4. Show that a uniformly continuous real-valued function on a bounded subset of $\mathbb{R}^2$ must have bounded range.

Solution:

Let $f : D \to \mathbb{R}$ and suppose not. Then there exists a sequence $\{f(x_n)\}$ such that $|f(x_n)| > n$. Since $D \subseteq \mathbb{R}^2$, its closure $\overline{D}$ is compact [Str00, thm. 9.2.7]. So, there exists a subsequence $\{x_{n_k}\}$ that converges that converges in D. Hence, $\{x_{n_k}\}$ is Cauchy, so for any given $\delta > 0$, there exists an N such that $\|x_{n_k} - x_{n_j}\| < \delta$ whenever $k > j \geq N$, and, in particular, $\|x_{n_N} - x_{n_{N+1}}\|$. But then

$$|f(x_{n_{N+1}}) - f(x_{n_N})| \geq |f(x_{N+1})| - |f(x_{n_N})| > n_{N+1} - n_N > 1.$$

Thus, f cannot be uniformly continuous.

□

5. Prove that there exists a unique continuous real-valued function f on $[0,1]$ such that

$$f(x) = \int_0^x \cos(tf(t))\,dt,$$

for all x in $[0,1]$.

Solution:

Let $T : C([0,1]) \to C([0,1])$ by $T[f](x) = \int_0^x \cos(tf(t))\,dt$. Observe

$$\begin{aligned}
\sup_{x\in[0,1]} |T[f](x) - T[g](x)| &= \sup_{x\in[0,1]} \left| \int_0^x \cos(tf(t))\,dt - \int_0^x \cos(tg(t))\,dt \right| \\
&= \sup_{x\in[0,1]} \left| \int_0^x \Big\{ \cos(tf(t)) - \cos(tg(t)) \Big\}\,dt \right| \\
&= \sup_{x\in[0,1]} \left| \int_0^x \int_{tg(t)}^{tf(t)} \sin(y)\,dydt \right| \\
&\leq \sup_{x\in[0,1]} \int_0^x \int_{tg(t)}^{tf(t)} |\sin(y)|\,dydt \\
&\leq \sup_{x\in[0,1]} \int_0^x |tf(t) - tg(t)| \cdot 1dt \\
&\leq \int_0^1 t|f(t) - g(t)|dt \\
&= \int_0^{1/2} t|f(t) - g(t)|dt + \int_{1/2}^1 t|f(t) - g(t)|dt \\
&\leq \frac{1}{4} \sup_{t\in[0,1]} |f(t) - g(t)| + \frac{1}{2} \sup_{t\in[0,1]} |f(t) - g(t)| \\
&= \frac{3}{4} \sup_{t\in[0,1]} |f(t) - g(t)|.
\end{aligned}$$

So, T as a map from $C([0,1])$ to itself under the sup metric (which is a complete space) is a contraction. By the contraction mapping principle, there is a unique f so that $T[f] = f$, yielding the desired equality.

$\square$

6. A real-valued function f on $\mathbb{R}$ is called Lipschitz if there exists a positive number L such that $|f(x) - f(y)| \leq L|x-y|$, for all $x, y \in \mathbb{R}$. The least number L for which the above inequality holds is called the Lipschitz constant for f.

(a) Show that $f(x) = \arctan(x)$ is a Lipschitz function, and determine its Lipschitz constant.

Solution:

Recall that

$$\frac{d}{dx} \arctan x = \frac{1}{1+x^2},$$

so

$$|\arctan x - \arctan y| = \left| \int_y^x \frac{1}{1+t^2}\, dt \right| \leq |x-y| \cdot 1$$

since

$$\left| \frac{1}{1+t^2} \right| < 1.$$

Hence $L \leq 1$, and in fact $L = 1$. To see this, observe that for $y = -x$,

$$\lim_{x\to 0} \frac{|\arctan(x) - \arctan(-x)|}{|x-(-x)|} = \lim_{x\to 0} \frac{\arctan x}{x} \stackrel{\text{l'Hop}}{=} \lim_{x\to 0} \frac{(1+x^2)^{-1}}{1} = 1$$

So, for any $L < 1$ then we can take x sufficiently small so that

$$\begin{aligned} &\left| \frac{|\arctan x - \arctan(-x)|}{|x-(-x)|} - 1 \right| < (1-L) \\ &\implies |\arctan x - \arctan(-x)| > (-(1-L)+1)|x-(-x)| = L|x-(-x)| \end{aligned}$$

so any $L < 1$ cannot be a Lipschitz constant, and thus $L \geq 1$.

