1. D'Angelo 2.6 Verify that $\mathbb{C}^n$ and ℓ^2 are complete.

Solution:

Following the method presented by Nhan Nguyen, we will first show that ℓ^2 is complete, then by noting that $\mathbb{C}^n$ can embedded as closed subspace of ℓ^2 via $(x_1, x_2, \dots, x_n) \to (x_1, x_2, \dots, x_n, 0, 0, \dots)$, completeness of $\mathbb{C}^n$ follows as a corollary.

We shall represent elements of ℓ^2 as functions from $\mathbb{N}$ to $\mathbb{C}$. Let $f_n : \mathbb{N} \to \mathbb{C}$ be Cauchy sequence in ℓ^2. For any fixed $i \in \mathbb{N}$, note that by adding squares,

$$|f_n(i) - f_m(i)|^2 \leq \left(\sum_{j=1}^{\infty} (f_n(j) - f_m(j))^2 \right) = \|f_n - f_m\|_{\ell^2}^2,$$

hence each sequence $\{f_n(i)\}_{n=1}^{\infty}$ is a Cauchy sequence in $\mathbb{C}$. By completeness there, for each $i \in \mathbb{N}$ there exists a complex number, say $f(i)$, such that $f_n(i) \to f(i)$ as $n \to \infty$. Take as our candidate limit of the Cauchy sequence $\{f_n\}$, and we must show that $f_n \to f$ in ℓ^2 and that $f \in \ell^2$.

Let $\varepsilon > 0$ be given, then

$$\|f_n - f_m\|_{\ell^2} = \left(\sum_{j=1}^{\infty} (f_n(j) - f_m(j))^2 \right)^{1/2} < \varepsilon/2$$

for $n > m \geq N$ for some given $N > 0$. By continuity of $\|\cdot\|_{\ell^2}$,

$$\|f_n - f\|_{\ell^2} = \lim_{m \to \infty} \|f_n - f_m\|_{\ell^2} \leq \varepsilon/2 < \varepsilon,$$

hence $f_n \to f$ in ℓ^2.

To see that $f \in \ell^2$, observe

$$\|f\|_{\ell^2} \leq \|f - f_n\|_{\ell^2} + \|f_n\|_{\ell^2} < 1 + \|f_n\|_{\ell^2}$$

for sufficiently large n.

$\square$

2. D'Angelo 2.9 Show that the set of bounded linear operators on a Hilbert space $\mathcal{H}$, $\mathcal{L}(\mathcal{H})$ is a complete normed vector space under the operator norm $\|L\|_{\mathcal{L}} = \sup_{\|z\|=1} \|L(z)\|$.

Solution:

We first verify the norm axioms for $\|\cdot\|_{\mathcal{L}}$. Clearly $\|L\|_{\mathcal{L}} \geq 0$ since $\|L(z)\| \geq 0$ for all $z \in \mathcal{H}$, and the operator norm of the zero operator is zero since the range is $\{0\}$. Suppose $\|L\|_{\mathcal{L}} = 0$, then for each $x \neq 0$ in $\mathcal{H}$

$$\frac{1}{\|x\|} \|L(x)\| = \left\| L\left(\frac{x}{\|x\|} \right) \right\| \leq \|L\|_{\mathcal{L}} = 0,$$

so $\|L(x)\| = 0$ which implies $L(x) = 0$ for each $x \neq 0$, and $L(0) = 0$ by linearity. Hence L is the zero map.

Clearly, $\|\alpha L\|_{\mathcal{H}} = \sup_{\|z\|=1} \|\alpha L(z)\| = |\alpha| \|L\|_{\mathcal{H}}$, and $\|L_1+L_2\|_{\mathcal{H}} = \sup_{\|z\|=1} \|L_1(z)+L_2(z)\| \leq \|L_1\|_{\mathcal{L}} + \|L_2\|_{\mathcal{L}}$ by the triangle inequality for $\|\cdot\|$ and properties of supremum.

For completeness, let $\{L_n\}$ be a Cauchy sequence in $\mathcal{L}(\mathcal{H})$. As in the previous problem, our candidate L is given by evaluation at each $x \in \mathcal{H}$; i.e. observe for each $x \in \mathcal{H}$

$$\|L_n(x) - L_m(x)\| = \|x\| \left\| L_n\left(\frac{x}{\|x\|}\right) - L_m\left(\frac{x}{\|x\|}\right) \right\| \leq \|x\| \|L_n - L_m\|_{\mathcal{L}}.$$

Thus, for each fixed $x \in \mathcal{H}$, $\{L_n(x)\}$ is a Cauchy sequence in $\mathcal{H}$, and thus has a unique limit, say $L(x)$. Take L to be the function that takes $x \to L(x)$. We must show that $L_n \to L$ with respect to $\|\cdot\|_{\mathcal{L}}$ and that L is a bounded linear function.

