1. Determine all positive real numbers a such athat $a^x \geq ax$ holds for all positive x.

Solution:

Let $f_a(x) = a^x - ax$ so $f_a'(x) = (\ln a)a^x - a$. So critical points are when

$$x_0(a) = \log_a\left(\frac{a}{\ln a}\right) = \frac{1}{\ln a}\ln\left(\frac{a}{\ln a}\right).$$

This point is unique when it exists since a^x is injective. In fact, when $a \leq 1$, $f_a'(x) < 0$ for all x so $f_a(x)$ is unbounded below, and thus $a^x < ax$ for some x and these a will not suffice.

When $a > 1$, note $f_a'(x) < 0$ when $x < x_0(a)$ and $f_a'(x) > 0$ when $x > x_0(a)$, thus $f(x_0(a))$ is a global minimum for f. Hence, $f(x) \geq 0$ is equivalent to

$$\begin{aligned}
& f(x_0(a)) \geq 0 \\
\iff & \frac{a}{\ln a} - \frac{a}{\ln a}\ln\left(\frac{a}{\ln a}\right) \geq 0 \\
\iff & 1 \geq \ln\left(\frac{a}{\ln a}\right) \\
\iff & e\ln a - a \geq 0
\end{aligned}$$

So we need only find a so that $e\ln a - a \geq 0$. Note that $g'(a) = \frac{e}{a} - 1$, and $g'(e) = 0$ with $g'(a) > 0$ when $a < e$ and $g'(a) < 0$ when $a > e$, so $g(a) \leq g(e) = 0$. Thus, the only value for which $g(a) \geq 0$, equivalent to the inequality, is when $a = e$.

□

2. Let $f : [0, \infty) \to \mathbb{R}$ be a continuous function with the property that there exist constants t_0, a and M such that for all $t \geq t_0, |f(t)| \leq Me^{at}$. Show that for all $s > a$,

$$\mathcal{L}[f](s) = \int_0^\infty e^{-st} f(t)\, dt$$

converges absolutely.

Solution:

Observe

$$\begin{aligned}
\left|\int_0^R e^{-st} f(t)\, dt\right| &\leq \int_0^R e^{-st}\left|f(t)\right|\, dt \\
&\leq M\int_0^R e^{-(s-a)t}\, dt \\
&= \frac{M}{s-a}\left(1 - e^{-(s-a)R}\right).
\end{aligned}$$

Taking limits as $R \to \infty$, we see that the absolute value of the integral is bounded above by $\frac{M}{s-a}$ since $s - a > 0$, thus the improper integral is absolutely convergent.

□

5. Let $C([0,1])$ denote the set of complex-valued continuous functions on the interval $[0,1]$. This is a complete metric space with the metric

$$d(f,g) = \sup_{x\in[0,1]} |f(x) - g(x)|.$$

Define $T : C([0,1]) \to C([0,1])$ by

$$Tf(x) = x + \int_0^x tf(t)\,dt$$

Show that T is a contractive map. Then show that the fixed point is a solution to the differential equation $f'(x) = 1 + xf(x)$.

Solution:

Observe

$$\begin{aligned}
d(Tf,Tg) &= \sup_{x\in[0,1]} |Tf(x) - Tg(x)| \\
&= \sup_{x\in[0,1]} \left| x + \int_0^x tf(t)\,dt - \left(x + \int_0^x tf(t)\,dt \right) \right| \\
&= \sup_{x\in[0,1]} \left| \int_0^x t(f(t) - g(t))\,dt \right| \\
&\le \sup_{x\in[0,1]} \int_0^x t|f(t) - g(t)|\,dt \\
&\le \int_0^1 t|f(t) - g(t)|\,dt \quad \text{since the integrand is positive } \int_0^x \cdot \text{ is increasing,} \\
&= \int_0^{1/2} t|f(t) - g(t)|\,dt + \int_{1/2}^1 t|f(t) - g(t)|\,dt \\
&\le \frac{1}{4} \sup_{x\in[0,1]} |f(t) - g(t)| + \frac{1}{2} \sup_{x\in[0,1]} |f(t) - g(t)| \\
&\le \frac{3}{4} d(f,g),
\end{aligned}$$

so T is a contraction. The contraction mapping theorem provides a unique f such that $Tf = f$. In particular,

$$f(x) = x + \int_0^x tf(t)\,dt \qquad \Longrightarrow \qquad f'(x) = 1 + xf(x)$$

by differentiating both sides and the fundamental theorem of calculus.

