# Discrete Boundary Value Problems

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# Neumann Boundary Value Problem

**Problem**

Find a function $u(x)$ satisfying

$$\begin{aligned} -u''(x) + \lambda u(x) = f(x), \quad x \in (0,1), \\ u'(0) = 0, \quad u'(1) = 0, \end{aligned} \tag{BVP}$$

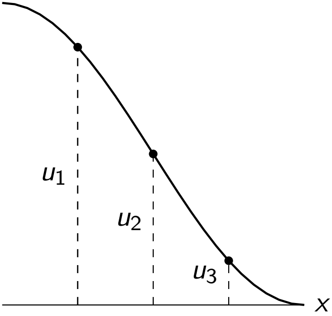
where $f \in C^0([0,1])$ and $\lambda \in \mathbb{R}$.

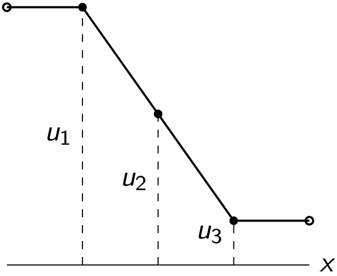
**Goal**

New understanding of Fredholm theory (existence and uniqueness of solutions) by constructing discrete approximations. We then take the limit as our approximations get better to learn about the original problem.

# Approximating Functions

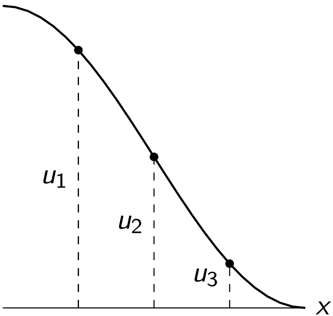
Approximate the function $u(x)$ by vector $\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$



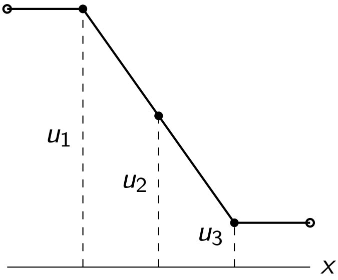
# Approximating Functions by Vectors

For each $n \in \mathbb{N}$ we set $\Delta x = \frac{1}{n+1}$ and construct the linear transformation $\varphi \colon C^0([0,1]) \to \mathbb{R}^n$, by defining $\mathbf{u} = \varphi(u) = (u^1, u^2, \ldots, u^n)^t \in \mathbb{R}^n$ with $u^k = u(k\Delta x)$.



# Constructing Functions from Vectors

We define the linear transformation $\psi\colon \mathbb{R}^n \to C^0([0,1])$ by linearly interpolating between the points $(k\Delta x, u^k)$, where we take $u^0 = u^1$ and $u^{n+1} = u^n$ in order to enforce the boundary conditions.



# Derivative and Antiderivative Matrices

The derivative matrix

$$D = \frac{1}{\Delta x}\begin{bmatrix} -1 & 1 & 0 & & \\ 0 & -1 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 1 & 0 \\ & & 0 & -1 & 1 \end{bmatrix}. \tag{1}$$

The antiderivative matrix

$$A = \Delta x \begin{bmatrix} 1 & 0 & 0 & & \\ 1 & 1 & 0 & & \\ \vdots & \ddots & \ddots & \ddots & \\ & & 1 & 0 & 0 \\ 1 & \cdots & 1 & 1 & 0 \end{bmatrix}. \tag{2}$$

# Approximating the Boundary Value Problem

- Approximate the boundary value problem by a linear algebra problem.
- Example: three point approximation

$$\begin{array}{ccccccc} -u''(x) & & + & \lambda u(x) & = & f(x) \\ \downarrow & & & \downarrow & & \downarrow \\ \underbrace{\frac{1}{\Delta x^2}\begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}}_{D_3^* D_3} & \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} & + & \lambda \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} & = & \begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix} \end{array}$$

# Generalized Approximate Problem

More generally, for each $n$ we obtain

$$D^*D\mathbf{u} + \lambda\mathbf{u} = \mathbf{f}, \qquad (\heartsuit)$$

where $D^*D$ is the $n \times n$ matrix

$$D^*D = \frac{1}{\Delta x^2}\begin{bmatrix} 1 & -1 & 0 & & \\ -1 & 2 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 2 & -1 \\ & & 0 & -1 & 1 \end{bmatrix}$$

and $\mathbf{u}, \mathbf{f} \in \mathbb{R}^n$.

