5. (a) Give a precise statement of the Contractive Mapping Principle.
Solution:

Let $T : M \to M$ be a map from a metric space (M, d) to itself such that

$$d(x, y) \leq kd(T(x), T(y))$$

for some $k < 1$. Then there exists a unique x_0 such that

$$T(x_0) = x_0.$$

□

5. (b) Consider the mapping $f : [1, \infty) \to [1, \infty)$ given by

$$f(x) = \frac{x}{2} + \frac{a}{2x}$$

for fixed a with $1 < a < 3$. Show that f is a contractive mapping. What is its fixed point?
Solution:

We take $M = ([1, \infty), |\ldots - \ldots|)$ to by the metric space with the standard metric from $\mathbb{R}$. Observe

$$\begin{aligned} |f(y) - f(x)| &= \left| \int_x^y f'(t)\, dt \right| \\ &\overset{\dagger}{\leq} \int_x^y |f'(t)|\, dt \\ &\leq |y - x| \cdot \max |f'(x)| \end{aligned}$$

where

$$f'(x) = \frac{1}{2} - \frac{a}{2}x^{-2}.$$

Note that $f''(x) = ax^{-3} > 0$ for $x > 1$, so f is increasing and thus $|f'(x)| \leq \max\{\lim_{x\to\infty} f'(x), |f'(1)|\} = \max\{\frac{1}{2}, |\frac{1}{2}(1-a)|\} < 1$. Letting k be this maximum, we have shown f to be a contraction.

A contraction satisfies

$$x = \frac{1}{2} + \frac{a}{2}x^{-1}$$

when

$$0 = x^2 - \frac{1}{2}x - \frac{a}{2}$$

when

$$x = \frac{1 \pm \sqrt{1 + 8a}}{4}.$$

So

$$x = \frac{1 + \sqrt{1 + 8a}}{4}.$$

□

6. (a) Let f be a continuous function on $[a,b]$. Show that if, for all $k \in \mathbb{N}$, $\int_a^b f(x)x^k\,dx = 0$, then $f \equiv 0$.

Solution:

Let $\varepsilon > 0$ be given. We invoke Weierstrauss to obtain a polynomial $p(x)$ so that

$$f(x) = p(x) + R_\varepsilon(x)$$

where $\sup|R_\varepsilon| < \varepsilon$. Since $\int_a^b x^n f(x) = 0$ for all n,

$$\begin{aligned} 0 = \int_a^b p(x)f(x)\,dx &= \int_a^b f^2(x) - R_\varepsilon(x)f(x)\,dx \\ \implies \int_a^b f^2(x) &= \int_a^b R_\varepsilon(x)f(x)\,dx < (b-a)\varepsilon \cdot \sup|f|. \end{aligned}$$

Hence $\int_a^b f^2 = 0$. We show that this implies that $f \equiv 0$ by contrapositve. That is, suppose $|f(x_0)| > 0$ for some $x_0 \in [a,b]$. Without loss of generality, we can take $a < x_0 < b$ by continuity. Now, continuity gives $\delta > 0$ with $x_0 - a > \delta$ and $b - x_0 < \delta$ so that $|f(x) - f(x_0)| < \frac{|f(x_0)|}{2}$, and, in particular, $|f(x)| > \frac{|f(x_0)|}{2}$ whenever $|x - x_0| < \delta$. Observe

$$\begin{aligned} \int_a^b f^2(x)\,dx &= \int_{x_0-\delta}^{x_0+\delta} f^2(x)\,dx + \int_{|x-x_0|\geq\delta} f^2(x)\,dx \\ &\geq \int_{x_0-\delta}^{x_0+\delta} |f(x)|^2\,dx \\ &\geq 2\delta \cdot \frac{|f(x_0)|^2}{4} \\ &> 0. \end{aligned}$$

□

6. (b) Let f be a continuous function on $[a,b]$. Show that if, for all $k \in \mathbb{N}$, $\int_{-a}^a f(x)x^{2k}\,dx = 0$, then $f \equiv 0$.

Solution:

Observe that

$$f(-x)(-x)^{2k+1} = -f(x)x^{2k+1},$$

so by linearity of integrals and symmetry of the interval, for any given polynomial

$$\int_{-a}^a p(x)f(x)\,dx = \int_{-a}^a q(x)f(x)\,dx$$

where q is a polynomial containing only the even monomial terms of p. The argument proceeds from here exactly as in (a).

