1. D'Angelo 1.51 Verify the formula $\Delta(u) = 4u_{z\overline{z}}$.
Solution:

We calculate

$$\begin{aligned}
\frac{\partial^2}{\partial z \partial \overline{z}} u &= \frac{1}{2}\frac{\partial}{\partial z}\left(\frac{\partial u}{\partial x} + i\frac{\partial u}{\partial y}\right) \\
&= \frac{1}{4}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)\left(\frac{\partial u}{\partial x} + i\frac{\partial u}{\partial y}\right) \\
&= \frac{1}{4}\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + i\left(\frac{\partial^2 u}{\partial y \partial x} - \frac{\partial^2 u}{\partial x \partial y}\right)\right) \\
&= \frac{1}{4}\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) \\
&= \frac{1}{4}\Delta(u).
\end{aligned}$$

□

2. D'Angelo 1.52 Show that the Laplacian in polar coordinates is given as follows:

$$\Delta(u) = u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta}$$

Solution:

Let $z = re^{i\theta}$, then by the chain rule

$$\begin{aligned}
\frac{\partial u}{\partial r} &= \frac{\partial u}{\partial z}e^{i\theta} + \frac{\partial u}{\partial \overline{z}}e^{-i\theta} \\
\frac{\partial u}{\partial \theta} &= \frac{\partial u}{\partial z}(ire^{i\theta}) + \frac{\partial u}{\partial \overline{z}}(-ire^{-i\theta}).
\end{aligned}$$

Hence,
$$\begin{aligned}
\frac{\partial^2 u}{\partial r^2} &= \left(\frac{\partial^2 u}{\partial z^2}e^{i\theta} + \frac{\partial^2 u}{\partial z \partial \overline{z}}e^{i\theta}\right)e^{i\theta} + \left(\frac{\partial^2 u}{\partial z \partial \overline{z}}e^{i\theta} + \frac{\partial^2 u}{\partial \overline{z}^2}e^{i\theta}\right)e^{-i\theta} \\
&= e^{2i\theta}u_{zz} + 2u_{z\overline{z}} + e^{-2i\theta}u_{\overline{z}\overline{z}},
\end{aligned}$$

and

$$\begin{aligned}
\frac{\partial^2 u}{\partial \theta^2} &= \frac{\partial}{\partial \theta}\left(\frac{\partial u}{\partial z}(ire^{i\theta})\right) + \frac{\partial}{\partial \theta}\left(\frac{\partial u}{\partial \overline{z}}(-ire^{-i\theta})\right) \\
&= \left(\frac{\partial^2 u}{\partial z^2}ire^{i\theta} + \frac{\partial^2 u}{\partial z \partial \overline{z}}(-ire^{i\theta})\right)(ire^{i\theta}) + \frac{\partial u}{\partial z}(-re^{i\theta}) + \dots \\
&\quad \dots \left(\frac{\partial^2 u}{\partial \overline{z} \partial z}ire^{i\theta} + \frac{\partial^2 u}{\partial \overline{z}^2}(-ire^{i\theta})\right)(-ire^{-i\theta}) + \frac{\partial u}{\partial \overline{z}}(-r\theta e^{-i\theta}) \\
&= r^2\left(-e^{2i\theta}u_{zz} + 2u_{z\overline{z}} - e^{-2i\theta}u_{\overline{z}\overline{z}}\right) - r\left(u_z e^{i\theta} + u_{\overline{z}}e^{-i\theta}\right).
\end{aligned}$$

Note $u_{rr} + \frac{1}{r^2}u_{\theta\theta} + \frac{1}{r}u_r = 4u_{z\overline{z}} = \Delta(u)$.

$\square$

3. D'Angelo 1.53 Use the previous exercise to show that the real and imaginary parts of z^n are harmonic for n a positive integer.

Solution:

Let $z^n = r^n e^{in\theta}$, then

$$\begin{aligned}\Delta(z^n) &= n(n-1)r^{n-2}e^{in\theta} + \frac{1}{r^2}(in)^2 r^n e^{in\theta} + \frac{1}{r} n r^{n-1} e^{in\theta} \\ &= n^2 r^{n-2} e^{in\theta} - n r^{n-2} e^{in\theta} - n^2 r^{n-2} e^{in\theta} + n r^{n-2} e^{in\theta} \\ &= 0.\end{aligned}$$

$\square$

4. D'Angelo 1.54 Given the series $\sum a_n z^n$, put $L = \limsup(|a_n|^{1/n})$. Show that the radius of convergence R satisfies $R = \frac{1}{L}$.

