathbf{E}$ is perpendicular to it.
Every plane orthogonal to (Oz) is a plane of symetry so $\mathbf{E}$ is inside this plane.
The problem is invariant by rotation along θ and by translation along z. So we have :

$$\mathbf{E} = E(r,t)\mathbf{e}_\theta$$

We have :

$$\mathrm{rot}(\mathbf{E}) = -\frac{\partial \mathbf{B}}{\partial t}$$

with the previous expression of the field we have :

$$\mathrm{rot}(\mathbf{E}) = \left[\frac{dE}{dr} + \frac{E(r)}{r}\right] \cancel{\sin(\omega t)}\mathbf{e}_z$$

on the other hand we have :

$$\frac{\partial \mathbf{B}}{\partial t} = \mu_0 \omega \frac{N}{L} I \sin(\omega t)\mathbf{e}_z$$

so we obtain the following ODE :

$$\frac{dE}{dr} + \frac{E(r)}{r} = \mu_0 \omega \frac{N}{L} I \ \sin(\omega t)$$

Along the z axis $\mathbf{j}(r=0)$ and $\mathbf{E} = 0$ so from the familiy of general solutions we select the case where $K = 0$. And we deduce :

$$\begin{aligned} \mathbf{E}(r,t) &= \frac{r\omega\mu_0}{2}\frac{N}{L} I \sin(\omega t)\mathbf{e}_\theta \\ \mathbf{j} = \sigma \mathbf{E} &= \sigma \frac{r\omega\mu_0}{2}\frac{N}{L} I \sin(\omega t)\mathbf{e}_\theta \end{aligned}$$

The power dissipated for ine period can be calculateg thanks to :

$$\frac{1}{T}\int_0^T \mathbf{J}\cdot\mathbf{E}\,dt$$

$$\begin{aligned} \frac{1}{T}\int_0^T \mathbf{J}\cdot\mathbf{E}\,dt &= \frac{\sigma}{4} r^2 \mu_0^2 \omega^2 \frac{N^2}{L^2} I^2 \int_0^T \sin^2 \omega t\,dt \\ &= \frac{\sigma}{8} r^2 \mu_0^2 \omega^2 \frac{N^2}{L^2} I^2 \end{aligned}$$

The power dissipated in the volume is equal to the integration over the volume of the previous term and leads for the considered cylinder :

$$\begin{aligned} P_{tot} &= \iiint_{cylinder} P\,d\tau \\ &= \frac{\sigma}{8} r^2 \mu_0^2 \omega^2 \frac{N^2}{L^2} I^2 \int_0^a r^2 2\pi r L\,dr \\ &= \frac{\sigma}{8} r^2 \mu_0^2 \omega^2 \frac{N^2}{L^2} I^2 a^4 \\ &\approx 2.8kW \end{aligned}$$

The total amount of energy required for the melting is :

$$E_{melting} = L_{heat} \times \pi a^2 L \times \rho_{Al} = t_{melting} \times P_{tot}$$

We deduce that :

$$t_{melting} \approx 321s$$

we have :

$$\mathrm{rot}(\mathbf{B}_1) = \mu_0 \mathbf{j}$$

with

$$\mathbf{B}_1 = B_1(r)\sin(\omega t)\mathbf{e}_z$$

From the expression of the curl we have :

$$\begin{aligned} -\frac{\partial B_1}{\partial r} &= \mu_0 j \\ &= \sigma \mathbf{E} \\ &= \sigma \frac{r\omega\mu_0}{2} \frac{N}{L} I \sin(\omega t) \end{aligned}$$

We deduce

$$B_1(r,t) = -\sigma \frac{r^2\omega\mu_0}{4} \frac{N}{L} I \sin(\omega t) + f(t)$$

And if $B_1(r=a,t) = 0$ we derive $f(t)$ and finally :

$$\mathbf{B}_1(r,t) = \sigma \frac{(a^2 - r^2)\omega\mu_0^2}{4} \frac{N}{L} I \sin(\omega t)$$

we deduce that

$$B_{1max} = \sigma \frac{a^2\omega\mu_0^2}{4} \frac{N}{L} I$$

with

$$B_0 = \mu_0 \frac{N}{L} I$$

in order to have $B_1 < B_0$ the radius a must satisfy :

$$\sigma \frac{(a^2 - r^2)\omega\mu_0^2}{4} \frac{N}{L} I << \mu_0 \frac{N}{L} I$$

which can be rewritten as :

$$a << \sqrt{\frac{4}{\mu_0 \sigma \omega}}$$

Here it yields $a << 1.610^{-2} m$ which is not respected in our case.

