1. D'Angelo 1.4 Find a sequence of complex numbers such that $\sum a_n$ converges but $\sum (a_n)^3$ diverges.

Solution:

Let $\omega = e^{2\pi i/3}$ and note $\omega^3 = 1$. Now define

$$a_n = \frac{\omega^n}{n^{1/3}},$$

and note $\{n^{-1/3}\}$ is a decreasing sequence converging to 0. Hence

$$\sum |n^{-1/3} - (n+1)^{-1/3}| = \sum (n^{-1/3} - (n+1)^{-1/3}) = 1 - 0.$$

Also,

$$\left|\sum_{n=1}^{N} \omega^n\right| = \left|\omega \frac{1-\omega^N}{1-\omega}\right| \leq \frac{2}{|1-\omega|}.$$

By Corollary 1.1, $\sum a_n$ converges. On the other hand,

$$\sum_{n=1}^{N} a_n^3 = \sum_{n=1}^{N} \frac{1}{n}$$

is the diverging harmonic series.

$\square$

2. D'Angelo 1.5 Modified The following statement is equivalent to that stated in the text. Let $(X, \|\cdot\|)$ be a normed vector space. Prove X is complete if and only if $\sum a_n$ converges whenever $\sum \|a_n\|$ converges.

Solution:

First suppose that X is complete. Now, take $a_n \in X$ such that $\sum \|a_n\|$ converges. Hence $\sum \|a_n\|$ is a Cauchy sequence in $\mathbb{R}$. We will show $\sum a_n$ is Cauchy. To that end, take $k > j$ and observe

$$\left|\sum_{n=j}^{k} a_n\right| \leq \sum_{n=j}^{k} \|a_n\| \quad \text{by the triangle inequality.}$$

Since $\sum \|a_n\|$ is Cauchy, the desired quantity can be made arbitrarily small.

On the other hand, suppose $\sum a_n$ converges whenever $\sum \|a_n\|$. Let $\{b_n\}$ be a Cauchy sequence in X, and we will show that it converges to some limit in X. First, define n_1 so that $|b_{n_1} - b_n| < 1/2$ whenever $n \geq n_1$ guaranteed by $\{b_n\}$ Cauchy. Now take $n_k > n_{k-1}$, so that $|b_{n_k} - b_n| < 2^{-k}$ whenever $n \geq n_k$. Observe

$$\sum_{k=1}^{N} |b_{n_k} - b_{n_{k+1}}| < \sum_{k=1}^{N} 2^{-k} < 1.$$

So $\sum |b_{n_k} - b_{n_{k+1}}|$ converges. Now, by assumption,

$$\sum_{k=1}^{N} b_{n_k} - b_{n_{k+1}} = b_{n_1} - b_{n_{N+1}} \to b$$

for some $b \in X$. Thus, b_{n_k} converges to $\widetilde{b} = b_{n_1} - b$. Now, for any $\varepsilon > 0$ given, take n and n_k sufficiently large so that

$$|b_n - \widetilde{b}| \leq |b_n - b_{n_k}| + |b_{n_k} - \widetilde{b}| < \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

□

3. D'Angelo 1.6. For $x < x < 2\pi$, show that $\sum_{n=0}^{\infty} \frac{\cos(nx)}{\log(n+2)}$ converges to a non-negative function.

Solution:

Denote

$$S_N = \sum_{n=0}^{N} \frac{\cos(nx)}{\log(n+2)} = \sum_{n=0}^{N} a_n b_n,$$

where $a_n = 1/\log(n+2)$ and $b_n = \cos(nx)$. $\{S_N\}$ converges by a similar argument to exercise 1.2; i.e. a_n is a decreasing sequence converging to 0 and $B_N = \sum b_n$ is bounded by the same constant that $\sum \sin(nx)$ is. It remains to show that this limit is non-negative.

Summing S_N by parts twice, we have

$$\begin{aligned} S_N &= a_N B_N - \sum_{j=0}^{N} (a_{j+1} - a_j) B_j \\ &= a_N B_N - \left((a_{N+1} - a_N) \sum_{j=0}^{N} B_j - \sum_{j=0}^{N} (a_{j+2} - 2a_{j+1} + a_j) \sum_{k=0}^{j} B_k \right) \\ &= a_N B_N + (a_N - a_{N+1}) \sum_{j=0}^{N} B_j + \sum_{j=0}^{N} (a_{j+2} - 2a_{j+1} + a_j) \sum_{k=0}^{j} B_j. \end{aligned}$$

Note that $a_N B_N \to 0$ as $N \to \infty$, since B_N is bounded and $a_N \to 0$. If we show that $(a_{N+1} - a_N) \sum_{j=0}^{N} B_j \geq 0$ and that $(a_{j+2} - 2a_{j+1} + a_j) \sum_{k=0}^{j} B_j \geq 0$ for all j, then for any given $\varepsilon > 0$ and sufficiently large N

$$S_N = \varepsilon + C_N, \quad \text{where } C_N = (a_N - a_{N+1}) \sum_{j=0}^{N} B_j + (a_{j+2} - 2a_{j+1} + a_j) \sum_{k=0}^{j} B_j.$$

Hence, $\lim S_N = \lim C_N$, which consists of non-negative terms, and thus, $\lim S_N$ will be shown to be non-negative.

