1. D'Angelo 1.13 Assume $a \in \mathbb{R}, b \in \mathbb{C}$, and $c > 0$. Find the minimum of the Hermitian polynomial R:

$$R(t, \bar{t}) = a + bt + \bar{b}\bar{t} + c|t|^2.$$

Solution:

Note

$$\begin{aligned} R(t, \bar{t}) &= c\left(t + \bar{b}/c\right)\overline{\left(t + \bar{b}/c\right)} - |b|^2/c + a \\ &\stackrel{*}{=} c|t + \bar{b}/c|^2 + (a - |b|^2/c) \\ &\geq (a - |b|^2/c) \end{aligned}$$

for all $t \in \mathbb{C}$. Moreover, if $t = -\bar{b}/c$, then $R(t, \bar{t}) \stackrel{*}{=} (a - |b|^2/c)$. Hence $\sup_{t \in \mathbb{C}} R(t, \bar{t}) = (a - |b|^2/c)$.

$\square$

2. D'Angelo 1.16 Prove the following statement from plane geometry. Let ξ be a point in the complex plane other than the origin, and let ω lie on the unit circle. Then every circle perpendicular to the unit circle, and containing both ξ and ω, also contains $(\bar{\xi})^{-1}$.

Solution:

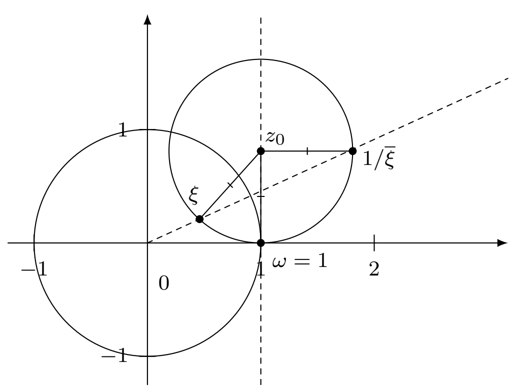

Figure 1: A diagram illustrating the statement.

Without loss of generality, we need only consider the case where $\omega = 1$, by transforming each point in the statement by the isometry given by $T(z) = e^{-i\theta}z$ where $\omega = e^{i\theta}$. Having proved the statement for $\omega = 1$, we transform back by $T^{-1}(z) = e^{i\theta}z$, and since this map is an isometry, the coincidence and geometric structure is preserved.

Any circle perpendicular to the unit circle at $\omega = 1$ has as its center $1 + ir$ for some $r > 0$. To see this, recall for every line tangent to a circle at some point ω, the line perpendicular to the tangent at ω passes through the center of the circle. Hence, for any circle, say $\mathcal{C}$, perpendicular to the unit circle at $\omega = 1$, $\mathcal{C}$ is tangent to the real axis at $\omega = 1$, and the perpendicular line $\{1+ir : r > 0\}$ passes through the center. See Figure 1.

Since $\xi \in \mathcal{C}$, we have

$$\begin{aligned} &|\xi - (1+ir)|^2 = r^2 \\ \iff &|\xi|^2 - \overline{\xi}(1+ir) - \xi(1-ir) + 1 + r^2 = r^2 \\ \iff &|\xi|^2 - \overline{\xi}(1+ir) - \xi(1-ir) + 1 \overset{\dagger}{=} 0. \end{aligned}$$

Now, it will suffice to show $|1/\overline{\xi} - (1+ir)|^2 = r^2$. Observe,

$$\begin{aligned} |1/\overline{\xi} - (1+ir)|^2 &= |1/\overline{\xi}|^2 - (1/\xi)(1+ir) - (1/\overline{\xi})(1-ir) + 1 + r^2 \\ &= \frac{1 - \overline{\xi}(1+ir) - \xi(1-ir) + |\xi|^2}{|\xi|^2} + r^2 \\ &\overset{\dagger}{=} 0 + r^2. \end{aligned}$$

□

3. D'Angelo 1.17 Prove that the series

$$e^M = \sum_{k=0}^{\infty} \frac{M^k}{k!}$$

converges for each square matrix of complex numbers.

(Please forgive the use of i as an integer index in the following solution.)

Solution:

Let M be a $k \times k$ matrix with entries $\{a_{ij}\}$. Take $m = \max |a_{ij}|$. We first find an upper bound on the absolute value of the terms of M^n in terms of $|m|$. We claim such an upper bound is $k^{n-1}m^n$ and prove it by induction. By definition, the entries of M^1 satisfy $|a_{ij}| \le k^0 m$. Now, denote the entries of M^n as $b_{ij}(n)$ and suppose $|b_{ij}(n)| \le k^{n-1}m^n$. The absolute value of the (i,j)th entry of M^{n+1} is

$$\left| \sum_{s=0}^{k} b_{is}(n) a_{sj} \right| \le \sum_{s=0}^{k} |b_{is}(n)||a_{sj}| \le \sum_{s=0}^{k} k^{n-1}m^n \cdot m = k^n m^{n+1}.$$

