Discrete Boundary Value Problems

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Consider the boundary value problem

$$-u''(x) + \lambda u(x) = f(x), \quad x \in (0, 1) \tag{1}$$ `General ODE`

where $\lambda < 0$ and $f(x)$ is a given function.

2.1 Dirichlet Boundary Value Problem

First we let $\lambda = -\omega^2$ and $f(x) = sin(\pi x)$ yielding

$$-u''(x) - \omega^2 u(x) = sin(\pi x), \quad x \in (0,1) \tag{2}$$

Dirichlet ODE

with Dirichlet boundary conditions

$$u(0) = 0, \quad u(1) = 0 \tag{3}$$

DBC

2.1.1 Homogeneous Solution

we guess solutions of the form $u_h(x) = e^{\delta x}$. Substituting into the homogeneous form of 2 we find that $\delta = \pm i\omega$. Thus, homogeneous solutions take the form

$$u_h(x) = Ae^{i\omega x} + Be^{-i\omega x} \tag{4}$$

Homo Soln

2.1.2 Particular Solution

We guess particular solutions of the form $u_p(x) = Csin(\pi x)$. Substituting into 2, we find that $C = \frac{1}{\pi^2 - \omega^2}$. Thus particular solutions take the form

$$u_p(x) = \frac{1}{\pi^2 - \omega^2} sin(\pi x)$$

Notice an issue when $\omega = \pi$. This will be addressed in the special case section.

2.1.3 Applying Boundary Conditions

By linearity, our general solution takes the form

$$u(x) = Ae^{i\omega x} + Be^{-i\omega x} + \frac{1}{\pi^2 - \omega^2} sin(\pi x).$$

when $\omega \neq \pi$. Applying boundary conditions 3, we conclude that when $\omega = k\pi$ and $k \geq 1$ for $k \in \mathbb{Z}$, there are infinitely many solutions of the form

$$u(x) = Ae^{ik\pi x} + Ae^{-ik\pi x} + \frac{1}{\pi^2(1 - k^2)} sin(\pi x).$$

When $\omega \neq k\pi$, there exists unique solutions that take the form

$$u(x) = \frac{1}{\pi^2 - \omega^2} sin(\pi x).$$

2.1.4 Special Case

We now consider the case where $\omega = \pi$. Since our previous particular solution fails, we look for a new particular solution of the form

$$u_h(x) = x * Csin(\pi x).$$

Substituting into 2 yields $C = \frac{1}{2\pi}$. However, boundary conditions 3 fail in this case. Thus, when $\omega = \pi$ there are no solutions.

2.2 Neumann Boundary Value Problem

First we let $\lambda = -\omega^2$ and $f(x) = cos(\pi x)$ yielding

$$-u''(x) - \omega^2 u(x) = cos(\pi x), \quad x \in (0,1) \tag{5}$$

Neumann ODE

with Neumann boundary conditions

$$u'(0) = 0, \quad u'(1) = 0. \tag{6}$$

NBC

Homogeneous solutions are the same as in the Dirichlet boundary value problem, thus homogeneous solutions take the same form as 4.

2.2.1 Particular Solution

We guess particular solutions of the form $u_p(x) = Ccos(\pi x)$. Substituting into 5, we find that $C = \frac{1}{\pi^2 - \omega^2}$. Thus particular solutions take the form

$$u_p(x) = \frac{1}{\pi^2 - \omega^2} cos(\pi x)$$

Notice an issue when $\omega = \pi$. This will be addressed in the special case section.