□

7. Let (X,d) be a compact metric space. Show that (X,d) is complete.

Solution:

Let $\{x_n\}$ be Cauchy sequence in (X, d). Since (X, d) is compact, there exists a limit point, say x, of $\{x_n\}$. Let $\varepsilon > 0$ be given. There exists $M > 0$ so that $d(x_m, x_n) < \varepsilon/2$ for $M \leq m < n$. Since x is a limit point, there exists $N \geq M$ so that $d(x_N, x) < \varepsilon/2$ for $n \geq N$. Combining these, we have that for $n \geq N \geq M$,

$$d(x_n, x) \leq d(x_n, x_N) + d(x_N, x) < \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

$\square$

8. Fix $N \in \mathbb{N}$ and a compact interval $[a, b]$, and let $\mathcal{F}$ be the family of all polynomials $\displaystyle\sum_{j=0}^{N} a_j x^j$on $[a, b]$ for which $|a_j| \leq 1$ for all j. Show that $\mathcal{F}$ is uniformly bounded and unofrmly equicontinous.

Solution:

For the even function $f(x) = |x|^n$ on $x \in [a, b]$, it attains its maximum at either $x = 0$, $x = a$ or $x = b$ since $f'(x) = 0$ or does not exist only when $x = 0$. Thus,

$$\begin{aligned} \left| \sum_{n=1}^{N} a_n x^n \right| &\leq \sum_{n=1}^{N} |a_n| \, |x|^n \\ &\leq N \max\{|a|, |b|, |a|^2, |b|^2, \ldots, |a|^N, |b|^N\}. \end{aligned}$$

So $\mathcal{F}$ is uniformly bounded.

Now, let $\varepsilon > 0$ be given. Since the monomial $f(x) = x^n$ is continuous (as a successive product of continuous functions), $f(x)$ is uniformly continuous on any given $[a, b]$ since the closed and bounded interval $[a, b]$ is compact. Thus, the distance $|x^n - t^n| < N\varepsilon$ for all $0 < n \leq N$, for sufficiently small $|x - t|$, by choosing a minimum. In particular, for a given representative of $\mathcal{F}$, $f(x) = \sum_{n=1}^{N} a_n x$, we have

$$\begin{aligned} \left| \sum_{n=1}^{N} a_n x^n - \sum_{n=1}^{N} a_n t^n \right| &\leq \sum_{n=1}^{N} |a_n| \, |x^n - t^n| \\ &\leq N \, |x^n - t^n| < \varepsilon. \end{aligned}$$

$\square$

9. (a) Show that $\sin\theta \geq \frac{2}{\pi}\theta$, for all $0 \leq \theta \leq \frac{\pi}{2}$.

Solution:

See Jordan's Lemma [BC04, p. 262].

Let $f(\theta) = \sin\theta$, and note $f''(\theta) = -\sin\theta < 0$ for $0 < \theta < \frac{\pi}{2}$, hence f is concave down there. Observe that the line defined by $\ell(\theta) = \frac{2}{\pi}\theta$ between 0 and $\frac{\pi}{2}$ is a line segment between $f(0)$ and $f(\pi/2)$. We will show that for such a concave down function, the line segment given by ℓ lies entirely below the graph of f.

Let $0 = a < x_0 < b = \frac{\pi}{2}$. By the mean value theorem then the fundamental theorem of calculus, there exists x_1, x_2 such that $a < x_1 < x_0 < x_2 < b$,

$$\begin{aligned}\frac{f(b)-f(x_0)}{b-x_0} - \frac{f(x_0)-f(a)}{x_0-a} &= f'(x_2) - f(x_1) \\ &= \int_{x_1}^{x_2} f''(t)dt \\ &< 0.\end{aligned}$$

Thus,

$$\begin{aligned} & (x_0-a)(f(b)-f(x_0)) < (b-x_0)(f(x_0)-f(a)) \\ \iff & (x_0-a)(f(b)-f(x_0)) + (f(x_0)-f(a))(x_0-a) < (b-x_0)(f(x_0)-f(a)) + (f(x_0)-f(a))(x_0-a) \\ \iff & (x_0-a)\Big(f(b)-f(a)\Big) < \Big(f(x_0)-f(a)\Big)(b-a) \\ \iff & \ell(x_0) = (x_0-a)\frac{f(b)-f(a)}{b-a} + f(a) < f(x_0). \\ \iff & \frac{2}{\pi} < \sin x_0. \end{aligned}$$

We remark that what we have shown is that for any function $f : [a,b] \to \mathbb{R}$ such that $f''(x) < 0$ there, $f(x) \geq \ell(x)$ where ℓ is the line between $(a, f(a))$ and $(b, f(b))$.

