1. The rise in abundance of algae in coastal waters is thought to be due to increase in nutrients such as nitrates and other forms of nitrogen. It is theorized that the excessive amounts of nitrate are due to human influences. Human populations can affect nitrogen inputs to rivers through industrial and automobile emissions to the atmosphere (causing the nitrogen to enter the river through rainfall), through fertilizer runnoff, through sewage discharge, and through watershed disturbance. Researchers gathered data from 42 rivers around the world to gauge the evidence that nitrates in the discharges of rivers around the world are associated with human population density. The data are on the course webpage under the filename `nitriver.txt`. Among the variables measured were:

$y =$ nitrate concentration (μM/l)

$x_1 =$ discharge: the estimated annual average discharge of the river into the ocean (l/sec $\times$ k $\times$ m^2)

$x_2 =$ runoff: the estimated annual average runoff from the watershed (l/sec $\times$ k $\times$ m^2)

$x_3 =$ precipitation (cm/year)

$x_4 =$ area of watershed (km^2)

$x_5 =$ human population density (people/km^2)

$x_6 =$ deposition: the product of precipitation times nitrate concentration

$x_7 =$ nitrate precipitation: the concentration of nitrate in wet precipitation at sites located near the watersheds (μmol NO_3/(sec $\times$ km^2)

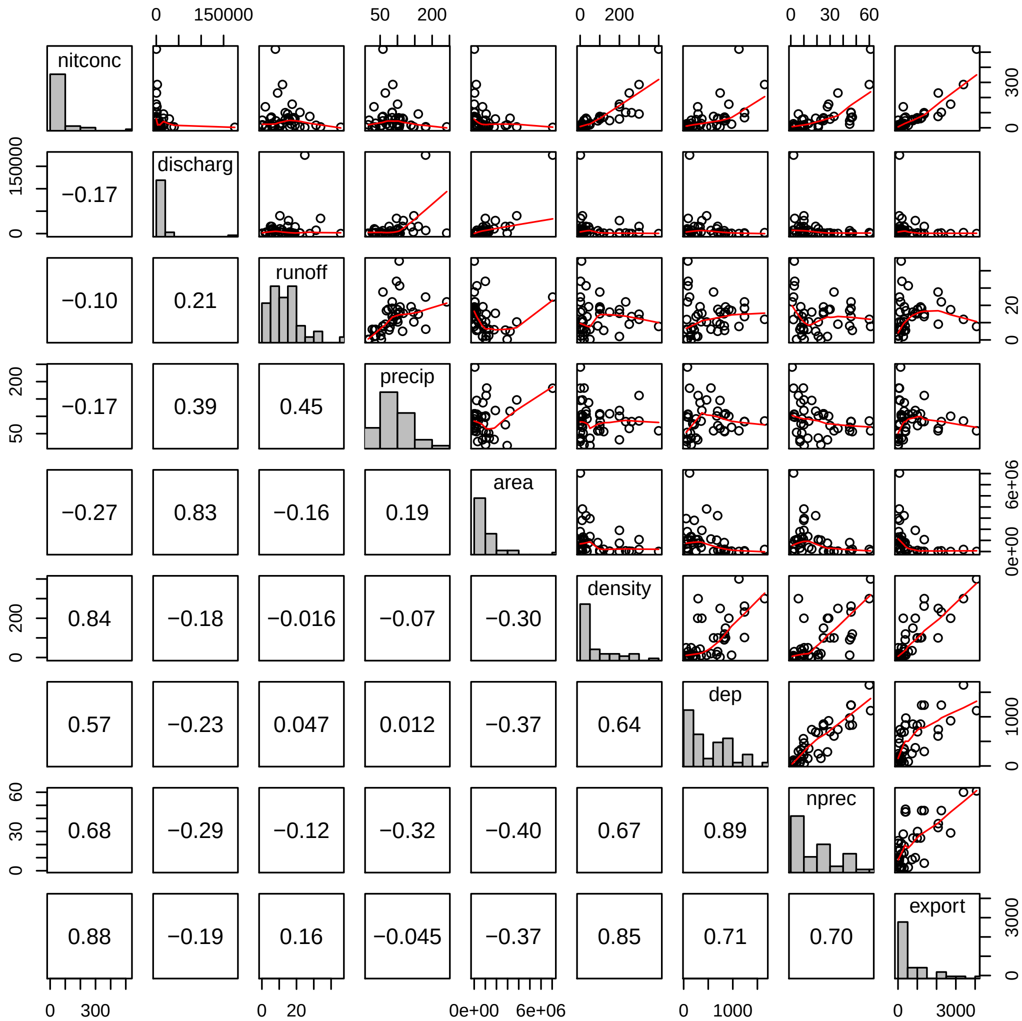
$x_8 =$ nitrate export: the product runoff times nitrate concentration.

