**4.** For $A$ and $B$ in $\mathcal{P}(\mathbb{Z}_+)$ (the power set of $\mathbb{Z}_+$, consisting of all sets of positive integers) define $d(A,B)$ as follows: If $A = B$, $d(A,B) = 0$. If $A \neq B$ and if $m$ is the smallest positive integer that is in $A$ or $B$ but not both, $d(A,B) = \frac{1}{m}$.

**(a)** Show that $d$ is a metric on $\mathcal{P}(\mathbb{Z}_+)$.

**Solution:**

Denote the set of all elements in $A$ or $B$ but not both, i.e. $(A \cup B) \setminus (A \cap B) = (A \setminus B) \cup (B \setminus A)$, as $A \triangle B$.

i) By definition, $d(A,A) = 0$. On the other hand, if $d(A,B) \neq 0$, then there exists an element $m \in A \triangle B = (A \setminus B) \cup (B \setminus A)$, and thus $A \neq B$.

ii) By symmetry of $\cup$ and $\cap$, $d$ is also symmetric.

iii) Suppose $d(A,C) > d(A,B) + d(B,C)$, then $d(A,C) > d(A,B)$ and $d(B,C)$. If $x = \min A \triangle C$, then $x < \min A \triangle B$ and $x < \min B \triangle C$, hence $x \in (A \triangle B)^c = (A \cup B)^c \cup (A \cap B)$ and $x \in (B \triangle C) = (B \cup C)^c \cup (B \cap C)$. If $x \in B$, then $x \in A \cup B$ and $B \cup C$, hence $x$ must be in $A \cap B \cap C \subseteq A \cap C$, a contradiction. Yet, if $x \not inB$ $x \in B$, then $x \in (A \cup B)^c = A^c \cap B^c$ and $x \in (B \cup C)^c = B^c \cap C^c$ implies $x \notin (A \cup C) \subseteq A \triangle C$ also a contradiction.

□

**(b)** If $\{A_n : n \in \mathbb{Z}_+\}$ is a sequence of subsets of $\mathcal{P}(\mathbb{Z}_+)$ satisfying $A_n \subseteq A_{n+1}$ for all $n$ and if $A = \bigcup_{n=1}^{\infty} A_n$, show that $A_n \to A$ in the metric space $(\mathcal{P}(\mathbb{Z}_+), d)$.

**Solution:**

Note that $A \triangle A_n = A \setminus A_n$ since $A_n \subseteq A$. If there exists an $m$ such that $A_m = A_n$ for all $n \geq m$, then $d(A, A_n) = d(A, A) = 0$ since $\bigcup_{n=1}^{\infty} A_n = A$ and proof is done.

Now, assume otherwise, i.e. for each $m > 0$, there exists $n > m$ such that $A_n \setminus A_m \neq \emptyset$, and since the sequence is nested, $A \setminus A_m \neq \emptyset$ for all $m$. Morever, if $m < n$ then $A \setminus A_n \subseteq A \setminus A_m$ imples $\min A \setminus A_m \leq \min A \setminus A_n$ which implies $d(A, A_n) \leq d(A, A_m)$. Hence, the sequence defined by $a_n = d(A, A_n)$ is decreasing.

So $a_n = (\min A \setminus A_n)^{-1}$ for all $n$. Since $A$ is the union of all $A_n$, there exists $(a_{n_1})^{-1} \in A_{n_1}$ and $(a_{n_1})^{-1} \notin A_1$. Proceed inductively to define a subsequence so that $(a_{n_{k-1}})^{-1} \notin A_{n_k}$. Hence, $(a_{n_k})^{-1} = \min A_{n_k} < \min A_{n_{k-1}} = (a_{n_{k-1}})^{-1}$. Hence $a_{n_k}$ is strictly decreasing sequence of positive integers and thus converges to 0, and since $a_n$ is monotone decreasing $a_n$ converges to zero. Hence $d(A_n, A)$ can be made arbitarily small and the proof is complete.

□

Show that the metric space $(\mathcal{P}(\mathbb{Z}_+), d)$ is compact.