2.2.2 Applying Boundary Conditions

By linearity, our general solution takes the form

$$u(x) = Ae^{i\omega x} + Be^{-i\omega x} + \frac{1}{\pi^2 - \omega^2} cos(\pi x).$$

when $\omega \neq \pi$. Applying boundary conditions 6, we conclude that when $\omega = k\pi$ and $k \geq 1$ for $k \in \mathbb{Z}$, there are infinitely many solutions of the form

$$u(x) = Ae^{ik\pi x} + Ae^{-ik\pi x} + \frac{1}{\pi^2(1 - k^2)} cos(\pi x).$$

When $\omega \neq k\pi$, there exists unique solutions that take the form

$$u(x) = \frac{1}{\pi^2 - \omega^2} cos(\pi x).$$

2.2.3 Special Case

We now consider the case where $\omega = \pi$. Since our previous particular solution fails, we look for a new particular solution of the form

$$u_h(x) = x * Ccos(\pi x).$$

Substituting into 5 yields $C = \frac{1}{2\pi}$. However, boundary conditions 6 fail in this case. Thus, when $\omega = \pi$ there are no solutions.

2.3 Fredholm Theory

The Fredholm Alternative Theorem states that for speicified values of λ, there will usually be a unique solution, with a few values of λ where there are issues with existence or uniqueness. These results are consistent with the Fredholm Alternative Theorem stated in Paul Allen's paper "Boundary Value Problems and Finite Differences."

3 Approximating a Boundary Value Problem

3.1 Functions to Vectors

We construct discrete approximations of functions using vectors. Let $u(x)$ be a function $u : [0,1] \to \mathbb{R}$ such that it satisfies either Neumann or Dirichlet boundary conditions. First we divide our domain $[0,1]$ into subintervals of size $\Delta x = \frac{1}{n+1}$ and set $x_k = k\Delta x$. The our function $u(x)$ is approximated by the vector $\mathbf{u} = (u_1, u_2, ..., u_n)$ where $u_k = u(x_k)$.

u_0 and u_{n+1} are the boundary points $u(0)$ and $u(1)$ respectively. They are not included in the vector for both the Dirichlet and Neumann case. In the Dirichlet case this is because $u_0 = 0$ and $u_{n+1} = 0$. In the Neumann case, we let $u_0 = u_1$ and $u_{n+1} = u_n$ which corresponds to satisfying Neumann boundary conditions.

The function $\varphi_n : C^0([0,1]) \to \mathbb{R}$ is the function that performs this transformation. It can be shown that φ_n is a continuous, linear transformation.

3.2 Vectors to Functions

3.3 Approximate Problem

$$M_n = \frac{1}{\Delta x^2}\begin{bmatrix} 2 & -1 & 0 & & \\ -1 & 2 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 2 & -1 \\ & & 0 & -1 & 2 \\ . & & & & \end{bmatrix} \tag{7}$$

Dirichlet 2nd Deriv Matrix

$$M_n = \frac{1}{\Delta x^2}\begin{bmatrix} 1 & -1 & 0 & & \\ -1 & 2 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 2 & -1 \\ & & 0 & -1 & 1 \\ . & & & & \end{bmatrix} \tag{8}$$

Neumann 2nd Deriv Matrix

3.4 Boundedness of $\mathbf{u_n}$ proof

Proof. Write our approximate problem as

$$\mathbf{u}_n = \mathcal{L}_n^{-1} F_n.$$

Thus,

$$\|\mathbf{u}_n\| \leq \|\mathcal{L}_n^{-1}\| \|\mathbf{f}_n\|.$$

We assume that $\|\mathbf{f}_n\|$ is bounded, thus it suffices to show that $\|\mathcal{L}_n^{-1}\|$ is bounded.

Because the matrix is symmetric, we know that $\mathcal{L}_n^{-1}$ has an orthonormal eigenbasis, call it $\mathbf{e}_1, ..., \mathbf{e}_n$, with corresponding eigenvalues $\sigma_1, ..., \sigma_n$ such that without loss of generality $\sigma_1 \leq \ ... \leq \sigma_n$. Now let $v \in \mathbb{R}^n$ be such that $\|\mathbf{v}\| = 1$‘. We write

$$\mathbf{v} = \alpha_1 \mathbf{e}_1 + ... + \alpha_n \mathbf{e}_n.$$

Then

$$\begin{aligned}\mathcal{L}_n^{-1}(\mathbf{v}) &= \mathcal{L}_n^{-1}(\alpha_1 \mathbf{e}_1) + ... + \mathcal{L}_n^{-1}(\alpha_n \mathbf{e}_n)\\ &= \alpha_1 \mathcal{L}_n^{-1}(\mathbf{e}_1) + ... + \alpha_n \mathcal{L}_n^{-1}(\mathbf{e}_n)\\ &= \alpha_1 \sigma_1 \mathbf{e}_1 + ... + \alpha_n \sigma_n \mathbf{e}_n.\end{aligned}$$

