4. Let $\{f_n\}$ and $\{g_n\}$ be real-valued functions on a set X. Assume that $f_n \to f$ and $g_n \to g$ uniformly on X.

(a) If λ, μ are scalars, show that $\lambda f_n + \mu g_n \to \lambda f + \mu g$ uniformly on X

Solution:

Observe

$$|\lambda f_n(x) + \mu g_n(x) - \lambda f(x) - \mu g(x)| \leq |\lambda||f_n(x) - f(x)| + |\mu||g_n(x) - g(x)|.$$

For a given $\varepsilon > 0$, take n sufficiently large so that $\sup_X |f_n - f| < \frac{\varepsilon}{|\lambda|}$ and $\sup_X |g_n - g| < \frac{\varepsilon}{|\mu|}$. Hence $\sup_X |\lambda f_n + \mu g_n - \lambda f - \mu g| < \varepsilon$, where we used the fact that the supremum is an upper bound on the right hand side and is least on the left.

□

(b) Is it true that $f_n g_n \to fg$ uniformly on X?

Solution:

This statement is false as it stands. Let $X = \mathbb{R}$ and consider $f_n(x) = \frac{1}{n}$ for all x and $g_n(x) = x$ for all x. It is clear that both $f_n \to 0$ and $g_n \to x$ both uniformly. The convergence of $f_n g_n x n \to 0$ is *not* uniform, however. To see this, let $\varepsilon = 1$ be fixed and consider $x_n = n$. Then

$$|f_n(x_n)| = 1 \geq \varepsilon$$

negating uniform convergence to 0.

□

(c) If the sequences are uniformly bounded (i.e., there exists M such that $|f_n(x)| \leq M$ and $|g_n(x)| \leq M$ for all $n \in \mathbb{N}$ and for all $x \in X$), show that $f_n g_n \to fg$ uniformly on X.

Solution:

Let M_f and M_g be the respective uniform bounds for f and g. Observe

$$\begin{aligned}
|f_n(x)g_n(x) - f(x)g(x)| &= |f_n(x)g_n(x) - f_n(x)g(x) + f_n(x)g(x) - f(x)g(x)| \\
&\leq |f_n(x)||g_n(x) - g(x)| + |g(x)||f_n(x) - f(x)| \\
&\leq M_f|g_n(x) - g(x)| + (|g(x) - g_n(x)| + |g_n(x)|)|f_n(x) - f(x)| \\
&\leq M_f|g_n(x) - g(x)| + (|g(x) - g_n(x)| + M_g)|f_n(x) - f(x)|.
\end{aligned}$$

□ Now, for a given $\varepsilon > 0$, take n sufficiently large so that $\sup_X |g_n - g| < \frac{\varepsilon}{2M_f}$ and $\sup_X |f_n - f| < \frac{\varepsilon}{2(\varepsilon + M_g)}$ and the convergence is established arguing as in part (a).

5. Define the distance between points (x_1, y_1) and (x_2, y_2) in the plane to be

$$|y_1 - y_2| \quad \text{if } x_1 = x_2, \qquad 1 + |y_1 - y_2| \quad \text{if } x_1 \neq x_2.$$

(a) Show that this defines a metric on the plane.

Solution:

Denote points in the plane as $p_i = (x_i, y_i)$.

i) If $p_1 = p_2$ then $x_1 = x_2$ and $|y_1 - y_2| = 0$. On the other hand, if $p_1 \neq p_2$ then either $x_1 \neq x_2$ or $y_1 \neq y_2$. In the first case, $d(p_1, p_2) = 1 + |y_1 - y_2| > 0$, and in the other case, both $|y_1 - y_2| > 0$ and $1 + |y_1 - y_2| > 0$, so $d(p_1, p_2) > 0$.

ii) Symmetry follows from symmetry of, $=$, $\neq$, and the distance metric on $\mathbb{R}$.