Write as $\mathcal{L}_n\mathbf{u} = \mathbf{f}$.

# Why Approximation is Helpful

There is Fredholm theory in linear algebra.

Write approximate problem as $\mathcal{L}_n \mathbf{u} = \mathbf{f}$.

Solvability depends on the null space of $\mathcal{L}_n$.

- Null space $= \mathbf{0} \implies$ There exists a unique solution.
- Null space $\neq \mathbf{0} \implies$ More complicated.

Null space of $\mathcal{L}_n$ depends on eigenspaces of $D_n^* D_n$.

# Main Theorem

### Theorem

*Let* $\lambda \in \mathbb{R} \setminus \{-(\pi k)^2 \mid k \in \mathbb{N}_0\}$ *and* $f \in C^0([0,1])$*. For large* $n$ *there exists a unique* $\mathbf{u}_n$ *satisfying the approximate problem. The corresponding sequence* $u_n = \psi(\mathbf{u}_n)$ *has a subsequence converging in* $C^0([0,1])$ *to a unique function* $u \in C^2([0,1])$ *satisfying* (BVP)*.*

# Proof of Main Theorem

- For large $n$ the only solution to $(D^*D + \lambda I)\mathbf{v} = 0$ is the zero vector.
- Thus we define a sequence of functions $u_n = \psi\left((D^*D + \lambda I)^{-1}\mathbf{f}_n\right) \in C^0([0,1])$.
- We use the Arzela-Ascoli theorem to obtain a subsequence that converges to $u \in C^0([0,1])$.
- Another application of the Arzela-Ascoli theorem shows $u \in C^1([0,1])$.
- Using Riemann sums we show that

$$-u'(x) + \lambda \int_0^x u(y)\,dy = \int_0^x f(y)\,dy.$$

- Thus $u \in C^2([0,1])$ and satisfies (BVP).

# Thank You

- Dr. Paul T Allen
- Roger's Program
- Lewis and Clark College

**Citation**


- Paul T Allen. Boundary Value Problems and Finite Differences. College Math Journal 47 (2016), no. 1, 34 – 41.

# Arzela-Ascoli Theorem

Suppose $\{f_k\}_{k=1}^{\infty} \subset C^0([0,1])$ is uniformly bounded and equicontinuous. Then $\{f_k\}_{k=1}^{\infty}$ contains a subsequence that converges to some function $f \in C^0([0,1])$.

# Fredholm Theorem

## Proposition (Fredholm Alternative for Linear Algebra)

*For a linear transformation* $\mathcal{L} : \mathbb{R}^n \mapsto \mathbb{R}^n$, *the solvability of* $\mathcal{L}\mathbf{u} = \mathbf{f}$ *is as follows.*

1. *If* $\ker(\mathcal{L}) = 0$, *then* $\mathcal{L}\mathbf{u} = \mathbf{f}$ *has a unique solution for all* $\mathbf{f} \in \mathbb{R}^n$.
2. $\ker(\mathcal{L}) \neq 0$, *then either*
   (a) $\mathbf{f} \in \ker(\mathcal{L}^*)^{\perp}$, *in the which case* $\mathcal{L}\mathbf{u} = \mathbf{f}$ *has multiple solutions, or*
   (b) $\mathbf{f} \notin \ker(\mathcal{L}^*)^{\perp}$, *in which case* $\mathcal{L}\mathbf{u} = \mathbf{f}$ *has no solution.*

# Fredholm Theorem

**Proposition (Fredholm Alternative for Linear Algebra)**

*If matrix $M$ is symmetric, the solvability of*

$$M\mathbf{u} = \mathbf{f} \tag{$\bigstar$}$$

*is determined by the solvability of*

$$M\mathbf{v} = 0. \tag{H}$$

1. *If $\mathbf{v} = 0$ is the only solution to (H) then there exists a unique solution to ($\bigstar$).*
2. *If (H) has multiple solutions then*
   (a) *if $\mathbf{f} \cdot \mathbf{v} = 0$ for all $\mathbf{v}$ solving (H) then ($\bigstar$) has infinite solutions,*
   (b) *otherwise ($\bigstar$) has no solutions.*