Problem 1.

Solution First we notice that if $z_2 = 0$ we get that

$$0 = n\left|\frac{z_2}{z_1}\right|^{n-1} < 1 = \frac{|z_1|}{|z_1| - |z_2|}$$

SO the result holds trivially for $z_2 = 0$ So now assume that $|z_2| > 0$ Then we have that

$$\frac{|z_1|}{|z_1| - |z_2|} = \frac{1}{1 - \left|\frac{z_2}{z_1}\right|} = \sum_{k=0}^{\infty}\left|\frac{z_2}{z_1}\right|^k > \sum_{k=0}^{n-1}\left|\frac{z_2}{z_1}\right|^k$$

Now since $\left|\frac{z_2}{z_1}\right| < 1$, we have that $\left|\frac{z_1}{z_2}\right|^{n-1} \leq \left|\frac{z_1}{z_2}\right|^k$ for all $0 \leq k \leq n-1$. Hence

$$\sum_{k=0}^{n-1}\left|\frac{z_2}{z_1}\right|^k \geq \sum_{k=0}^{n-1}\left|\frac{z_2}{z_1}\right|^{n-1} = n\left|\frac{z_2}{z_1}\right|^{n-1}$$

and thus

$$\frac{|z_1|}{|z_1| - |z_2|} > n\left|\frac{z_2}{z_1}\right|^{n-1}$$

Problem 2

Solution Let x_0 be a discontinuity of f. Then for any increasing sequence a_n such that $a_n \to x_0$ then by virtue of the increasing nature of f we get $f(a_n) \leq f(a_{n+1}) \leq f(x_0)$ and thus $f(x_0-) - \lim_{x\to x_0^-} f(x)$ exists. We can similarly show that $f(x_0+) = \lim_{x\to x_0^+} f(x)$ exists by considering decreasing sequences to x_0. Now since x_0 is a discontinuity, and since f is increasing we must have that $f(x_0-) < f(x_0+)$. Now let A be the setr of discontinuities of f in $[a, b]$. Define, for $n \geq 1$:

$$B_n = \left\{x \in A : \frac{1}{n+1} < f(x+) - f(x-) \leq \frac{1}{n}\right\}$$

and

$$B_0 = \{x \in A : 1 < f(x+) - f(x-)\}$$

Now assume that A is uncountable. Notice that

$$A = \bigcup_{k=0}^{\infty} B_k$$

Since A is uncountable, there must be a B_j that is infinite, else A is the countable union of finite sets, and would thus have to be countable. Since B_j is infinite then it has a subset $\{x_i\}_{i=1}^{\infty}$ such that $a < x_1 < x_2 < \ldots < b$. See that for ever n we have, by virtue of f being increasing

$$\sum_{k=1}^{n} f(x_k+) - f(x_k-) \leq \sum_{k=1}^{n} f(x_{k+1}+) - f(x_k-) = f(x_n+) - f(x_1-) \leq f(b+) - f(a-)$$

So then the partial sums of

$$\sum_{k=1}^{\infty} f(x_k+) - f(x_k-)$$

are bounded and increasing, which means the series converges. But, since each $x_k \in B_j$ we have that

$$\sum_{k=1}^{\infty} f(x_k+) - f(x_k-) > \sum_{k=1}^{\infty} \frac{1}{j+1} = \infty$$

which implies that the series should diverge. This is a contradiction, and so then A cannot be uncountable.

Problem 3

Solution Suppose there is a $y \in [a, b)$ such that $f(y) < f(b)$, noting that they cannot be equal else the one to one condition fails. Consider a $x_1 \in (y, b]$. If $f(x_1) < f(y) < f(b)$, then, by the intermediate value theorem, there is a $x_2 \in (x_1, b)$ such that $f(x_2) = f(y)$, which contradicts that f is one to one. So then we must have that $f(y) < f(x)$ for all $x \in (y, b)$. With this fact we can complete the proof. Suppose that $f(a) < f(b)$, then by the above fact we have that $f(a) < f(x)$ for all $x \in (a, b]$. Now if $f(x) > f(b)$, then we would have that $f(a) < f(b) < f(x)$, and the IVT would give that there is a $x_1 \in (a, x)$, such that $f(x_1) = f(b)$, a contradiction. So we have that $f(a) < f(x) < f(b)$ for all $x \in (a, b)$. Now we can apply the above fact with any $y \in (a, b)$ instead of a, to get that $f(y) < f(x)$ for all $x \in (y, b)$. This gives that f is increasing. Now if $f(a) > f(b)$, then we need consider $g = -f$ to get that $g(a) < g(b)$, and by the previous argument we have that g is increasing, and thus f is decreasing.