(a) Perform some exploratory data analysis on these variables and write a brief (no more than 2 paragraphs) summary of your findings. Since nitrate concentration is the response variable, your analysis should focus on the relationship between nitrate concentration and the other variables, as well as any relationships among the explanatory variables. Also include in your discussion what problems you might expect to encountering trying to determine a best model according to some selection criterion. [Do NOT actually look for a "best" model - I am just looking for what problem(s) you might anticipate in the process of finding such a model based on your EDA.]

Solution:

Each variable is quantitative, hence, comparisons can be made with scatter plots between each variable. The histograms in the following plot indicate that each variable is right skewed to varying degrees. There is a data point (index 34) that has very large `nitconc`, `discharge` and `runoff` indicating a potential outlier.

In the comparisons with `nitconc`, it appears that discharge, run-off, precipitation, and area are weakly, if at all, correlated with the response. Population density, deposition, nitrate precipitation, and nitrate export are moderately correlated (positively) with nitrate concentration. The linearity of the relationships are not perfect, and a transformation (perhaps log-log due to each being right skewed) may be appropriate. Additionally, there may be collinearity issues among density, deposition, nitrate precipitation, and nitrate export, and also among discharge, runoff, precipitation, and area due to their apparent correlation (although discharge and runoff do not appear to be correlated).



$\square$

(b) Suppose after examining the numerous models, you decide that the general linear model with explanatory variables (x_2, x_6, x_7, x_8) is the best model. Using the "best" model, give a set of Bonferroni joint confidence intervals for the model parameters. Explain in a sentence or two why it is important to use a Bonferroni correction here.

Solution:

The 95% Bonferroni corrected confidence intervals are given in the following table:

$$
\begin{array}{lrl}
\widehat{\beta}_0 \pm t_{1-.05/(2\cdot 5)}(n-5)\cdot \mathrm{SE}(\beta_0) = & 25.198 \pm 35.592 & \text{or } (-10.394, 60.790) \\
\widehat{\beta}_2 \pm t_{1-.05/(2\cdot 5)}(n-5)\cdot \mathrm{SE}(\beta_2) = & -1.850 \pm 1.774 & \text{or } (-3.625, -0.076) \\
\widehat{\beta}_6 \pm t_{1-.05/(2\cdot 5)}(n-5)\cdot \mathrm{SE}(\beta_6) = & -0.094 \pm 0.088 & \text{or } (-0.182, -0.007) \\
\widehat{\beta}_7 \pm t_{1-.05/(2\cdot 5)}(n-5)\cdot \mathrm{SE}(\beta_7) = & 2.077 \pm 2.226 & \text{or } (-0.149, 4.302) \\
\widehat{\beta}_8 \pm t_{1-.05/(2\cdot 5)}(n-5)\cdot \mathrm{SE}(\beta_8) = & 0.091 \pm 0.024 & \text{or } (0.067, 0.115)
\end{array}
$$

Since there is joint uncertainty in estimating $(\beta_0, \beta_2, \beta_6, \beta_7, \beta_8)$ simultaneously, the Bonferroni correction protects against falsely rejecting one of the tests $H_0 : \beta_k = 0,\ k = 0, 2, 6, 7, 8$. That is, this protects against Type I errors.