5.2 General solutions of wave propagation

We start from the following equation :

$$\frac{\partial^2 f}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 f}{\partial t^2} = 0$$

We set :

$$\begin{aligned} u &= x + ct \\ v &= x - ct \end{aligned}$$

we have :

$$\begin{aligned} df &= \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial t}dt \\ df &= \frac{\partial f}{\partial u}du + \frac{\partial f}{\partial v}dv \end{aligned}$$

By adding and substracting both equations we obtain :

$$\begin{aligned} df &= \frac{\partial f}{\partial u}(dx - cdt) + \frac{\partial f}{\partial v}(dx + cdt) \\ &= \left(\frac{\partial f}{\partial u} + \frac{\partial f}{\partial v}\right) dx + c\left(\frac{\partial f}{\partial v} - \frac{\partial f}{\partial u}\right) dt \end{aligned}$$

By identification we obtain :

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial u} + \frac{\partial f}{\partial v}$$

$$\frac{\partial f}{\partial t} = c\left(\frac{\partial f}{\partial v} - \frac{\partial f}{\partial u}\right)$$

At second order of derivation we obtain :

$$\begin{aligned} \frac{\partial^2 f}{\partial x^2} &= \frac{\partial^2 f}{\partial u^2} + \frac{\partial^2 f}{\partial v^2} + 2\frac{\partial^2 f}{\partial u \partial v} \\ \frac{\partial^2 f}{\partial t^2} &= c^2\left(\frac{\partial^2 f}{\partial u^2} + \frac{\partial^2 f}{\partial v^2} - 2\frac{\partial^2 f}{\partial u \partial v}\right) \end{aligned}$$

We finally obtain :

$$\frac{\partial^2 f}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 f}{\partial t^2} = 4\frac{\partial^2 f}{\partial u \partial v}$$

A first integartion with respect to u gives :

$$\frac{\partial f}{\partial u} = f_1'(v)$$

And

$$f(u, v) = F_1(v) + F_2(u)$$

This solution is a wave which propagates along $+x$ and $-x$ at speed c. In a framework of a spherical coordinate system, we have for a radial function :

$$\Delta f = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right)$$

So we set $f(r, t) = \frac{1}{r}g(r, t)$ and we can remark that :

$$\Delta f = \frac{1}{r}\frac{\partial^2 g}{\partial r^2}$$

We can deduce that g verifies :

$$\frac{\partial^2 g}{\partial r^2} - \frac{1}{c^2}\frac{\partial^2 g}{\partial t^2} = 0$$

So :

$$g(r, t) = F(r + ct) + G(r - ct)$$

and

$$f(r, t) = \frac{1}{r}\left[F(r + ct) + G(r - ct)\right]$$

we have a wave with propagating speed c and a decreasing amplitude in $\frac{1}{r}$.

Chapter 6

Propagation-Interferences

6.1 Interferences between plane waves

Following the correpsonding skecth we have :

$$\begin{aligned}\mathcal{E}_1 &= E_1 \exp^{i[k_1 y \sin(\theta) + k_1 z \cos(\theta)]} \mathbf{e}_x \\ \mathcal{E}_2 &= E_2 \exp^{i[-k_2 y \sin(\theta) + k_2 z \cos(\theta) + 2\varphi]} \mathbf{e}_x\end{aligned}$$

We are setting $k_1 = k_2 = k$ and

$$\begin{aligned}\mathcal{B}_1 &= \frac{1}{i\omega}\mathrm{rot}(\mathcal{E}_1) \\ \mathcal{B}_2 &= \frac{1}{i\omega}\mathrm{rot}(\mathcal{E}_2)\end{aligned}$$