$\square$

(b) If f_n is a sequence of real-valued Lipschitz functions on $\mathbb{R}$ uniformly converging to a real-valued function f on $\mathbb{R}$, then prove that f is uniformly continuous.

Solution:

Take $\{L_n\}$ the sequence of Lipschitz constants for $\{f_n\}$, then $|f_n(x) - f_n(y)| \leq L_n|x-y|$. Let $N > 0$ so that

$$\sup_{x\in\mathbb{R}} |f_n(x) - f(x)| < \varepsilon/3 \quad \text{for } n \geq N.$$

Observe

$$\begin{aligned} |f(x) - f(y)| &\leq |f(x) - f_N(x)| + |f_N(x) - f_N(y)| + |f_N(y) - f(y)| \\ &< \frac{\varepsilon}{3} + L_N|x-y| + \frac{\varepsilon}{3} \\ &< \varepsilon \quad \text{for } |x-y| < \varepsilon/(3L_N). \end{aligned}$$

□

7. Let f be a continuously differentiable real-valued function on $[0, b]$, where $b > 0$, with $f(0) = 0$.

(a) Show that

$$f(x)^2 \leq 2x^{1/2} \int_0^x t^{1/2} f'(t)^2 \, dt,$$

for all x in $[0, b]$.

Solution:

Let $g(t) = t^{-1/4}$ and $h(t) = t^{1/4} f'(t)$, then with the observation that $\frac{d}{dt} g^2(t) = 2x^{1/2}$, Cauchy-Schwartz-Bunyakovski says

$$\begin{aligned} 2x^{1/4} \int_0^x t^{1/2} f'(t) \, dt &= \int_0^x g(t)^2 \, dt \int_0^x h(t)^2 \, dt \\ &\geq \left(\int_0^x g(t) h(t) \right)^2 \\ &= \left(\int_0^x f'(t) \, dt \right)^2 \\ &= f(x)^2. \end{aligned}$$

□

(b) Prove that

$$\int_0^b \frac{f(x)^2}{x^2} dx \leq 4 \int_0^b f'(x)^2 dx.$$

Solution:

Dividing both sides of the last result by x^2 and integrating both sides over $(0, b]$, we have

$$\begin{aligned}
\int_0^b \frac{f(x)^2}{x^2}dx &\le \int_0^b \left\{2x^{-3/2}\cdot \int_0^x t^{1/2}f'(t)^2\,dt\right\}dx \\
&= \left\{-4x^{-1/2}\int_0^x t^{1/2}f'(t)\,dt\right\}\Bigg|_{x=0}^{x=b} + \int_0^b 4x^{-1/2}x^{1/2}f'(x)^2dx \\
&= 4\int_0^b f'(x)dx - 4b^{-1/2}\int_0^b t^{1/2}f'(t)^2\,dt + 4\lim_{x\to 0}\frac{\int_0^x t^{1/2}f'(t)^2\,dt}{\sqrt{x}} \\
&= 4\int_0^b f'(x)dx - 4b^{-1/2}\int_0^b t^{1/2}f'(t)^2\,dt + \cancel{4\lim_{x\to 0}\frac{x^{1/2}f'(x)^2}{\frac{1}{2}x^{-1/2}}} \\
&\le 4\int_0^b f'(x)dx.
\end{aligned}$$

□

8. Let (X,d) be a metric space, and assume that d is bounded on X, that is, there exists a number M such that $d(x,y)\le M$, for all $x,y\in X$. For a nonempty subset A of X let f_A denote the function defined by

$$f_A(x)=\inf\{d(x,a): a\in A\},\quad x\in X.$$

Let $\mathcal{C}$ denote the nonempty subsets of X which are closed.

(a) If $x\in X$ and $A\in\mathcal{C}$, then show that $f_A(x)=0$ if and only if $x\in A$.

Solution:

If $x\in A$, then $d(a,a)=0$ and this is the minimum value of $d(x,a)$ hence $f_A(x)=0$. On the other hand, if $x\notin A$ then there exists an open ball, say of radius ε, such that $B_\varepsilon(x)\cap A=\emptyset$ since A is closed. So for each $a\in A$, $d(x,a)\ge\varepsilon>0$, and thus $\inf\{d(x,a): a\in A\}\ge\varepsilon>0$. Hence $f_A(x)\ne 0$.