Problem 4

Solution For any $\beta \in \mathcal{A}$, define $a_\beta = \{x \in E : f_\beta(x) > a\}$, and let

$$A_a = \{x \in E : f(x) = \sup_\alpha f_\alpha(x) > a\}.$$

First we claim that $a_\beta \subset A_a$ for all a and $\beta \in \mathcal{A}$. To show this let $y \in a_\beta$. Then we have that $f_\beta(y) > a$, and thus we have that $\sup_\alpha f_\alpha(y) > a$, and thus $y \in A_a$. Next we show that a_β is open, whcih follows from the fact that

$$a_\beta = f_\beta^{-1}(a, \infty)$$

And since (a, ∞) is open and f_β is continuous, then we must have that its inverse image is open, hence a_β is open. Now we prove that f is lsc. THis amounts to showing that A_a is open. If $x \in A_a$, then we have that $\sup_\alpha f_\alpha(y) > a$, and thus we must have that there is a β such that $f_\beta(y) > a$, else a would be greater than the supremum. This gives that $y \in a_\beta \subset A_a$, and so there is an open subset of A_a that contains y, and since all our choices were arbitrary, we have there every point of A_a has an open neighborhood contained in A_a, and thus A_a is open.

Now since $f(x) > 0$ we have that $E = f^{-1}(0, \infty)$. Furthermore we have that

$$E = f^{-1}(0, \infty) - \bigcup_{a>0} f^{-1}(a, \infty)$$

Now, since f is lsc, we have that each of the $f^{-1}(a, \infty)$ is open, and the above union constitutes an open cover of E, and since E is compact, there must be a finite subcover, i.e. there are $a_n > 0$ such that

$$E = \bigcup_{k=1}^{n} f^{-1}(a_n, \infty)$$

If we let $\delta = \min a_n$, then we have that $E = f^{-1}(\delta, \infty)$, and thus we have that $f(x) > \delta > 0$.

Now, this result need not be true for the case that f is usc. Let $E = [0, 1]$ with the subspace topology. Consider the following function

$$f(x) = \begin{cases} 1 & x = 0 \\ x & 0 < x \leq 1 \end{cases}$$

It is easily seen that this does not satsfy the desired condition fromthe previous part. We want to show that it is USC. Now for $a \leq 0$ we have that

$$\{x \in E : f(x) > a\} = \emptyset$$

For $0 < a \leq 1$:

$$\{x \in E : f(x) > a\} = (0, a)$$

and for $a > 1$:

$$\{x \in E : f(x) > a\} = [0, 1] = E$$

In all cases, these sets are open, and thus f is usc.

Problem 5

Solution We can give such a sequence by usind a doubly indexed sequence, which is still a countable list, and thus essentially still a sequence, when considered under a lexicograpghical ordering. Define for any n and $0 \leq m < n$:

$$f_{n,m}(x) = \chi_{[\frac{m}{n}, \frac{m+1}{n}]}(x)$$

the characteristic function of the interval $[\frac{m}{n}, \frac{m+1}{n}]$. First note that since these are all characteristic functions of measurables sets, they are themselves trivially measurable. Moreover

$$\int_0^1 \chi_{[\frac{m}{n}, \frac{m+1}{n}]} = \frac{1}{n} \to 0$$

However, let x be an arbitrary element of $[0, 1]$. Then for any n, there is an m such that $\frac{m}{n} \leq x \leq \frac{m+1}{n}$, which is found by choosing the largest m such that $\frac{m}{n} \leq x$. In other words, for any n there is an m such that $f_{n,m}(x) = 1$, and thus $\{f_{n,m}(x)$ cannot be convergent.