Thus,

$$\begin{aligned}\|\mathcal{L}_n^{-1}(\mathbf{v})\| &\leq \|\alpha_1 \sigma_1 \mathbf{e}_1\| + ... + \|\alpha_n \sigma_n \mathbf{e}_n\|\\ &= |\alpha_1 \sigma_1| + ... + |\alpha_n \sigma_n|\end{aligned}$$

Since $\Sigma|\alpha_i|^2 = 1$, the largest $|\mathcal{L}_n^{-1}((v))|$ can be is when $\alpha_n = 1$ and for all $i \neq n$, $\alpha_i = 0$. Thus, $|\mathcal{L}_n^{-1}((v))| \leq |\sigma_n|$ which is the largest eigenvalue of $\mathcal{L}_n^{-1}$. This is equivalent to the inverse of the smallest eigenvalue of $\mathcal{L}_n$. Thus $\mathbf{u}_n$ is bounded by the smallest eigenvalue of $\mathcal{L}_n$.

□

3.5 Lemmas

Reconstruction Lemma

Lemma 3.1 (Reconstruction Lemma)**.** *Let $u \in C^0([0,1]$. Let vector $\mathbf{u}_n = (u_n^1, \ldots, u_n^n)$ be defined by $u_n^k = u(x^k)$. Then $\psi_n(\mathbf{u}_n) \to u$ in $C^0([0,1])$.*

Proof. Let $u_n = \psi_n(\mathbf{u}_n)$. Let $\epsilon > 0$. Since $[0,1]$ is compact, u is uniformly continuous. Thus there exists a $\delta > 0$ such that $|x-y| < \delta$ implies $|u(x)-u(y)| < \frac{\epsilon}{2}$. Fix n such that $\frac{1}{n+1} < \delta$. It suffices to show that for all $x \in [0,1]$ we have $|u_n(x) - u(x)| < \epsilon$. Let $x \in [0,1]$. Choose k such that $\frac{k}{n+1} \leq x \leq \frac{k+1}{n+1}$. Using the formula that defines ψ_n we see that

$$\begin{aligned}|u_n(x) - u(x)| &= \left| u_n^k + \frac{u_n^{k+1} - u_n^k}{\frac{1}{n+1}} \left(x - \frac{k}{n+1} \right) - u(x) \right|\\ &\leq \left| u_n^k - u(x) \right| + \left| (u_n^{k+1} - u_n^k) \left(\frac{x - \frac{k}{n+1}}{\frac{1}{n+1}} \right) \right|\end{aligned}$$

Since

$$\left| \frac{x - \frac{k}{n+1}}{\frac{1}{n+1}} \right| < 1$$

we have

$$|u_n(x) - u(x)| \le \left|u_n^k - u(x)\right| + \left|u_n^{k+1} - u_n^k\right| < \epsilon. \qquad \square$$

u_Convergence_Lemma

Lemma 3.2. *Suppose we have a sequence of vectors $\mathbf{u}_n$ and suppose there exists a constant c such that $\|\mathbf{u}_n\| \le c$ and $\|D_n \mathbf{u}_n\| \le c$ then we see that $u_n = \psi_n(\mathbf{u}_n)$ has a convergent subsequence that converges to a continuous function in $C^0([0,1])$.*