We deduce :

$$\mathcal{B}_1 \ \frac{ik}{i\omega}\left|\begin{array}{l} 0 \\ \cos(\theta) E_1 \exp^{i[ky\sin(\theta)+kz\cos(\theta)]} \\ -\sin(\theta) E_1 \exp^{i[ky\sin(\theta))+kz\cos(\theta)]} \end{array}\right.$$

And :

$$\mathcal{B}_2 \ \frac{ik}{i\omega}\left|\begin{array}{l} 0 \\ \cos(\theta) E_2 \exp^{i[ky\sin(\theta)+kz\cos(\theta)+2\varphi]} \\ \sin(\theta) E_2 \exp^{i[ky\sin(\theta))+kz\cos(\theta)+2\varphi]} \end{array}\right.$$

We are going to evaluate :

$$\begin{aligned}\mathcal{P} &= \frac{1}{2\mu_0}\{\mathcal{E}_t \times \mathcal{B}_t^*\} \\ &= \frac{1}{2\mu_0}\{(\mathcal{E}_1 + \mathcal{E}_2) \times (\mathcal{B}_1 + \mathcal{B}_2)^*\} \\ &= \frac{1}{2\mu_0}\{\underbrace{\mathcal{E}_1 \times \mathcal{B}_1^*}_{\mathcal{P}_1} + \underbrace{\mathcal{E}_2 \times \mathcal{B}_2^*}_{\mathcal{P}_2} + \underbrace{\mathcal{E}_1 \times \mathcal{B}_2^*}_{\mathcal{P}_3} + \underbrace{\mathcal{E}_2 \times \mathcal{B}_1^*}_{\mathcal{P}_4}\}\end{aligned}$$

On a donc :

$$\mathcal{P}_1 \ \frac{k}{2\mu_0\omega}\left|\begin{array}{l} 0 \\ \sin(\theta) E_1^2 \\ \cos(\theta) E_1^2 \end{array}\right.$$

$$\mathcal{P}_2 \ \frac{k}{2\mu_0\omega}\left|\begin{array}{l} 0 \\ -\sin(\theta) E_2^2 \\ \cos(\theta) E_2^2 \end{array}\right.$$

$$\mathcal{P}_3 \; \frac{k}{2\mu_0\omega}\left| \begin{array}{l} 0 \\ -\sin(\theta)E_1E_2^* \exp^{i[2ky\sin(\theta)-2\varphi]} \\ \cos(\theta)E_1E_2^* \exp^{i[2ky\sin(\theta)-2\varphi]} \end{array} \right.$$

$$\mathcal{P}_4 \; \frac{k}{2\mu_0\omega}\left| \begin{array}{l} 0 \\ \sin(\theta)E_1 * E_2 \exp^{-i[2ky\sin(\theta)-2\varphi]} \\ \cos(\theta)E_1 * E_2 \exp^{-i[2ky\sin(\theta)-2\varphi]} \end{array} \right.$$

Finally we have :

$$\mathcal{P} \; \frac{k}{2\mu_0\omega}\left| \begin{array}{l} 0 \\ \sin(\theta)\left[E_1^2 - E_2^2 \overbrace{-E_1E_2^* \exp^{i[2ky\sin(\theta)-2\varphi]} + E_1^*E_2 \exp^{-i[2ky\sin(\theta)-2\varphi]}}^{-2iE_1E_2\sin(2ky\sin(\theta)-2\varphi)}\right] \\ \cos(\theta)\left[E_1^2 + E_2^2 + \underbrace{E_1E_2^* \exp^{i[2ky\sin(\theta)-2\varphi]} + E_1^*E_2 \exp^{-i[2ky\sin(\theta)-2\varphi]}}_{2E_1E_2\cos(2ky\sin(\theta)-2\varphi)}\right] \end{array} \right.$$