Problem 6

Solution

Problem 7

Solution We shall first show that f is uniformly continuous and thus continuous. First note that since X is compact, then we must have that f_n is uniformly equicontinuous. this follows from noting that the negation of uniform equicontinuity is that there is one f_n is the sequence that fails to be uniformly continuous, but since X compact, and each f_n is continuous, they all must be uniformly continuous. Now then we can go on to show the uniform continuity of f. Let $\epsilon < 0$ be given, and let δ be such that $d(x,y) < \delta$ implies $|f_n(x) - f_n(y)| < \frac{\epsilon}{2}$ for all n (we appeal to the uniform equicontinuity to get such a δ). Now let $n \to \infty$, and since we have pointwise convergence, and absolute value is continuous, we get that

$$|f(x) - f(y)| = \lim_{n\to\infty} |f_n(x) - f_n(y)| \leq \frac{\epsilon}{2} < \epsilon$$

Hence f is uniformly continuous.

We can now move on to proving that the convergence is uniform. let $\epsilon > 0$ be given, and suppose that δ_1 is such that $d(x,y) < \delta_1 \Rightarrow |f_n(x) - f_n(y)| < \frac{\epsilon}{3}$, we are appealing to the uniform equicontinuity to get δ_1. Similarly, let δ_2 be such that $d(x,y) < \delta_2 \Rightarrow |f(x) - f(y)| < \frac{\epsilon}{3}$, and we are appealing to the uniform continuity of f to get such a δ_2. Now let $\delta = \min\{\delta_1, \delta_2\}$. We note that

$$\bigcup_{x\in X} B(x,\delta)$$

forms an open cover of X, and by compactness, we must have a finite subcover i.e. there are finitely many $x_i \in X$ such that

$$X = \bigcup_{i=1}^{k} B(x_i,\delta).$$

Now let N_i be such that $|f_n(x_i) - f(x_i)| < \frac{\epsilon}{3}$ for all $n \geq N_i$, and let $N = \max_{1\leq i\leq k} N_i$. Now, for any x there is an x_i such that $x \in B(x_i,\delta)$, and thus for all $n \geq N$, we have, forr any $x \in X$:

$$|f_n(x) - f(x)| = |f_n(x) - f_n(x_i) + f)n(x_i) - f(x_i) + f(x_i) - f(x)|$$

$$\leq |f_n(x) - f_n(x_i)| + |f_n(x_i) - f(x_i)| + |f(x_i) - f(x)|$$

Since $d(x,x_i) < \delta < \delta_1$ we get that $|f_n(x) - f_n(x_i)| < \frac{\epsilon}{3}$, and similarly, since $d(x,x_i) < \delta < \delta_2$ we get that $|f(x_i) - f(x)| < \frac{\epsilon}{3}$. Because $n \geq N \geq N_i$ we have that $|f_n(x) - f_n(x_i)| < \frac{\epsilon}{3}$. Combining all these we get that for all $n \geq N$ and any $x \in X$:

$$|f_n(x) - f(x)| < \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon$$

So then f_n converges uniformly to f.

Problem 8

Solution By the maximum modulus theorem we know that it will attain the maximum modula of the boundary of this sqaure. So then we need only consider the 4 sides. Let $0 \leq x, y \leq 2\pi$.
In the case where $z = x$ we see that $\max|\sin z| = \max|\sin x| = 1$.
When $z = iy$ we have that

$$\max|\sin z| = \max|\sin iy| = \max\frac{|e^{i(iy)} - e^{-i(iy)}|}{2} = \max\frac{|e^{-y} - e^{y}|}{2} = \max\frac{e^{y} - e^{-y}}{2}$$

Now since

$$\frac{\partial}{\partial y}\frac{e^{y} - e^{-y}}{2} = \frac{e^{y} + e^{-y}}{2} > 0$$

we have that this function is an increasing one. So then

$$\max|\sin iy| = \max\frac{e^{y} - e^{-y}}{2} = \frac{e^{2\pi} - e^{-2\pi}}{2}$$

