Problem 1.

Solution We will prove that this is an increasing sequence by induction. Notice that

$$a < a + \sqrt{a} \Rightarrow \sqrt{a} < \sqrt{a + \sqrt{a}} \Rightarrow x_1 < x_2$$

Now assume that $x_{n-1} < x_n$, Then

$$x_{n+1} = \sqrt{a + x_n} > \sqrt{a + x_{n-1}} = x_n$$

And so strict increasingness holds by induction. Now we show that it is bounded. Notice that the graph of $y = \sqrt{x+a}$ and the graph of $y = x$ has a single intersection, which is found to be at $x = \frac{1+\sqrt{1+4a}}{2}$, and since

$$\frac{d}{dx}\sqrt{x+a} = \frac{1}{2\sqrt{x+a}} < 1 \text{ for } x > \frac{1}{4} - a$$

Notice that

$$\frac{1+\sqrt{1+4a}}{2} > \frac{1}{4} - a$$

And so if $x > \frac{1+\sqrt{1+4a}}{2}$, then we have that the slope of $\sqrt{x+a}$ is less than 1. this means that the graph of $y = x$ passes through the graph of $y = \sqrt{x+a}$ as we move through the intersection. So then if the was an x_n such that $x_n > \frac{1+\sqrt{1+4a}}{2}$, then we would have that $x_{n+1} = \sqrt{x+a} < x_n$, and the sequence would fail to be increasing. So then we have that the sequence is bounded above by $\frac{1+\sqrt{1+4a}}{2}$. hence it must converge. Morover, this value is the limit, since the limit of any convergent recursive sequence must also be a fixed point of the generating function.

Problem 2

Solution For the first part, suppose that there is a positive decreasing function $f(x)$ such that $\lim_{x\to\infty} xf(x) \neq 0$ and

$$\int_0^\infty f(x)dx < \infty$$

Since $\lim_{x\to\infty} xf(x) \neq 0$ and f is positive, we must have that

$$\lim_{x\to\infty} xf(x) = \alpha > 0$$

and thus there is an x_0 such that $xf(x) > \frac{\alpha}{2}$ for all $x > x_0$, and so $f(x) > \frac{\alpha}{2x}$ for $x > x_0$. This gives that

$$\int_1^\infty f(x)dx \geq \int_{x_0}^\infty f(x)dx \geq \frac{\alpha}{2}\int_{x_0}^\infty \frac{1}{x}dx = \lim_{t\to\infty} \ln(t) - \ln(x_0) = \infty$$

But this contradicts that $\int_1^\infty f(x)dx < \infty$.

Part(b) is a millenium problem I'm sure...

Problem 3

Solution Let g be defined as the hint. then we have that

$$g(x) = \int_0^x |f'(t)|dt \geq \left|\int_0^x f'(t)dt\right| = |f(x) - f(0)| = |f(x)|, \text{ since } f(0) = 0$$

Additionally, notice that

$$g'(x) = |f'(x)|$$

Lastly, consider the follwing calculation, using integration by parts

$$\int_0^1 g(x)g'(x)dx = g^2(x)\big|_0^1 - \int_0^1 g(x)g'(x)dx$$

which implies that

$$2\int_0^1 g(x)g'(x)dx = g^2(1) - g^2(0)$$

Since $g(0) = 0$ we get that $\int_0^1 g(x)g'(x)dx = \frac{1}{2}g^2(1)$. bringing these three facts together we get that

$$\int_0^1 |f(x)f'(x)|dx = \int_0^1 |f(x)|g'(x)dx \leq \int_0^1 g(x)g'(x)dx = \frac{1}{2}g^2(1) = \left(\int_0^1 |f'(x)|dx\right)^2 \leq \int_0^1 |f'(x)|^2dx$$

where the last inequality holds due to Cauchy-Bunakovsky-Schwarz.

Problem 4

Solution Notive that for $x = 1$ the sequence is just $(1^n f(1))_n = f(1)$, and thus it converges. For a fixed $x \in [\frac{1}{2}, 1)$ we have that $x^n \to 0$, and thus $x^n f(x) \to 0$. So our pointwise limit is given by

$$\lim x^n f(x) = \begin{cases} 0 & \frac{1}{2} \leq x < 1 \\ f(1) & x = 1 \end{cases}$$

Now suppose that the convergence is uniform. Then we must have that $\lim x^n f(x)$ is continuous at 1, since each $x^n f(x)$ is continuous at 1. But since $x^n f(x)$ coverges to 0 uniformly on $[\frac{1}{2}, 1)$, we would have to have that $f(1) = 0$. So then the uniform limit is 0, which means that f must be bounded.

Now suppose that f is bounded and that $f(1) = 0$. Let ϵ be given. Then since f is continuous at 1, there is a delta such that for all $x \in (1-\delta, 1]$ we have that $|f(x) - f(1)| < \epsilon$, and since $f(1) = 0$, we get that $|f(x)| < \epsilon$ for all $x \in (1-\delta, 1]$. Now since f is bounded, let $\alpha = \max_{x\in[\frac{1}{2},1-\delta]} |f(x)|$. Let N be such that $(1-\delta)^n \leq \frac{\epsilon}{\alpha}$. Now x^n is increasing on $[\frac{1}{2}, 1-\delta]$, and thus we have that, for any $x \in [\frac{1}{2}, 1-\delta]$, for all $n \geq N$:

$$x^n f(x) \leq (1-\delta)^n \alpha < \frac{\epsilon}{\alpha}\alpha = \epsilon$$

and thus the convergence is uniform.