**Solution:**

Since this is a metric space, it suffices to prove that it is complete and totally bounded. We first show completeness. Let $A_n$ be a given Cauchy sequence. Then, for sufficiently large $m > n$, we have

$$d(A_n, A_m) < \frac{\varepsilon}{2}.$$

Let

$$B_n = \bigcap_{k=n}^{\infty} A_k.$$

and note that $B_n$ is nested as in part **(b)**. So $B_n \to B = \bigcup B_n$ in this space. Note $A_n \supseteq B_n$ so $A_n \triangle B_n = A_n \setminus B_n$. Moreover, if

$$J = \min A_n \setminus B_n = \min A_n \setminus \bigcap_{k=n}^{\infty} A_k$$

then $J \in A_n$ and there exists an $m \geq n$ such that $J \notin A_m$. Hence $J \in (A_n \cup A_m) \setminus (A_n \cap A_m) = A_n \triangle A_m$. So

$$J \geq \min A_n \triangle A_m > \frac{2}{\varepsilon} \iff d(A_n, B_n) = \frac{1}{J} < \frac{\varepsilon}{2}.$$

Finally, using the converges of $B_n$ and the above inequality, we have

$$d(A_n, B) \leq d(A_n, B_n) + d(B, B_n) < \varepsilon$$

So $A_n$ converges to $B$ in this space.

We now show that the space is totally bounded. Let $\varepsilon > 0$ be a given radius. There exists an $N \in \mathbb{N}$ such that $\frac{1}{N} \leq \varepsilon$. Consider the collection of balls

$$\Big\{ B(\varepsilon, D) : D \in \mathcal{P}(\{1, \ldots, N\}) \Big\}.$$

Now, for any $A \in \mathcal{P}(\mathbb{N})$, the set $D_0 = \{x \in A : x \leq N\} \in P(\{1, \ldots, N\})$. Observe $A \triangle D_0 = A \setminus D_0 = \{x \in A : x > N\}$ implies $d(A, D_0) < \frac{1}{N} \leq \varepsilon$. Hence, $A$ is covered in the collection above, and since it is indexed by the finite set $P(\{1, \ldots, N\})$, the cover is as desired and the space is totally bounded.

$\square$

**6.** The *diameter* of a nonempty set $E$ in a metric space $(X, d)$ is defined to be

$$\operatorname{diam}(E) = \sup\{d(x, y) : x, y \in E\}.$$

Show that if $\{E_k : k \in \mathbb{Z}_+\}$ is a decreasing sequence of non-empty closed subsets of $(X, d)$ whose diameters tend to 0 and if $(X, d)$ is complete, the $\bigcap_{k=1}^{\infty} E_k$ is a singleton (i.e., consists of a single point).

**Solution:**

Let $x_k \in E_k$, and we show that $\{x_k\}$ is a Cauchy sequence. Let $\varepsilon > 0$ be given, then there exists an $N > 0$ such that $\operatorname{diam} E_N < \varepsilon$. Note that $E_n \subseteq E_m \subseteq E_N$ for $n > m \geq N$ since $E_k$ are decreasing. Hence $x_n, x_m \in E_N$ and $d(x_n, x_m) < \varepsilon$.

Since $\{x_n\}$ was shown to be Cauchy, there exists a unique limit, say $x$, that they converge to.

We now show $x \in \bigcap_{k=1}^{\infty} E_k$. Fix an arbitrary $k$, then $E_k \supseteq E_n$ for each $n \geq k$ since the sets are decreasing. The subsequence $\{x_n\}_{n=k}^{\infty}$ converges to $x$, and since $E_k$ is closed, $x \in E_k$. Since $k$ was arbitrarly choosen, $x \in \bigcap_{k=1}^{\infty} E_k$.

It remains to show that $\bigcup_{k=1}^{\infty} E_k$ is a singleton. Let $\varepsilon > 0$ be given, then there exists a $n$ such that $\operatorname{diam} E_n < \varepsilon$. Since $\bigcap_{k=1}^{\infty} E_k \subseteq E_n$, we have $\operatorname{diam} \bigcap_{k=1}^{\infty} E_k \leq \operatorname{diam} E_n < \varepsilon$. Hence $\operatorname{diam} \bigcap_{k=1}^{\infty} E_k = 0$. This implies $\bigcap_{k=1}^{\infty} E_k$ has only one element, for if otherwise then it has two distinct elements for which $d(x, y) > 0$.

