1. D'Angelo 1.32. Define a sequence of functions $\{f_n\}$ on $[0,1]$ as follows: $f_n(x) = 0$ for $9 \leq x \leq \frac{1}{n}$ and $f_n(x) = -\log(x)$ otherwise.
Solution:

Note that $\lim_{n\to\infty} f_n(x) = -\log x$ if $x > 0$ and $\lim_{n\to\infty} f_n(0) = 0$. The resulting function is unbounded, hence it is not Riemann integrable. However, for $m < n$,

$$\|f_n - f_m\|_{L_2}^2 = \int_0^1 (f_n(x) - f_m(x))^2 dx = \int_{1/n}^{1/m} \log^2(x) dx.$$

By the mean value theorem,

$$\begin{aligned}\int_{1/n}^{1/m} \log^2(x)dx &= \left(\frac{1}{m} - \frac{1}{n}\right)\log^2(c) \quad \text{for some } c \in \left(\frac{1}{n}, \frac{1}{m}\right)\\ &\leq \left(\frac{1}{n}\right)\log^2\left(\frac{1}{n}\right).\end{aligned}$$

To evaluate this limit, we reparameterize and apply L'Hopital's rule twice

$$\begin{aligned}\lim_{x\to 0} y\log^2(y) &= \lim_{x\to 0}\frac{2x^{-1}\log x}{-x^{-2}}\\ &= \lim_{x\to 0}\frac{-2\log x}{x^{-1}}\\ &= \lim_{x\to 0}\frac{-2x^{-1}}{-x^{-2}}\\ &= \lim_{x\to 0} 2x\\ &= 0\end{aligned}$$

Hence $\{f_n\}$ is Cauchy with respect to the L_2 norm, yet does not converge to a Riemann integrable function.

If we take $m < n$ sufficiently small so that $|\log(1/m)| > 1$, then $|\log(x)| < |\log(x)|^2$ for $1/n < x < 1/m$ and

$$\|f_n - f_m\|_{L_1} = \int_0^1 |f_n(x) - f_m(x)|dx = \int_{1/n}^{1/m} |\log(x)|dx \leq \int_{1/n}^{1/m} \log^2(x)dx = \|f_n - f_m\|_{L_2}.$$

So the sequence is Cauchy in L_1 as well.

□

2. D'Angelo 1.35. For $0 \leq r < 1$, define $P_r(\theta)$ as follows. Put $z = re^{i\theta}$ and put $P_r(\theta) = \frac{1-|z|^2}{|1-z|^2}$. Show

$$P_r(\theta) = \sum_{n\in\mathbb{Z}} r^{|n|}e^{in\theta}.$$

Solution:

We calculate

$$
\begin{aligned}
P_r(\theta) &= \sum_{n\in\mathbb{Z}} r^{|n|}e^{in\theta} \\
&= \sum_{n=0}^{\infty} r^{|n|}e^{in\theta} + \sum_{n=1}^{\infty} r^{|-n|}e^{-in\theta} \\
&= \sum_{n=0}^{\infty} z^n + \sum_{n=1}^{\infty} \overline{z}^n \\
&= \frac{1}{1-z} + \frac{\overline{z}}{1-\overline{z}} \\
&= \frac{(1-\overline{z}) + \overline{z}(1-z)}{(1-z)(1-\overline{z})} \\
&= \frac{1-\overline{z}z}{(1-z)\overline{(1-z)}} \\
&= \frac{1-|z|^2}{|1-z|^2}.
\end{aligned}
$$

$\square$

3. D'Angelo 1.36. For $0 < t < \infty$, put $\mathcal{G}_t(x) = \sqrt{\frac{t}{\pi}}e^{-tx^2}$. Then $\mathcal{G}_t$ defines an approximate idenity.

