1. D'Angelo 1.56 Let p be a polynomial. Show that the series $\sum(-1)^n p(n)$ is Abel summable. More generally, for $|z| < 1$, show that $\sum_0^\infty p(n)z^n$ is a polynomial in $\frac{1}{1-z}$ with no constant term. Hence, the limit, as we approach the unit circle from within, exists at every point except 1.

Solution:

It suffices to prove only the general case since $z = -1$ is defined for any polynomial in $\frac{1}{1-z}$. In fact, given a polynomial $p(n) = c_0 + c_1 n + \cdots + c_d n^d$, by linearity it further suffices only to show that $\sum_{n=0}^\infty n^k z^n$ can be written as a polynomial in $\frac{1}{1-z}$.

Observe that $z\frac{d}{dz}z^n = nz^n$. Proceeding inductively, we have

$$\left[z\frac{d}{dz}\right]^k z^n = \left[z\frac{d}{dz}\right]^{k-1} nz^n = n\left[z\frac{d}{dz}\right]^{k-1} z^n = \cdots = n^k z^n.$$

Now, we evaluate

$$\begin{aligned}
\sum_{n=0}^\infty n^k z^n &= \sum_{n=0}^\infty \left[z\frac{d}{dz}\right]^k z^n \\
&\overset{*}{=} \left[z\frac{d}{dz}\right]^k \sum_{n=0}^\infty z^n \\
&= z^n \frac{d^n}{dz^n}(1-z)^{-1} \\
&= z^n(1-z)^{-n-1} \\
&= \frac{1}{1-z}\left(\frac{z}{1-z}\right)^n \\
&= \frac{1}{1-z}\left(\frac{1}{1-z} - 1\right)^n.
\end{aligned}$$

In $\overset{*}{=}$ we used the fact that the sum is a power series to interchange the order of $\frac{d^n}{dz^n}$ and $\sum_{n=1}^\infty$.

$\square$