Section 2. Incorporating Count Distributions in Copula-based fGARCH

Table .1: Summary of Notations Used in Copula-fGARCH and Count Modeling Framework

Symbol	Description
X_{ij}	Observed count for day i and interval j
Y_{ij}	Count increment: $X_{i,j+1} - X_{ij}$
Y_{ij}^2	Squared increment (raw volatility measure)
$Y_i^2(t)$	Smoothed functional volatility trajectory for day i
$\mu(t)$	Mean functional volatility curve
$\phi_k(t)$	k-th FPCA eigenfunction
Z_{ik}	FPCA score for day i and component k
p	Number of retained FPCA components
$F_{Z_k}(z)$	Marginal cumulative distribution function of Z_k
U_{ik}	Transformed uniform variable: $U_{ik} = F_{Z_k}(Z_{ik})$
$C(u_1, \ldots, u_p)$	Copula function modeling dependence among FPCA scores
σ_t^2	Latent volatility at time t (GARCH process)
$\hat{\sigma}_t^2$	Estimated latent volatility proxy used in count modeling and monitoring
$\lambda, \mu, \theta, \nu$	Parameters of count distributions (Poisson, NB, CMP)
$f(\cdot)$	Link function mapping latent volatility to conditional mean/rate
Combined effect	Average effect from Gaussian, Clayton, and empirical copulas

Table .2: Notation for Control Chart of Volatility Time Series

Symbol	Description
$\hat{\sigma}_t^2$	Estimated latent volatility at time t from Copula-fGARCH
T	Total number of time points in the volatility series
μ_{vol}	Mean of the volatility series $\hat{\sigma}_t^2$
σ_{vol}	Standard deviation of $\hat{\sigma}_t^2$
U	Upper control limit, $U = \mu_{\text{vol}} + 2\sigma_{\text{vol}}$
L	Lower control limit, $L = \mu_{\text{vol}} - 2\sigma_{\text{vol}}$
t	Index of the time point in the volatility series, $t = 1, \ldots, T$
i_k	Index of the k-th out-of-control point
m	Total number of out-of-control points
Δ_k	Interval between consecutive out-of-control points: $\Delta_k = i_{k+1} - i_k$
ARL	Average Run Length: mean interval between out-of-control signals
ARL_{SD}	Standard deviation of run lengths between out-of-control signals

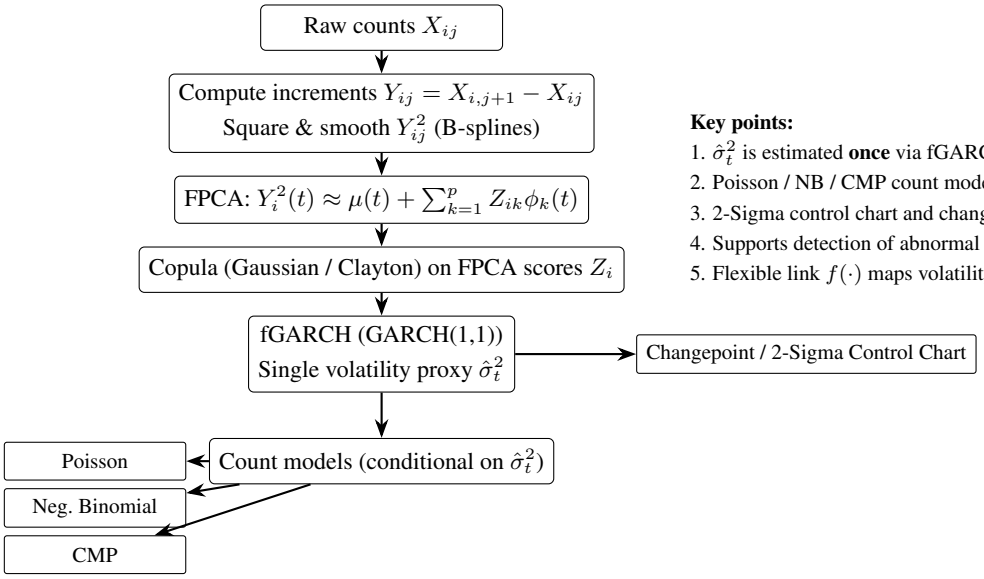
Table .3: Summary of Changepoint Detection Methods.