Now, when we consider the case that $z = 2\pi + iy$ note that

$$\sin(2\pi + iy) = \cos(2\pi)\sin(iy) + \cos(iy)\sin(2\pi) = \sin(iy)$$

So then the prefvious case applies.
When we look at $z = x + 2\pi i$, note that, using the triangle inequality

$$|\sin(x + 2\pi i)| = \frac{|e^{ix}e^{-2\pi} - e^{-ix}e^{2\pi}|}{2} \leq \frac{e^{-2\pi} + e^{2\pi}}{2}$$

and so it cannot exceeed the maximum given on the line $z = iy$. In summary the maximum occurs when $z = 2\pi i$ and it is equal to $\frac{e^{-2\pi}+e^{2\pi}}{2}$.

Problem 9

Solution Assume that f is non constant. Since Ω is connected and open, the open mapping theorem applies, which says that f must map Ω to an open subset of $\mathbb{C}$. But since $|f| = \alpha$ is constant then we have that the image of f is contained in the circle $|z| = \alpha$, and thus cannot be an open subset of $\mathbb{C}$ (this is because every neighborhood of a point on a circle, by definition, contains a point not on the circle). This is a contradiction, and so f must be constant.

Problem 10

Solution So, first notice that $sin(z)$ is nonzero off of the real line. So then we have isolated singularities at $\pm n\pi$, and at 0. We will calculate the integral using these residues in a standard way. The most taxing of these residues will be the one at 0. We will be using the laurent expansion to find it. As a first step we will find the laurent expansion of $\frac{1}{\sin(z)}$. Notice that

$$\lim_{z \to 0}\frac{z}{\sin(z)} = 1$$

and thus $\frac{1}{\sin(z)}$ has a simple pole at 0. This gives that, plugging is the series expansion for $\sin(z)$:

$$\frac{1}{z - \frac{z^3}{3!} + \frac{z^5}{5!} - \dots} = \frac{a_{-1}}{z} + a_0 + a_1 x + \dots$$

Multplying both sides by z gives

$$\frac{1}{1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \dots} = a_{-1} + a_0 z + a_1 z^2$$

Letting $z = 0$ gives that $a_{-1} = 1$. Differentiating the above expansion gives that

$$\frac{-(-\frac{2z}{3!} + \frac{4z^3}{5!} - \dots)}{(1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \dots)^2} = a_0 + 2a_1 z + \dots$$

Again let $z = 0$ to get that $a_0 = 0$. Differentiate one more time (left as an exercise for the reader), and plug in $z = 0$ again to get that

$$\frac{1}{3} = 2a_1 \Rightarrow a_1 = \frac{1}{6}$$

This should come as no surprise. This gives the series exapansion

$$\frac{1}{z^2 \sin(z)} = \frac{1}{z^3} + \frac{1}{6}\frac{1}{z} + \dots$$

and thus we have that $Res(\frac{1}{z^2 \sin(z)}, 0) = \frac{1}{6}$. Further more we can see that since we have simple poles at $\pm n\pi$

$$Res(\frac{1}{z^2 \sin(z)}, \pm n\pi) = \frac{\frac{1}{(\pm n\pi)^2}}{\sin'(\pm n\pi)} \frac{(-1)^n}{n^2\pi^2}$$

So then the value of the integral is $2\pi i$ time the sum of all the residues, noting that for negative n the residues are the same as positive n, and so we can just double them. This gives the resulting formula

$$\int_{C_N} \frac{1}{z^2 \sin(z)} dz = 2\pi i \left[\frac{1}{6} + 2 \sum_{n=1}^{N} \frac{(-1)^n}{\pi^2 n^2} \right]$$

It is easily apparent that this intergral converges to 0 as $N \to \infty$ (Consider the modulus, and the fact that z^2 and $\sin(z)$ take on maximum values on the boundary). Hence

$$\frac{1}{6} + 2\sum_{n=1}^{\infty} \frac{(-1)^n}{\pi^2 n^2} = 0 \Rightarrow -\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} = \frac{\pi^2}{12} \Rightarrow \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} = \frac{\pi^2}{12}$$