The induction on the claim is complete. Now, the absolute value of the entries of $\sum_{n=0}^{N} M^n/(n!)$ satisfy

$$\begin{aligned} \left| \sum_{n=0}^{N} \frac{b_{ij}(n)}{n!} \right| &\le \sum_{n=0}^{N} \frac{b_{ij}(n)}{n!} \\ &\le \sum_{n=0}^{N} \frac{k^{n-1}m^n}{n!} \\ &\le \sum_{n=0}^{N} \frac{(km)^n}{n!}. \end{aligned}$$

This last sequence of partial sums converges to e^{km}, and thinking of $b_{ij}(n)$ as a function of (i, j), the Weierstrass M-test gives the convergence of the entries of the partial matrix sums of $\sum M^n/(n!)$.

$\square$

4. D'Angelo 1.22 Find e^{At} if

$$A = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$$

Solution:

Let $A = \Lambda + N$ where

$$\Lambda = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} \quad \text{and} \quad N = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

Note that N^2 is the zero matrix, so $N^j = 0$ for $j > 1$. Now, observe for $n > 0$

$$\begin{aligned} A^n &= (\Lambda + N)^n \\ &\overset{\dagger}{=} \sum_{j=0}^{n} \binom{n}{j} \Lambda^{N-n} N^j \\ &= \Lambda^n + \Lambda^{n-1} N. \end{aligned}$$

Both Λ^n and Λ^{n-1} are diagonal matrices with λ^n and λ^{n-1} on each diagonal, respectively. We compute

$$A^n = \Lambda^n + \Lambda^{n-1} N = \begin{pmatrix} \lambda^n & n\lambda^{n-1} \\ 0 & \lambda^n, \end{pmatrix}$$

so

$$\begin{aligned} e^{At} &= \sum_{n=0}^{\infty} \frac{(At)^n}{n!} \\ &= I + \sum_{n=1}^{\infty} \frac{t^n}{n!} \left(\Lambda^n + \Lambda^{n-1} N \right) \\ &= \begin{pmatrix} \displaystyle\sum_{n=0}^{\infty} (\lambda t)^n/n! & \displaystyle\sum_{n=1}^{\infty} n t^n \lambda^{n-1}/n! \\ 0 & \displaystyle\sum_{n=0}^{\infty} (\lambda t)^n/n! \end{pmatrix} \\ &= \begin{pmatrix} e^{\lambda t} & t e^{\lambda t} \\ 0 & e^{\lambda t} \end{pmatrix}. \end{aligned}$$

We remark that for a matrix in Jordan form, one proceeds as above, but when one expands the binomial in †, higher order powers of N appear. One need only compute a closed form for each $\binom{n}{j} \Lambda^{n-j} \sum N^n$, which is relatively easy yet tedious and is not done here.

□

5. D'Angelo 1.19 If B is invertible, prove that $Be^M B^{-1} = e^{BMB^{-1}}$.
Solution:

First note that $(BMB^{-1})^n = BM^nB^{-1}$. This can be seen by induction. That is, in the $n = 1$ case it is given, and if $(BMB^{-1})^n = BM^nB^{-1}$, then $(BMB^{-1})^{n+1} = (BM^nB^{-1})(BMB^{-1}) = BM^{n+1}B^{-1}$, and the induction is complete. Now, let us compute

$$\begin{aligned} e^{BMB^{-1}} &= \sum_{n=0}^{\infty}(BMB^{-1})^n/n! \\ &= \sum_{n=0}^{\infty} BM^nB^{-1}/n! \\ &= B\left(\sum_{n=0}^{\infty} M^n/n!\right)B^{-1} \\ &= Be^M B. \end{aligned}$$

□

6. D'Angelo 1.20 Find a simple expression for $\det(e^M)$ in terms of a trace.
Solution:

Recall from linear algebra, that for every $k \times k$ matrix M, there exists an invertible $k \times k$ matrix P, and an upper triangular matrix U so that

$$M = PUP^{-1}.$$

Using the result in 1.19, we have

$$e^M = Pe^U P^{-1}.$$

Note that for U upper triangular with diagonal elements $(\lambda_1, \dots, \lambda_k)$, U^n is also upper triangular with diagonal elements $(\lambda_1^n, \dots, \lambda_k^n)$. Hence, e^U is upper triangular with diagonal elements $(e^{\lambda_1}, \dots, e^{\lambda_k})$. Using the fact that the determinant of an upper triangular matrix is given by the product of the diagonal elements, and that determinants respect multiplication and inversion, we have

$$\det(e^M) = \det(P)\det(e^U)\det(P)^{-1} = \prod_{j=1}^{k} e^{\lambda_j} = \exp\left(\sum_{j=1}^{k}\lambda_j\right) = \exp(\mathrm{Tr(M)}),$$

where $\mathrm{Tr}(M)$ denotes the trace of M.

□