Method	Optimization	Complexity	Remarks
PELT	Penalized cost	$\mathcal{O}(T)$	Exact, scalable
BinSeg	Greedy splitting	$\mathcal{O}(T \log T)$	Fast, approximate
SegNeigh	Exhaustive search	$\mathcal{O}(KT^2)$	Accurate, slow for large T

Table .4: Notation for Changepoint Detection in Volatility Series

Symbol	Description
$\hat{\sigma}_t^2$	Estimated latent volatility proxy at time t
T	Total number of time points in the series
τ_k	Index of the k-th changepoint, $k = 1, \ldots, K$
K	Total number of changepoints detected
$\mathcal{C}(\cdot)$	Segment-specific cost function (e.g., based on mean/variance)
β	Penalty for adding a changepoint, controls model complexity
τ^*	Greedy estimate of changepoint in BinSeg algorithm
L_i	Cost associated with segment i: $L_i = \mathcal{C}(\hat{\sigma}^2_{t_{i-1}+1:t_i})$
t_i	Segment boundary index for segment i

Figure .1: Compact flowchart from raw counts X_{ij} to a single volatility proxy $\hat{\sigma}_t^2$, which is then used for alternative count distributions and monitored via control chart and changepoint detection.



Key points:

1. $\hat{\sigma}_t^2$ is estimated **once** via fGARCH.
2. Poisson / NB / CMP count models are conditional on $\hat{\sigma}_t^2$ (no re-fit).
3. 2-Sigma control chart and changepoint detection applied to $\hat{\sigma}_t^2$.
4. Supports detection of abnormal volatility periods and structural shifts.
5. Flexible link $f(\cdot)$ maps volatility to conditional mean/rate.

Section 3. Real Data Analysis

Table .5: Comparison of count models for Buy and Sell transactions.

Model	AIC (Buys)	BIC (Buys)	LogLik (Buys)	AIC (Sells)	BIC (Sells)	LogLik (Sells)
Poisson	-6080.416	-6078.641	3049.208	-6080.416	-6078.641	3049.208
NB	-18001.758	-17999.983	9009.879	-17989.336	-17987.561	9003.668
CMP	-17205.478	-17203.703	8611.739	-17222.045	-17220.270	8620.023

Table .6: ARL statistics and changepoint detection results for BTC Buy and Sell transactions using Poisson, NB, and CMP models. Changepoints are identified from the latent volatility proxy $\hat{\sigma}_{ij}^2$ using three detection methods: PELT, BinSeg, and SegNeigh.

Data	Model	ARL Mean	ARL SD	Changepoints (PELT)	Changepoints (BinSeg)	Changepoints (SegNeigh)
Buys	Poisson	6.19	12.45	7, 77, 99, 109, 137, 179	7, 77, 103, 137, 179	77, 105, 137, 179
Buys	NB	5.82	12.29	13, 49, 75, 99, 109, 133, 179	81, 102, 109, 135, 179	78, 109, 133, 179
Buys	CMP	6.06	13.71	10, 52, 76, 102, 112, 124, 126, 136, 182	6, 80, 102, 136, 182	79, 107, 136, 182
Sells	Poisson	1.74	2.05	41, 43, 139, 220	40, 47, 139, 223	41, 43, 139, 220
Sells	NB	1.78	2.37	6, 34, 44, 64, 86, 139, 204, 224	6, 42, 139, 204, 224	44, 139, 204, 224
Sells	CMP	1.65	2.68	7, 34, 44, 64, 87, 137, 208, 224	7, 43, 137, 208, 223	44, 137, 208, 224

