1. D'Angelo 2.25. Let $L : \ell^2 \to \ell^2$ be defined by

$$L(z_1, z_2, \dots) = (0, z_1, z_2, \dots).$$

Show that $\|Lz\|_2 = \|z\|_2$ for all z but that L is not unitary.

Solution:

Observe

$$\|Lz\|_2^2 = 0 + \sum_{n=1}^{\infty} |z_n|^2 = \|z\|_2^2,$$

so $\|Lz\|_2 = \|z\|_2$.

But, this map is not surjective since $(1, 0, 0 \dots)$ is not in its range, i.e. hence it is not invertible.

$\square$

2. D'Angelo 2.26 Give an example of a bounded linear $L : \mathcal{H} \to \mathcal{H}$ that is injective but not surjective and an example that is surjective but not injective.

Solution:

Take $\mathcal{H} = \ell^2$ with L as in **D'Angelo 2.25.** which was shown not to be surjective. Observe that

$$\langle Lz, v \rangle = 0 + \sum_{n=2}^{\infty} z_{n-1} v_n = \sum_{n=1}^{\infty} z_n v_{n+1}.$$

Hence L^* is given by

$$L(v_1, v_2, \dots) = (v_2, v_3, \dots),$$

and note $L^* L v = v$. Hence L has a left inverse, so it is injective. That is, if $Lv = Lw$ then $v = w$ since L^* was shown to be a well defined map.

Now, L^* is surjective since it has a right inverse; i.e. for any given $z \in \ell^2$, (Lz) maps to z under L^*. Note that $L^*(k, v_2, v_3, \dots) = (v_2, v_3, \dots)$ for any k, so L^* is not injective.

$\square$

3. D'Angelo 2.28 Give an example of an operator L for which $\|L^2\| \neq \|L\|^2$. Suppose $L = L^*$; show that $\|L^2\| = \|L\|^2$.

Solution:

Let $L : \mathbb{C}^2 \to \mathbb{C}^2$ by $Lz = L(z_1, z_2) = (z_2, 0)$. Note $L^2 z = L(z_2, 0) = (0, 0)$, hence $L^2 = 0$ the operator, and thus, $\|L^2\| = 0$. On the other hand, $\|L(0, 1)\| = \|(1, 0)\| = 1$, hence $\|L\| > 0$ implies $\|L\|^2 = 0$.

Now for a general $L \in \mathcal{L}(\mathcal{H})$, observe that

$$\|L^2 z\| \leq \|L\| \, \|Lz\| \leq \|L\|^2 \, \|z\|.$$

Hence, $\|L^2\| \leq \|L\|^2$. Now, suppose $L = L^*$. Then

$$\|Lz\|^2 = |\langle Lz, Lz \rangle| = |\langle z, L^* L z \rangle = \langle z, L^2 z \rangle| \leq \|z\| \, \|L^2 z\|.$$

$\square$

4. D'Angelo 2.26

5. D'Angelo 2.26

6. D'Angelo 2.26

7. D'Angelo 2.26