□

9. (b) By using part (a), or by any other method, show that if $\lambda < 1$, then

$$\lim_{R\to\infty} R^\lambda \int_0^{\pi/2} e^{-R\sin\theta}\, d\theta = 0.$$

Solution:

By part (a)

$$-R\sin\theta \leq \theta - \frac{2R}{\pi}$$

so

$$\begin{aligned}0 < R^\lambda \int_0^{\pi/2} e^{-R\sin\theta}\, d\theta &\leq R^\lambda \int_0^{\pi/2} e^{-2R\theta/\pi} \\ &= R^{\lambda-1}\frac{\pi}{2}(1-e^{-R})\end{aligned}$$

for all $R > 0$. Letting $R \to \infty$, both sides of the inequality go to 0, hence by squeezing, the integral goes to 0.

□

10. Show that if f and g are analytic functions that both have a zero of order $n \geq 0$ at z_0, then

$$\lim_{z\to z_0} \frac{f(z)}{g(z)} = \frac{f^{(n)}(z)}{g^{(n)}(z)}.$$

Solution:

Since f and g are analytic, they both can be represented as power series centered at z_0

$$f(z) = \sum_{k=0}^{\infty} a_k (z - z_0)^k \quad \text{and} \quad g(z) = \sum_{k=0}^{\infty} b_k (z - z_0)^k.$$

Moreover, $a_0, \ldots, a_{n-1}$ and $b_0, \ldots, b_{n-1}$ are each zero since both have zeros of order n. Thus

$$\begin{aligned} \lim_{z \to z_0} \frac{f(z)}{g(z)} &= \lim_{z \to z_0} \frac{(z - z_0)^{-n} \cdot \sum_{k=n}^{\infty} a_k (z - z_0)^k}{(z - z_0)^{-n} \cdot \sum_{k=n}^{\infty} b_k (z - z_0)^k} \\ &= \lim_{z \to z_0} \frac{\sum_{k=n}^{\infty} a_k (z - z_0)^{k-n}}{\sum_{k=n}^{\infty} b_k (z - z_0)^{k-n}} \\ &= \frac{a_n}{b_n}. \end{aligned}$$

Now,

$$\left(\frac{d}{dz}\right)^n f(z) = n! a_n \quad \text{and} \quad \left(\frac{d}{dz}\right)^n g(z) = n! b_n$$

So

$$\frac{f^{(n)}(z)}{g^{(n)}(z)} = \frac{a_n}{b_n}$$

and the equality is established.

$\square$

11. Evaluate

$$\int_0^{\infty} \left(\frac{\sin x}{x}\right)^3 dx.$$

Justify each step in your calculation.

Solution:

See [BC04, ch. 7.75 p. 269]. First note

$$\sin^3(x) = \left(\frac{e^{ix} - e^{-ix}}{2i}\right)^3 = \frac{1}{8i^3}(e^{3ix} - 3e^{ix} + 3e^{-ix} - e^{-3ix}).$$

Now, let

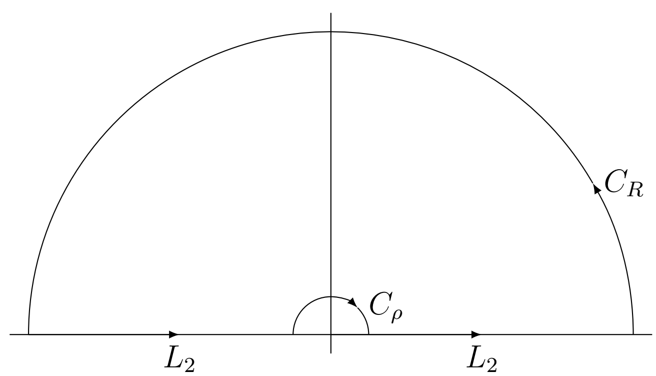
$$f(z) = \frac{e^{3iz} - 3e^{iz} + 2}{z^3},$$

and observe that $f(x) + f(-x) = 8i^3 \left(\frac{\sin x}{x}\right)^3$. Moreover, f has a simple pole at $z = 0$ since