Solution:

Since $\mathcal{G}_t(x) > 0$, we need only to show that $\int_{-\infty}^{\infty} \mathcal{G}_t(x)dx = 1$ and, that for $\delta > 0$,

$$\lim_{t\to\infty} \int_{|x|\geq\delta} \mathcal{G}_t(x)dx = 0.$$

First, we calculate

$$
\begin{aligned}
\int_{-\infty}^{\infty} \mathcal{G}_t(x)dx &= \pi^{-1/2}\int_{-\infty}^{\infty} e^{-tx^2}\sqrt{t}dx \\
&= \pi^{-1/2}\int_{\infty}^{\infty} e^{-u^2}dx \\
&\overset{*}{=} \pi^{-1/2}\sqrt{\pi} \\
&= 1.
\end{aligned}
$$

The equality in $*$ can be seen by noting the convergence of $\int u^{-2} \geq \int e^{-u^2}$, and evaluating the square of the integral by polar coordinates. I.e.

$$\left(\int_{-\infty}^{\infty} e^{-u^2}du\right)^2 = \int_0^{2\pi}\int_0^{\infty} e^{-r^2}rdrd\theta = \int_0^{2\pi} 2d\theta = \pi.$$

Now, since $\mathcal{G}_t$ is even, and using the same change of variables as before

$$\begin{aligned}\lim_{t\to\infty}\int_{|x|\geq\delta}\mathcal{G}_t(x)dx &= 2\int_{-\infty}^{-\delta}\mathcal{G}_t(x)dx\\ &= \frac{2}{\sqrt{\pi}}\int_{-\infty}^{-\delta\sqrt{t}} e^{-u^2}dx = 2F(-\delta\sqrt{t}),\end{aligned}$$

where $F:(-\infty,0]\to(0,1/2]$, defined by

$$F(y)\frac{1}{\sqrt{\pi}}\int_{-\infty}^{y} e^{-u^2}dx.$$

To see that F does indeed map into $(0,1/2]$, recall that the integral over the whole real line was shown to be 1, and since the integrand is even, the integral over $(-\infty,0]$ is $1/2$. Since $e^{-u^2}>0$, F is increasing, and by the fundamental theorem of calculus, F is continuous. Moreover, $\lim_{y\to\infty}F(y)=0$. Hence, for a given $\varepsilon>0$there exists a y^* so that $F(y^*)=\varepsilon/2$, by the intermediate value theorem, . Now, choose $t>\frac{y^*}{\delta}$ then $-\delta\sqrt{t}<y^*$ implies

$$\int_{|x|\geq\delta}\mathcal{G}_t(x)dx = 2F(-\delta\sqrt{t})<\varepsilon.$$

$\square$

4. D'Angelo 1.38. Find the Fourier series for $\cos^{2N}(\theta)$.
Solution:

Using the complex identity for $\cos(\theta)$, we calculate

$$\begin{aligned}\cos^{2N}(\theta) &= \frac{1}{2^{2N}}(e^{i\theta}+e^{-i\theta})^{2N}\\ &= \frac{1}{2^{2N}}\sum_{j=0}^{2N} e^{i\theta j}e^{-i\theta(2N-j)}\\ &= \frac{1}{2^{2N}}\sum_{j=0}^{2N} e^{i\theta(2N-2j)}\\ &= \frac{1}{2^{2N}}\left(\sum_{j=0}^{N} e^{2i\theta(N-j)}+\sum_{j=N+1}^{2N} e^{2i\theta(N-j)}\right)\\ &\overset{\dagger}{=} \frac{1}{2^{2N}}\left(\sum_{k=0}^{N} e^{2i\theta k}+\sum_{k=1}^{N} e^{-2i\theta k}\right)\\ &= \sum_{k=-N}^{N} c_k e^{i\theta k} \quad \text{where } c_k=\frac{1}{2^{2N}} \text{ if } k \text{ is even, 0 otherwise.}\end{aligned}$$

Incidentally, we can easily calculate $\int \cos^{2N}$ from † by

$$\begin{aligned}
\int \cos^{2N}(\theta)\,d\theta &\stackrel{\dagger}{=} \frac{1}{2^{2N}}\left(\sum_{k=1}^{N}\int e^{2i\theta k}d\theta + \sum_{k=1}^{N}\int e^{-2i\theta k}d\theta + \theta\right) + c \\
&= \frac{1}{2^{2N}}\left(\sum_{k=1}^{N}\frac{1}{2ik}e^{2i\theta k} + \sum_{k=1}^{N}\frac{-1}{2ik}e^{-2i\theta k} + \theta\right) + c \\
&= \frac{1}{2^{2N}}\left(\sum_{k=1}^{N}\frac{1}{2ik}\left(e^{2i\theta k} - e^{-2i\theta k}\right) + \theta\right) + c \\
&= \frac{1}{2^{2N}}\left(\sum_{k=1}^{N}\frac{1}{k}\sin(2k\theta) + \theta\right) + c.
\end{aligned}$$

□