Solution:

We prove the statement for both R and L in $(0, \infty)$. By Theorem 1.10, $R = \sup\{r : |a_n|r^n \text{ is a bounded sequence}\}$.

Let $\varepsilon > 0$ be given, and $0 < R - \varepsilon < r < R$ so $|a_n|r^n \leq M$ for some $M < \infty$. Hence, $|a_n| \leq \frac{M}{r^n}$, and thus $|a_n|^{1/n} \leq \frac{M^{1/n}}{r}$. Since the sequence $M^{1/n} \to 1$, there exists an $N > 0$ so that $|a_n|^{1/n} \leq \frac{1+\varepsilon}{r}$ when $n \geq N$. Since $\sup_{n \geq m} |a_n|^{1/n}$ is decreasing,

$$\begin{aligned} L \leq \sup_{n \geq m} |a_n|^{1/n} &\leq \frac{1+\varepsilon}{r} \\ \implies R < r - \varepsilon &\leq \frac{1}{L}. \end{aligned}$$

On the other hand, let $m > 0$ so that $\sup_{n \geq m} |a_n|^{1/n} - L \leq \varepsilon$, so $\sup_{n \geq m} |a_n| \leq (L+\varepsilon)^n$. Hence, $|a_n| \left(\frac{1}{L+\varepsilon}\right)^n \leq 1$ for $n \geq m$, and $|a_n| \left(\frac{1}{L+\varepsilon}\right)^n$ is bounded for $n < m$, since it is a finite list. Thus

$$\frac{1}{L+\varepsilon} \leq R \implies \frac{1}{R} \leq L + \varepsilon \implies \frac{1}{R} \leq L \implies \frac{1}{L} \leq R.$$

$\square$

5. D'Angelo 1.55 Give three examples of power series with radius of convergence 1 with the following true. The first series converges at no points of the unit circle, the second series converges at some but not all points of the unit circle, and the third series converges at all points of the unit circle.

Solution:

Denote the power series as $\sum a_n z^n$. In the first case, consider $a_n = n$. Note that $|\frac{a_{n+1}z^{n+1}}{a_n z^n}| = \frac{(n+1)r^{n+1}}{nr^n} = \frac{n+1}{n} r$. So the power series converges when $r < 1$, and since

$a_n = n$ is unbounded, diverges on the circle. If $a_n = \frac{1}{n}$, then $|\frac{a_{n+1}z^{n+1}}{a_n z^n}| = \frac{n}{n+1}r$, so the power series converges when $r < 1$ and diverges for $r > 1$. When $r = 1$, it converges for $z = -1$, but diverges for $z = 1$. In the last case, let $a_n = \frac{1}{n^2}$. Note $|\frac{a_{n+1}z^{n+1}}{a_n z^n}| = \frac{n^2}{(n+1)^2}r$. So the power series converges for $r < 1$ and diverges for $r > 1$. Moreover, when $r = 1$, the serices converges as a p-series.

□

6. D'Angelo 1.56 Let p be a polynomial. Show that the series $\sum(-1)^n p(n)$ is Abel summable. More generally, for $|z| < 1$, show that $\sum_0^\infty p(n)z^n$ is a polynomial in $\frac{1}{1-z}$ with no constant term. Hence, the limit, as we approach the unit circle from within, exists at every point except 1.

Solution:

We only prove the general case and proceed by induction on the degree of p, say d. When $d = 0$, it is clear that $\sum c_0 n z^n = \frac{c}{1-z}$. Now, observe for a dth degree polynomial

$$\begin{aligned}\sum p(n)z^n &= \sum \left(c_d n^{d+1} + c_{d-1}n^d + \dots c_0\right) z^n \\ &= \sum c_n n^{d+1} z^n + \sum (c_{n-1}n^{d-1} + \cdots + c_0) z^n \\ &= \sum c_n n^d \frac{d}{dz} z^{n+1} + \sum (c_{n-1}n^{d-1} + \cdots + c_0) z^n.\end{aligned}$$

We invoke the induction hypothesis, and the fact that the first sum is a power series to interchange the order of $\frac{d}{dz}$ and $\sum$.

□