iii) Let p_1, p_2 and p_3 be points in the plane. If $x_1 = x_3$, then

$$d(x_1, x_3) = |y_1 - y_3| \leq |y_1 - y_2| + |y_2 - y_3| \leq 1 + |y_1 - y_2| + |y_2 - y_3|.$$

If $x_1 \neq x_3$, then it is not the case that $x_1 = x_2 = x_3$, thus $x_1 \neq x_2$ or $x_2 \neq x_3$ (or both), and in each of these cases

$$d(x_1, x_3) = 1 + |y_1 - y_3| \leq 1 + |y_1 - y_2| + |y_2 - y_3| \leq 2 + |y_1 - y_2| + |y_2 - y_3|.$$

$\square$

(b) Show that $\{0\} \times (-1/2, 1/2)$ is open and that $\{0\} \times [-1/2, 1/2]$ is compact.

Solution:

Let $p_0 \in \{0\} \times (-1/2, 1/2)$, then $p_0 = (0, y_0)$ where $|y_0| < 1/2$. A ball of radius $r < 1$ about p_0 has points $p = (x, y) \in B(r, p_0)$ satisfying $x = 0$ since $1 + |y - y_0| \geq 1$. Hence,

$$d(p, p_0) = |y - y_0| < r \implies |y| < r + |y_0|,$$

and if we take $r = \min\{1, 1/2 - |y_0|\}$, then $|y| < 1/2$ and thus $y \in \{0\} \times (-1/2, 1/2)$. Hence each point p_0 of the set is an interior point, so the set is open.

It suffices to show that the set $\{0\} \times [-1/2, 1/2]$ satisfies sequential compactness since it is a subset of a metric space. Let $p_n \in \{0\} \times [-1/2, 1/2]$ then $x_n = 0$ and $y_n \in [-1/2, 1/2]$. By compactness of $[-1/2, 1/2]$ in $\mathbb{R}$, there exists a subsequence $\{y_{n_k}\}$ and a point y so that $|y - y_{n_k}| \to 0$ in $\mathbb{R}$. Choose $p_{n_k} = (0, y_{n_k})$ and $p = (0, y)$ and note $d(p, p_{n_k}) = |y - y_{n_k}| \to 0$, hence $p_{n_k} \to p$ in this space.

$\square$

(c) Is $Y = [-1, 1] \times [-1/4, 1/4]$ compact?

Solution:

The set is *not* compact. It suffices to provide a sequence with no converging subsequences. Consider $p_n = (1/n, 0)$. Then for any n_k, $d(p_{n_k}, p_{n_j}) = 1$ for $k \neq j$, hence any p_{n_k} is not Cauchy and does not converge.

$\square$

6.

(a) Show that

$$F_n(x) = \left(1 + \frac{x}{n}\right)^n e^{-x}$$

is bounded above and below by constants independent of n for $x \in [0, \infty)$.

Solution:

Note first that that $F_n(x) > 0$ since both factors are.

$$\left(1 + \frac{x}{n}\right)^n e^{-x} \geq e^{-x} = 1 - e^{-n} \to 1 \quad \text{as } n \to \infty.$$

Now,

$$\begin{aligned} F_n(x) &= \sum_{k=0}^{n} \binom{n}{k} \left(\frac{x}{n}\right)^k e^{-x} \\ &= \sum_{k=0}^{n} \frac{n \cdot (n-1) \cdot \cdots \cdot (n-k+1)}{k! n^k} x^k e^{-x} \\ &\leq \sum_{k=0}^{n} \frac{x^k}{k!} e^{-x} \\ &\leq 1. \end{aligned}$$

□

(b) Evaluate, with justifications,

$$\lim_{n\to\infty} \int_0^n \left(1 + \frac{x}{n}\right)^n e^{-2x}\, dx$$

Solution:

