1. D'Angelo 1.38. Find the Fourier series for $\cos^{2N}(\theta)$.
Solution:

Using the complex identity for $\cos(\theta)$, we calculate

$$\begin{aligned}
\cos^{2N}(\theta) &= \frac{1}{2^{2N}}(e^{i\theta}+e^{-i\theta})^{2N} \\
&= \frac{1}{2^{2N}}\sum_{j=0}^{2N}\binom{2N}{j}e^{i\theta j}e^{-i\theta(2N-j)} \\
&= \frac{1}{2^{2N}}\sum_{j=0}^{2N}\binom{2N}{j}e^{i\theta(2N-2j)} \\
&= \frac{1}{2^{2N}}\left(\sum_{j=0}^{N}\binom{2N}{j}e^{2i\theta(N-j)}+\sum_{j=N+1}^{2N}\binom{2N}{j}e^{2i\theta(N-j)}\right) \\
&\overset{\dagger}{=} \frac{1}{2^{2N}}\left(\sum_{k=0}^{N}\binom{2N}{N-k}e^{2i\theta k}+\sum_{k=1}^{N}\binom{2N}{N+k}e^{-2i\theta k}\right) \\
&= \frac{1}{2^{2N}}\sum_{k=-N}^{N}\binom{2N}{N-k}e^{2i\theta k} \\
&= \sum_{k=-N}^{N}c_k e^{i\theta k} \qquad \text{where } c_k=\frac{\binom{2N}{N-k}}{2^{2N}} \text{ if } k \text{ is even, 0 otherwise.}
\end{aligned}$$

Incidentally, we can easily calculate $\int\cos^{2N}$ from $\dagger$ by

$$\begin{aligned}
\int\cos^{2N}(\theta)\,d\theta &\overset{\dagger}{=} \frac{1}{2^{2N}}\sum_{k=-N}^{N}\binom{2N}{N-k}\int e^{2i\theta k}d\theta \\
&= \frac{1}{2^{2N}}\left(\theta+\sum_{k=1}^{N}\binom{2N}{N-k}\frac{1}{2ik}e^{2i\theta k}-\sum_{k=1}^{N}\binom{2N}{N+k}\frac{1}{2ik}e^{-2i\theta k}\right)+c \\
&= \frac{1}{2^{2N}}\left(\theta+\sum_{k=1}^{N}\binom{2N}{2N-(N+k)}\frac{1}{2ik}e^{2i\theta k}-\sum_{k=1}^{N}\binom{2N}{N+k}\frac{1}{2ik}e^{-2i\theta k}\right)+c \\
&= \frac{1}{2^{2N}}\left(\theta+\sum_{k=1}^{N}\frac{1}{2ik}\binom{2N}{N+k}\frac{1}{2ik}\left(e^{2i\theta k}-e^{-2i\theta k}\right)\right)+c \\
&= \frac{1}{2^{2N}}\left(\theta+\sum_{k=1}^{N}\frac{1}{2ik}\binom{2N}{N+k}\frac{1}{k}\sin(2k\theta)\right)+c
\end{aligned}$$

$\square$