If we want to calculte the intensity measured by the detector we have to calculate :

$$I = \mathfrak{Re}(\mathcal{P})$$

we obtain :

$$I = \left[(I_1 - I_2)^2 \sin^2(\theta) + (I_1 + I_2 + 2\sqrt{I_1I_2}\cos(2\varphi))^2 \cos^2(\theta)\right]^{\frac{1}{2}}$$

6.2 Polarisation state of a plane wve

The wave vector $\mathbf{k}$ can be recovered in the phasor term of tyhe electric field, since we have :

$$\mathcal{E}(\mathbf{r}) = \mathcal{E}_0 \exp^{i\mathbf{K}\cdot\mathbf{r}}$$

By identification we have :

$$\mathbf{k}\left| \begin{array}{l} \frac{-40\pi}{\sqrt{3}} \\ 0 \\ \frac{-40\pi}{3} \end{array} \right.$$

The frequency of the signal is determined thanks to :

$$\|\mathbf{k}\| = \frac{\omega}{c}$$

It yields

$$f = 4GHz$$

The field can be decomposed in two separated fields :

$$\mathcal{E}(\mathbf{r}) = \mathcal{E}_1(\mathbf{r}) + \mathcal{E}_2(\mathbf{r})$$

with

$$\begin{array}{rcl} \mathcal{E}_1(\mathbf{r}) & = & [E_1(0.71 - j0.71)\vec{a}_y] \exp^{\left(-j\left[(\frac{40\pi}{\sqrt{3}})x + (\frac{40\pi}{3})z\right]\right)} \\ \mathcal{E}_1(\mathbf{r}) & = & [jE_2(0.5\vec{a}_x - 0.866\vec{a}_z)] \exp^{\left(-j\left[(\frac{40\pi}{\sqrt{3}})x + (\frac{40\pi}{3})z\right]\right)} \end{array}$$

We can remark that for both fields the direction of the electric field is prependicular to (k), both polarisation states are linear.

Radiation

We start from the expresssion :

$$\begin{aligned}
\mathcal{H}(\mathbf{r}) &= \frac{jk}{4\pi}\int_\Omega \left(1+\frac{j}{kR}\right) G(R)\mathbf{u}(\mathbf{r},\mathbf{r}')\wedge \mathbf{j}(\mathbf{r}')d\mathbf{r}' \\
\mathcal{E}(\mathbf{r}) &= \frac{jk\eta}{4\pi}\int_\Omega \left(\frac{3}{k^2R^2}-\frac{3j}{kR}-1\right) G(R)\,\mathbf{j}_u(\mathbf{r},\mathbf{r}')d\mathbf{r}' \\
&\quad -\frac{jk\eta}{4\pi}\int_\Omega \left(-1-\frac{j}{kR}+\frac{1}{k^2R^2}\right) G(R)\mathbf{j}(\mathbf{r}')d\mathbf{r}'
\end{aligned}$$

If we consider that $R >> \lambda$ we deduce that $\frac{2\pi R}{\lambda} >> 1$, $kR >> 1$, so :

$$\begin{aligned}
\frac{1}{kR} &<< 1 \\
\frac{1}{k^2R^2} &<< 1
\end{aligned}$$

If

$$\mathbf{u}(\mathbf{r},\mathbf{r}') \approx \mathbf{u}_0(\mathbf{r})$$

In the far zone we have $r' << r$ and as usual we do a taylor expansion :

$$\begin{aligned}
\|\mathbf{r}-\mathbf{r}'\| &= \sqrt{r + r'^2 - 2\mathbf{r}\cdot\mathbf{r}'} \\
&= r\sqrt{1-\frac{2\mathbf{r}\cdot\mathbf{r}'}{\|\mathbf{r}\|}} \\
&\approx 1-\mathbf{u}_0(\mathbf{r})\cdot\mathbf{r}'
\end{aligned}$$