□

(c) In the "best" model of part (b), perform a permutation test of $H_0 : \beta_7 = 0$ where β_7 is the parameter corresponding to x_7. Report the permutation p-value and a clear conclusion to the test. Does this agree with the normal-based p-value reported in the **Coefficients** table?

Solution:

We fit the "best" model with 4999 permutations of the `nprec` variable and compare the computed F-statistic for significance of β_7 between the permuted and un-permuted model. The ratio of the number of those where the permuted model resulted in a more significant statistic and the total number of permutations (counting the original ordering) was found to be $p = (67 + 1)/(4999 + 1) = 0.0136$. This is comparable with the p-value (0.0156) given by the normal based test.

□

2. As part of a study of the effects of predatory intertidal crab species on snail populations, researchers measured the mean closing forces and the propodus heights of the claws on several crabs of three species. Specifically, there were three crab species in the study (*Hemigrapsus nudus, Lophopanopeus bellus, Cancer productus*) with between 12-14 crabs of each type on which measurements were taken. The data are given below and can be found on the course webpage under the filename `crab.txt`.

Let:

$$\begin{aligned} y &= \text{ the log mean closing force of a crab,} \\ x_1 &= \text{ the log mean propodus height of a crab claw,} \\ x_2 &= \begin{cases} 1 & \text{if the crab species is } \textit{H. nudus} \\ 0 & \text{otherwise} \end{cases} \\ x_3 &= \begin{cases} 1 & \text{if the crab species is } \textit{L. bellus} \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

(a) Consider the multiple linear regression model given by:

$$y_i = \beta_0 + \beta_1 x_{i1} + \beta_2 x_{i2} + \beta_3 x_{i3} + \beta_4 x_{i1} x_{i2} + \beta_5 x_{i1} x_{i3} + \epsilon_i.$$

Interpret the parameters $\beta_1, \beta_2, \&\beta_4$ clearly *in the context of the problem.* For example, do not just say: "β_1 is the partial slope of y on x_1." Explain its meaning in terms of the mean respons y.

Solution:

Throughout this analysis, we use base 10 logarithms. For each ith response, there are three cases for (x_{i2}, x_{i3}): $(x_{i2}, x_{i3}) = (0, 0)$, in which case the species is species *C. productus*; $(x_{i2}, x_{i3}) = (0, 1)$, in which case the species is *L. bellus*; or $(x_{i2}, x_{i3}) = (1, 0)$, and the species is *H. nudus.*

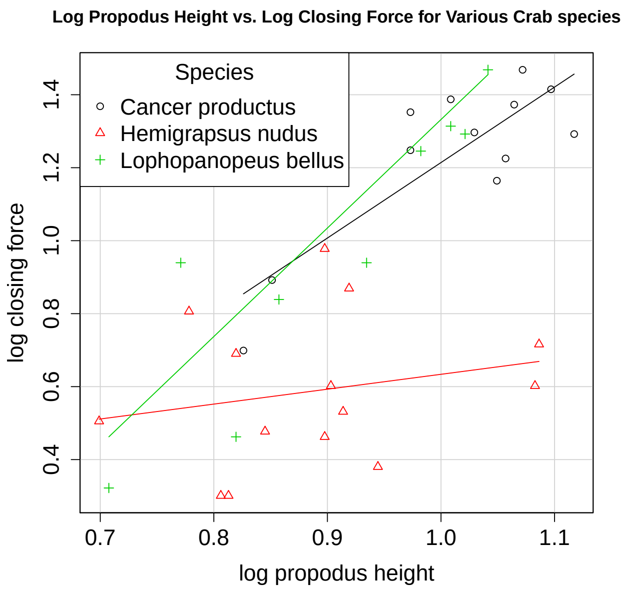
When $(x_{i2}, x_{i3}) = (0,0)$, $y_i = \beta_0 + \beta_1 x_{i1} + \epsilon_i$, whose expected value transformed back is $10^{\beta_0+\beta_1 x_{i1}}$. From this we see that β_0 is the (not so informative) order of 10 when the propodus height is 1. The interpretation of β_1 is for a 10-fold increase in propodus height, we expect β_1 10-fold increases in mean closing force.