$\square$

6. Fix $N \in \mathbb{N}$ and a compact interval $[a,b]$ and let $\mathcal{F}$ be the family of all polynomials $\displaystyle\sum_{j=0}^N a_j x^j$ on $[a,b]$ for which $a_j| \le 1$ for all j. Show that $\mathcal{F}$ is uniformly bounded and uniformly equicontinuous.

Solution:

This is also on Spring 2007.

For the even function $f(x) = |x|^n$ on $x \in [a, b]$, it attains its maximum at either $x = 0$, $x = a$ or $x = b$ since $f'(x) = 0$ or does not exist only when $x = 0$. Thus,

$$\begin{aligned}\left|\sum_{n=1}^{N} a_n x^n\right| &\le \sum_{n=1}^{N} |a_n|\,|x|^n \\ &\le N \max\{|a|, |b|, |a|^2, |b|^2, \ldots, |a|^N, |b|^N\}.\end{aligned}$$

So $\mathcal{F}$ is uniformly bounded.

Now, let $\varepsilon > 0$ be given. Since the monomial $f(x) = x^n$ is continuous (as a successive product of continuous functions), $f(x)$ is uniformly continuous on any given $[a, b]$ since the closed and bounded interval $[a, b]$ is compact. Thus, the distance $|x^n - t^n| < N\varepsilon$ for all $0 < n \le N$, for sufficiently small $|x - t|$, by choosing a minimum. In particular, for a given representative of $\mathcal{F}$, $f(x) = \sum_{n=1}^{N} a_n x$, we have

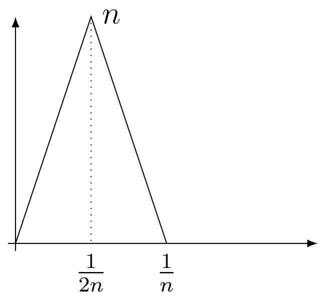
$$\begin{aligned}\left|\sum_{n=1}^{N} a_n x^n - \sum_{n=1}^{N} a_n t^n\right| &\le \sum_{n=1}^{N} |a_n|\,|x^n - t^n| \\ &\le N\,|x^n - t^n| < \varepsilon.\end{aligned}$$

□

7. Prove or disprove: If f_n is a sequence of continuous functions on $[0, 1]$ such that $f_n(x) \to 0$ for every $x \in [0, 1]$, then $\lim_{n\to\infty} \int_0^1 f(x)\,dx = 0$. Discuss the situation.

Solution:

The claim is false as stated. Consider $f_n \in C([0, 1])$ such that the graphs of f_n form a family of isosceles triangles centered at $\frac{1}{2n}$ with base width $1/n$ and height n. I.e.



$$f_n(x) = \begin{cases} n - 2n^2\left|x - \frac{1}{2n}\right| & x < \frac{1}{n} \\ 0 & x \ge \frac{1}{n} \end{cases}.$$

Via the geometric description, $\int_0^1 f_n(x)\,dx = \frac{1}{2}\frac{1}{n}\,n = \frac{1}{2}$ for all n, and thus

$$\lim_{n\to\infty} \int_0^1 f_n(x)\,dx = \frac{1}{2}.$$

Yet, for each fixed $x \in (0, 1]$, there exists an integer $N > \frac{1}{x}$, so $x > \frac{1}{N}$ which implies $|f_n(x)| = 0$. This with $f_n(0) = 0$ for all n implies that $f_n(x) \to 0$.

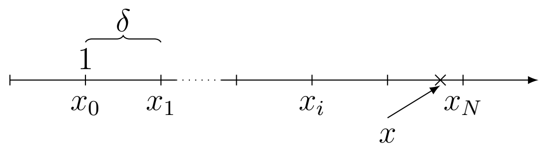
If we additionally require that $\{f_n\}$ be uniformly convergent over $[0, 1]$, then the result holds, or if $\{f_n\}$ can be uniformly bounded then Lebesgue's dominated convergence theorem guarantees the equality.

$\square$

8. Show that if f is uniformly continuous on $[1, \infty)$, then there exists $C \geq 0$ such that $|f(x)| \leq Cx$ for sufficiently large x.

Solution:

Let $\delta > 0$ such that $|f(x) - f(t)| < 1$ when $|x - t| \leq \delta$ given by uniform continuity.