We derive :

$$\begin{aligned}
\mathcal{H}(\mathbf{r}) &= \frac{jk}{4\pi}\frac{\exp^{jkr}}{r}\mathbf{u}_0(\mathbf{r})\wedge\int_\Omega \exp^{-jk\mathbf{u}_0(\mathbf{r})\cdot\mathbf{r}'}\,\mathbf{j}(\mathbf{r}')d\mathbf{r}' \\
\mathcal{E}(\mathbf{r}) &= \frac{-jk\eta}{4\pi}\frac{\exp^{jkr}}{r}\int_\Omega \exp^{-jk\mathbf{u}_0(\mathbf{r})\cdot\mathbf{r}'}\,\mathbf{j}(\mathbf{r}')d\mathbf{r}' + \frac{-jk\eta}{4\pi}\frac{\exp^{jkr}}{r}\int_\Omega [\mathbf{j}(\mathbf{r}')\cdot\mathbf{u}_0(\mathbf{r})]\exp^{-jk\mathbf{u}_0(\mathbf{r})\cdot\mathbf{r}'}\,\mathbf{u}_0(\mathbf{r})d\mathbf{r}'
\end{aligned}$$

By virtue of :

$$\mathbf{a}\wedge(\mathbf{b}\wedge\mathbf{c}) = (\mathbf{a}\cdot\mathbf{c})\mathbf{b} - (\mathbf{a}\cdot\mathbf{b})\mathbf{c}$$

we deduce that :

$$\mathcal{E}(\mathbf{r}) = -\eta\mathbf{u}_0(\mathbf{r})\wedge\mathcal{H}(\mathbf{r})$$

So we locally observe a plane wave strucutre.

Hertzian dipole

In that case the integration is very simple since the dipole is located at the origin and is ponctual. So we directly derive :

$$\mathcal{H}(\mathbf{r}) = \frac{jk}{4\pi}\frac{\exp^{jkr}}{r}\mathbf{u}_0(\mathbf{r})\wedge Il\mathbf{e}_z$$

The averaged power dissipated is :

$$\begin{aligned}
\mathcal{P} &= \frac{1}{2}\mathcal{E} \wedge \mathcal{H}^* \\
&= -\frac{\eta}{2}(\mathbf{u}_0(\mathbf{r}) \wedge \mathcal{H}(\mathbf{r}) \wedge \mathcal{H}^*(\mathbf{r})) \\
&= \frac{\eta}{2}\|\mathcal{H}^2(\mathbf{r})\|\mathbf{u}_0(\mathbf{r}) \\
&= \frac{\eta}{2}\frac{k^2}{16\pi^2}\frac{I^2 l^2 \sin^2(\theta)}{r^2}\mathbf{u}_0(\mathbf{r}) \\
&= \frac{1}{8}\sqrt{\frac{\mu_0}{\varepsilon_0}}\left[\frac{Il}{\lambda r}\sin(\theta)\right]^2 \mathbf{u}_0(\mathbf{r})
\end{aligned}$$

The power is dissipated along $\mathbf{u}_0(\mathbf{r})$ (radially) and varies as $\sin^2(\theta)$ (null along the axis of the dipole and maximum in the azimutal plane).

Dipole of finite length

We have to evaluate :

$$\mathcal{H}(\mathbf{r}) = \frac{jk}{4\pi}\frac{\exp^{jkr}}{r}\mathbf{u}_0(\mathbf{r}) \wedge \int_\Omega \exp^{-jk\mathbf{u}_0(\mathbf{r})\cdot\mathbf{r}'}\mathbf{j}(\mathbf{r}')d\mathbf{r}'$$

with

$$\mathbf{j}(\mathbf{r}') = I_0\cos(kz)\mathbf{e}_z$$

$$\begin{aligned}
\int_\Omega \exp^{-jk\mathbf{u}_0(\mathbf{r})\cdot\mathbf{r}'}\mathbf{j}(\mathbf{r}')d\mathbf{r}' &= I_0\int_{-\frac{\lambda}{4}}^{\frac{\lambda}{4}} \exp^{-jkz\cos(\theta)}\cos(kz)\,dz\mathbf{e}_z \\
&= I_0\mathbf{e}_z\int_{-\frac{\lambda}{4}}^{\frac{\lambda}{4}} \exp^{-jkz\cos(\theta)}\left[\frac{\exp^{jkz}+\exp^{-jkz}}{2}\right]dz
\end{aligned}$$