When $(x_{i2}, x_{i3}) = (0,1)$, $y_i = (\beta_0 + \beta_3) + (\beta_1 + \beta_5)x_{i1} + \epsilon_i$, and we see that β_3 is the increase in the intercept for the log model when switching from *C. productus* to *L. bellus* and β_5 is the increase in the effect of log propodus height on log force. Similarly when $(x_{i2}, x_{i3}) = (0,1)$, $y_i = (\beta_0 + \beta_2) + (\beta_1 + \beta_4)x_{i1} + \epsilon_i$, and β_2 is the increase in the intercept when *C. productus* switches to *H. nudus*, and β_4 is the increase in the effect of the log height on the log force for a similar species change.

$\square$

(b) Construct a scatterplot of log force vs. log height, with a different symbol for each crab species. In a few sentences, describe the associations between the three variables (log force, log height, crab species).

Solution:



Each species has a positive association between log force and log height. There appears to be an effect of species on this association as the slope of *H. nudus* appears to be less than those for *C. propodus* and *L. bellus*. Moreover, the average force appears to be less overall for *H. nudus* than both *C. productus* and *L. bellus*.

$\square$

(c) Fit the multiple linear regression model $y_i = \beta_0 + \beta_1 x_{i1} + \beta_2 x_{i2} + \beta_3 x_{i3} + \beta_4 x_{i1} x_{i2} + \beta_5 x_{i1} x_{i3} + \epsilon_i$. Report the **ANOVA** and **Coefficients** table. Test simultaneously whether or not either of the interactions are significant. [As always, give the hypotheses, test statistic, p-value, and a clear conclusion based on this p-value.] Do the results of this test confirm or refute what is seen in the scatter plot? Explain.

Solution:

Coefficients Table

Predictor	$\widehat{\beta}_k$	SE($\widehat{\beta}_k$)	t	p-val
Intercept	-0.8544	0.6293	-1.358	0.18405
x_1	2.0685	0.6208	3.332	0.00219
x_2	1.0798	0.7646	1.412	0.16752
x_3	-0.7873	0.8049	-0.978	0.33536
x_1x_2	-1.6601	0.7889	-2.104	0.04330
x_1x_3	0.9052	0.8302	1.090	0.28368

Analysis of Variance Table

Source	df	SS	MS	F	p-val
Regression	5	4.3740	0.875	24.75	3.94e-10
Error	32	1.1311	0.0353		
Total	37	5.5054	0.149		

We test $\beta_4 = \beta_5 = 0$ by computing

$$F = \frac{R(\beta_4, \beta_5 | \beta_0, \dots, \beta_3)}{\text{MSE}} \approx \frac{0.4497}{0.0353} \approx 6.36,$$

for which $p < .005$. Hence, there is significant evidence for both interaction effects in the model; i.e. the increase in mean closing force due to log propodus height depends on the species. This agrees with the qualitative analysis in the last part since we observed that *not* all slopes are the same.

□

(d) Predict the mean closing force using a 95% prediction interval for a crab of species *L. bellus* with a propodus height of 11.5. Would you expect the prediction interval to be wider or narrower if the height had been 8? Explain in a sentence or two.

Solution:

We are predicting for the vector $\boldsymbol{x_0}' = (1, \log 11.5, 0, 1, 0, \log 11.5)$, so the 95% prediction interval for the *log* mean closing force is given by

$$\begin{aligned}\boldsymbol{x_0}'\widehat{\boldsymbol{\beta}} \pm t_{.975}(n-p) \cdot & \sqrt{\text{MSE} \cdot (1 + \boldsymbol{x_0}'(\boldsymbol{X'X})^{-1}\boldsymbol{x_0})} \\ & \approx 1.513 \pm 0.435 \\ & \text{or } (1.077, 1.948).\end{aligned}$$

Transforming back to mean force, we have

$$32.55 \pm 2.72 \text{ or } (29.823, 35.27).$$

The correpsonding interval for 8 would be narrower for two reasons: first, the inflation due to the reverse transformation 10^x would be less, and second, $\log 8 \approx 0.94$ is closer to the mean log force value ($\approx .90$) than $\log 11.5 \approx 1.06$. In fact, if we look at the scatterplot in (b), we see that $\log 11.5$ is larger than any *L. bellus* observation, hence the width will be wider than any observation within the range of *L. bellus* data.