Let $\varepsilon > 0$ be given. Let $n > m$ both greater than or equal to a sufficiently large N so that $\|L_n - L_m\|_{\mathcal{L}} < \varepsilon/2$. By the generic continuity of norms

$$\|L_n - L\|_{\mathcal{L}} = \lim_{m\to\infty} \|L_n - L_m\|_{\mathcal{L}} \leq \frac{\varepsilon}{2} < \varepsilon.$$

For linearity, we must show that $L(x+y) = L(x) + L(y)$ and proceed by approximating L with L_n. That is,

$$\begin{aligned}\|L(x+y) - (L(x)+L(y))\| &= \|L(x+y) - L_n(x+y) + L_n(x+y) - (L(x)+L(y))\| \\ &\leq \|L(x+y) - L_n(x+y)\| + \|L_n(x) - L(x)\| + \|L_n(y) - L(y)\|,\end{aligned}$$

where each term can be made arbitrarily small.

Again, approximating by L_n, we have that for a sufficiently large n

$$\|L\|_{\mathcal{L}} \leq \|L - L_n\|_{\mathcal{L}} + \|L_n\|_{\mathcal{L}} \leq 1 + \|L_n\|_{\mathcal{L}},$$

so L is bounded. Moreover, by continuity of norms, we now have that $\|L\|_{\mathcal{L}} = \lim \|L_n\|_{\mathcal{L}}$.

□

3. D'Angelo 2.11. Let P be a projection. Verify that $I - P$ is a projection, such that $\mathcal{R}(P) = \mathcal{N}(I-P)$, and that $\mathcal{H} = \mathcal{R}(P) + \mathcal{N}(P)$.

Solution:

By definition, $P = P^2$. Observe that $(I-P)^2 = I - 2P + P^2 = I - 2P + P = I - P$. If $y \in \mathcal{R}(P)$, then there exists an x such that $Px = y$, so

$$(I-P)y = y - Py = y - P^2x = y - Px = y - y = 0$$

. Hence $y \in \mathcal{N}(I-P)$. On the other hand, if $y \in N(I-P)$, then $0 = y - Py$, so $Py = y$, hence $y \in \mathcal{R}(P)$. We remark that a symmetric identity holds by replacing P with $I - P$, that is $\mathcal{R}(I-P) = \mathcal{N}(P)$. Now, take any $x \in \mathcal{H}$, and let $x - Px = y$, hence $x = Px + y$. By the previous remark, $y \in \mathcal{N}(P)$, as desired.

We also remark that if $\mathcal{R}(P)$ has a complete orthonormal set, say $\{y_n\}$, then $Px = \sum \langle x, y_k \rangle y_k$ for each $x \in \mathcal{H}$. Using the symmetric identity $\mathcal{N}(P) = \mathcal{R}(I-P)$, we have that for any $x - \sum \langle x, y_k \rangle \in \mathcal{R}(I-P)$ and any fixed y_n,

$$\left\langle x - \sum_k \langle x, y_k \rangle y_k, y_n \right\rangle = \langle x, y_n \rangle - \sum_k \langle x, y_k \rangle \langle y_k, y_n \rangle = \langle x, y_n \rangle - \langle x, y_n \rangle = 0$$

Hence $\mathcal{H} = \mathcal{R}(P) \oplus \mathcal{N}(P)$.

□

4. D'Angelo 2.17. Find the orthogonal projection of the function given by x^2 onto the span of the functions 1 and x in $L^2[0,1]$.

Solution:

We will first provide a general method (alternative to Gram-Schmidt) for finding projections onto *finite* dimensional subspaces in a Hilbert space. Let V be a finite dimensional subspace of a Hilbert space H with basis $\{b_1, \ldots, b_n\}$ and $w \notin V$. It suffices to minimize the quantity

$$\left\| w - \sum_{i=1}^{n} \alpha_i b_i \right\|^2 = \left\langle w - \sum_{i=1}^{n} \alpha_i b_i,\ w - \sum_{i=1}^{n} \alpha_i b_i \right\rangle$$
$$= \|w\|^2 - \sum_{i=1}^{n} \alpha_i \langle b_i, w\rangle - \sum_{j=1}^{n} \overline{\alpha}_j \langle w, b_j\rangle + \sum_{i=1}^{n}\sum_{j=1}^{n} \alpha_i \overline{\alpha}_j \langle b_i, b_j\rangle.$$