Since $|F_n(x)| \leq 1$, we have

$$\int_0^n \left(1 + \frac{x}{n}\right)^n e^{-2x}\, dx \leq \int_0^n e^{-x}\, dx = 1 - e^{-n} \leq 1.$$

Recall that $(1 + x/n)^n \to e^x$ as $n \to \infty$ for each real x. Moreover, the convergence is monotone increasing since $F_n(x)$ is monotone increasing and $F_n(x)e^x = \left(1 + \frac{x}{n}\right)^n$. Let $\varepsilon > 0$ be given, then

$$\begin{aligned} \int_0^n \left(1 + \frac{x}{n}\right)^n e^{-2x}\, dx &= \int_0^n \left[\left(1 + \frac{x}{n}\right)^n - e^x\right] e^{-2x} + e^{-x}\, dx \\ &= -\int_0^n \left|e^x - \left(1 + \frac{x}{n}\right)^n\right| e^{-2x} + e^{-x}\, dx \\ &> -\varepsilon \int_0^n e^{-2x}\, dx + \int_0^n e^{-x}\, dx \end{aligned}$$

for sufficiently large n. Since $\int_0^n e^{-2x}\, dx \to \frac{1}{2}$ and $\int_0^n e^{-x}\, dx \to 1$, we have

$$\int_0^n \left(1 + \frac{x}{n}\right)^n e^{-2x}\, dx > \frac{\varepsilon}{2} + 1 \implies \int_0^n \left(1 + \frac{x}{n}\right)^n e^{-2x}\, dx = 1.$$

$\square$

7. Let (X, d) be a metric space. A function $f : X \to \mathbb{R}$ is called Lipschitz if there is a constant L such that $|f(x) - f(y)| \leq Ld(x, y)$, for al $x, y \in X$.

(a) Show that the sum of two real-valued Lipschitz functions is Lipschitz.

Solution:

Let f and g be two Lipschitz functions with respective Lipschitz constants L_f and L_g. Observe

$$|f(x)+g(x)-f(y)-g(y)| \leq |f(x)+f(y)|+|g(x)-g(y)| \leq (L_f+L_g)|x-y|.$$

$\square$

(b) Prove that the product of two bounded real-valued Lipschitz functions is again Lipschitz.

Solution:

Let M_f and M_g be the respective bounds for f and g with Lipschitz constants as above. Observe

$$|f(x)g(x)-f(y)g(y)| \leq |f(x)g(x)-f(x)g(y)|+|f(x)g(y)-f(y)g(y)| \leq (M_f L_g+M_g L_f)|x-y|.$$

$\square$

(c) Show that the product of two real-valued Lipschitz functions need not be Lipschitz.

Solution:

Consider $f(x) = x$ on $\mathbb{R}$ and it is clear that f is Lipschitz. However, f^2 is not. To see this, let $L > 0$ be given and observe

$$|x^2 - y^2| = |x + y||x - y| > L|x - y|$$

for $x > L$ and $y = 0$.

$\square$

8. Consider the mapping

$$\omega = J(z) = \frac{1}{2}\left(z + \frac{1}{z}\right).$$

For each $r > 0$, describe the image of the circle $|z| = r$ under this mapping (e.g., as a certain line, circle, ellipse, etc.).

Solution:

We can parametrize the circle $|z| = r$ as $re^{i\theta} = r\cos\theta + ir\sin\theta$ for $-\pi \leq \theta \leq \pi$. Let $x = \operatorname{Re}\omega$ and $y = \operatorname{Im}\omega$, then

$$\begin{aligned} x &= \frac{1}{2}\left(re^{i\theta} + r^{-1}e^{-i\theta} + re^{-i\theta} + r^{-1}e^{i\theta}\right) \\ &= \frac{1}{2}\left(e^{i\theta} + e^{-i\theta}\right)(r + r^{-1}) \\ &= \cos\theta(r + r^{-1}) \end{aligned}$$

and

$$\begin{aligned} y &= \frac{1}{2i}\left(re^{i\theta} + r^{-1}e^{-i\theta} - re^{-i\theta} - r^{-1}e^{i\theta}\right) \\ &= \frac{1}{2i}\left(e^{i\theta} - e^{-i\theta}\right)\left(r - r^{-1}\right) \\ &= \sin\theta(r + r^{-1}). \end{aligned}$$

If we let $a = r + r^{-1}$ and $b = r - r^{-1}$, then

$$\frac{x^2}{a} + \frac{y^2}{b} = 1$$

and such points describe an ellipse.

