4. Let f be a function defined on $[1/2, 1]$ which is continuous at 1. Prove that the sequence $(x^n f(x))_n$ converges for every $x \in [1/2, 1]$, and that this convergence is uniform if and only if f is bounded and $f(1) = 0$.

Solution:

First if $x = 1$, then $x^n f(x) = f(1)$ for all n, and thus converges to $f(1)$. Fix $x \in [1/2, 1)$, then for a given $\varepsilon > 0$,

$$n > \ln\left(\frac{\varepsilon}{|f(x)|}\right) / \ln(x) \iff n \ln x < \left(\frac{\varepsilon}{|f(x)|}\right) \implies x^n < \frac{\varepsilon}{|f(x)|}$$

by applying the increasing function e^x to both sides of the inequality. Hence $|x^n f(x)| = x^n|f(x)| < \varepsilon$ implies $(x^n f(x))_n$ converges to 0. and thus $(x^n f(x))_n$ converges to the function g where $g(x) = 0$ when $1/2 \leq x < 1$ and $g(1) = 1$.

Now, suppose $|f(x)| \leq M$ for each x and $f(1) = 1$. Then if we take

$$n > \ln\left(\frac{\varepsilon}{M}\right) / \ln(x)$$

a similar calculation replacing $|f(x)|$ by M yields $|x^n f(x)| \leq x^n M < \varepsilon$ implies uniform convergence to the constant 0. And, in the case where $f(1) = 0$, is equal to g.

On the other hand, if we assume that the convergence is uniform, then since each function g_n where $g_n(x) = x^n f(x)$ are continuous on $[1/2, 1]$. Hence the point-wise limit $g(x)$ must be continuous which implies $f(1) = \lim f(1/n) = 0$.

□

5. Suppose $f : [0, \infty) \to \mathbb{R}$ is decreasing, and that $\int_0^\infty f(x)dx$ converges. Prove that for every $\delta > 0$ the series $\sum_{n=1}^\infty f(n\delta)$ converges and

$$\lim_{\delta \to 0^+} \delta \sum_{n=1}^{\infty} f(n\delta) = \int_0^\infty f(x)\, dx$$

Solution:

We first show that $f(x) \geq 0$ for all x. Assume otherwise, then there exists an $x_0 < 0$ and since f is decreasing, then $f(x) < f(x_0) < 0$ for all $x > x_0$. Thus

$$\int_0^N f(x)\, dx = \int_0^{x_0} f(x)\, dx + \int_{x_0}^N f(x)\, dx \leq \int_0^{x_0} f(x)\, dx - f(x_0)(x_0 - N) \to -\infty$$

as $N \to \infty$.

Thus, it suffices to to prove that $\sum_{n=1}^\infty f(n\delta)$ is bounded above. For each $\delta > 0$, $(0, \delta, 2\delta, \ldots, N\delta)$ is a partition of $[0, N\delta]$. Since f is decreasing and positive

$$\delta \sum_{n=1}^{N} f(n\delta) = \sum_{P} \inf_{[x_i - x_{i-1}]} \{f(t)\}\Delta_i = L(P, f) \leq \int_0^{N\delta} f(x)\, dx \leq \int_0^\infty f(x)\, dx.$$

Hence $\sum_{n=1}^N f(n\delta) \leq \frac{1}{\delta} \int_0^\infty f(x)\, dx$.

Now, let $\varepsilon > 0$ be given, and let $R > 0$ such that

$$\int_0^\infty f(x)\,dx - \varepsilon < \int_0^R f(x)\,dx.$$

For the family of partitions $(0, \delta, 2\delta, \ldots, N_R\delta, R)$, indexed by all $\delta > 0$, we have that the maximum width of each inteval goes to 0 as $\delta \to 0$, hence, for sufficiently small $\delta > 0$, we have

$$\int_0^R f(x)\,dx - \varepsilon < L(P, f) = \delta \sum_{n=1}^{N} f(n\delta).$$

Hence

$$\int_0^\infty f(x)\,dx - 2\varepsilon < \delta \sum_{n=1}^{\infty} f(n\delta) = L(P, f) \leq \int_0^R f(x)\,dx \leq \int_0^\infty f(x), dx$$

and the convergence is evident.

□

7. Let (X, d) be a metrix space. A function $f : X \to \mathbb{R}$ is called Lipschitz if there is a constant L such that $|f(x) - f(y)| \leq Ld(x, y)$, for all $x, y \in X$.

(a) Show that every Lipschitz function is uniformly continuous and give an example of a uniformly continuous function that is not Lipschitz.

