Estimating Variation in a Repeated Measurement Model

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Introduction

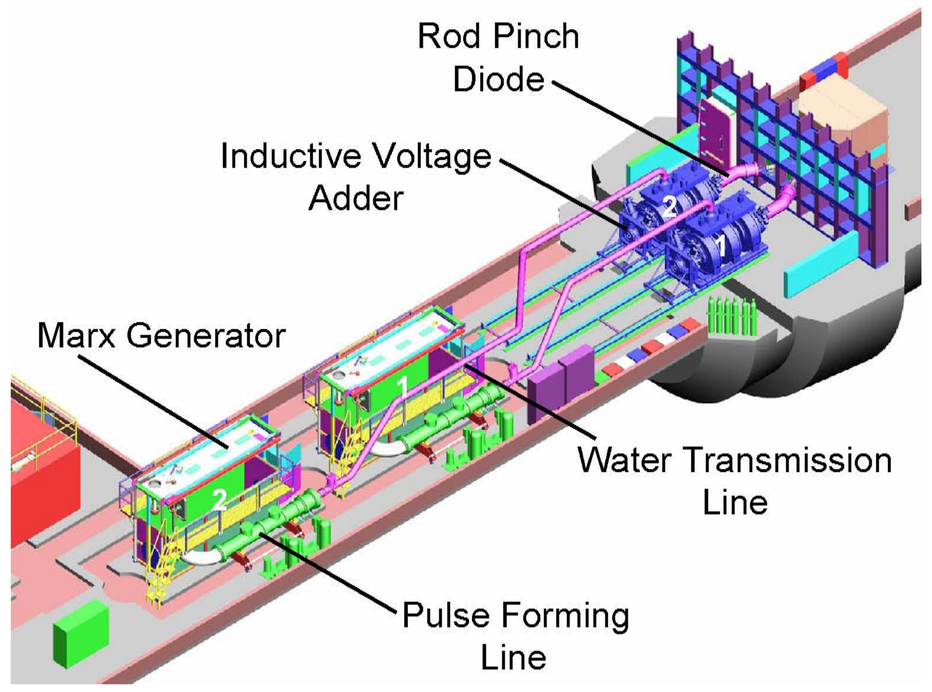

Figure 1: Cygnus Pulse Power Machine

The Cygnus pulse power machine and imaging system is a large scale sub-critical experiment diagnostic jointly owned and operated by Los Alamos National Labs, Sandia National Labs, and National Securities Technologies. An extensive data record of thermoluminescent dosimeter (TLD) readings has been kept as a performance diagnostic since 2009. From the reported dose measurements, an estimate of the radiation dose at one meter (DAOM) from the imaged package is extrapolated based on a model of inverse square radiation decay.

We investigate a model of non-identical measurements for estimating the mean and variation in the DOAM of multiple radiation sources. Our model also includes variation due to measurement error.