□

9. Let $m > 2$ be a positive integer, and let ω be a primitive m-th root of unity. For complex numbers α and β evaluate

$$\frac{1}{m}\sum_{k=0}^{m-1}|\alpha + \omega^k\beta|^2\omega^k.$$

Solution:

First observe

$$\begin{aligned} \frac{1}{m}\sum_{k=0}^{m-1}|\alpha + \omega^k\beta|^2\omega^k &= \frac{1}{m}\sum_{k=0}^{m-1}(\alpha + \omega^k\beta)(\overline{\alpha} + \omega^{-k}\overline{\beta})\omega^k \\ &= \frac{1}{m}\sum_{k=0}^{m-1}(\alpha + \omega^k\beta)(\omega^k\overline{\alpha} + \overline{\beta}) \\ &= \frac{1}{m}\sum_{k=0}^{m-1}\left(\omega^k|\alpha|^2 + \omega^{2k}\beta\overline{\alpha} + \alpha\overline{\beta} + \omega^k|\beta|^2\right). \end{aligned}$$

To simplify this sum, we use the fact that

$$(\omega - 1)\sum_{k=0}^{m-1}\omega^k = \omega^m - 1 = 0 \implies \sum_{k=0}^{m-1}\omega^k = 0$$

since $m > 2$ and ω is primitive. Similarly

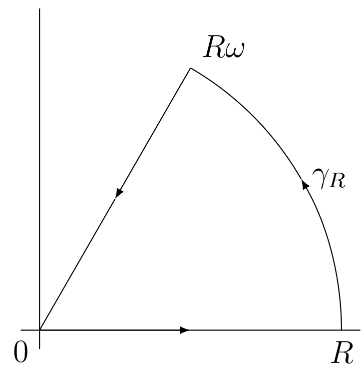
$$(\omega^2 - 1)\sum_{k=0}^{m-1}\omega^{2k} = \omega^{2m} - 1 = (\omega^m - 1)(\omega^m + 1) = 0 \implies \sum_{k=0}^{m-1}\omega^{2k} = 0.$$

Hence, the sum above simplifies to $\alpha\overline{\beta}$.

□

10. Let $b > 2$ and put $\omega = e^{2\pi i/b}$. Define z^b with respect to the principal branch of the logarithm on $G = \mathbb{C} \setminus \{x \in \mathbb{R} : x \leq 0\}$.

(a) Show that
$$\int_{[0,R\omega]} \frac{dz}{1+z^b} = e^{2\pi i/b} \int_0^R \frac{dx}{1+x^b}.$$
For $R > 1$ let γ_R be the contour sketched in the following figure



Solution:

Parametrize $[0, R\omega]$ by $z(t) = t\omega$ for $0 \le t \le R$, then
$$\int_{[0,R\omega]} \frac{dz}{1+z^b} = \int_0^R \frac{\omega dt}{1+(t\omega)^b} = \omega \int_0^R \frac{dt}{1+t^b}$$
since $(t\omega)^b = \exp(\log(te^{2\pi i/b})b) = \exp(\log(t)b + 2\pi i) = t^b$.