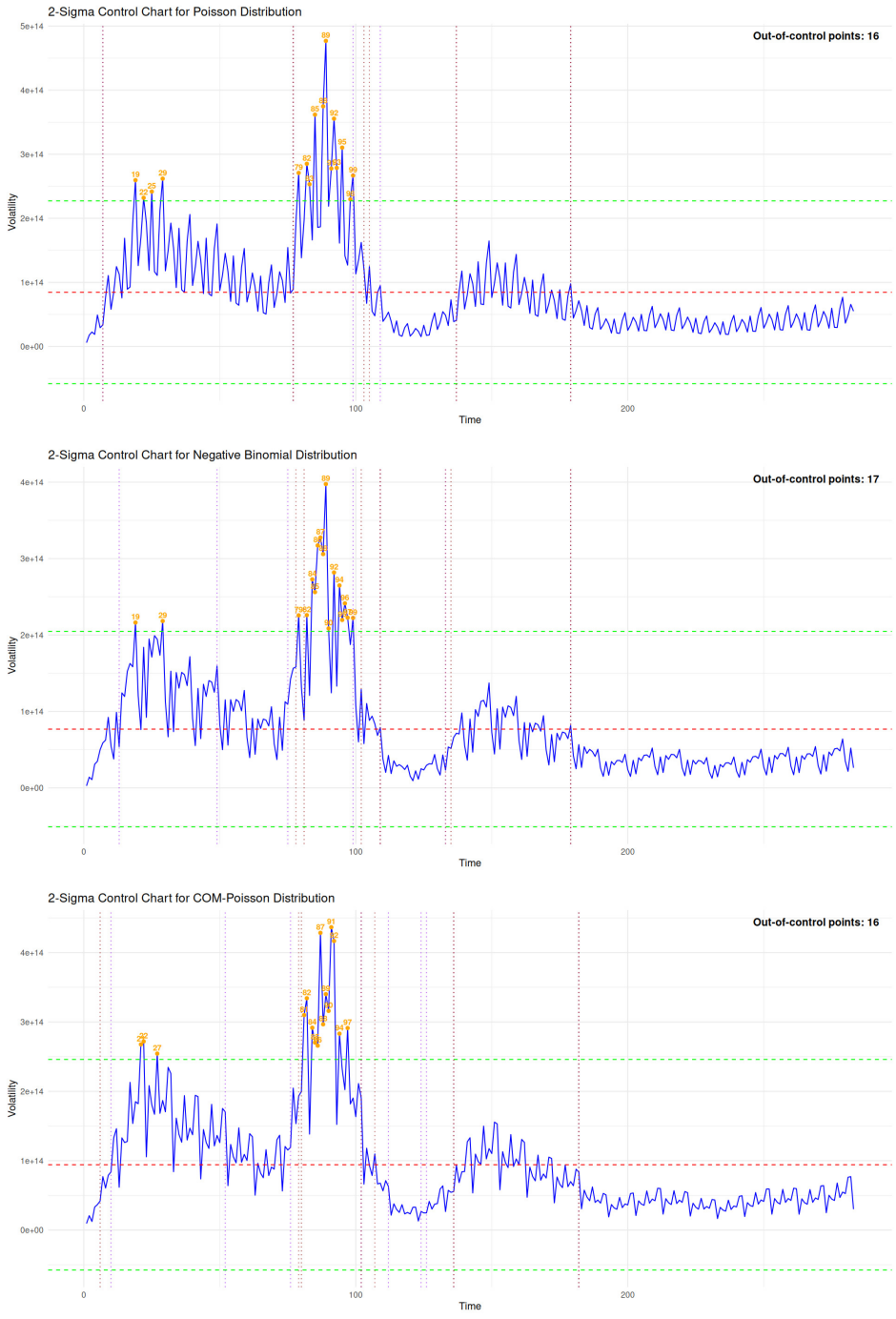


Figure .2: Intra-day latent volatility for BTC buy transactions (11-day average perspective).