□

(b) Prove that ρ defined by

$$\rho(A,B)=\sup\{|f_A(x)-f_B(x)|\}: x\in X\}$$

defines a metric on $\mathcal{C}$.

Solution:

i) Clearly, $\rho(A,A)=\sup\{|f_A(x)-f_A(x)|\}=\sup\{0\}=0$. Now, if $\rho(A,B)=0$, then

$$|f_A(x)-f_B(x)|=0$$

for all $x\in X$. In particular whenever $x\in A$, we have

$$0=|0-f_B(x)|=|f_B(x)|\implies x\in B.$$

A symmetric argument proves that for $x\in B$, then $x\in A$. We have shown $A=B$.

ii) The symmetry of $|\cdot|$ gives the symmetry $\rho(A,B) = \rho(B,A)$.

iii) Observe

$$\rho(A,C) = \sup_{x\in X}\{|f_A(x) - f_B(x) + f_B(x) - f_C(x)|\} \leq \sup_{x\in X}\{|f_A(x) - f_B(x)|\} + \sup_{x\in X}\{|f_B(x) - f_C(x)|\}$$

by the triangle inequality in $\mathbb{R}$ and since the terms in the sup are positive.

$\square$

(c) Show that the map $x \mapsto \{x\}$ is an isometry of X into C.

Solution:

Observe

$$\begin{aligned}\rho(\{x\},\{y\}) &= \sup_{z\in X}\{|f_{\{x\}}(z) - f_{\{y\}}(z)|\} \\ &= \sup_{z\in X}\{|d(x,z) - d(y,z)|\}.\end{aligned}$$

The reverse-triangle inequality gives $|d(x,z) - d(y,z)| \leq d(x,y)$ and $|d(x,x) - d(y,x)| = d(x,y)$ so the supremum is acheived at $z = x$. Hence $\rho(\{x\},\{y\}) = d(x,y)$.

$\square$

9. Let X be the set of all bounded functions $f : \mathbb{N} \to \mathbb{R}$.

(a) Show that ρ defined by

$$\rho(f,g) = \sup\{|f(n) - g(n)| : n \in \mathbb{N}\}$$

is a metric on X.

Solution:

i) Clearly $\rho(f,f) = \sup\{|f(n) - f(n)|\} = \sup\{0\} = 0$. If $\rho(f,g) = 0$ then $|f(n) - g(n)| = 0$ for all n, and thus $f(n) = g(n)$.

ii) The symmetry of $|\cdot|$ gives the symmetry $\rho(f,g) = \rho(g,f)$.

iii) Observe

$$\rho(f,h) = \sup_{n\in\mathbb{N}}\{|f(n) - g(n) + g(n) - h(n)|\} \leq \sup_{n\in\mathbb{N}}\{|f(n) - g(n)|\} + \sup_{n\in\mathbb{N}}\{|g(n) - h(n)|\}$$

by the triangle inequality in $\mathbb{R}$ and since the terms in the sup are positive.

$\square$

(b) Prove that the set K consisting of all $f : \mathbb{N} \to \mathbb{R}$ such that $0 \leq f(n) \leq 1/n$, fora ll $n \in \mathbb{N}$ is a compact subset of X.[Hint: Prove that K is closed and totally bounded.]

Solution:

Suppose $g \notin K$ then $g(N) \notin [0, 1/N]$ for some $N \in \mathbb{N}$. So $|g(N) - \frac{1}{2N}| \geq \frac{1}{2N}$. Now consider a ball of radius $\frac{1}{2N}$ about g. If f is in such a ball, then $\rho(f, g) < \frac{1}{4N}$, and thus $|g(N) - f(N)| < \frac{1}{4N}$. Observe

$$\begin{aligned} \left| f(N) - \frac{1}{2N} \right| &= \left| g(N) - \frac{1}{2N} - (g(N) - f(N)) \right| \\ &\geq |g(N) - \frac{1}{2N}| - |g(N) - f(N)| \\ &> \frac{1}{2N} - \frac{1}{4N} = \frac{1}{2N} \end{aligned}$$

So $f \notin K$, and we've shown the complement of K is open, and thus K is closed.