The first integral gives

$$\begin{aligned}
\int_{-\frac{\lambda}{4}}^{\frac{\lambda}{4}} \frac{\exp^{jkz(1-\cos(\theta))}}{2}dz &= \left[\frac{\exp^{jkz(1-\cos(\theta))}}{2jk(1-\cos(\theta))}\right]_{-\frac{\lambda}{4}}^{\frac{\lambda}{4}} \\
&= \frac{\cos(\frac{\pi}{2}\cos(\theta))}{k(1-\cos(\theta))}
\end{aligned}$$

while the second gives :

$$\begin{aligned}
\int_{-\frac{\lambda}{4}}^{\frac{\lambda}{4}} \frac{\exp^{-jkz(1+\cos(\theta))}}{2}dz &= -\left[\frac{\exp^{-jkz(1+\cos(\theta))}}{2jk(1+\cos(\theta))}\right]_{-\frac{\lambda}{4}}^{\frac{\lambda}{4}} \\
&= \frac{\cos(\frac{\pi}{2}\cos(\theta))}{k(1+\cos(\theta))}
\end{aligned}$$

After summation of both terms we have :

$$\mathcal{H}(\mathbf{r}) = \frac{jI_0}{2\pi}\frac{\cos(\frac{\pi}{2}\cos(\theta))}{\sin^2(\theta)}\mathbf{u}_0(\mathbf{r}) \wedge \mathbf{e}_z$$

The power dissipated is :

$$\mathcal{P} = \frac{\eta}{2}\frac{I_0^2}{4\pi^2}\left[\frac{\cos(\frac{\pi}{2}\cos(\theta))}{\sin^2(\theta)}\right]^2\mathbf{u}_0(\mathbf{r})$$

Magnetic dipole

We have to reconsider the approximation of the integral :

$$\int_\Omega \exp^{-jk\mathbf{u}_0(\mathbf{r})\cdot\mathbf{r}'}\mathbf{j}(\mathbf{r}')d\mathbf{r}'$$

If we do a power expansion of the phase :

$$\exp^{-jk\mathbf{u}_0(\mathbf{r})\cdot\mathbf{r}'} \approx 1 - jk\mathbf{u}_0(\mathbf{r})\cdot\mathbf{r}'$$

and we have for the wire loop :

$$\int_\Omega (1 - jk\mathbf{u}_0(\mathbf{r})\cdot\mathbf{r}')I_0\mathbf{dl}'$$

This calculation is the same as the one performed for the static case :

$$-jk\int_\Omega (\mathbf{u}_0(\mathbf{r})\cdot\mathbf{r}')I_0\mathbf{dl}' = -jkI_0\frac{\pi a^2}{2}(\mathbf{u}_0 \wedge \mathbf{e}_z)$$

we finally have :

$$\mathcal{H}(\mathbf{r}) = \frac{k^2}{4\pi}(\mathbf{u}_0(\mathbf{r}) \wedge \mathbf{m}) \wedge \mathbf{u}_0(\mathbf{r})\frac{\exp^{jkr}}{r}$$

with

$$\mathbf{m} = \frac{\pi a^2}{2}I_0\mathbf{e}_z$$

We can remark the symetric relations of $\mathcal{H}$ and $\mathcal{E}$ with respect to the fields found the electric dipole.

Chapter 7

Tutorial 8 : Reflection-Refraction

Reflection-Refraction at normal incidence

we have

$$\mathbf{k}_i \left| \begin{array}{l} 0 \\ 0 \\ k \end{array} \right.$$

and $k = \frac{\omega}{c}$, here $k = 20\pi$ so we deduce :

$$\mathcal{E}_i(\mathbf{r}) = \mathcal{E}_0 \exp^{j20\pi z} \mathbf{e}_x$$

By applyation the Fresnel coefficients, we have :

$$\begin{aligned} r &= \frac{1-5}{1+5} = -\frac{4}{6} \\ t = &= \frac{2 \times 1}{1+5} = \frac{2}{6} \end{aligned}$$

