Boundary Value Problems via Finite Difference Approximations

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Abstract

The standard approach to proving existence and uniqueness of solutions to boundary value problems requires graduate level theory from functional analysis. We present an elementary proof of existence and uniqueness of solutions to a simple class of boundary value problems using only tools from linear algebra and introductory analysis courses.

A Neumann Boundary Value Problem

We focus on the boundary value problem

$$\begin{aligned} -u''(x) + \lambda u(x) &= f(x), \quad x \in (0,1), \\ u'(0) = 0, &\quad u'(1) = 0, \end{aligned} \qquad \text{(BVP)}$$

where $f \in C^0([0,1])$ and $\lambda \in \mathbb{R}$.

Approximate Problem

We approximate a function u by a vector $\mathbf{u}_n$ (see following section).
The standard central finite difference approximation of (BVP) yields a system of linear equations

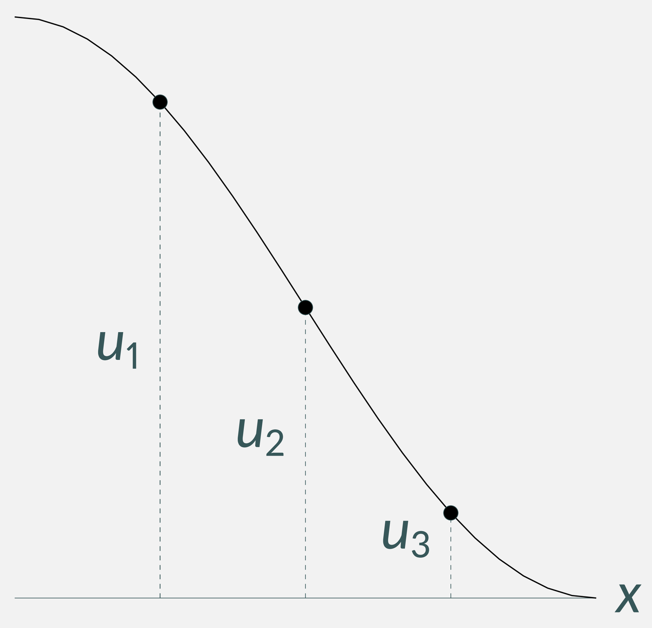
$$D^* D\mathbf{u}_n + \lambda \mathbf{u}_n = \mathbf{f}_n \qquad (\heartsuit)$$

where D is the $(n-1) \times n$ matrix

$$D = \frac{1}{\Delta x}\begin{bmatrix} -1 & 1 & & \\ & & \ddots & \\ & & -1 & 1 \end{bmatrix}.$$

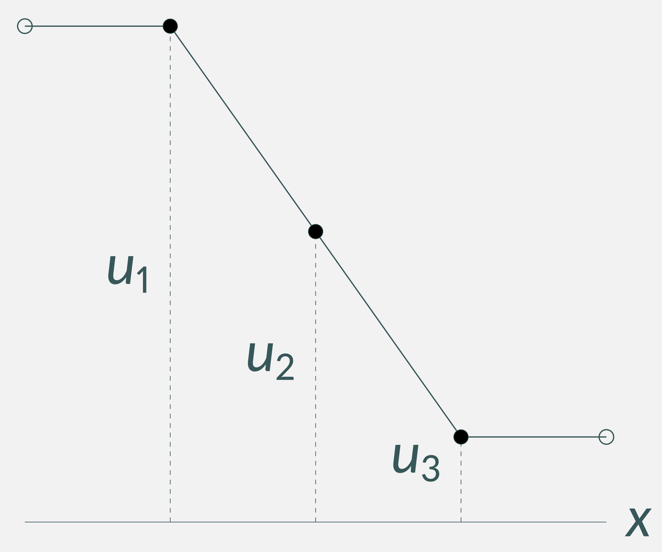
Approximating Functions by Vectors

For each $n \in \mathbb{N}$ we set $\Delta x = \frac{1}{n+1}$ and construct the linear transformation $\varphi \colon C^0([0,1]) \to \mathbb{R}^n$, by defining $\mathbf{u} = \varphi(u) = (u^1, u^2, \dots, u^n)^t \in \mathbb{R}^n$ with $u^k = u(k\Delta x)$.



Constructing Functions from Vectors

We define the linear transformation $\psi \colon \mathbb{R}^n \to C^0([0,1])$ by linearly interpolating between the points $(k\Delta x, u^k)$, where we take $u^0 = u^1$ and $u^{n+1} = u^n$ in order to enforce the boundary conditions.



Main Theorem

Let $\lambda \in \mathbb{R} \setminus \{-(\pi k)^2 \mid k \in \mathbb{N}_0\}$ and $f \in C^0([0,1])$. For large n there exists a unique $\mathbf{u}_n$ satisfying ($\heartsuit$). The corresponding sequence $u_n = \psi(\mathbf{u}_n)$ has a subsequence converging in $C^0([0,1])$ to a unique function $u \in C^2([0,1])$ satisfying (BVP).

Proof of Main Theorem, Part I

- For large n the only solution to $(D^*D + \lambda I)\mathbf{v} = 0$ is the zero vector.
- Thus we define a sequence of functions $u_n = \psi\left((D^*D + \lambda I)^{-1}\mathbf{f}_n\right) \in C^0([0,1])$.
- We use the Arzela-Ascoli theorem to obtain a subsequence that converges to $u \in C^0([0,1])$.

Proof of Main Theorem, Part II

- Another application of the Arzela-Ascoli theorem shows $u \in C^1([0,1])$.
- Using Riemann sums we show that
$$-u'(x) + \lambda \int_0^x u(y)\,dy = \int_0^x f(y)\,dy.$$
- Thus $u \in C^2([0,1])$ and satisfies (BVP).

Tool from Linear Algebra

If matrix M is symmetric, the solvability of

$$M\mathbf{u} = \mathbf{f} \qquad (\bigstar)$$

is determined by the solvability of

$$M\mathbf{v} = 0. \qquad \text{(H)}$$

1. If $\mathbf{v} = 0$ is the only solution to (H) then there exists a unique solution to ($\bigstar$).
2. If (H) has multiple solutions then
   (a) if $\mathbf{f} \cdot \mathbf{v} = 0$ for all $\mathbf{v}$ solving (H) then ($\bigstar$) has infinite solutions,
   (b) otherwise ($\bigstar$) has no solutions.

Tool from Analysis: Arzela-Ascoli Thm

Suppose $\{f_k\}_{k=1}^\infty \subset C^0([0,1])$ is uniformly bounded and equicontinuous. Then $\{f_k\}_{k=1}^\infty$ contains a subsequence that converges to some function $f \in C^0([0,1])$.

Future Work

The next steps of this project are

- analyzing the case when there are either multiple or no solutions to (BVP), and
- repeating our analysis in the case of Dirichlet boundary conditions.

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