$$\begin{aligned}
f(z) &= z^{-3} \sum_{n=0}^{\infty} \left\{ \frac{(3iz)^n}{n!} - \frac{3(iz)^n}{n!} \right\} + 2z^{-3} \\
&= \cancel{\left\{ \frac{1}{0!z^3} - \frac{3}{0!z^3} + 2 \right\}} + \cancel{\left\{ \frac{3i}{1!z^2} - \frac{3iz}{1!z^2} \right\}} + \left\{ \frac{-9}{2!z} - \frac{-3}{2!z} \right\} + \sum_{n=0}^{\infty} \{(3i)^n - 3i^n\} \frac{z^n}{n!} \\
&\overset{\dagger}{=} \frac{-3}{z} + g(z).
\end{aligned}$$

where $g(z)$ is the analytic function given by the power series of positive power terms in the Laurent series for $f(z)$.

Let Γ be the indented contour given by the path sum of the paths indicated below in the complex plane.



Since f is analytic in the interior of Γ, the Cauchy-Gorsat theorem gives

$$0 \overset{*}{=} \int_{\Gamma} f(z)\,dz = \int_{L1} f(z)\,dz + \int_{L2} f(z)\,dz + \int_{C_R} f(z)\,dz + \int_{C_\rho} f(z)\,dz.$$

If we parametrize L_1 by $z(t) = t$ for $\rho \le t \le R$ and $-L_2$ by $z(t) = -t$ for $\rho \le t \le R$, then

$$\begin{aligned}
\int_{L1} f(z)\,dz + \int_{L2} f(z)\,dz &= \int_{\rho}^{R} f(t)dt + \int_{R}^{\rho} f(-t)(-dt) \\
&= \int_{\rho}^{R} f(t) + f(-t)dt \\
&= 8i^3 \int_{\rho}^{R} \left(\frac{\sin t}{t}\right)^3 dt
\end{aligned}$$

Now, note

$$\left| \int_{C_R} f(z)\,dz \right| \le \pi R \frac{e^{-3\mathrm{Im}\, z} + e^{-\mathrm{Im}\, z} + 2}{R^3}$$

where $z \in C_R$, and if $R \to \infty$ the modulus goes to 0.

For the last path integral we parametrize $-C_\rho$ with $z(t) = \rho e^{it}$ for $0 \le t \le \pi$, so

$$\begin{aligned}\int_{C_\rho} f(z)\,dz &\overset{\dagger}{=} \int_{C_\rho} \frac{-3}{z}\,dz + \int_{C_\rho} g(z)\,dz \\ &= 3\int_0^\pi \frac{i\rho e^{it}}{\rho e^{it}}\,dt + \int_{C_\rho} g(z)\,dz \\ &= 3\pi i + \int_{C_\rho} g(z)\,dz.\end{aligned}$$

Finally note,

$$\left|\int_{C_\rho} g(z)\,dz\right| \le \pi\rho \max_{|z|\le 1} |g(z)|$$

when $\rho \le 1$, and when $\rho \to 0$ the modulus goes to 0. We now have

$$\begin{aligned}0 &\overset{*}{=} \lim_{\substack{\rho\to 0\\ R\to\infty}} \left\{\int_{L_1+L_2} f(z)\,dz + \int_{C_R+C_\rho} f(z)\,dz\right\} \\ 0 &= 8i^3 \int_0^\infty \left(\frac{\sin t}{t}\right)^3 dt + 3\pi i\end{aligned}$$

if and only if

$$\int_0^\infty \left(\frac{\sin t}{t}\right)^3 dt = \frac{3\pi}{8}.$$

□

12. Suppose f is an entire function and that $u(x,y) = \mathrm{Re}\,[f(z)]$ is bounded above (i.e., there exists u_0 such that $u(x,y) \le u_0$). Show that $u(x,y)$ is constant.

Solution:

Let $g(z) = e^{f(z)} = e^{u(x,y)} \cdot e^{i\mathrm{Im}\,[f(z)]}$. Note $|g(z)| = e^{u(x,y)} < e^{u_0}$ hence g is bounded. Since the composition of entire functions are entire, g is entire, and by the Maximum Modulus Principle, must be constant. Thus $u(x,y) = \ln(|g(z)|)$ is constant.

□

13. (a) Let $P(z)$ and $Q(z)$ be polynomials of degrees n and m correspondingly, and let $m \ge n+2$. Write the expression

$$\frac{1}{z^2} \cdot \frac{P(1/z)}{Q(1/z)} \quad (z \ne 0)$$

as the quotient of two polynomials, and point out why $z = 0$ is a removable singular point of that quotient.

