1. D'Angelo 1.27. We wish to find a particular solution to $(D-\lambda)y = g$, when g is a polynomial of degree m. Identify the coefficiients of g as a vector in $\mathbb{C}^{m+1}$. Assuming $\lambda \neq 0$, show that there is a unique particular solution y that is a polynomial of degree m. Write explicitly the matrix of the linear transformation that sends y to g and note that it is invertible. Explain precisely what happens when $\lambda = 0$.

Solution:

Recall that complex polynomials of degree m, say P_m, form an $(m+1)$-dimensional vector space with the basis $\{1, z, z^2, \dots, z^m\}$. Hence, the linear map $(D-\lambda) : P_m \to P_m$ can represented as a matrix by observing its action on this basis, and writing the resulting vector in this basis as column of the matrix. Using $(D-\lambda)z^n = nz^{n-1} - \lambda z^n$, the result is an upper-triangular $(m+1) \times (m+1)$-matrix, with $-\lambda$ on the diagonal, 1 on the upper diagonal, and all other entries zero. I.e.

$$M = \begin{pmatrix} -\lambda & 1 & 0 & \cdots & 0 \\ 0 & -\lambda & 2 & \ddots & \vdots \\ 0 & 0 & \ddots & \ddots & 0 \\ \vdots & \vdots & \ddots & \ddots & m \\ 0 & \cdots & \cdots & 0 & -\lambda \end{pmatrix}.$$

Note that the corresponding matrix equation $My = g$ can easily by solved by inductively backsolving when $\lambda \neq 0$. That is, if $g(z) = c_0 + c_1 z + \dots c_m z_m$, and the coefficients to be determined are given by $y = d_0 + d_1 z^1 + \dots d_m z^m$, then the last coefficent is $d_m = -1/\lambda$, and d_{m-k} is obtained by solving

$$-\lambda d_{m-k} + (m-k+1)d_{m-k+1} = c_{m-k} \iff d_{m-k} = \frac{((m-k+1)d_{m-k+1} - c_{m-k}}{\lambda}.$$

If $\lambda = 0$, then there are two cases to consider. If $d_m \neq 0$, then g is an mth degree polynomial, for which no mth degree polynomial y satisfies $\frac{d}{dx}y = g$. If $d_m = 0$, then g is $(m-1)$th degree, and there are infinitely many polynomials satisfying $\frac{d}{dx}y = g$. Namely, $\int g + c_0$.

□

2. D'Angelo 1.28. Consider the equation $(D-\lambda)^m y = 0$. Prove by induction that $x^j e^{\lambda x}$ for $0 \leq j \leq m-1$ form a linearly independent set of solutions.

Solution:

Consider the case where $m = 1$. Then $(D-\lambda)^m e^{\lambda x} = \lambda e^{\lambda x} - \lambda e^{\lambda x} = 0$. Now, suppose the claim holds for m. If $0 \leq j \leq m$, then

$$\begin{aligned} (D-\lambda)^{m+1} x^j e^{\lambda x} &= (D-\lambda)^m (D-\lambda) x^j e^{\lambda x} \\ &= (D-\lambda)^m (j x^{j-1} e^{\lambda x} + \lambda x^j e^{\lambda x} - \lambda x^j e^{\lambda x}) \\ &= (D-\lambda)^m (j x^{j-1} e^{\lambda x}) \\ &= 0, \end{aligned}$$

where the last equality follows from the induction hypothesis.

□

3. ODE Primer 1. Consider the ODE $y'' + 4y = 0$. This is an example of a *second order ODE with constant coefficients.*

(a) Verify that $y(t) = A\sin(2t) + B\cos(2t)$ is a solution for any constants A and B.

Using $(\frac{d^2}{dt^2}\sin)(t) = (\frac{d}{dt}\cos)(t) = -\sin t$ and $(\frac{d^2}{dt^2}\cos)(t) = (-\frac{d}{dt}\sin)(t) = -\cos t$, we calculate

$$\begin{aligned}\left(\frac{d^2}{dt^2} + 4\right) y(t) &= \frac{d^2}{dt^2}y(t) + 4y(t)\\ &= -4A\sin(2t) - 4B\cos(2t) + 4A\sin(2t) + 4B\cos(2t)\\ &= 0\end{aligned}$$

(b) Verify that $y(t) = Ce^{2it} + De^{-2it}$ is a solution for any C and E.

Using $\frac{d^2}{dt^2}e^{ct} = c^2e^{ct}$,

$$\begin{aligned}\left(\frac{d^2}{dt^2}^2 + 4\right) y(t) &= \frac{d^2}{dt^2}^2 y(t) + 4y(t)\\ &= (2i)^2Ce^{2it} + (-2i)^2De^{-2it} + 4Ce^{2it} + 4De^{-2it}\\ &= -4Ce^{2it} + -4De^{-2it} + 4Ce^{2it} + 4De^{-2it}\\ &= 0.\end{aligned}$$

(c) Show that the above two solutions are really the same. Let Euler help

Observe

$$Ce^{2it} + De^{-2it} = (C + D)\cos(2t) + i(C - D)\sin(2t).$$

So, by taking $A = C + D$ and $B = i(C - D)$, the solution in (b) is a form of (a). Moreover, if

$$C = \frac{A - iB}{2} \quad \text{and} \quad D = \frac{A + iB}{2}$$

then, using the fact that $1/i = -i$,

$$Ce^{2it} + De^{-2it} = \frac{A}{2}(e^{2it} + e^{-2it}) + \frac{B}{2i}(-e^{2it} + e^{-2it}) = A\cos(2t) + B\sin(2t).$$

Hence the solution in (a) is a form of (b).