For each fixed x, let $N_x = \min\limits_{N \in \mathbb{N}} \{1 + N\delta \geq x\}$ and $x_i = 1 + i\delta$ for $0 \leq i \leq N_x$. So

$$|f(x) - f(1)| \leq |f(x) - f(x_{N_x - 1})| + |f(x_{N_x-1}) - f(x_{N_x-2})| + \ldots |f(x_1) - f(x_0)| < N_x.$$

Since N_x is a minimum, then $1 + (N_x - 1)\delta < x$ which implies $N_x < \frac{1}{\delta}(x - 1) + 1$. Now

$$\begin{aligned} |f(x)| - |f(1)| \leq |f(x) - f(1)| &< \frac{1}{\delta}(x-1) + 1 \\ \iff |f(x)| &< \frac{1}{\delta}x + \left(1 + |f(1)| - \frac{1}{\delta}\right) \\ &= \frac{2}{\delta}x + \left(-\frac{x}{\delta} + \underbrace{1 + |f(1)| - \frac{1}{\delta}}_{K}\right). \end{aligned}$$

Let $K = 1 + |f(1)| - \frac{1}{\delta}$, and take $x > K\delta$ so $-\frac{x}{\delta} + K < 0$, and thus

$$|f(x)| < \frac{2}{\delta}x.$$

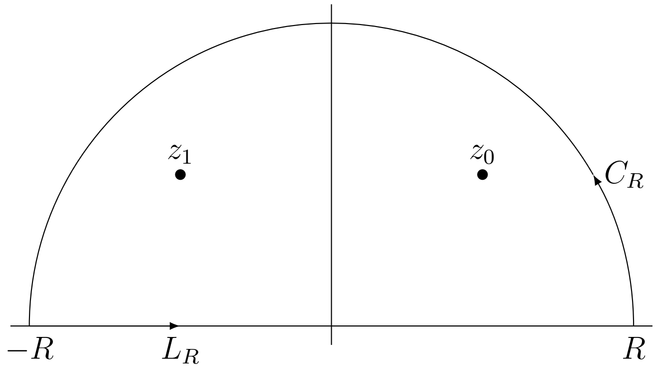
$\square$

9. Show that

$$\int_0^\infty \frac{x^2+1}{x^4+1}\,dx = \frac{\pi}{\sqrt{2}}.$$

Solution:

Let $f(z) = \frac{z^2+1}{z^4+1} = \frac{p(z)}{q(z)}$. Observe that $q(z) = \prod_{j=0}^{3}(z - z_j)$ where $z_j = e^{\pi i/4 + j\pi/2}$. So $f(z)$ is analytic except at each isolated simple pole z_j (since $p(z_j) \neq 0$). Consider the contour consisting of a line segment from $-R$ to R, say L_R, followed by the upper half circle from R to $-R$, say C_R. If $R > \sqrt{2}$, then f is analytic on $L_R + C_R$ and each z_j is interior to this contour.



The residue theorem gives

$$\int_{L_R+C_R} f(z)\,dz = 2\pi i \left\{ \operatorname*{Res}_{z=z_0} f + \operatorname*{Res}_{z=z_1} f \right\}.$$

Observe

$$\begin{aligned} q'(z_0) &= \prod_{j=1}^{3}(z-z_j) + \cancel{(z-z_0)\,\frac{d}{dz}\prod_{j=1}^{3}(z-z_j)}\Bigg|_{z=z_0} \\ &= \frac{q(z)}{(z-z_0)}, \end{aligned}$$

hence

$$\operatorname*{Res}_{z=z_0} f = \lim_{z\to z_0} \frac{f(z)}{z-z_0} = \frac{p(z_0)}{q'(z_0)}.$$

A similar argument gives $\operatorname{Res}_{z=z_1} = \frac{p(z_1)}{q'(z_1)}$. Note also that $z_1 = z_0^3$,

$$\begin{aligned} \int_{L_R+C_R} f(z)\,dz &= 2\pi i \left(\frac{z_0^2+1}{4z_0^3} + \frac{z_0^6+1}{4z_0^9} \right) \\ &= 2\pi i \left(\frac{z_0^2+1}{4z_0^3} + \frac{z_0^{-2}+1}{4z_0} \right) \\ &= \frac{\pi i}{2} \left(\frac{z_0^2+1+1+z_0^2}{z_0^3} \right) \\ &= -\pi i \left(z_0^3 + z_0 \right) \\ &= \pi\sqrt{2}. \end{aligned}$$

Using the triangle inequality and the reverse triangle inequality, we have

$$\left| \int_{C_R} f(z)\,dz \right| \le \frac{R^2+1}{R^4-1} \cdot \pi R = \pi \frac{R^{-1}+R^{-4}}{1-R^{-4}} \to 0$$

as $R \to \infty$. Hence,

$$\int_{-\infty}^{\infty} \frac{x^2+1}{x^4+1}\,dx = \lim_{R\to\infty} \int_{L_R} f(z)\,dz = \lim_{R\to\infty} \int_{L_R+C_R} f(z)\,dz = \pi\sqrt{2},$$

and by even-symmetry, $\int_0^\infty \frac{x^2+1}{x^4+1}\,dx = \pi/\sqrt{2}$.