Solution:

Let

$$P(z) = a_0 + \cdots + a_n z^n \quad \text{and} \quad Q(z) = b_0 + \cdots + b_m z^m \quad (a_n \neq 0 \text{ and } b_m \neq 0).$$

Observe,

$$\begin{aligned} \frac{1}{z^2} \cdot \frac{P(1/z)}{Q(1/z)} &= \frac{1}{z^2} \frac{a_0 + a_1 z^{-1} \cdots + a_n z^{-n}}{b_0 + b_1 z^{-1} \cdots + b_m z^{-m}} \\ &= \frac{a_0 z^{m-2} + \cdots + a_n z^{n-m-2}}{b_0 z^m + \cdots + b_m}. \end{aligned}$$

Taking the limit as $z \to 0$ gives $0/b_m = 0$ when $m > n+2$ and a_n/b_m when $m \geq n+2$. In either case, the limit exists. This new expression is the quotient of two analytic functions with the denominator non-zero, hence it is analytic. Thus, the expression can be extended analytically to $z = 0$ so the singularity is removable.

□

13. (b) Use the final result in part (a) to show that if all the zeros of $Q(z)$ are interior to a given simple closed curve C, then

$$\int_C \frac{P(z)}{Q(z)}\, dz = 0$$

Solution:

See [BC04, ch. 6.64, p. 228]. Let $f(z) = \frac{P(z)}{Q(z)}$ which has finitely many isolated singularities. Hence, there is an annular region with repsective radii $R_1 < R_2$ containing the contour C for which f is analytic. By Laurent's theorem, we may write f as the series

$$f(z) = \sum_{k=-\infty}^{\infty} b_k z^k \quad \text{with} \quad b_k = \frac{1}{2\pi i} \int_C \frac{f(z)}{z^{k+1}}\, dz$$

for $R_1 < |z| < R_2$. If we evaluate the quantity in part (a),

$$\frac{1}{z^2} f(z^{-1}) = \sum_{k=-\infty}^{\infty} b_k z^{-k-2}$$

for $R_1 < |z^{-1}| < R_2$ if and only if $R_2^{-1} < |z| < R_1^{-1}$. Since this function has a removable singularity at $z = 0$, this series may only have terms with non-negative powers of z. Hence $b_{-1} = b_0 = b_1 = \cdots = 0$. In particular

$$0 = b_{-1} = \frac{1}{2\pi i} \int_C f(z)\, dz.$$

We have established the desired equality.

□

14. Evaluate the integral

$$\int_{|z|=2} \frac{\sin \pi z\, dz}{(2z+1)^3}.$$

Solution:

Let D be the annular region $1 < |z + 1/2| < 3$ and note that the contour $|z| = 2$ is entirely within D. Let $f(z) = \frac{\sin \pi z}{(z+1/2)^3}$, then f is analytic when $z \neq -1/2$ as the quotient of two entire functions and, in particular, is analytic in D. By Laurent's theorem, f can be expressed in D as

$$f(z) = \sum_{k=-\infty}^{\infty} b_k (z + 1/2)^k$$

and the integral of f around $|z| = 2$ is given by $2\pi i b_{-1}$. Using the uniqueness of representation and the Taylor series of $\sin \pi z$ centered at $-1/2$, we have

$$\begin{aligned} f(z) &= (z + 1/2)^{-3} \sum_{n=0}^{\infty} (-1)^n \left. \frac{d^n}{dz^n} \sin \pi z \right|_{z=-1/2} (z + 1/2)^n \\ &= \frac{-1}{(z + 1/2)^3} + \frac{0}{(z + 1/2)^2} + \frac{\pi^2}{z + 1/2} + \sum_{n=3}^{\infty} (-1)^n \left. \frac{d^n}{dz^n} \sin \pi z \right|_{z=-1/2} (z + 1/2)^{n-3} \end{aligned}$$

so

$$\int_{|z|=2} \frac{\sin \pi z \, dz}{(2z + 1)^3} = 8 \int_{|z|=2} \frac{\sin \pi z \, dz}{(z + 1/2)^3} = 16\pi^3 i$$

$\square$

References

[BC04] James Ward Brown and Ruel V. Churchill. *Complex Variables and Applications*. McGraw-Hill, 7th edition, 2004.

[Str00] Robert S. Strichartz. *The Way of Analysis Revised Edition*. Jones and Bartlett Publishers, 2000.