□

**7.** Let $(X, d)$ be a complete metric space and suppose that $T : X \to X$ is a continuous mapping such that

$$d(T(x), T(y)) \leq \alpha[d(T(x), x) + d(T(y), y) + d(x, y)]$$

for all $x, y \in X$ and for $\alpha$ independent of $x$ and $y$ satisfying $0 < \alpha < 1/3$. Prove that $f$ has a unique fixed point.

**Solution:**

Let $x_0$ be some fixed element in $X$ and $x_n = T(x_{n-1})$. We will show that this sequence is Cauchy. Observe, first

$$\begin{aligned}
d(T(x_n), T(x_n)) &\leq \alpha[d(T(x_n), x_n) + d(T(x_{n-1}), x_{n-1}) + d(x_n, x_{n-1})] \\
d(x_{n+1}, x_n) &\leq \alpha[d(x_{n+1}, x_n) + d(x_n, x_{n-1}) + d(x_n, x_{n-1})] \\
(1 - \alpha)d(x_{n+1}, x_n) &\leq 2\alpha\, d(x_n, x_{n-1}) \\
d(x_{n+1}, x_n) &\leq \frac{2\alpha}{1 - \alpha} d(x_n, x_{n-1}) \\
d(x_{n+1}, x_n) &\leq \beta^n d(x_1, x_0)
\end{aligned}$$

where $\beta = \frac{2\alpha}{1-\alpha} < \frac{2/3}{1-1/3} = 1$. Now, for $n > m$

$$\begin{aligned}
d(x_n, x_m) &\leq d(x_n, x_{n-1}) + d(x_{n-1}, x_{n-2}) + \cdots + d(x_{m+1}, x_m) \\
&\leq \beta^{n-1} d(x_1, x_0) + \beta^{n-2} d(x_1, x_0) + \cdots + \beta^m d(x_1, x_0) \\
&= d(x_1, x_0) \frac{\beta^m - \beta^n}{1 - \beta}
\end{aligned}$$

which can be made arbitrarily small by making $n > m \geq \log_\beta \frac{d(x_1, x_0)}{\varepsilon(1-\beta)}$ for an arbitrary $\varepsilon > 0$.

Since $X$ is complete, there exists a limit to this sequence, say $x$. Observe,

$$\begin{aligned}
d(T(x), x) &\leq d(T(x), x_n) + d(x_n, x) \\
&\leq \alpha[d(T(x), x) + d(x_n, x_{n-1}) + d(x, x_{n-1})] + d(x_n, x) \\
d(T(x), x) &\leq (1 - \alpha)^{-1}[\alpha d(x_n, x_{n-1}) + \alpha d(x, x_{n-1}) + d(x_n, x)] \\
&\leq (1 - \alpha)^{-1}[(1 + \alpha)d(x_n, x) + 2\alpha d(x_{n-1}, x)].
\end{aligned}$$

Hence, we need only choose $N$ such that for each $n > N + 1$ implies $d(x_n, x) < (1 - \alpha)/(1 + \alpha)\varepsilon < \varepsilon$ since $\frac{2\alpha}{1-\alpha} < 1$. So, $d(T(x), x) = 0$ and thus $T(x) = x$.

$\square$

**8.** A function $f : \mathbb{C} \to \mathbb{R}$ is called *lower semicontinuous* if, for every sequence $(z_n)$ in $\mathbb{C}$ converging to $z$ we have

$$\liminf_{n\to\infty} f(z_n) \geq f(z).$$

Prove that a lower semicontinuous function $f : \mathbb{C} \to \mathbb{R}$ which is bounded below and satisfies $\lim\limits_{|z|\to\infty} = \infty$, has a minimum value.

