**3. D'Angelo 1.6.** For $x < x < 2\pi$, show that $\sum_{n=0}^{\infty} \frac{\cos(nx)}{\log(n+2)}$ converges to a non-negative function.

**Solution:**

Denote
$$S_N = \sum_{n=0}^{N} \frac{\cos(nx)}{\log(n+2)} = \sum_{n=0}^{N} a_n b_n,$$
where $a_n = 1/\log(n+2)$ and $b_n = \cos(nx)$. $\{S_N\}$ converges by a similar argument to exercise 1.2; i.e. $a_n$ is a decreasing sequence converging to 0 and $B_N = \sum b_n$ is bounded by the same constant that $\sum \sin(nx)$ is. It remains to show that this limit is non-negative.

Summing $S_N$ by parts twice, we have

$$\begin{aligned}
S_N &= a_N B_N - \sum_{j=0}^{N}(a_{j+1} - a_j)B_j \\
&= a_N B_N - \left((a_{N+1} - a_N)\sum_{j=0}^{N} B_j - \sum_{j=0}^{N}(a_{j+2} - 2a_{j+1} + a_j)\sum_{k=0}^{j} B_k\right) \\
&= a_N B_N + (a_N - a_{N+1})\sum_{j=0}^{N} B_j + \sum_{j=0}^{N}(a_{j+2} - 2a_{j+1} + a_j)\sum_{k=0}^{j} B_j.
\end{aligned}$$

Note that $a_N B_N \to 0$ as $N \to \infty$, since $B_N$ is bounded and $a_N \to 0$. If we show that $(a_{N+1} - a_N)\sum_{j=0}^{N} B_j \geq 0$ and that $(a_{j+2} - 2a_{j+1} + a_j)\sum_{k=0}^{j} B_j \geq 0$ for all $j$, then for any given $\varepsilon > 0$ and sufficiently large $N$

$$S_N = \varepsilon + C_N, \quad \text{where } C_N = (a_N - a_{N+1})\sum_{j=0}^{N} B_j + (a_{j+2} - 2a_{j+1} + a_j)\sum_{k=0}^{j} B_j.$$

Hence, $\lim S_N = \lim C_N$, which consists of non-negative terms, and thus, $\lim S_N$ will be shown to be non-negative.

We first show

$$\begin{aligned}
\sum_{j=0}^{N} B_j &= \sum_{j=0}^{N} \sum_{n=0}^{j} \cos(nx) \\
&= \sum_{j=0}^{N} \sum_{n=0}^{j} \frac{\omega^n + \overline{\omega}^n}{2}, \qquad \text{where } \omega = e^{ix} \\
&= \sum_{j=0}^{N} \frac{1}{2} \left( \frac{1-\omega^{j+1}}{1-\omega} + \frac{1-\overline{\omega}^{j+1}}{1-\overline{\omega}} \right) \\
&= \Re \left[ \sum_{j=0}^{N} \frac{1-\omega^{j+1}}{1-\omega} \right] \\
&= \frac{1}{|1-\omega|^2} \Re \left[ \sum_{j=0}^{N} 1 - \omega^{j+1} - \overline{\omega} + \omega^j \right] \\
&= \frac{1}{|1-\omega|^2} \Re \left[ N(1-\overline{\omega}) + (1-\omega) \sum_{j=0}^{N} \omega^j \right] \\
&= \frac{N(1-\cos x) + \Big[1 - \cos\Big((N+1)x\Big)\Big]}{|1-\omega|^2} \\
&\overset{\dagger}{\geq} 0.
\end{aligned}$$

Now let $f(x) = 1/\log x$ for $x > 1$ and note $a_N - a_{N+1} = f(N+2) - f(N+3)$. Observe for $x > 1$,

$$\begin{aligned}
f(x) - f(x+1) &= -\int_x^{x+1} f'(t)dt \\
&= -\int_0^1 f'(u+x)du
\end{aligned}$$

and

$$f'(x) = \frac{-1}{x \log x}.$$

Hence $0 \leq f(N+2) - f(N+3) = a_N - a_{N+1}$. This together with $\dagger$, we have that the first term in $C_N$ is non-negative.

Now, take $f(x)$ as before, and observe

$$\begin{aligned}
f(x) - 2f(x+1) + f(x+2) &= \big(f(x+2) - f(x+1)\big) - \big(f(x+1) - f(x)\big) \\
&= \int_{x+1}^{x+2} f'(t)dt - \int_{x}^{x+1} f'(t)dt \\
&= g(x+1) - g(x), \qquad \text{where } g(x) = \int_x^{x+1} f'(t)dt \\
&= \int_x^{x+1} g'(t)dt \\
&= \int_0^1 g'(u+x)du.
\end{aligned}$$

So it suffices to show that $g'(x) \geq 0$. Well,

$$\begin{aligned}
g'(x) &= \frac{d}{dx}\left[\int_a^{x+1} f'(t)dt - \int_a^x f'(t)dt\right] \quad \text{for } a > 1 \\
&= f'(x+1) - f'(x) \\
&= \int_x^{x+1} f''(t)dt.
\end{aligned}$$

Finally we calculate

$$f''(c) = \frac{\log(c)(\log(c)+2)}{\left(c\log(c)^2\right)^2} \geq 0.$$

Again, this with †, shows that the terms of the partial sum of the second term in in $C_N$ are non-negative. We have shown $C_N$ to be non-negative, and thus, the assertion is proved.

□

**4. D'Angelo 1.7** Put $f(\theta) = 1 + a\cos(\theta)$. Note that $f \geq 0$ if and only if $|a| \leq 1$. In this case, find $p$ such that $|p(e^{i\theta})|^2 = |f(\theta)|$.

**Solution:**

Note that if $z$ is such that $|z| = 1$, then for $z = e^{i\theta}$, the trigonometric polynomial

$$\frac{a}{2}z^{-1} + 1 + \frac{a}{2}z = 1 + a\cos(\theta) = f(\theta).$$

Take

$$\begin{aligned}
q(z) &= z\left(\frac{a}{2}z^{-1} + 1 + \frac{a}{2}z\right) \\
&= \frac{a}{2}\left(z^2 + \frac{2}{a}z + 1\right).
\end{aligned}$$

The two roots of $q$ are given by

$$-\frac{1}{a} \pm \sqrt{\frac{1}{a^2} - 1},$$

and note that they are multiplicative inverses of each other. Hence, denote them as

$$\xi = -\frac{1}{a} + \sqrt{\frac{1}{a^2} - 1}, \quad \xi^{-1} = -\frac{1}{a} - \sqrt{\frac{1}{a^2} - 1}.$$

Writing $q$ in factored form, we have

$$\begin{aligned} q(z) &= \frac{a}{2}(z-\xi)(z-\xi^{-1}) \\ &= \frac{a}{2}(z-\xi)\left(\frac{1}{\overline{z}} - \frac{1}{\xi}\right) \\ &= \frac{a}{2}(z-\xi)\frac{-1}{\overline{z}\xi}(\overline{z}-\xi) \\ &= \frac{-a}{\overline{z}\xi}|z-\xi|^2 \end{aligned}$$

Therefore, if

$$p(z) = \left|\frac{a}{2\xi}\right|^{1/2} \cdot (z-\xi),$$

then $|p(z)|^2 = |q(z)| = |z| \left|\frac{a}{2}z^{-1} + 1 + \frac{a}{2}z\right| = |1 + a\cos\theta|$.

$\square$