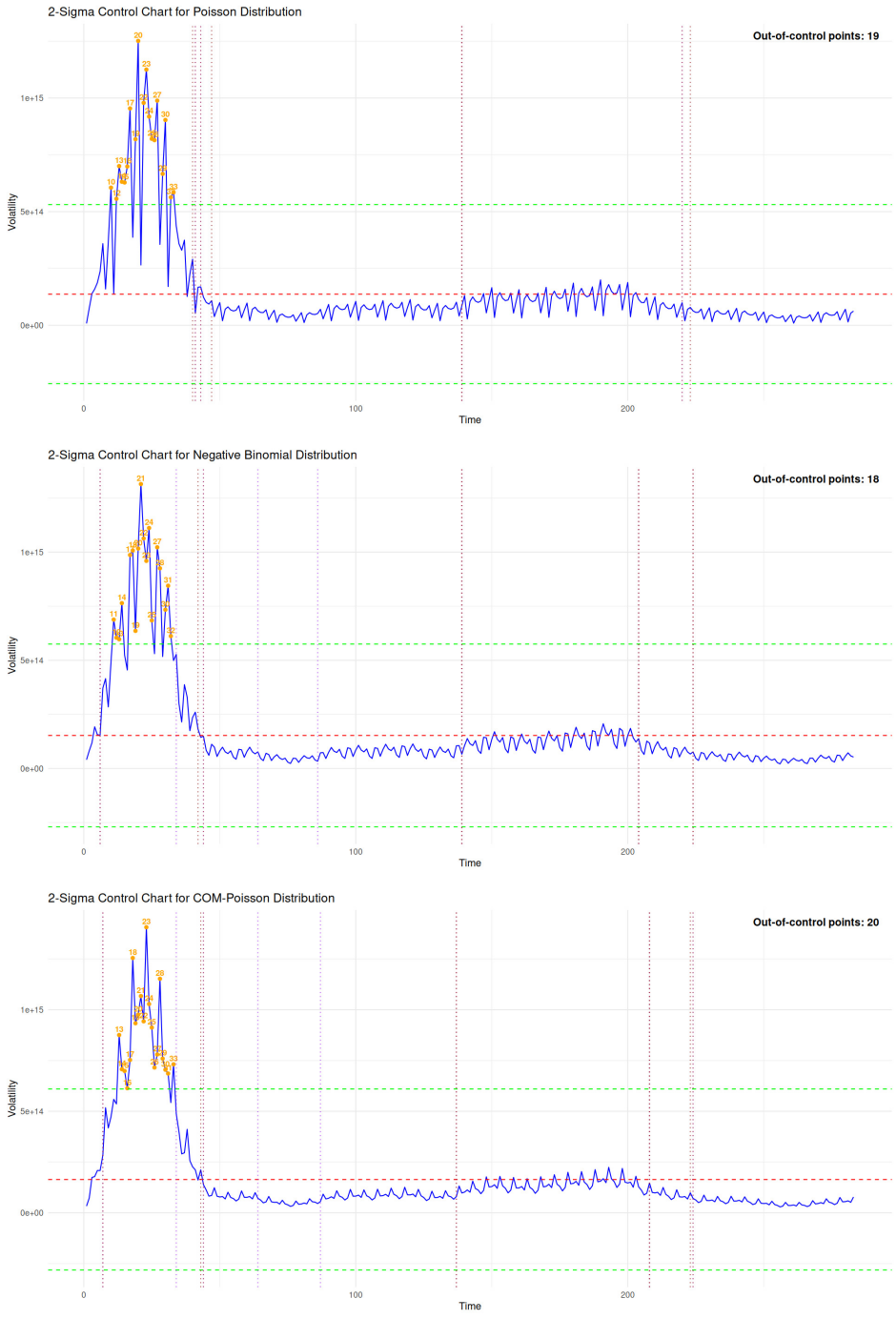


Figure .3: Intra-day latent volatility for BTC sell transactions (11-day average perspective).