Let $\varepsilon > 0$ be given. For K to be totally bounded, we must provide a finite cover of ε balls. Let $N > 0$ such that $\frac{1}{N} < \varepsilon$.

□

10. Let $a_0, a_1, a_2, \ldots$ be complex numbers such that $\sum_{n=2}^{\infty} n|a_n| < |a_1|$.

(a) Show that the series $\sum_{n=0}^{\infty} a_n z^n$ convertes absolutely for all $|z| < 1$.

Solution:

We first show that a_n is bounded. This follows since the sequence of partial sums $S_N = \sum_{n=2}^{N} n|a_n| < |a_1|$ converges as a monotone increasing, bounded sequence. Hence,

$$S_N - S_{N-1} = N|a_N| < 1$$

for sufficiently large N, and thus

$$|a_n| \leq M = \max\{1, |a_0|, |a_1|\}.$$

Hence,

$$\sum_{n=0}^{N} |a_n z^n| \leq M \sum_{n=0}^{N} |z|^n = M \frac{1 - |z|^{N+}}{1 - |z|} \to \frac{M}{1 - |z|}$$

for all $|z| < 1$.

□

(b) Prove that the analytic function f defined by $f(x) = \sum_{n=0}^{\infty} a_n z^n$ for $|z| < 1$, is injective on the unit disk $\{z \in \mathbb{C} : |z| < 1\}$.

Solution:

Suppose $f(z) = f(w)$. Then

$$\begin{aligned}
0 &= |f(z) - f(w)| \\
&= \left| \sum_{n=1}^{\infty} a_n (z^n - w^n) \right| \\
&= |z - w| \left| \sum_{n=1}^{\infty} a_n \sum_{k=0}^{n-1} z^{n-1-k} w^k \right| \\
&= |z - w| \left| a_1 + \sum_{n=2}^{\infty} a_n \sum_{k=0}^{n-1} z^{n-1-k} w^k \right| \\
&\geq |z - w| \left(|a_1| - \sum_{n=2}^{\infty} |a_n| n \right)
\end{aligned}$$

by the reverse triangle inequality and since the inner sum contains n terms less than 1. By assumption, the second factor is greater than zero, hence $|z - w| = 0$, and $z = w$ implies f is injective.

□

11. Let T be defined by $T(z) = \dfrac{1-z}{1+z}$, for all $z \neq 1$. Show that

$$\operatorname{Re} T(z) = \frac{1 - |z|^2}{|1 + z|^2}.$$

Solution:

Observe

$$\begin{aligned}
\operatorname{Re} T(z) &= \frac{1}{2}\left(T(z) + \overline{T(z)}\right) = \frac{1}{2}\left(\frac{1-z}{1+z} + \frac{1-\overline{z}}{1+\overline{z}}\right) = \frac{1}{2}\left(\frac{(1-z)(1+\overline{z}) + (1-\overline{z})(1+z)}{|1+z|^2}\right) \\
&= \frac{1}{2}\left(\frac{1 - 2\operatorname{Re} z - |z|^2 + 1 + 2\operatorname{Re} z - |z|^2}{|1+z|^2}\right) = \frac{1 - |z|^2}{|1+z|^2}.
\end{aligned}$$

□

(b) Show that T induces an analytic bijection between theunit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ and the right half plane $H = \{z \in \mathbb{C} : \operatorname{Re} z > 0\}$.

Solution:

By part **(a)**, $T(\mathbb{D}) \subseteq H$. Observe for $w \in H$,

$$\frac{1-z}{1+z} = w \iff 1 - z = w + wz \iff 1 - w = z(1 + w) \iff z = \frac{1-w}{1+w}.$$

Moreover,

$$\left| \frac{1-w}{1+w} \right|^2 = \frac{(1-w)(1-\overline{w})}{(1+w)(1+\overline{w})} = \frac{1 - 2\operatorname{Re} w + |w|^2}{1 + 2\operatorname{Re} w + |w|^2} < 1$$

since $\mathrm{Re}\, w > 0$ implies $2\mathrm{Re}\, w > -2\mathrm{Re}\, w$. Hence if we define $\widetilde{T} : H \to \mathbb{D}$ by $\widetilde{T}(w) = T(w)$ then $\widetilde{T}Tz = z$ and $T\widetilde{T}w = w$, and thus T is a bijection. Moreover, both are analytic since $-1 \notin H \cup \mathbb{D}$.