Proof. Let $u_n = \psi_n(\mathbf{u_n})$. By the Arzela-Ascoli Theorem it suffices to show that the sequence u_n is bounded and equicontinuous. To show u_n is bounded, we consider

$$\begin{aligned}
|u_n(x)| &= \left| u_n^k + \frac{u_n^{k+1} - u_n^k}{\frac{1}{n+1}} \left(x - \frac{k}{n+1} \right) \right| \\
&= \left| u_n^{k+1} \left(\frac{x - \frac{k}{n+1}}{\frac{1}{n+1}} \right) + u_n^k \left(1 - \left(\frac{x - \frac{k}{n+1}}{\frac{1}{n+1}} \right) \right) \right| \\
&\le \left| c \left(\frac{x - \frac{k}{n+1}}{\frac{1}{n+1}} \right) + c \left(1 - \left(\frac{x - \frac{k}{n+1}}{\frac{1}{n+1}} \right) \right) \right| \\
&\le c.
\end{aligned}$$

We now consider the equicontinuity of u_n. Let $\epsilon > 0$. Take $\delta = \epsilon/c$. Let $x, y \in (0,1)$ such that $|x - y| < \delta$. We compute

$$\begin{aligned}
|u_n(x) - u_n(y)| &\le \max_k \left| \frac{u_n^{k+1} - u_n^k}{\frac{1}{n+1}} \right| |x - y| \\
&\le \|D_n \mathbf{u_n}\| \, |x - y| \\
&\le c|x - y| < \epsilon. \qquad \square
\end{aligned}$$

ntegral_convergence_lemma

Lemma 3.3. *Suppose we have a sequence of vectors $\mathbf{u}_n$ and suppose there exists a constant c such that $\|\mathbf{u}_n\| \le c$ and $\|D_n \mathbf{u}_n\| \le c$. Then*

$$u_n(x) - u_n(0) - \int_0^x \psi_n'(\mathbf{u}_n)(y)\, dy \to 0 \tag{9}$$

in $C^0([0,1])$.

• is the boundedness of u_n needed?

Proof. Let $\mathbf{u}_n$ be a sequence of vectors such that $\|D_n \mathbf{u}_n\| \le c$. Let $\mathbf{v}_n = D_n \mathbf{u}_n$ and $v_n = \psi_n'(\mathbf{u}_n)$. Recall that the components of $\mathbf{v}_n$ are given by

$$v_n^k = v_n \left(\frac{k + 1/2}{n+1} \right) \qquad k = 1, \ldots, n-1$$

and $v_n^0 = 0 = v_n^n$. Let $w_n = \overline{\psi'}(\mathbf{u}_n)$. For each $x \in [0,1]$ we have

$$u_n(x) = u_n(0) + \int_0^x w_n(y)\, dy.$$

Therefore it suffices to show

$$\int_0^x (w_n(y) - v_n(y))\, dy \to 0.$$

There exists an integer K such that $\frac{K-1/2}{n+1} \leq x < \frac{K+1/2}{n+1}$. Let $x_* = \frac{K-1/2}{n+1}$ and, for each integer k, let I_k be the interval given by

$$I_k = \left[\frac{k-1/2}{n+1}, \frac{k+1/2}{n+1}\right).$$

Since both $v_n(x) = 0$ and $w_n(x) = 0$ when $x < \frac{1}{2(n+1)}$ we have

$$\int_0^x (w_n(y) - v_n(y))\, dy = \sum_{k=1}^{K-1} \int_{I_k} (w_n(y) - v_n(y))\, dy + \int_{x_*}^x (w_n(y) - v_n(y))\, dy. \tag{10}$$

integral break

Recall the definition of ψ', which implies that on the interval I_k we have

$$v_n(x) = v_n^{k-1} + (v_n^k - v_n^{k-1})(n+1)(x - \frac{k-1/2}{n+1}).$$

Similarly, recall that the definition of $\overline{\psi'}$ gives

$$w_n(x) = \begin{cases} v_n^{k-1} & \text{if} \quad \dfrac{k-1/2}{n+1} \leq x < \dfrac{k}{n+1} \\ \\ v_n^k & \text{if} \quad \dfrac{k}{n+1} \leq x < \dfrac{k+1/2}{n+1}. \end{cases}$$