$\square$

(b) Show that
$$\lim_{R\to\infty} \int_{\gamma_R} \frac{dz}{1+z^b} = (1 - e^{2\pi i/b}) \int_0^\infty \frac{dx}{1+x^b}.$$

Solution:

Denote the arc from R to ωR on γ_R as C_R. Note that the integral along this contour is bounded above by
$$\left| \int_{C_R} \frac{dz}{1+z^b} \right| \le \frac{1}{1-|z^b|} \cdot \frac{2\pi R}{b},$$
and $|z^b| = |\exp(\log(Re^{i\theta})b)| = |\exp(\log(R)b + i\theta)| = R^b$ implies
$$\frac{1}{1-|z^b|} \cdot \frac{2\pi R}{b} \to 0 \quad \text{as } R \to \infty$$
since $b > 2$. Now
$$\begin{aligned} \lim_{R\to\infty} \int_{\gamma_R} \frac{dz}{1+z^b} &= \lim_{R\to\infty} \left\{ \int_0^R \frac{dx}{1+x^b} + \int_{C_R} \frac{dz}{1+z^b} + \int_{[R\omega,0]} \frac{dz}{1+z^b} \right\} \\ &= \int_0^\infty \frac{dx}{1+x^b} + 0 - \omega \int_0^\infty \frac{dx}{1+x^b} \\ &= (1-\omega) \int_0^\infty \frac{dx}{1+x^b}. \end{aligned}$$

$\square$

(c) Use the Residue Theorem to evaluate $\displaystyle\int_{\gamma_R} \frac{dz}{1+z^b}$.

Solution:

The integrand is singular provided $1+z^b = 1+\exp(\log(z)b) = 0$. Observe

$$\begin{aligned}
\exp(\log(z)b) &= \exp(\ln|z|b) + ib\arg z) \\
&= |z|^b \exp(ib\arg z) \\
&= -1
\end{aligned}$$

implies $|z| = 1$ and $\arg z = \pi/b$. Hence, $e^{i\pi/b}$ is an isolated singularity interior to γ_R when $R > 1$.

Since $\frac{d}{dz}(1+z^b) = bz^{b-1}$ which is non-zero at $e^{i\pi/b}$, the singularity is a simple pole, and the residue of the integrand is given by

$$\frac{1}{b\exp\left(\pi i \frac{b-1}{b}\right)}$$

The Residue Theorem indicates that for such R,

$$\int_{\gamma_R} \frac{dz}{1+z^b} = \frac{2\pi i}{b\exp\left(\pi i\frac{b-1}{b}\right)}$$

$\square$

(d) Use (b) and (c) to evaluate $\displaystyle\int_0^\infty \frac{dx}{1+x^b}$.

Solution:

Well,

$$\begin{aligned}
\int_0^\infty \frac{dx}{1+x^b} &= \frac{2\pi i}{(1-\omega)b\exp\left(\pi i\frac{b-1}{b}\right)} \\
&= \frac{2\pi i}{b}\left(\exp\left(\pi i\frac{b-1}{b}\right) - \exp\left(\pi i\frac{b-1}{b} + \frac{2\pi i}{b}\right)\right)^{-1} \\
&= \frac{2\pi i}{b}\left(\exp\left(\pi i\frac{b-1}{b}\right) - \exp\left(\pi i\frac{1-b}{b} - \frac{2b\pi i}{b}\right)\right)^{-1} \\
&= \frac{2\pi i}{b}\left(\sin\left(\pi\frac{b-1}{b}\right)\right)^{-1}.
\end{aligned}$$

$\square$

References

[BC04] James Ward Brown and Ruel V. Churchill. *Complex Variables and Applications.* McGraw-Hill, 7th edition, 2004.

[Boa10] Harold P. Boas. Julius and Julia: Mastering the art of the Schwarz lemma. *The American Mathematical Monthly*, 117(9):770–785, 2010.

[Ste04] J Michael Steele. *The Cauchy-Schwarz master class: an introduction to the art of mathematical inequalities*. Cambridge University Press, 2004.

[Str00] Robert S Strichartz. *The way of analysis*. Jones & Bartlett Learning, 2000.