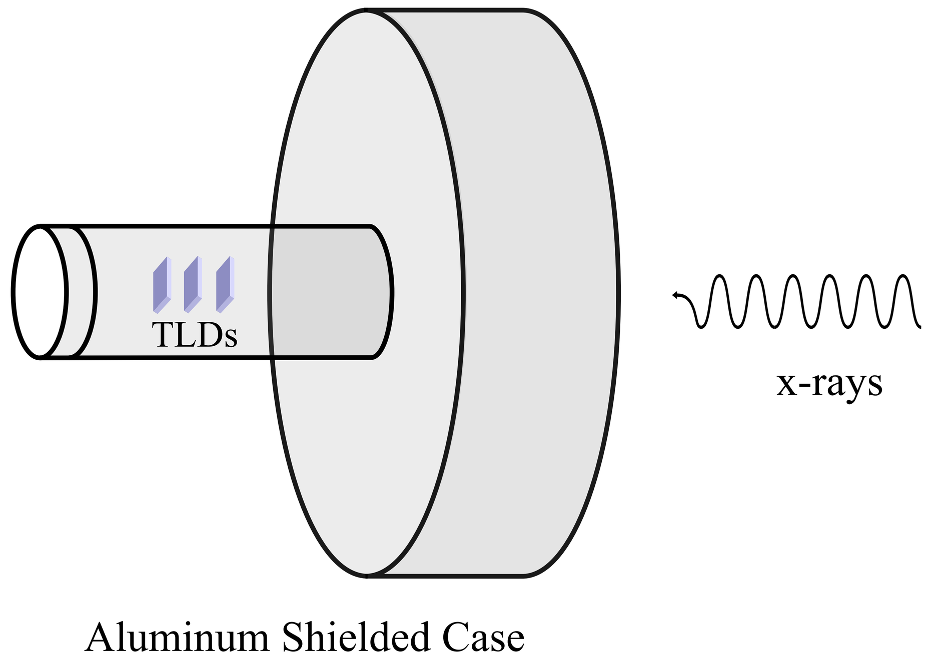

Figure 2: TLD Measurement Device

Data

The TLD chips are stacked facing the radiation source so that the absorbed ionizing radiation is attenuated by consecutive chips. That is, chips stacked behind other chips absorb less radiation and under-report dose.

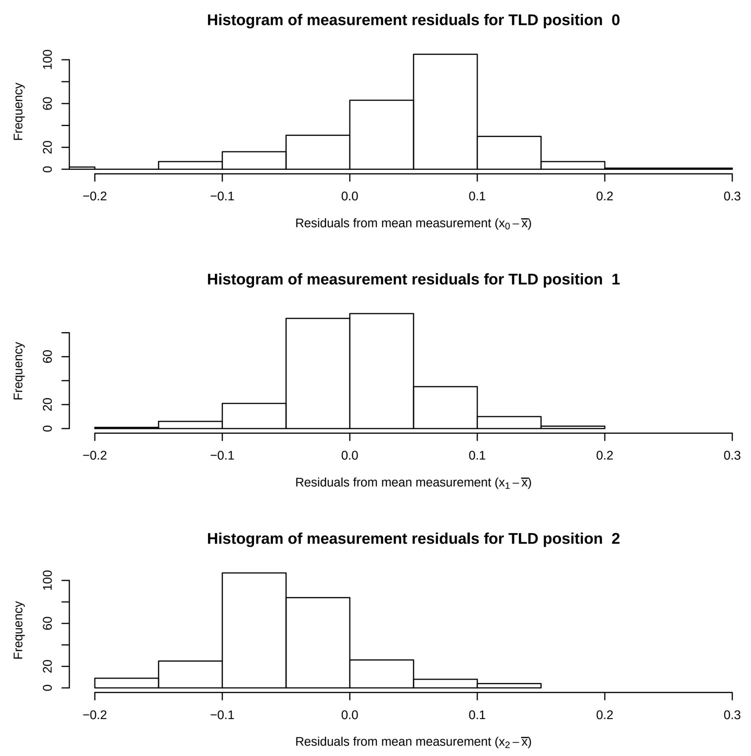

Figure 3: Evidence of Attenuation

Model

We model the TLD measurements as proportional to the integrated energy intensity of the x-rays absorbed by the TLD with exponential attenuation through the chips.

$$\tilde{y}_{ijk} = \exp\left(\beta_{0k} + \beta_1 x_{i1} + \beta_2 x_{i2} + \eta_{ik} + \epsilon_{ijk}\right). \tag{1}$$

$\epsilon_{ijk} \overset{iid}{\sim} N(0, \sigma_\epsilon^2)$ is the random error due to TLD measurement; and $\eta_{ik} \sim N(0, \sigma_{\eta k}^2)$ is the random variation in each machine source.

A log transformation of (1) results in a linear model in the parameters with two random components. The matrix-vector form is

$$\boldsymbol{y} = \boldsymbol{X}\boldsymbol{\beta} + \boldsymbol{P}\boldsymbol{\eta} + \boldsymbol{\epsilon}, \tag{2}$$

where $\boldsymbol{P}$ is a $6n \times 2n$ block matrix with 3×1 blocks of ones along the diagonal.