We compute

$$\int_{I_k} w_n(y)\, dy = \frac{1}{2(n+1)}(v_n^{k-1} + v_n^k),$$

and

$$\int_{I_k} v_n(y)\, dy = \frac{1}{2(n+1)}(v_n^{k-1} + v_n^k)$$

Thus

$$\int_{I_k} (v_n(y) - w_n(y))\, dy = 0. \tag{11}$$

area equivalence

Thus using (10) and (11) we have

$$\begin{aligned}
\left|\int_0^x w_n(y)\,dy - v_n(y)\,dy\right| &= \left|\int_{x_*}^x w_n(y)\,dy - v_n(y)\,dy\right| \\
&\le \int_{x_*}^x \big|w_n(y)\big| + \big|v_n(y)\big|\,dy \\
&\le \frac{2c}{n+1}.
\end{aligned}$$

Thus, as $n \to \infty$ we we see that it must be the case that

$$u_n(x) - u_n(0) - \int_0^x \psi_n'(\mathbf{u}_n)(y)\,dy \to 0. \qquad \square$$

v_convergence_lemma

Lemma 3.4. *Suppose we have a sequence of vectors $\mathbf{u}_n$ and suppose there exists a constant c such that $\|\mathbf{u}_n\| \le c$, $\|D_n\mathbf{u}_n\| \le c$ and $\|D_n^* D_n \mathbf{u}_n\| \le c$. Then $\psi_n'(\mathbf{u}_n)$ has a subsequence that converges to a function v in $C^0([0,1])$.*

Proof. Let the function $v_n = \psi_n'(\mathbf{u}_n)$. Let the vector $\mathbf{v_n} = D_n\mathbf{u_n}$. By the Arzela-Ascoli Theorem it suffices to show that the sequence v_n is bounded and equicontinuous. To show v_n is bounded we consider the formula

$$\begin{aligned}
\|v_n(x)\| &= \left| v_n^{k-1} + \frac{v_n^k - v_n^{k-1}}{\frac{1}{n+1}}\left(x - \frac{k-1/2}{n+1}\right)\right| \\
&= \left| v_n^{k-1}\left(1 - \left(x - \frac{k-1/2}{n+1}\right)\right) + v_n^k\left(x - \frac{k-1/2}{n+1}\right)\right| \\
&\le \left| c\left(1 - \left(x - \frac{k-1/2}{n+1}\right)\right) + c\left(x - \frac{k-1/2}{n+1}\right)\right| \\
&\le c.
\end{aligned}$$

We now consider the equicontinuity of v_n. Let $\epsilon > 0$. Take $\delta = \epsilon/c$. Let $x, y \in (0,1)$ such that $|x-y| < \delta$. We compute

$$\begin{aligned}
|v_n(x) - v_n(y)| &\le \max_k \left|\frac{v_n^{k+1} - v_n^k}{\frac{1}{n+1}}\right| |x-y| \\
&\le \|D_n^* D_n \mathbf{u_n}\|\,|x-y| \\
&\le c|x-y| < \epsilon. \qquad \square
\end{aligned}$$

u_C1_convergence_lemma

Lemma 3.5. *Suppose we have a sequence of vectors $\mathbf{u}_n$ and suppose there exists a constant c such that $\|\mathbf{u}_n\| \le c$, $\|D_n\mathbf{u}_n\| \le c$ and $\|D_n^* D_n\mathbf{u}_n\| \le c$ then we see that $\psi_n(\mathbf{u_n})$ has a convergent subsequence that converges to a continuous function in $C^1([0,1])$.*

Proof. Consider

$$\lim_{x,y\to 0}\left|(u(x) - u_n(x)) + \left(u_n(x) - \left(u_n(0) + \int_0^x u_n(y)\,dy\right)\right) + (u_n(0) - u(0)) + \left(\int_0^x (v_n(y) - v(y)\,dy)\right)\right|.$$

By 3.2, 3.3, 3.4, this limit converges to 0. This can be rewritten as

$$\lim_{x,y\to 0}\left|u(x)-u(0)-\int_0^x v(y)\,dy\right|\to 0,$$

thus

$$u(x)-u(0)-\int_0^x v(y)\,dy=0.$$

By the fundamental theorem of calculus we conclude that $u'(x)=v(x)$, thus $u\in C^1([0,1])$. □

4 Function Land

This section is the proof of our main theorem. Throughout this section let $\lambda\in\mathbb{R}\setminus\{-(\pi k)^2\mid k\in\mathbb{N}_0\}$ be fixed. Fix a function f such that f is continuous and bounded. Using those we build and approximate problem. Proposition 1: Solutions to the approximate problem give us a sequence of functions that converges subsequentially to a function $u\in C^1$.

