1. D'Angelo 2.2. Prove the Cauchy-Schwarz inequality in $\mathbb{R}^n$ by writing $\|x\|^2\|y\|^2-|\langle x,y\rangle|^2$ as a sum of squares. Give the analogous proof in $\mathbb{C}^n$.

Solution:

For $x=(x_1,\ldots,x_n)$, $y=(y_1,\ldots,y_n)$, the inequality is equivalent to

$$\left(\sum_{j=1}^n x_j^2\right)\left(\sum_{k=1}^n y_k^2\right)-\left(\sum_{j=1}^n x_jy_j\right)^2\geq 0.$$

We prove the following identity by induction.

$$\left(\sum_{j=1}^n x_j^2\right)\left(\sum_{k=1}^n y_k^2\right)-\left(\sum_{j=1}^n x_jy_j\right)^2\overset{*}{=}\sum_{k=1}^{n-1}\sum_{j=1}^{k}(x_ky_i-y_kx_i)^2.$$

In the case $n=1$, it is clear that both the left and right hand side of the identity are 0. Observe

$$\begin{aligned}\left(\sum_{j=1}^{n+1} x_j^2\right)\left(\sum_{k=1}^{n+1} y_k^2\right)&=\left(\sum_{j=1}^n x_j^2+x_{n+1}^2\right)\left(\sum_{k=1}^n y_k^2+y_{n+1}^2\right)\\&=\sum_{j=1}^n x_j^2\sum_{k=1}^n y_k^2+x_{n+1}^2\sum_{k=1}^n y_k^2+y_{n+1}^2\sum_{j=1}^n x_j^2+x_{n+1}^2y_{n+1}^2,\end{aligned}$$

and

$$\left(\sum_{j=1}^{n+1} x_jy_j\right)^2=\left(\sum_{j=1}^n x_jy_j\right)^2+2x_{n+1}y_{n+1}\sum_{j=1}^n x_jy_j+x_{n+1}^2y_{n+1}^2.$$

Subtracting these quantities and invoking the induction hypothesis, the left hand side of * is

$$\begin{aligned}&\overset{*}{=}\left(\sum_{k=1}^{n-1}\sum_{j=1}^{k}(x_ky_i-y_kx_i)^2\right)+x_{n+1}^2\sum_{k=1}^n y_k^2-2x_{n+1}y_{n+1}\sum_{j=1}^n x_jy_j+y_{n+1}^2\sum_{j=1}^n x_j^2\\&=\left(\sum_{k=1}^{n-1}\sum_{j=1}^{k}(x_ky_i-y_kx_i)^2\right)+\sum_{j=1}^n(x_{n+1}y_j)^2-2x_{n+1}y_{n+1}x_jy_j+(y_{n+1}x_j)^2\\&=\left(\sum_{k=1}^{n-1}\sum_{j=1}^{k}(x_ky_i-y_kx_i)^2\right)+\sum_{j=1}^n(x_{n+1}y_j-y_{n+1}x_j)^2\\&=\sum_{k=1}^{n}\sum_{j=1}^{k}(x_ky_i-y_kx_i)^2.\end{aligned}$$

The analogous proof in $\mathbb{C}^n$ would be to prove the similarly indexed identity $\sum|x_j|^2\sum|y_k|^2-|\sum x_jy_j|=\sum\sum|x_ky_i-y_kx_i|^2$ in a completely analogous fashion, only differing by writing $|\sum x_jy_j|^2=\sum x_jy_j\sum\overline{x}_j\overline{y}_j$, and factoring a similar expression.

$\square$

2. D'Angelo 2.23. Prove the Cauchy-Schwarz inequality in $\mathbb{R}^n$ using Lagrange multipliers.

Solution:

For $y = 0$, it is clear that $\|x\|\|y\| = 0 = \langle x, y\rangle$, so consider the case where $\|y\| > 0$. For $z = \frac{y}{\|y\|}$, consider the function $g_x : \mathbb{R}^n \to \mathbb{R}$ by $g_x(z) = \langle x, z\rangle$. Note that $\|z\| = 1$. We now solve the optimization problem:

$$\text{maximize } g_x(z) \text{ subject to } f(z) := \|z\|^2 = 1;$$

by Lagrange multipliers. That is, a solution z^* satisfies

$$\nabla g_x(z^*) = -\lambda \nabla f(z^*) \iff x_i = -2\lambda z_i^*$$

for all $i = 1 \ldots n$. So a solution z^* has coordinates satisfying $z_i^* = \frac{-x_i}{2\lambda}$. For z^* to satisfy the constraint $f(z^*) = 1$,

$$1 = \sum_{i=0}^{n} (z_i^*)^2 = \sum_{i=1}^{n} \frac{x_i^2}{4\lambda^2} \iff \lambda = \pm\frac{1}{2}\sqrt{\sum_{i=1}^{n} x_i^2}.$$

Since $\{f(z)\}$ is compact, and g_x is continuous on this set, we have that g_x attains a maximum and minimum there, hence the two points classified above give the global maximum and minimum. So

$$|\langle x, z\rangle| = |g_x(z)| \le |g_x(z^*)| = \left|\left(\sum_{i=1}^{n} x_i \frac{-x_i}{2\lambda}\right)\right| = \left(\sum_{i=1}^{n} x_i^2\right)^{1/2} = \|x\|.$$

Substituting $z = \frac{y}{\|y\|}$ and multiplying both sides by $\|y\|$ yields the desired inequality.

$\square$