□

Let f be an analytic function on the unit disk $\mathbb{D}$, such that $f(z)$ has positive real part for each $z \in \mathbb{D}$ and $f(0) = 1$. Prove that

$$\frac{1-|z|}{1+|z|} \le |f(z)| \le \frac{1+|z|}{1-|z|}$$

Solution:

The Schwarz Lemma states that if $g : \mathbb{D} \to \mathbb{D}$ with $g(0) = 0$, then $|g(z)| \le |z|$ [Boa10]. Note that $T \circ f : \mathbb{D} \to \mathbb{D}$ is analytic with $T \circ f(0) = T(1) = 0$. Hence

$$|T \circ f(z)| \le |z| \implies \left|\frac{1-f(z)}{1+f(z)}\right| \overset{\dagger}{\le} |z|.$$

Using the reverse triangle inequality in the numerator and standard triangle inequality in the numerator

$$\frac{1-|f(z)|}{1+|f(z)|} \le |z| \iff |f(z)| \ge \frac{1-|z|}{1+|z|}$$

and, similarly

$$\frac{|f(z)|-1}{1+|f(z)|} \le |z| \iff |f(z)| \le \frac{1+|z|}{1-|z|}.$$

□

12. Consider the function $f(z) = \dfrac{ee^{iz}}{\cosh z}$ for all $x \in \mathbb{R}$.

(a) Show that $f(x + ipi) = -e^{-\pi}\frac{e^{ix}}{\cosh x}$.

Solution:

Observe

$$f(x+i\pi) = \frac{e^{ix}e^{-\pi}}{\frac{1}{2}(e^{x+i\pi} + e^{-x-i\pi})} = \frac{e^{ix}e^{-\pi}}{e^{i\pi}\cosh x} = -e^{-\pi}\frac{e^{ix}}{\cosh x}.$$

□

(b) By integrating f around rectangles with vertices $-R, R, R+\pi i$, and $R - \pi i$ for arbitrary large positive numbers R, evaluate the integral

$$\int_{-\infty}^{\infty} \frac{\cos x}{\cosh x}\, dx$$

Solution:

Note $\cosh(i\pi/2) = 0$ so $z_0 = i\pi/2$ is an isolated singularity of f. Moreover $\frac{d}{dz}\cosh(z)\big|_{z_0} = \sinh(z_0) = i$, so f has a simple pole there. If we denote the contour as C, then

$$\int_C f(z)\,dz = 2\pi i \operatorname*{Res}_{z_0} f = 2\pi i\left(\frac{e^{iz_0}}{\sinh(z_0)}\right) = 2\pi e^{-\pi/2}$$

Parametrize the right vertical segment, say L_1, by $z(t) = it + R$ for $0 \le t \le \pi$. Then

$$\left|\int_{L_1} f(z)\,dz\right| = \left|\int_0^\pi \frac{e^{-t}e^{iR}}{\frac{1}{2}(e^R e^{it} + e^{-R}e^{-it})}\,idt\right| \le \pi\frac{1}{\frac{1}{2}(e^R - e^{-R})} \to 0 \quad \text{as } R \to \infty$$

Similarly, if we parametrize the the left vertical segment backwards by $z(t) = it - R$ for $0 \le t \le \pi$, then

$$\left|\int_{L_2} f(z)\,dz\right| = \left|\int_\pi^0 \frac{e^{-t}e^{-iR}}{\frac{1}{2}(e^{-R}e^{it} + e^{R}e^{-it})}\,idt\right| \le \pi\frac{1}{\frac{1}{2}(e^R - e^{-R})} \to 0 \quad \text{as } R \to \infty$$

Hence

$$\begin{aligned}
\lim_{R\to\infty}\int_C f(z)\,dz &= \lim_{R\to\infty}\int_{-R}^{R} f(x)\,dx + \int_R^{-R} f(x+i\pi)\,dx \\
&= \lim_{R\to\infty}\int_{-R}^{R}\frac{e^{ix}}{\cosh x} + e^{-\pi}\frac{e^{ix}}{\cosh x}\,dx \\
&= 2\pi e^{-\pi/2}
\end{aligned}$$

Equating real parts and dividing, we have

$$\int_{-\infty}^{\infty}\frac{\cos x}{\cosh x}\,dx = \frac{2\pi e^{-\pi/2}}{1+e^{-\pi}}.$$

□

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