We first show

$$\begin{aligned}
\sum_{j=0}^{N} B_j &= \sum_{j=0}^{N}\sum_{n=0}^{j}\cos(nx) \\
&= \sum_{j=0}^{N}\sum_{n=0}^{j}\frac{\omega^n+\overline{\omega}^n}{2}, \qquad \text{where } \omega = e^{ix} \\
&= \sum_{j=0}^{N}\frac{1}{2}\left(\frac{1-\omega^{j+1}}{1-\omega}+\frac{1-\overline{\omega}^{j+1}}{1-\overline{\omega}}\right) \\
&= \Re\left[\sum_{j=0}^{N}\frac{1-\omega^{j+1}}{1-\omega}\right] \\
&= \frac{1}{|1-\omega|^2}\Re\left[\sum_{j=0}^{N}1-\omega^{j+1}-\overline{\omega}+\omega^j\right] \\
&= \frac{1}{|1-\omega|^2}\Re\left[N(1-\overline{\omega})+(1-\omega)\sum_{j=0}^{N}\omega^j\right] \\
&= \frac{N(1-\cos x)+\Big[1-\cos\Big((N+1)x\Big)\Big]}{|1-\omega|^2} \\
&\overset{\dagger}{\geq} 0.
\end{aligned}$$

Now let $f(x) = 1/\log x$ for $x > 1$ and note $a_N - a_{N+1} = f(N+2) - f(N+3)$. Observe for $x > 1$,

$$\begin{aligned}
f(x) - f(x+1) &= -\int_x^{x+1} f'(t)dt \\
&= -\int_0^1 f'(u+x)du
\end{aligned}$$

and

$$f'(x) = \frac{-1}{x\log x}.$$

Hence $0 \leq f(N+2) - f(N+3) = a_N - a_{N+1}$. This together with $\dagger$, we have that the first term in C_N is non-negative.

Now, take $f(x)$ as before, and observe

$$\begin{aligned} f(x) - 2f(x+1) + f(x+2) &= \big(f(x+2) - f(x+1)\big) - \big(f(x+1) - f(x)\big) \\ &= \int_{x+1}^{x+2} f'(t)dt - \int_{x}^{x+1} f'(t)dt \\ &= g(x+1) - g(x), \qquad \text{where } g(x) = \int_{x}^{x+1} f'(t)dt \\ &= \int_{x}^{x+1} g'(t)dt \\ &= \int_{0}^{1} g'(u+x)du. \end{aligned}$$

So it suffices to show that $g'(x) \geq 0$. Well,

$$\begin{aligned} g'(x) &= \frac{d}{dx}\left[\int_{a}^{x+1} f'(t)dt - \int_{a}^{x} f'(t)dt\right] \quad \text{for } a > 1 \\ &= f'(x+1) - f'(x) \\ &= \int_{x}^{x+1} f''(t)dt. \end{aligned}$$

Finally we calculate

$$f''(c) = \frac{\log(c)(\log(c)+2)}{\left(c\log(c)^2\right)^2} \geq 0.$$

Again, this with †, shows that the terms of the partial sum of the second term in in C_N are non-negative. We have shown C_N to be non-negative, and thus, the assertion is proved.

□

4. D'Angelo 1.7 Put $f(\theta) = 1 + a\cos(\theta)$. Note that $f \geq 0$ if and only if $|a| \leq 1$. In this case, find p such that $|p(e^{i\theta})|^2 = |f(\theta)|$.

Solution:

Note that if z is such that $|z| = 1$, then for $z = e^{i\theta}$, the trigonometric polynomial

$$\frac{a}{2}z^{-1} + 1 + \frac{a}{2}z = 1 + a\cos(\theta) = f(\theta).$$

Take

$$\begin{aligned} q(z) &= z\left(\frac{a}{2}z^{-1} + 1 + \frac{a}{2}z\right) \\ &= \frac{a}{2}\left(z^2 + \frac{2}{a}z + 1\right). \end{aligned}$$

The two roots of q are given by

$$-\frac{1}{a} \pm \sqrt{\frac{1}{a^2} - 1},$$

and note that they are multiplicative inverses of each other. Hence, denote them as

$$\xi = -\frac{1}{a} + \sqrt{\frac{1}{a^2} - 1}, \quad \xi^{-1} = -\frac{1}{a} - \sqrt{\frac{1}{a^2} - 1}.$$

Writing q in factored form, we have

$$\begin{aligned} q(z) &= \frac{a}{2}(z-\xi)(z-\xi^{-1}) \\ &= \frac{a}{2}(z-\xi)\left(\frac{1}{\overline{z}} - \frac{1}{\xi}\right) \\ &= \frac{a}{2}(z-\xi)\frac{-1}{\overline{z}\xi}(\overline{z}-\xi) \\ &= \frac{-a}{\overline{z}\xi}|z-\xi|^2 \end{aligned}$$

Therefore, if

$$p(z) = \left|\frac{a}{2\xi}\right|^{1/2} \cdot (z-\xi),$$

then $|p(z)|^2 = |q(z)| = |z| \left|\frac{a}{2}z^{-1} + 1 + \frac{a}{2}z\right| = |1 + a\cos\theta|$.

$\square$