□

(e) Still considering the full model, test for a difference in the slopes between log force and log height for the *H. nudus* and *C. productus* crab species. That is, test whether or not the slopes resulting from a regression of log force on log height are different for the two crab species.

Solution:

As per the interpretation in part (a), this amounts to testing the significance of the second interaction term; $H_0 : \beta_4 = 0$. From the coefficients table, the t-test for $t = -2.1$ gives $p = 0.043$ and suggests a possible difference from zero. So there is some evidence for differing associations between mean closing force and height between *H. nudus* and *C. productus*

□

(f) Another way we might test for difference in the slopes of the regression lines between log force and log height for the three crab species is to run separate regression for the three species, and compare the slopes using 3 independent t-tests. Discuss in a short paragraph which of these two methods (3 independent t-test, multiple regression) would be better and why? Think about degrees of freedom, and the mean squared error. [Do NOT perform the tests to answer this question.]

Solution:

The two sample t-test comparing the normally distributed coefficients indicated would test a t-statistic with $n_1 + n_2 - 2$ degrees of freedom (n_i depending on which populations are in consideration), where the tests in multiple linear regression on the same hypothesis (either $\beta_4 = 0$ or $\beta_5 = 0$) would have $n - 5$ degrees of freedom. For these sample sizes, the t-statistics for multiple linear regression will have more degrees of freedom than the ones for the pairwise comparisons. Moreover, in each case to calculate the t-statistics, we divide by standard errors for the estimates we are testing. In the multiple linear regression case, the squared standard error has $n - 5$ in the denominator and the pair-wise tests will have $n_1 + n_2 - 2$ in the denominator. Assuming that the algebra for the numerators yields numbers on the same order ($(\boldsymbol{X'X})^{-1}_{ii}$ in the first case and $(n_1 - 1)s_1^2 + (n_2 - 1)s_2^2$ in the other), we would expect the standard error for multiple linear regression to be less. Both of these factors (more dfs and smaller standard error) makes the multiple linear regression method a more powerful test for finding significance.

□

3. A study was undertaken in major US cities to examine the effects of pollution levels on mortality, adjusting for climate and socioeconomic information. Specifically, data were collected at each city on the following variables:

y = mortality rate (deaths per 100,000 people over a 3-year period),
x_1 = `precip` = mean annual precipitation (inches)
x_2 = `education` = the mean number of school years completed, for persons of age 25 years or older,
x_3 = `nonwhite` = the percentage of the population that is non-white,
x_4 = `NOx` = the relative pollution potential of oxides and nitrogen,
x_5 = `SO2` = the relative pollution potential of sulfer dioxide.

The primary goal of the study was to investigate whether or not mortality is associated with either of the two pollution variables after accounting for the effects of the climate and socioeconomic variables on mortality. Suppose a sequence of models was fit with this in mind, resulting in the output tables given below.

Coefficients Table

Predictor	$\widehat{\beta}_k$	SE($\widehat{\beta}_k$)	t	p-val
Intercept	1000.1021	92.3981	*10.824*	3.85e-15
x_1	1.3792	0.7000	*1.970*	0.053943
x_2	-15.0790	7.0706	*-2.133*	0.037519
x_3	3.1602	*0.6287*	5.026	5.84e-06
x_4	-0.1076	0.1359	*-0.792*	0.432066
x_5	*0.3555*	0.0914	*3.889*	*2.78e-04*