`u satisfies NBC` **Proposition 4.1.** *Let* $\mathbf{u}_n$ *be the solutions to* $M_n\mathbf{u}_n+\lambda\mathbf{u}_n=\mathbf{f}_n$. *The corresponding sequence of functions* $u_n=\psi_n(\mathbf{u}_n)$ *has a subsequence that converges to a function* $u\in C^1([0,1])$ *and satisfies the Neumann boundary condition.*

Proof. Using Corollary **??**, Lemma **??**, and Lemma **??**, we implement Lemma 3.5 to conclude that $u\in C^1([0,1])$. In order to show that u satisfies the Neumann boundary condition at u(0), it suffices to show that $\left|\frac{u(h)-u(0)}{h}\right|\to 0$. Let $\epsilon>0$. Fix $\delta=1/2(n+1)$. Let $h<\delta$ and $x<h$. We know that $v_N(x)$ satisfies Neumann boundary conditions, thus $v_N(x)=0$ for $x<h$. By Lemma 3.4, we know that there exists an N such that $\|u'(x)-v_N(x)\|<\epsilon$. Thus

$$\begin{aligned}\left|\frac{u(h)-u(0)}{h}\right| &\le |u'(x)-v_N(x)\|+\|v_N(x)\|\\ &<\epsilon+\|v_N(x)\|\\ &<\epsilon\end{aligned}$$

□

Now we show that this function u satisfies the integral equation

$$u'(x)=\int_0^x\lambda u(y)-f(y)dy \tag{12}$$ `integral equation`

Proposition 4.2. *Integral equation is satisfied.*

Proof. Needed. □

We explain why integral equation means that $u \in C^2$ and satisfies BVP. The right hand side of Equation 12 is differentiable, thus the left hand side is also differentiable. By 4.1 we know that $u \in C^1([0,1])$. Since u' is differentiable, we conclude that $u \in C^2([0,1])$,

We now need to examine the special case where $\gamma \pm 1$. First consider the case where $\gamma = 1$. Then we have

$$\tilde{d}_n = 2\tilde{d}_{n-1} - \tilde{d}_{n-2}$$

with initial conditions of

$$\begin{aligned}\tilde{d}_1 &= 1\\ \tilde{d}_2 &= 1\end{aligned}$$

First we guess $\tilde{d}_n = Z^n$. Substituting this in yields

$$\begin{aligned}Z^n &= 2Z^{n-1} - Z^{n-2}\\ Z^2 - 2Z + 1 &= 0\\ (Z-1)^2 &= 0\end{aligned}$$

Thus our only solution is that $Z = 1$.
To find our second solution we guess $\tilde{d}_n = nZ^n$. Then we have

$$\begin{aligned}nZ^n &= 2(n-1)Z^{n-1} - (n-2)z^{n-2}\\ nZ - 2(n-1)Z + (n-2) &= 0\\ Z = \frac{2(n-1) \pm \sqrt{4(n-1)^2 - 4n(n-2)}}{2n}\\ Z &= \frac{(n-1) \pm 1}{n}.\end{aligned}$$

