1. D'Angelo 1.40. Assume that $f : \mathcal{S}^1 \to \mathbb{C}$ is k times continuously differentiable. Show that there is a constant C such that for $n > 0$

$$|\widehat{f}(n)| \leq \frac{C}{n^k}.$$

Solution:

First, observe

$$\begin{aligned}
\mathcal{F}(f^{(j)})(n) &= \int_0^{2\pi} f^{(j)}(x)e^{-inx}\,dx \\
&= f^{(j-1)}(x)e^{-inx}\Big|_{x=0}^{2\pi} + \frac{in}{2\pi}\int_0^{2\pi} f^{(j-1)}e^{-inx}\,dx \\
&= 0 + in\mathcal{F}(f^{(j-1)})(n).
\end{aligned}$$

After $j-1$ more integration by parts, we have

$$\mathcal{F}(f^{(j)})(n) = (in)^j\mathcal{F}(f)(n).$$

Hence,

$$\|\mathcal{F}(f)\|_{\sup n} = \frac{\|\mathcal{F}(f^{(j)})\|_{\sup n}}{n^k} \leq \frac{\|f^{(j)}\|_{L_1}}{n^k}$$

which is finite since $f^{(j)}$ is continuous on the compact set $\mathcal{S}^1$.

$\square$