Analysis of Variance Table

Source	df	SS	MS	F	p-val
Regression	*6-1*	*152888*	*30577.6*	*21.91*	0.000
Error	*54*	75385	*1395.77*		
Total	*60-1*	*2228273*	*3868.17*		

Sequential Sums of Squares

Source	df	Seq. SS
x_1	1	59256
x_2	1	20492
x_3	1	51163
x_4	1	867
x_5	1	21110

(a) The R^2-statistic for the full model was $R^2 = 0.6698$ and the residual standard error was 37.36. Using this information and the information given in the tables above, fill in all of the missing information in the **Coefficients** and **ANOVA** tables. If you cannot figure out what some value is, just make up a value, let me know what value you made up, and complete the problem with your values

Solution:

The full SSReg is given by the sum of the entries in the **Sequential Sums of Squares** table – SSReg $= 152888$ and since $p = 6$, MSR $=$ SSReg$/(6-1) = 30577.6$. We can recover TSS from $R^2 =$ SSReg/TSS, i.e. TSS $= 152888/0.6698 = 2228273$. Since $\sqrt{\text{MSE}} = 37.36$, MSE $= 1395.77$. Also MSE $=$ RSS$/(n-p)$ implies $n =$ RSS/MSE $+ p = 60$. Now MST $=$ TSS$/(n-1) = 3868.17$ and $F =$ MSR/MSE $= 21.91$.

In the **Coefficients** table, each test is for $H_0 : \beta_k = 0$. Hence we can recover reach missing value for x_1 through x_4 by $t = \widehat{\beta}_k/\text{SE}(\widehat{\beta}_k)$. We can recover the t-statistic for $H_0 : \widehat{\beta}_5 = 0$ by taking the square root of the F-statistic $F = \text{SSReg}(x_5|x_1, x_2, x_3, x_4)/\text{MSE}$ for the equivalent test, i.e. $t = \sqrt{F} = \sqrt{21110/1395.77} \approx 3.889$. Upon verifying that each test was conducted at $\alpha = .05$, we can complete the table.

□

(b) The researcher's goal was to test the significance of the two pollution variables simultaneously after accounting for the effects of the three climate and socioeconomic variables (x_1, x_2, x_3). Conduct such a test using the information in these tables. State your hypothesis clearly, give the test statistic and p-value, and state a conclusion *in context of the problem.*

Solution:

We test the hypothesis $H_0 : \beta_4 = \beta_5 = 0$, given x_1, x_2, x_3 are in the model by comparing the models with and without the terms using the appropriate F-statistic. That is

$$F = \frac{R(\beta_4, \beta_5|\beta_0, \ldots, \beta_3)/2}{\text{MSE}} = \frac{(867 + 21110)/2}{1395.77} \approx 7.87$$

for which the p-value is 9.99×10^{-4}. Hence we have very strong evidence that both pollutant explanatory variables are significant in the model (holding all other variables constant).

□

(c) Test the model with only x_1 against the model with variables (x_1, x_2, x_3). Again, state the hypotheses, give the test statistic & p-value, and give a clear conclusion *in the context of the problem.*

Solution:

We test $H_0 : \beta_2 = \beta_3 = 0$ given x_1 is in the model. The numerator of the F-test is $\text{SSReg}(x_2, x_3|x_1)/2 = (\text{SSReg}(x_2|x_1) + \text{SSReg}(x_3|x_1, x_2))/2$. The denominator is

$$\text{MSE}(x_1, x_2, x_3) = \frac{\text{RSS}(x_1, x_2, x_3)}{n-4} = \frac{\text{RSS}(x_1, x_2, x_3, x_4) + \text{SSReg}(x_4|x_1, \ldots, x_3) + \text{SSReg}(x_5|x_1, \ldots, x_4)}{n-4}.$$

All together,

$$F = \frac{(20492 + 51163)/2}{(75385 + 867 + 21110)/56} \approx 11.61 \quad \text{and } p \approx 6.03 \times 10^{-5}.$$

We conclude that in the model without pollution variables, the socioeconomic predictors education and non-white proportion add significant explanatory power to only having precipitation in the model.

□