**Solution:**

Since $f(z)$ is bounded below, $\inf\{f(z) : z \in \mathbb{C}\} = M > -\infty$. Also, for each $n \in \mathbb{N}$, since $M-1/n$ is not a lower bound, there exists $z_n$ such that $M \leq f(z_n) < M-1/n$. Since $\{f(z_n)\}$ converges, it is bounded. Thus $\{|z_n|\}$ is also bounded, otherwise $\{f(z_n)\}$ would be unbounded. Hence, there exists a subsequence $z_{n_k}$ that converges to some $z_0 \in \mathbb{C}$. Since $f(z_n) \to M$, it follows that $f(z_{n_k}) \to M$ also. Observe now

$$M = \lim_{n\to\infty} f(z_n) = \liminf_{k\to\infty} f(z_{n_k}) \geq f(z_0).$$

Hence, $f(z_0) = M$ since $M \leq f(z_0)$ as it is a lower bound.

$\square$

**9.** Suppose $f$ is analytic in the annulus $\{z : 1 \leq |z| \leq 2\}$. Suppose that $|f(z)| \leq 3$ if $|z| = 1$ and that $|f(z)| \leq 12$ if $|z| = 2$. Show that $|f(z)| \leq 3|z|^2$ for $1 \leq |z| \leq 2$.

**Solution:**

Consider the function $g(z) = \frac{f(z)}{z^2}$ which is analytic on the annulus $1 \leq |z| \leq 2$. By the maximum modulus principle, $|g(z)|$ is bounded on the boundary of the annulus, $|z| = 1$ or $|z| = 2$. When $|z| = 1$ then $|g(z)| \leq \frac{3}{1} = 3$ and when $|z| = 2$, $|g(z)| \leq \frac{12}{2^2} = 3$. In either case, $|g(z)| \leq 3$, and thus for all $z$ in the annulus. Hence

$$\left|\frac{f(z)}{z^2}\right| \leq 3 \implies |f(z)| \leq 3|z^2| = 3|z|^2.$$

$\square$

**10.** Compute

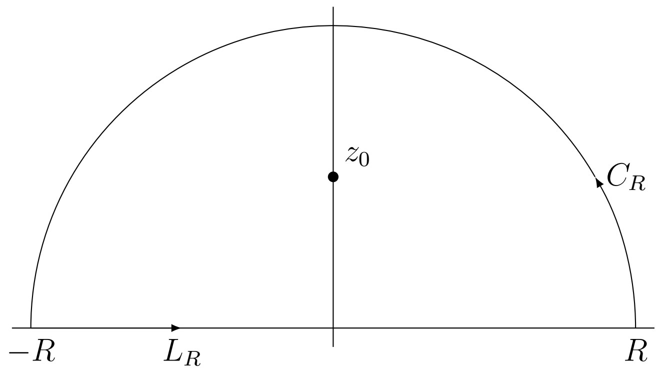
$$\int_{-\infty}^{\infty} \frac{\cos x}{(1+x^2)^2}\, dx.$$

**Solution:**

Let

$$f(z) = \frac{e^{iz}}{(1+z^2)^2}$$

Let $C_R$ be a contour whose image is the upper half circle of radius $R$ centered at $z = 0$ oriented positively, and $L_R$ be along the real axis from $-R$ to $R$. Note that $(1+z^2)^2$ is a fourth degree polynomial with two roots $\pm i$, both of order two. Hence $f$ has exactly one isolated singularity in the interior of the closed contour $C_R + L_R$, which is a pole of order two at $z = i$. Thus




$$\frac{1}{2\pi i}\int_{C_R+L_R} \frac{f(z)}{(1+z^2)^2} = \operatorname*{Res}_{z=i} f(z) = \frac{d}{dz}\frac{e^{iz}}{(1+i)^2}\bigg|_{z=i} = \frac{(2i)^2 ie^{-1} - e^{-1}2(2i)}{(2i)^4} = \frac{1}{2ie}.$$

Now, for $z = x + iy$,

$$\left|\int_{C_R} \frac{e^{iz}}{(1+z^2)^2}\, dz\right| \leq \frac{e^{-y}}{R^4 - 2R^2 - 1} \cdot \pi R \to 0 \quad \text{ as } R \to \infty.$$

Hence

$$\frac{\pi}{e} = \lim_{R\to\infty} \int_{C_R+L_R} f(z)\, dz = \int_{-\infty}^{\infty} \frac{e^{ix}}{(1+x^2)^2}\, dx$$

and equating real parts gives the desired result.