Estimation Method 1

If we consider both random pieces as one random vector $\boldsymbol{\xi} = \boldsymbol{\epsilon} + \boldsymbol{P}\boldsymbol{\eta}$, then we view the model as

$$\boldsymbol{y} = \boldsymbol{X}\boldsymbol{\beta} + \boldsymbol{\xi}, \tag{3}$$

where $\boldsymbol{\xi} \sim \mathcal{N}(\boldsymbol{0}, \boldsymbol{\Sigma})$ with $\boldsymbol{\Sigma}$ a with a specific block matrix structure. We use an iterative generalized least squares (GLS) to estimate each of the parameters.

1. Initially estimate $\widehat{\beta}_0$ and $\widehat{\boldsymbol{\Sigma}}_0$ with standard least squares.
2. Estimate $\widehat{\boldsymbol{\Sigma}}_{\boldsymbol{k}}$.
3. Estimate $\widehat{\beta}_{\boldsymbol{k}}$.
4. If convergence in $\widehat{\boldsymbol{\xi}}_k^T \widehat{\boldsymbol{\xi}}_k$ is reached, stop, else set $k = k + 1$ and continue from step 2.

Estimation Method 2

We can view (2) as a mixed effects model and estimate each of the parameters of interest using restricted maximum likelihood (REML) techniques [2]. That is, we view the variation shot to shot as a random effect.

This method was implemented from codes from the `R` library `lme4`, and the results from it are very recent. Hence, the details of the implementation are not known well by us, but seem to produce a less biased estimate for the measurement variation.

Simulation Results

TLD measurements		
Variation	σ_ϵ^2	1.00e-06
Attenuation 1	β_1	–0.002
Attenuation 2	β_2	–0.004

Cygnus - 1 DAOM		
Mean DOAM	$\boldsymbol{\beta_{01}}$	1.39
Variation	$\sigma_{\eta\,1}^2$	4.00e-04

Cygnus - 2 DAOM		
Mean DOAM	$\boldsymbol{\beta_{01}}$	1.10
Variation	$\sigma_{\eta\,1}^2$	0.0016

Table 1: Simulation Coefficients.

TLD measurements		
Est. Variation	$\widehat{\sigma_\epsilon}^2$	6.73e-07
Est. Atten. 1	$\widehat{\beta_1}$	–1.98e-03
Est. Atten. 2	$\widehat{\beta_2}$	–3.90e-03

Cygnus - 1 DAOM		
Est. Mean	$\widehat{\boldsymbol{\beta}}_{\boldsymbol{01}}$	1.38
Variation	$\widehat{\sigma}_{\eta,1}^2$	4.42e-04

Cygnus - 2 DAOM		
Est. Mean	$\widehat{\boldsymbol{\beta}}_{\boldsymbol{02}}$	1.10
Variation	$\widehat{\sigma}_{\eta,1}^2$	1.91e-3

Table 2: Method 1 estimates.

TLD measurements		
Est. Variation	$\widehat{\sigma_\epsilon}^2$	1.01e-06
Est. Atten. 1	$\widehat{\beta_1}$	–1.98e-03
Est. Atten. 2	$\widehat{\beta_2}$	–3.90e-03

Cygnus - 1 DAOM		
Est. Mean	$\widehat{\boldsymbol{\beta}}_{\boldsymbol{01}}$	1.40
Variation	$\widehat{\sigma}_{\eta,1}^2$	4.40e-04

Cygnus - 2 DAOM		
Est. Mean	$\widehat{\boldsymbol{\beta}}_{\boldsymbol{02}}$	1.10
Variation	$\widehat{\sigma}_{\eta,1}^2$	1.90e-03

Table 3: Method 2 estimates.

References

[1] G. Corrow, et.al. Cygnus Performance in Subcritical Experiments. No. DOE/NV/25946–294. National Security Technologies, LLC, 2008.

[2] J. Faraway (2005) *Linear Models with R*. Chapter 6, pages 90-92.

Future Research

Standard errors for the estimates on the real data still need to be estimated to obtain confidence intervals for each estimate. These can be obtained through bootstrapping will address questions about whether the parameters actually differ from one source to the other. More questions like these can be explored through the random effects veiwpoint.

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