1. D'Angelo 1.40. Assume that $f : \mathcal{S}^1 \to \mathbb{C}$ is k times continuously differentiable. Show that there is a constant C such that for $n > 0$

$$|\widehat{f}(n)| \leq \frac{C}{n^k}.$$

Solution:

First, observe

$$\begin{aligned}
\mathcal{F}(f^{(j)}) &= \int_0^{2\pi} f^{(j)} e^{-inx}\, dx \\
&= f^{(k-1)} e^{-inx}\Big|_{x=0}^{2\pi} + \frac{in}{2\pi} \int_0^{2\pi} f^{(j-1)} e^{-inx}\, dx \\
&= 0 + in\mathcal{F}(f^{(j-1)}).
\end{aligned}$$

Proceeding inductively from k, we have

$$\mathcal{F}(f^{(j)}) = (in)^k \mathcal{F}(f).$$

Hence,

$$|\mathcal{F}(f)| = \frac{|\mathcal{F}(f^{(j)})|}{n^k} \leq \frac{\|f^{(j)}\|_{L_1}}{n^k}$$

which is finite since $f^{(j)}$ is continuous on the compact set $\mathcal{S}^1$.

$\square$

2. D'Angelo 1.41. Assume that $f(x) = -1$ for $-\pi < x < 0$ and $f(x) = 1$ for $0 < x < \pi$. Compute the Fourier series for f.

Solution:

We calculate

$$\begin{aligned}
\frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-inx}\, dx &= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) \cos(nx)\, dx - \frac{i}{2\pi} \int_{-\pi}^{\pi} f(x) \sin(nx)\, dx \\
&= 0 - \frac{i}{\pi} \int_0^{\pi} \sin(nx)\, dx \\
&= \frac{-i(\cos(n\pi) - 1)}{\pi n} \\
&= \begin{cases} \dfrac{2i}{\pi n} & \text{if } n \text{ is even} \\ 0 & \text{otherwise.} \end{cases}
\end{aligned}$$

$\square$

3. D'Angelo 1.44. Put $D_k = \sum_{-k}^{k} e^{inx}$. Define F_N by

$$F_N = \frac{D_0(x) + D_1(x) + \cdots + D_{N-1}(x)}{N},$$

and show that

$$F_N = \frac{1}{N}\frac{\sin^2\left(\frac{Nx}{2}\right)}{\sin^2\left(\frac{x}{2}\right)}.$$

Solution:

Denote $\omega = e^{inx}$, and note that $\omega^{-1} = \overline{\omega}$. Expanding F_N, we have

$$\begin{aligned}
F_n(x) = \frac{1}{N}\sum_{k=0}^{N-1}\sum_{n=-k}^{k}\omega^n &= \frac{1}{N}\sum_{k=0}^{N-1}\omega^{-k}\sum_{n=-k}^{k}\omega^{k+n} \\
&= \frac{1}{N}\sum_{k=0}^{N-1}\omega^{-k}\sum_{n=0}^{2k}\omega^{n} \\
&= \frac{1}{N}\sum_{k=0}^{N-1}\omega^{-k}\frac{1-\omega^{2k+1}}{1-\omega} \\
&= \frac{1}{N}\frac{1}{1-\omega}\sum_{k=0}^{N-1}\overline{\omega}^k - \omega^{k+1} \\
&= \frac{1}{N}\frac{1}{1-\omega}\left(\frac{1-\overline{\omega}^N}{1-\overline{\omega}} - \frac{\omega-\omega^{N+1}}{1-\omega}\right) \\
&= \frac{1}{N}\frac{1}{|1-\omega|^2}\left(1-\overline{\omega}^N - (1-\overline{\omega})\frac{\omega-\omega^{N+1}}{1-\omega}\right) \\
&= \frac{1}{N}\frac{1}{|1-\omega|^2}\left(1-\overline{\omega}^N - \frac{-(1-\omega)+\omega^N(1-\omega)}{1-\omega}\right) \\
&= \frac{1}{N}\frac{1}{|1-\omega|^2}\left(2-\overline{\omega}^N-\omega^N\right) \\
&= \frac{1}{N}\frac{2-2\cos(Nx)}{(1-\cos x)^2+\sin^2 x} \\
&= \frac{1}{N}\frac{2-2\cos(Nx)}{2-2\cos x}.
\end{aligned}$$

The result follows upon using the following identity,

$$\frac{1-\cos(\alpha)}{2} = \sin^2\left(\frac{\alpha}{2}\right).$$

□

4. D'Angelo 1.50. If $\{s_n\}$ is a monotone sequence of real numbers, show that the averages $\sigma_N = \frac{1}{N}\sum_{j=1}^{N}$ also define a monotone sequence. Give an example where the converse is false.

Solution:

Without loss of generality, assume that $s_n \leq s_{n+1}$. We must show that $\sigma_N \leq \sigma_{N+1}$. Observe

$$\begin{aligned}
\sigma_{N+1} - \sigma_N &= \frac{s_1 + \cdots + s_{N+1}}{N+1} - \frac{s_1 + \cdots + s_N}{N} \\
&= \frac{N(s_1 + \cdots + s_{N+1}) - (N+1)(s_1 + \cdots + s_N)}{N(N+1)} \\
&\overset{\dagger}{=} \frac{Ns_{N+1} - (s_1 + \cdots + s_N)}{N(N+1)} \\
&= \frac{(s_{N+1} - s_1) + (s_{N+1} - s_2) + \cdots + (s_{N+1} - s_N)}{N(N+1)} \\
&\geq 0
\end{aligned}$$

by monotonicity.

For the converse assertion, consider the sequence $\{s_n\}$ given by $s_n = \left(-\frac{1}{2}\right)^n$. This sequence is clearly non-monotonic, yet continuing from †

$$\begin{aligned}
\sigma_{N+1} - \sigma_N &\overset{\dagger}{=} \frac{N\left(-\frac{1}{2}\right)^{N+1} - \displaystyle\sum_{n=1}^{N}\left(-\frac{1}{2}\right)^n}{N(N+1)} \\
&= \frac{N\left(-\frac{1}{2}\right)^{N+1} + \frac{1}{3}\left(1 - \left(-\frac{1}{2}\right)^N\right)}{N(N+1)} \\
&= \frac{\frac{1}{3}\left(1 - (\frac{3}{2}N + 1)\left(-\frac{1}{2}\right)^N\right)}{N(N+1)}.
\end{aligned}$$

It suffices to show that $1 - (\frac{3}{2}N + 1)\left(-\frac{1}{2}\right)^N$ is non-negative. Let $f(x) = 1 - (\frac{3}{2}x + 1)2^{-x}$ and observe that $f(N) \leq 1 - (\frac{3}{2}N + 1)\left(-\frac{1}{2}\right)^N$. Note that $f'(x) = -\frac{3}{2}2^{-x} + (\ln 2)2^{-x}(\frac{3}{2} + 1) = \ell(x)2^{-x}$. Since ℓ is a line with positive slope, f has a global minimum either at $x = 0$ or the root of ℓ. Now, note $f(0) = 1$, $f(1) = -\frac{1}{4}$, and $f(2) = 0$, so by the continuity of f', the global minimum of f occurs in $[0, 2)$. Thus, the quantity is non-negative for $N \geq 2$. Finally, observe for $N = 1$, $1 - (\frac{3}{2}1 + 1)\left(-\frac{1}{2}\right)^1 \geq 0$.

□