1. D'Angelo 2.2. Prove the Cauchy-Schwarz inequality in $\mathbb{R}^n$ by writing $\|x\|^2\|y\|^2 - |\langle x, y\rangle|^2$ as a sum of squares. Give the analogous proof in $\mathbb{C}^n$.

Solution:

For $x = (x_1, \ldots, x_n)$, $y = (y_1, \ldots, y_n)$, the inequality is equivalent to

$$\left(\sum_{j=1}^n x_j^2\right)\left(\sum_{k=1}^n y_k^2\right) - \left(\sum_{i=1}^n x_i y_i\right)^2 \geq 0.$$

To proceed, we prove the following identity by induction.

$$\left(\sum_{j=1}^n x_j^2\right)\left(\sum_{k=1}^n y_k^2\right) - \left(\sum_{i=1}^n x_i y_i\right)^2 = \sum_{k=1}^n \sum_{j=1}^{k-1} (x_k y_j - y_k x_j)^2.$$

where summation $\sum_{j=1}^0$ is taken to be zero. When $n = 1$, both the left hand and right hand side are zero of the equality above. Now, Observe

$$\begin{aligned}\left(\sum_{j=1}^{n+1} x_j^2\right)\left(\sum_{k=1}^{n+1} y_k^2\right) &= \left(\sum_{j=1}^n x_j^2 + x_{n+1}^2\right)\left(\sum_{k=1}^n y_k^2 + y_{n+1}^2\right)\\ &= \sum_{j=1}^n x_j^2 \sum_{k=1}^n y_k^2 + x_{n+1}^2 \sum_{k=1}^n y_k^2 + y_{n+1}^2 \sum_{j=1}^n x_j^2 + x_{n+1}^2 y_{n+1}^2,\end{aligned}$$

and

$$\left(\sum_{j=1}^{n+1} x_j y_j\right)^2 = \left(\sum_{j=1}^n x_j y_j\right)^2 + 2x_{n+1}y_{n+1}\sum_{j=1}^n x_j y_j + x_{n+1}^2 y_{n+1}^2.$$

Subtracting these quantities and invoking the induction hypothesis, results in

$$\begin{aligned}&\left(\sum_{k=1}^n \sum_{j=1}^{k-1} (x_k y_j - y_k x_j)^2\right) + x_{n+1}^2 \sum_{k=1}^n y_k^2 - 2x_{n+1}y_{n+1}\sum_{j=1}^n x_j y_j + y_{n+1}^2 \sum_{j=1}^n x_j^2\\ &= \left(\sum_{k=1}^n \sum_{j=1}^{k-1} (x_k y_j - y_k x_j)^2\right) + \sum_{j=1}^n (x_{n+1}y_j)^2 - 2x_{n+1}y_{n+1}x_j y_j + (y_{n+1}x_j)^2\\ &= \left(\sum_{k=1}^n \sum_{j=1}^{k-1} (x_k y_j - y_k x_j)^2\right) + \sum_{j=1}^{(n+1)-1} (x_{n+1}y_j - y_{n+1}x_j)^2\\ &= \sum_{k=1}^{n+1} \sum_{j=1}^{k-1} (x_k y_j - y_k x_j)^2.\end{aligned}$$

The analogous proof in $\mathbb{C}^n$ would be to prove the similarly indexed identity $\sum |x_j|^2 \sum |y_k|^2 - |\sum x_j y_j| = \sum\sum |x_k \overline{y}_i - y_k \overline{x}_i|^2$ in a completely analogous fashion, only differing by writing $|\sum x_j y_j|^2 = \sum x_j y_j \sum \overline{x}_j \overline{y}_j$, and factoring a similar expression.

$\square$