1. D'Angelo 2.5. Let V be a real or complex vector space with a norm. Show that this norm comes from an inner product if and only if it satisfies the parallelogram law; i.e. for each vector z and w,

$$\|u+v\|^2 + \|u-v\|^2 = 2\|u\|^2 + 2\|v\|^2.$$

Solution:

It has already been shown that the parallelogram law is satisfied in an inner product space, so we proceed in showing that the parallelogram law induces an inner product. For a complex vector space, define for each vector u and v

$$f(u,v) = \frac{1}{4}\left(\|u+v\|^2 - \|u-v\|^2 + i\|u+iv\|^2 - i\|u-iv\|^2\right).$$

Note that

$$f(u,u) = \frac{1}{4}\left(\|2u\|^2 - 0 + |1+i|^2\|u\|^2 - |1+i|^2\|u\|^2\right) = \|u\|^2,$$

hence, f is positive definite and if f can be shown to satisfy the definition of an inner product, it induces the given norm. (For reference, we prove the properties in the order (4)-done (3) (1) (2) of the text.)

First, note

$$\begin{aligned}
\overline{f(u,v)} &= \frac{1}{4}\left(\|u+v\|^2 - \|u-v\|^2 - i\|u+iv\|^2 + i\|u-iv\|^2\right)\\
&= \frac{1}{4}\left(\|v+u\|^2 - |-1|\,\|v-u\|^2 - i|-i|\,\|v-iu\|^2 + i|-i|\,\|v+iu\|^2\right)\\
&= f(v,u).
\end{aligned}$$

Now, to see linearity in the first slot, consider the following calculation,

$$\begin{aligned}
\operatorname{Re} 4f(u+v,w) &= \|u+v+w\|^2 - \|u+v-w\|^2\\
&= \|u+v+w\|^2 + \|u-(v+w)\|^2 - (\|(u-w)-v\|^2 + \|(u-w)+v\|^2)\\
&= 2\|u\|^2 + 2\|v+w\|^2 - (2\|(u-w)\|^2 + 2\|v\|^2).
\end{aligned}$$

Using symmetry, we also have

$$\operatorname{Re} 4f(v+u,w) = 2\|v\|^2 + 2\|u+w\|^2 - (2\|(v-w)\|^2 + 2\|u\|^2).$$

Adding these, dividing by 8, and rearranging terms we have

$$\operatorname{Re} f(u+v,w) = \frac{1}{4}(\|u+w\|^2 - \|u-w\|^2 + \|v+w\|^2\|v-w\|^2) = \operatorname{Re} f(u,w) + \operatorname{Re}(f,w).$$

Note,

$$\operatorname{Im} 4f(u+v,w) = \|(u+v)+iw\|^2 - \|(u+v)-iw\|^2 = \operatorname{Re} 4f(u+v,iw),$$

hence,

$$\operatorname{Im} 4f(u+v,w) = \operatorname{Re}\Big[4f(u,iw) + 4f(v,iw)\Big] = \operatorname{Im}\Big[4f(u,w) + 4f(v,w)\Big],$$

by linearity of both Re and Im. Having equated real and imaginary parts, the identity $f(u+v,w) = f(u,w) + f(v,w)$ holds. (Credit and thanks is due to Nhan Nyugen for a very helpful suggestion that greatly simplified the calculation above!)

We proceed in showing $f(zu,v) = zf(u,v)$ for $z \in \mathbb{C}$ and fixed $u, v \in V$ by cases. First note,

$$f(-u,v) + f(u,v) = f(-u+u,v) = f(0,v) = 0 + 0 = 0.$$

and

$$\begin{aligned} f(iu,v) &= \frac{1}{4}\left(\|iu+v\|^2 - \|iu-v\|^2 + i\|iu+iv\|^2 - i\|iu-iv\|^2\right) \\ &= \frac{1}{4}\left(|i|\,\|u-iv\|^2 - \|iu-v\|^2 + i|i|\,\|u+v\|^2 - i|i|\,\|u-v\|^2\right) \\ &= \frac{i}{4}\left(-i\|u-iv\|^2 + i\|iu-v\|^2 + \|u+v\|^2 - \|u-v\|^2\right) \\ &= if(u,v). \end{aligned}$$

For any rational number, say $\frac{p}{q}$, we use can use linearity as follows

$$f\left(\frac{p}{q}u,v\right) = f\left(\sum_{i=1}^{p} \frac{1}{q}u,v\right) = \sum_{i=1}^{p} f\left(\frac{1}{q}u,v\right) = pf\left(\frac{1}{q}u,v\right).$$

Also

$$qf\left(\frac{1}{q}u,v\right) = \sum_{i=1}^{q} f\left(\frac{1}{q}u,v\right) = f\left(\sum_{i=1}^{q} \frac{1}{q}u,v\right) = f(u,v),$$

hence $f\left(\frac{1}{q},v\right) = \frac{1}{q}f(q,v)$ and with the last result shows $f\left(\frac{p}{q}u,v\right) = \frac{p}{q}f(u,v)$.

The last three results combined show that for any complex numbers of the form $z_n = x_n + iy_n$ where x_n, y_n are rational, we have

$$f(z_n u,v) = f((x_n + iy_n)u,v) = x_n f(u,v) + iy_n f(u,v) = z_n f(u,v).$$

Now, we remark that f is a continuous composition of norms, so it is clearly continuous. Moreover, for any complex number, say $z = x + iy$, by the density of the rationals there exists sequences of rational numbers $x_n \to x$ and $y_n \to y$, and thus

$$f(zu,v) = f(\lim z_n u,v) = \lim f(z_n u,v) = \lim z_n f(u,v) = zf(u,v).$$

$\square$