□

10. Prove the following analytic version of l'Hospital's rule: Let f and g be analytic functions in a neighborhood of z_0, not identically zero. Show that if $\lim_{z\to z_0} f(z) = \lim_{z\to z_0} g(z) = 0$, then $f(z)/g(z)$ has a limit (possibly infinite) when z approaches z_0 and $\lim_{z\to z_0} f(z)/g(z) = \lim_{z\to z_0} f'(z)/g'(z)$.

Solution:

This problem is similar to Spring 2007 10, and we modify the argument there.

Since f and g are analytic, they both can be represented as power series centered at z_0

$$f(z) = \sum_{k=0}^{\infty} a_k (z-z_0)^k \quad \text{and} \quad g(z) = \sum_{k=0}^{\infty} b_k (z-z_0)^k.$$

where $a_n = \frac{f^{(n)}(z_0)}{n!}$ and $b_n = \frac{g^{(n)}(z_0)}{n!}$. Moreover, since each are not identically zero, there exist n and m such that $|a_n| > 0$ and $|b_m| > 0$ and $a_i = 0$ and $b_j = 0$ for $i < n$ and $j < m$. When $n = m$

$$\begin{aligned}
\lim_{z\to z_0} \frac{f(z)}{g(z)} &= \lim_{z\to z_0} \frac{(z-z_0)^{-n} \cdot \sum_{k=n}^{\infty} a_k (z-z_0)^k}{(z-z_0)^{-n} \cdot \sum_{k=n}^{\infty} b_k (z-z_0)^k} \\
&= \lim_{z\to z_0} \frac{\sum_{k=n}^{\infty} a_k (z-z_0)^{k-n}}{\sum_{k=n}^{\infty} b_k (z-z_0)^{k-n}} \\
&= \frac{a_n}{b_n} = \frac{f^{(n)}(z_0)}{g^{(n)}(z_0)}.
\end{aligned}$$

When $n \geq m$, multiply the top and bottom by $(z-z_0)^m$ and observe

$$\begin{aligned}
\lim_{z\to z_0} \frac{f(z)}{g(z)} &= \lim_{z\to z_0} \frac{\sum_{k=n}^{\infty} a_k (z-z_0)^{k-m}}{b_m + \sum_{k=m+1}^{\infty} b_k (z-z_0)^{k-m}} \\
&= 0.
\end{aligned}$$

And when $n < m$,

$$\begin{aligned}
\lim_{z\to z_0} \frac{f(z)}{g(z)} &= \lim_{z\to z_0} \frac{a_n + \sum_{k=n+1}^{\infty} a_k (z-z_0)^{k-n}}{\sum_{k=m}^{\infty} b_k (z-z_0)^{k-n}} \\
&= \infty.
\end{aligned}$$

Proceed inductively on $1 \ldots \min\{n, m\}$ to establish the claim.

$\square$

11. Suppose f is an entire function and that $u(x, y) = \mathrm{Re}\,[f(z)]$ is bounded above (i.e., there exists u_0 such that $u(x, y) \leq u_0$). Show that $u(x, y)$ is constant.

Solution:

Let $g(z) = e^{f(z)} = e^{u(x,y)} \cdot e^{i\mathrm{Im}\,[f(z)]}$. Note $|g(z)| = e^{u(x,y)} < e^{u_0}$ hence g is bounded. Since the composition of entire functions are entire, g is entire, and by the Maximum Modulus Principle, must be constant. Thus $u(x, y) = \ln(|g(z)|)$ is constant.

$\square$

12. Let n be an integer with $n > 1$ and let ω be a primitive nth root of unity (i.e., $\omega^n = 1$ but $\omega^k \neq 1$ if $1 \leq k < n$). Show that

$$\sum_{k=0}^{n-1} \omega^k = 0$$

Solution:

Observe

$$(1 - \omega) \sum_{k=0}^{n-1} \omega^k = \sum_{k=0}^{n-1} \omega^k - \sum_{k=1}^{n} \omega^k = -\omega^n + 1 = 0.$$

Since $(1 - \omega^1) \neq 0$ by assumption, the equality is established.

$\square$

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