We proceed by viewing the quantity above as a function in n complex variables, say $F(\alpha_1, \ldots, \alpha_n)$. A local minimum of this function satisfies $\frac{\partial}{\partial \alpha_i} F = 0$ and $\frac{\partial}{\partial \overline{\alpha}_j} F = 0$ for all i and j. We calculate this, and move the negative quantity to the left hand side, so

$$\langle b_i, w\rangle = \sum_{j=1}^{n} \overline{\alpha}_j \langle b_i, b_j\rangle \quad \text{and} \quad \langle w, b_j\rangle = \sum_{i=1}^{n} \alpha_i \langle b_i, b_j\rangle.$$

Note that these identities are equivalent by conjugate symmetry. If we put $\langle b_i, b_j\rangle$ into the i,j elements of an $n \times n$ matrix B, α_i into the $n \times 1$ vector α, and $\langle w, b_i\rangle$ into the $n \times 1$ vector w_b, then the above identities are equivalent to the matrix-vector equation

$$B\alpha = w_b.$$

Since B is self-adjoint and $\{b_i\}$ are linearly independent, the above matrix equation is guaranteed to have a unique solution. Since the global minimum guaranteed by Theorem 2.3 is also a local minimum, the solution to the matrix equation above is the unique local minimum.

We now apply this method to the example above. That is, we solve

$$\begin{bmatrix} \int_0^1 1 \cdot 1 & \int_0^1 1 \cdot x \\ \int_0^1 x \cdot 1 & \int_0^1 x \cdot x \end{bmatrix} \begin{bmatrix} \alpha_0 \\ \alpha_1 \end{bmatrix} = \begin{bmatrix} \int_0^1 x^2 \cdot 1 \\ \int_0^1 x^2 \cdot x \end{bmatrix}$$
$$\iff \begin{bmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{3} \end{bmatrix} \begin{bmatrix} \alpha_0 \\ \alpha_1 \end{bmatrix} = \begin{bmatrix} \frac{1}{3} \\ \frac{1}{4} \end{bmatrix}$$
$$\iff \begin{bmatrix} \alpha_0 \\ \alpha_1 \end{bmatrix} = \frac{1}{\frac{1}{3} - \frac{1}{4}} \begin{bmatrix} \frac{1}{3} & -\frac{1}{2} \\ -\frac{1}{2} & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{3} \\ \frac{1}{4} \end{bmatrix} = \begin{bmatrix} -\frac{1}{6} \\ 1 \end{bmatrix}.$$

$\square$

5. D'Angelo 2.20. Assume $\mathcal{H}$ is infinite dimensional. Show that a sequence of orthonormal vectors does not converge, but does converge weakly to 0.

Solution:

Let $\{y_k\}$ be a sequence of orthonormal vectors in $\mathcal{H}$ and $g \in \mathcal{H}$ be given. Bessel's inequality states

$$\sum_{k=0}^{\infty} |\langle g, y_k \rangle|^2 \leq \|g\|^2.$$

Hence, the sequence $\{|\langle g, y_k \rangle|^2\}$ converges to 0 as $k \to \infty$, which implies $\langle g, y_k \rangle \to 0$. Thus y_k converges weakly to 0.

However, observe

$$\|y_n - y_m\|^2 = \langle y_n - y_m, y_n - y_m \rangle = \|y_n\|^2 - \langle y_n, y_m \rangle - \langle y_n, y_m \rangle + \|y_m\|^2 = 2.$$

Thus, y_n is not a Cauchy sequence in $\mathcal{H}$, and thus, does not converge.

$\square$

6. D'Angleo 2.21. Give an example of a linear map of $\mathbb{R}^2$ such that $\langle Lu, u \rangle = 0$ for all u but $L = 0$.

Solution:

We can represent $L : \mathbb{R}^2 \to \mathbb{R}^2$ as a matrix, say

$$[L] = \begin{bmatrix} x & y \\ z & w \end{bmatrix}.$$

If we require $\langle Lu, u \rangle_{\mathbb{R}^2} = 0$ for all $u = (a, b) \in \mathbb{R}^2$, then

$$\left\langle \begin{bmatrix} x & y \\ z & w \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix}, \begin{bmatrix} a \\ b \end{bmatrix} \right\rangle_{\mathbb{R}^2} = a^2 x + ab(y + z) + b^2 w = 0.$$

So, if $x = w = 0$ and $y = 1$ and $z = -1$, then the above equality is satisfied. That is, the non-zero linear operator defined by $L(a, b) = (-b, a)$, has $\langle L(a, b), (a, b) \rangle_{\mathbb{R}^2} = -ab + ba = 0$.

$\square$