$\square$

**11.** Let $\omega = -\frac{1}{2} + \frac{1}{2}i\sqrt{3}$.

**(a)** If $g$ is an entire function such that $g(z) = g(\omega z)$ for all $z \in\in \mathbb{C}$, show that the power series expansion for $g$ contains only powers of $z$ with exponents divisible by 3.

**Solution:**

Since $g$ is entire, for any $|z| > 0$ we can represent both $g(z)$ and $g(z\omega)$ as power series centered at 0, and thus

$$0 = g(z) - g(\omega z) = \sum_{k=0}^{\infty} a_k z^k - \sum_{k=0}^{\infty} a_k (z\omega)^k = \sum_{k=1}^{\infty} a_k z^k (1 - \omega^k).$$

By the uniqueness of power series representation, each $a_k z^k (1 - \omega^k) = 0$. Note that $\omega = e^{2\pi i/3}$ is a primative third root of unity. So for each coefficient where 3 does not divide $k$, $a_k = 0$.

$\square$

**(b)** If $f$ is an entire function, show that there exist entire functions $g, h$, and $k$ such that

- $f(z) = g(z) + h(z) + k(z)$ for all $z \in\in \mathbb{C}$,
- $g(z) = g(\omega z)$ for all $z \in \mathbb{C}$,
- $h(\omega z) = \omega h(z)$ for all $z \in \mathbb{C}$, and
- $k(\overline{\omega} z) = \omega k(z)$ for all $z \in \mathbb{C}$.

**Solution:**

Let

$$g(z) = \sum_{n=0}^{\infty} a_{3n} z^{3n}, \quad h(z) = \sum_{n=0}^{\infty} a_{3n+1} z^{3n+1}, \quad k(z) = \sum_{n=0}^{\infty} a_{3n+2} z^{3n+2}$$

Since $g, h$, and $k$ are composed of the each of the terms of $f$ counting by three, it is clear that their sum is $f$. Moreover, since $f$ was assumed to be entire, each of the series above converges for all $z$ since they have fewer terms. Calculating similarly to **(a)**, $g(z) = g(\omega z)$. Moreover,

$$h(\omega z) = \sum_{n=0}^{\infty} a_{3n+1} (e^{2\pi i/3} z)^{3n+1} = \omega h(z)$$

and

$$g(\omega z) = \sum_{n=0}^{\infty} a_{3n+1} (e^{2\pi i/3} z)^{3n+2} = e^{4\pi i/3} h(z) = \overline{\omega} h(z)$$

since $e^{4\pi i/3} \cdot e^{2\pi i/3} = 1$.

$\square$

**12.** Use complex numbers to find (and proov) a closed-form expression for

$$\sum_{k=0}^{n} \cos(k\theta).$$

**Solution:**

Well,

$$\begin{aligned}
\sum_{k=0}^{n} \cos(k\theta) &= \sum_{k=0}^{n} \frac{e^{ik\theta} + e^{-ik\theta}}{2} \\
&= \frac{1}{2} \left( \sum_{k=0}^{n} e^{ik\theta} + \sum_{k=0}^{n} e^{-ik\theta} \right) \\
&= \frac{1}{2} \left( \frac{1 - e^{i\theta(n+1)}}{1 - e^{i\theta}} + \frac{1 - e^{-i\theta(n+1)}}{1 - e^{-i\theta}} \right) \\
&= \frac{1}{2} \left( \frac{1 - e^{i\theta(n+1)} - e^{-i\theta} + e^{i\theta n} + 1 - e^{-i\theta(n+1)} - e^{i\theta} + e^{i\theta n}}{|1 - e^{i\theta}|^2} \right) \\
&= \frac{1 - \cos(n+1)\theta - \cos\theta + \cos n\theta}{2 + 2\cos\theta}
\end{aligned}$$

$\square$

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