Solution:

Let f be a Lipschitz function with Lipschitz constant L and $\varepsilon > 0$ be given, then observe

$$|f(x) - f(y)| \leq Ld(x, y) < \varepsilon$$

by choosing x, y such that $d(x, y) < \varepsilon/L$.

On the other hand, consider $f : [0, 1] \to \mathbb{R}$ by $f(x) = \sqrt{x}$. This function is uniformly continuous as a continuous function with a compact domain. However, for any given $L \geq 0$, choose $x < \frac{1}{L^2}$ and $y = 0$, then

$$|f(x) - f(y)| = \left|\frac{1}{L}\right| = L\left|\frac{1}{L^2} - 0\right| > L|x - y|.$$

□

Let f be a bounded uniformly continuous real-valued function on X. For each positive integer n, consider the function f_n defined by

$$f_n(x) = \inf\{f(y) + nd(x, y) : y \in X\},$$

for $x \in X$. Prove the following statements:

(b) Each f_n is a Lipschitz function on X.

Solution:

Let $x_1 \neq x_2$ in X labeled such that $f(x_1) \leq f(x_2)$ so that $|f(x_2)-f(x_1)| = f(x_2) - f(x_1)$. Let $\varepsilon > 0$ be given. There exists a $y_1 \in X$ so that

$$f(y_1) + nd(x_1, y_1) \leq f_n(x) + \varepsilon$$

since $f_n(x)$ is the greatest of the upper bounds. Now, for $f_n(x_2)$ we have

$$\begin{aligned} f_n(x_2) &\leq f(y_1) + nd(x_2, y_1) \\ &= f(y_1) + nd(x_2, y_1) - nd(y_1, x_1) + nd(y_1, x_1) \\ &\overset{*}{\leq} f(y_1) + nd(x_2, x_1) + nd(y_1, x_1) \\ &\leq f_n(x_1) + \varepsilon + nd(x_2, x_1). \end{aligned}$$

Hence, $|f_n(x_2) - f_n(x_1)| \leq nd(x_2, x_1) + \varepsilon$ for an arbitrary $\varepsilon > 0$, which suffices for f_n to be Lipschitz.

$^*nd(x_2, y_1) \leq nd(y_1, x_1) + nd(x_2, x_1) \implies nd(x_2, y_1) - nd(y_1, x_1) \leq nd(x_2, x_1)$

$\square$

(c) $f_n \to f$ uniformly on X as $n \to \infty$.

Solution:

Let $|f(x)| \leq M$ and $\varepsilon > 0$ be given. First, observe $f_n(x) \leq f(x) + nd(x, x) = f(x)$, hence $|f(x) - f_n(x)| = f(x) - f_n(x)$. Let $\delta > 0$ be such that $|f(x) - f(y)| < \frac{\varepsilon}{2}$ whenever $d(x, y) < \delta$. Choose $n \geq \frac{2M}{\delta}$. Now, for any $x \in X$, there exists y_x such that

$$f(y_x) + nd(x, y_x) \leq f_n(x) + \frac{\varepsilon}{2} \iff f(y_x) - f_n(x) + nd(x, y_x) < \varepsilon.$$

So

$$\begin{aligned} f(x) - f_n(x) &= f(x) - f(y_x) + \big(f(y_x) - f_n(x) + nd(x, y_x)\big) - nd(x, y_x) \\ &< f(x) - f(y_x) - nd(x, y_x) + \frac{\varepsilon}{2}. \end{aligned}$$

We proceed in two cases, when $d(x, y_x) \geq \delta$ or when $d(x, y_x) < \delta$. In the first case, $n > \frac{2M}{\delta}$ implies $nd(x, y_x) > \frac{2M}{\delta}\delta = 2M$, so

$$\begin{aligned} f(x) - f_n(x) &< f(x) - f(y_x) - 2M + \frac{\varepsilon}{2} \\ &\leq (f(x) - M) - (f(y_x) + M) + \frac{\varepsilon}{2} \\ &\leq \frac{\varepsilon}{2}, \end{aligned}$$

where we used the facts that $f(x) \leq M$ and $f(y_x) \geq -M$. In the other case where $d(x, y_x) < \delta$,

$$\begin{aligned} f(x) - f_n(x) &< \frac{\varepsilon}{2} - nd(x, y_x) + \frac{\varepsilon}{2} \\ &\leq \varepsilon, \end{aligned}$$

since $nd(x, y_x) \geq 0$. Since our choice of n did not depend on x, the convergence is uniform.

$\square$

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