4. ODE Primer 3. Consider $y'' - 4y = \sin t + \cos t$.

(a) Find a function $y_p(t)$ that is a *particular solution.* One method is to assume it has the form $y_p(t) = a\sin t + b\cos t$ and figure out what a and b need to be.

Rather than the proposed method, we employ the method outlined in 4.1 of D'Angelo. Using the same method as (c) of problem 3,

$$\sin t + \cos t = Ae^{it} + Be^{-it} \qquad \text{where } A = (1 - i)/2 \text{ and } B = (1 + i)/2.$$

Assuming a solution of the form $c(t)e^{\lambda t}$ for some differentiable c, we have

$$\begin{aligned}(D^2 - 4)y &= Ae^{it} + Be^{-it}\\ (D - 2)(D + 2)y &= Ae^{it} + Be^{-it}\\ (D + 2)y &= e^{2t}\int_{\infty}^{t}(Ae^{ix} + Be^{-ix})e^{-2x}dx\end{aligned}$$

$$
\begin{aligned}
(D+2)y &= e^{2t}\left(\frac{A}{-2+i}e^{it-2t} + \frac{B}{-2-i}e^{-it-2t}\right)\\
y &= e^{-2t}\int_{-\infty}^{t}\left(\frac{A}{-2+i}e^{ix} + \frac{B}{-2-i}e^{-ix}\right)e^{2ix}dx\\
&= e^{-2t}\left(\frac{A}{-5}e^{it+2t} + \frac{B}{-5}e^{-it+2t}\right)\\
&= -\frac{1}{5}\left(Ae^{it} + Be^{-it}\right)\\
&= -\frac{1}{5}\mathrm{Re}\big[(1+i)e^{it}\big] \quad \text{since } \overline{Ae^{it}} = \frac{1+i}{2}e^{-it} = Be^{-it},\\
&= -\frac{1}{5}(\cos t + \sin t).
\end{aligned}
$$

(b) Show that if y_h is a solution to $y'' - 4y = 0$ (the *homogeneous equation*), and if y_p is a particular solution to $y'' - 4y = \sin t + \cos t$, then $y_h + y_p$ is also a solution to $y'' - 4y = \sin t + \cos t$.

Note that if $y_p = c_1e^{2t} + c_2e^{-2t}$, then using linearity and factoring the operator $D^2 - 4$, we have

$$
\begin{aligned}
(D^2-4)y_p &= c_1(D+2)(D-2)e^{2t} + c_2(D-2)(D+2)e^{-2t}\\
&= c_1(D+2)0 + c_2(D-2)0 = 0.
\end{aligned}
$$

Again, using linearity

$$
\begin{aligned}
(D^2-4)(y_h+y_p) &= (D^2-4)y_h + (D^2-4)y_p\\
&= 0 + (\sin x + \cos x).
\end{aligned}
$$

We remark that if we define the appropriate domain and range for $(D^2 - 4)$ and invoke a result from linear algebra, one can show that *all* such solutions are of the form $y_p + y_h$.

5. D'Angelo 1.29. Give an example of a function on the real line that is differentiable (at all points) but not continuously differentiable.

Solution:

Consider the function $f : \mathbb{R} \to \mathbb{R}$ defined by

$$
f(x) = \begin{cases} x^2\sin\left(\dfrac{1}{x}\right) & \text{if } x \neq 0\\ 0 & \text{otherwise.}\end{cases}
$$

For $x \neq 0$, we can calculate

$$
f'(x) = 2x\sin\left(\frac{1}{x}\right) + x^2\cos\left(\frac{1}{x}\right)(-x^{-2}) = 2x\sin\left(\frac{1}{x}\right) + \cos\left(\frac{1}{x}\right).
$$

When $x = 0$, the derivative is given by

$$\lim_{h\to 0} \frac{f(h) - f(0)}{h} = \lim_{h\to 0} \frac{h^2 \sin\left(\frac{1}{h}\right)}{h} = 0,$$

since $\sin(\frac{1}{h})$ is bounded. We have shown that f is differentiable. Yet, $\lim_{x\to 0} f'(x)$ does not exist. To see this, consider $x_n = (n\pi)^{-1}$, and note $\{x_n\}$ converges to 0, yet $f'(x_n) = (-1)^n$ which defines a diverging sequence. Hence f' is not continuous.

□

6. D'Angelo 1.31 Give a suggestive argument why $e^{Dt} f(x) = f(x+t)$.

Solution:

Let us assume that f has sufficiently many derivatives so that

$$e^{Dt} f(x) := \sum_{n=0}^{\infty} \frac{f^{(n)}(x)}{n!} t^n$$

exists for all x and t. If we approximate f by its Taylor series centered at t, and evaluate at $x+t$, we have

$$f(x+t) = \sum_{n=0}^{N} \frac{f^{(n)}(t)}{n!} ((x+t) - t)^n + R_N(x+t).$$

Using the integral form of the remainder term, we have $R_N(x+t) = \int_a^{x+t} f^{(N+1)}(s) s^n/(N!)\, ds$. For a fixed x and t, the integrand can be shown to pointwise converge to zero.

□