So we finally deduce the expression for the reflected and transmitted field :

$$\begin{aligned} \mathcal{E}_r(\mathbf{r}) &= -\frac{4}{6}\mathcal{E}_0 \exp^{-j20\pi z} \mathbf{e}_x \\ \mathcal{E}_t(\mathbf{r}) &= \frac{2}{6}\mathcal{E}_0 \exp^{j100\pi z} \mathbf{e}_x \end{aligned}$$

Reflection-Refraction : TM case

Due to the general expression of $\mathcal{H}$ and $\mathcal{E}$ we can deduce from the phasir term the value of the wave vector; here we have :

$$\mathbf{k}_i \left| \begin{array}{l} \frac{\sqrt{2}\pi}{4} \\ 0 \\ \frac{\sqrt{6}\pi}{4} \end{array} \right.$$

we have

$$\|\mathbf{k}_i\| = \frac{\omega}{c}$$

here, $\|\mathbf{k}_i\| = \sqrt{2}\pi$
we deduce that :

$$\nu = \frac{310^8}{\sqrt{2}} \approx 212\,\mathrm{MHz}$$

The expression of the electric field can be found thanks to Maxwell-Ampere equation :

$$\mathrm{rot}(\mathcal{H}_i) = -j\omega\varepsilon_0\mathcal{E}_i$$

we find :

$$\mathcal{E}_i \quad - \frac{H_0}{j\omega\varepsilon_0}\left| \begin{array}{l} \frac{-3j\pi}{4}\exp^{j\sqrt{2}\pi\left(\frac{\sqrt{3}}{2}z+\frac{1}{2}x\right)} \\ 0 \\ \frac{\sqrt{3}j\pi}{4}\exp^{j\sqrt{2}\pi\left(\frac{\sqrt{3}}{2}z+\frac{1}{2}x\right)} \end{array} \right.$$

For the reflection and transmission coefficients, we must first determine the value of the transmitted angle in the second medium, thanks to Descartes relations :

$$\sqrt{\varepsilon_{r1}}\sin(\theta_1) = \sqrt{\varepsilon_{r2}}\sin(\theta_2)$$

we deduce :

$$\theta_2 = \arcsin(\frac{\sqrt{\varepsilon_{r1}}\sin(\theta_1)}{\sqrt{\varepsilon_{r2}}})$$

$$\theta_2 = 19.47°$$

We deduce

$$\begin{aligned} r &= \frac{\cos(60) - \frac{3}{9}\cos(19.47)}{\cos(60) + \frac{3}{9}\cos(19.47)} = 0.228 \\ t &= \frac{2\cos(60)}{\cos(60) + \frac{3}{9}\cos(19.47)} = 1.22 \end{aligned}$$

Value of the Brewster angle :

$$\theta_b = \arctan(3) = 71.56°$$

Mixed polarisation case

The incident field can be written as :

$$\mathcal{E}_i(\mathbf{r}) = \vec{E}_i \exp j\mathbf{k}_i \cdot \mathbf{r}$$

here it gives :

$$\mathcal{E}_i(\mathbf{r}) = \vec{E}_i \exp jk(\frac{\sqrt{3}}{2}x + \frac{z}{2}$$

with $k = \frac{\omega}{c} = 40\pi$ The incident field can be seen as the sum of two planes waves with different polarisations :

- A TE wave with an amplitude $E_1 0.707\mathbf{e}_y$
- A TM wave with an amplitude $E_2(0.5\vec{a}_x - 0.866\vec{a}_z)$

We have to calculate the transmission coefficients for both plane waves and sum up hereafter. The refraction angle is found thanks to Descartes relations :

$$\theta_2 = \arcsin(\frac{\sin(\theta_i)}{5}) \approx 10°$$

we deduce

$$\begin{aligned} r_{TE} &= -0.81 \\ t_{TE} &= 0.19 \\ r_{TM} &= 0.434 \\ t_{TM} &= 1.434 \end{aligned}$$