Problem 5

Solution Since $\int_0^\infty f(x)dx$ converges, then we must have that $f(x) to 0$ as $x \to \infty$. Since f is decreasing, and converging to 0 we must have that $f(x) \geq 0$. Now then, since f is nonnegative and decreasing we have that

$$\sum_{n=1}^{\infty} f(n\delta) \le \int_0^{\infty} f(\delta x)dx = \frac{1}{\delta}\int_0^{\infty} f(u)du$$

which converges, and thus we have that $\sum_{n=1}^{\infty} f(n\delta)$ converges. Furthermore, notice that

$$\int_0^{\infty} f(x)dx = \int_0^{\delta} f(x)dx + \sum_{n=1}^{\infty}\int_{n\delta}^{(n+1)\delta} f(x)dx = \int_0^{\delta} f(x)dx + \sum_{n=1}^{\infty}\delta\int_n^{n+1} f(x\delta)dx \le \delta f(0) + \delta\sum_{n=1}^{\infty} f(n\delta)$$

Subtracting $\delta f(0)$, and using the previous comparison we get that

$$\int_0^{\infty} f(x)dx - \delta f(0) \le \delta\sum_{n=1}^{\infty} f(n\delta) \le \int_0^{\infty} f(x)dx$$

Letting $\delta \to 0$, and applying the squeeze theorem, gives the result.

Problem 6.

Solution

Problem 7.

Solution Part (a) is easy. Let f be lipschitz, with lipschitz constant M. Given $\epsilon > 0$ for any $x, y \in X$ such that $d(x, y) < \frac{\epsilon}{M}$ we have

$$|f(x) - f(y)| \le Md(x, y) < M\frac{\epsilon}{M} = \epsilon$$

Since our choices of x and y were arbitrary we have that f is uniformly continuous.
However, not ever uniformly continuous function need be lipschitz. Let $X = [0, 1]$, and let $f = \sqrt{x}$, then since f is continuous, and X compact, we must have that f is uniformly continuous. However, notice that

$$\lim_{x\to 0} f'(x) = \lim_{x\to 0}\frac{1}{2\sqrt{x}} = \infty$$

and so f has an unbounded derivative, and thus cannot be lipschitz.
The remaining parts are not obvious. Also, they're dumb.

Problem 8.

Solution Consider that

$$\cos^n(\theta) = \left(\frac{e^{i\theta} + e^{-i\theta}}{2}\right)^n = \frac{1}{2^n}\sum_{k=0}^{n}\binom{n}{k}e^{i(n-k)\theta}e^{-ik\theta}$$

$$= \frac{1}{2^n}\sum_{k=0}^{n}\binom{n}{k}e^{i(n-2k)\theta}$$

Notice that $\cos^n(\theta)$ is real. Thus we have that

$$\cos^n(\theta) = Re(\cos^n(\theta)) = Re\left(\frac{1}{2^n}\sum_{k=0}^{n}\binom{n}{k}e^{i(n-2k)\theta}\right) = \frac{1}{2^n}\sum_{k=0}^{n}\binom{n}{k}Re\left(e^{i(n-2k)\theta}\right)$$

$$= \frac{1}{2^n} \sum_{k=0}^{n} \binom{n}{k} \cos((n-2k)\theta)$$

Problem 9

Solution First we note that the Laurent expansion must converge uniformly on the annulus. The we must also have that

$$\left| \sum_{n \in \mathbb{Z}} c_n z^n \right|$$

Also converge uniformly. This follows from the obervation that, for an z and n:

$$\left| \left| \sum_{-n \le k \le n} c_k z^k \right| - |f(z)| \right| \le \left| \sum_{-n \le k \le n} c_n z^n - f(z) \right|$$

Anyway, we get that, from the hint

$$\int_0^{2\pi} |f(re^{i\theta})|^2 d\theta = \int_0^{2\pi} \left(\sum_{n \in \mathbb{Z}} c_n r^n e^{in\theta} \right) \left(\sum_{m \in \mathbb{Z}} \overline{c_m} r^m e^{-im\theta} \right) d\theta$$

$$= \int_0^{2\pi} \sum_{n \in \mathbb{Z}} \sum_{m \in \mathbb{Z}} c_n \overline{c_m} r^{m+n} e^{i(n-m)\theta} d\theta = \sum_{n \in \mathbb{Z}} \sum_{m \in \mathbb{Z}} c_n \overline{c_m} r^{m+n} \int_0^{2\pi} e^{i(n-m)\theta} d\theta$$

It is easy to see that

$$\int_0^{2\pi} e^{i(n-m)\theta} d\theta = \begin{cases} 0 & n \neq m \\ 2\pi & n = m \end{cases}$$

So then all the terms where $n \neq m$ in the above expression drop out. This gives that

$$\int_0^{2\pi} |f(re^{i\theta})|^2 d\theta = \sum_{n \in \mathbb{Z}} 2\pi c_n \overline{c_n} r^{2n} = 2\pi \sum_{n \in \mathbb{Z}} |c_n|^2 r^{2n}$$

Notice now that

$$\int_\Omega |f(x+iy)|^2 dx dy = \int_0^1 \int_0^{2\pi} r |f(re^{i\theta})|^2 d\theta dr$$

$$= \int_0^1 2\pi \sum_{n \in \mathbb{Z}} |c_n|^2 r^{2n+1} dr = 2\pi \sum_{n \in \mathbb{Z}} |c_n|^2 \int_0^1 r^{2n+1} dr$$

Now if $|c_n| > 0$ for any $n \le -1$, then we would have that

$$\int_0^1 r^{2n+1} dr = \infty$$

for $n \le -1$. This would contradict that $\int_\Omega |f|^2 dx dy < \infty$. So then $c_n = 0$ for $n \le -1$. This implies that the singularity at 0 is removable, since the negative coefficients of the laurent series are all 0.

Problem 10

Solution I am not going to do all the work here, but it is easy to show that the value of the integral is $\frac{\pi}{4e}$.