Therefore, either $Z = \frac{n-2}{n}$ which is absurd, or $Z = 1$. This means our two solutions are $\tilde{d}_n = 1$ and $\tilde{d}_n = n1^n$. Thus our general solution is

$$\tilde{d}_n = \alpha(1) + \beta n.$$

Applying our initial conditions we find that

$$\begin{aligned}1 &= \alpha + \beta(1)\\ 1 &= \alpha + \beta(2)\end{aligned}$$

which implies that $\beta = 0$ and $\alpha = 1$. Our final solution is $\tilde{d}_n = 1$. Thus it is always true that

$$d_n = 0.$$

We now consider the case where $\gamma = -1$. Then we have

$$d_n = -3\tilde{d}_{n-1} - \tilde{d}_{n-2}$$

and

$$\tilde{d}_n = -2\tilde{d}_{n-1} - \tilde{d}_{n-2}$$

with initial conditions

$$\tilde{d}_0 = 1$$
$$\tilde{d}_1 = -3$$

. First we guess that $\tilde{d}_n = Z^n$. Substituting this in yields

$$Z^n = -2Z^{n-1} - Z^{n-2}$$
$$Z^2 + 2Z + 1 = 0$$

Thus our only solution is that $Z = -1$. To find our second solution we guess that $\tilde{d}_n = nZ^n$. Then we have

$$nZ^n = -2(n-1)Z^{n-1} - (n-2)Z^{n-2}.$$

Dividing by Z^{n-2} yields

$$nZ^2 + 2(n-1)Z + (n-2) = 0.$$

Thus

$$Z = \frac{-(n-1) \pm 1}{n}.$$

Therefore, either $Z = \frac{-(n-1)}{n}$ which is absurd, or $Z = -1$. This means our solutions are $\tilde{d}_n = (-1)^n$ and $\tilde{d}_n = n(-1)^n$ with a general solution of

$$\tilde{d}_n = \alpha(-1)^n + \beta n(-1)^n.$$

Applying our initial conditions we find that

$$1 = \alpha$$
$$-3 = -\alpha - \beta$$

which implies that $\alpha = 1$ and $\beta = 2$. Thus our final solution is $\tilde{d}_n = (-1)^n + 2n(-1)^n$. Thus

$$d_n = 4n(-1)^n.$$

4.1 Eigenvalues

In order to find the eigenvalues of M_n, we set $d_n = 0$. We start with the case where $\gamma \neq \pm 1$. Then

$$d_n = \frac{1}{2\sqrt{\gamma^2 - 1}} \left[\omega_+^{n-1}(\omega_+ - 1)^2 - \omega_-^{n-1}(\omega_- - 1)^2 \right] = 0.$$

We know that the denominator is nonzero, thus we conclude that the numerator must be equal to zero. Thus

$$\omega_+^{n-1}(\omega_+ - 1)^2 - \omega_-^{n-1}(\omega_- - 1)^2 = 0.$$

We know that $\omega_- = \frac{1}{\omega_+}$. Thus

$$\omega_+^{n-1}(\omega_+ - 1)^2 - \frac{1}{\omega_+^{n-1}} \left(\frac{1}{\omega_+} - 1 \right)^2 = 0$$

$$(\omega_+^{2n} - 1)(\omega_+ - 1)^2 = 0$$

which implies $\omega_+ = 1$ and $\omega_+^{2n} = 1 = e^{i2\pi k}$. Thus $\omega_+ = e^{i\pi k/n}$ for $k = 0, 1, 2, ..., 2n - 1$. We also know that $\omega_+ = \gamma + \sqrt{\gamma^2 - 1}$. Thus

$$\begin{aligned} \gamma &= \frac{\omega_+^2 + 1}{2\omega_+} \\ &= \frac{\omega_+ + \frac{1}{\omega_+}}{2} \\ &= \frac{e^{i(\pi k/n)} + e^{-i(\pi k/n)}}{2} \\ &= \cos\left(\frac{\pi k}{n}\right) \end{aligned}$$

for $k = 1, ..., n - 1$. We also know that $\gamma = \frac{2-(\Delta x)^2 \mu}{2}$. Thus

$$\cos\left(\frac{\pi k}{n}\right) = \frac{2 - (\Delta x)^2 \mu}{2}$$

and solving for μ we find

$$\mu = \frac{2 - 2\cos(\frac{\pi k}{n})}{(\Delta x)^2}$$

for $k = 1, ..., n - 1$.

4.1.1 $\gamma = -1$

$d_n = 4n(-1)^n$ will never equal zero when $n > 0$. Thus there are no solutions.