1. Check the possibility of reducing the Sturm-Liouville problem

$$\begin{cases} y'' + \lambda e^{2x} y = 0, & 0 < x < 1 \\ y(0) = 0, & y'(1) = 0 \end{cases}$$

to a Fredholm equation with symmetric kernel and find this equation.

Solution:

Consider the corresponding homogeneous problem

$$\begin{cases} y'' = Ly = 0, & 0 < x < 1 \\ y(0) = 0, & y'(1) = 0. \end{cases} \tag{1}$$

which implies $y(x) = c_1 x + c_2$, for which $y(0) = c_2 = 0$ and $y'(1) = c_1 = 0$, so $y \equiv 0$. Thus, the solution to (1) is uniquely 0 which implies that there is a Green's function, $G(x, s)$, satisfying

$$y(x) = -\lambda \int_0^1 G(x, s) e^{2s} y(s)\, ds. \tag{2}$$

Such a $G(x, s)$ can be written

$$G(x, s) = \begin{cases} y_1(x), & 0 \le x \le s \le 1 \\ y_2(x), & 0 \le s < x \le 1. \end{cases}$$

$G(x, s)$ must satisfy (1) in x, so $y_1(x) = c_1(s)x + c_2(s)$ and $y_2(x) = c_3(s)x + c_4(s)$. The boundary conditions imply $0 = G(0, s) = y_1(0) = c_2(s)$ and $0 = G_x(1, s) = y_2'(1) = c_3(s)$. In summary,

$$G(x, s) = \begin{cases} c_1(s)x, & 0 \le x \le s \le 1 \\ c_4(s), & 0 \le s < x \le 1. \end{cases}$$

The "diagonal-jump" condition implies $G_x(x, s)|_{x=s^+} - G_x(x, s)|_{x=s^-} = c_1(s) - 0 = 1$. Symmetry of $G(x, s)$ implies that $y_1(s) = y_2(s) = c_4(s)$, so $c_4(s) = s$. In summary,

$$G(x, s) = \begin{cases} x, & 0 \le x \le s \le 1 \\ s, & 0 \le s < x \le 1. \end{cases}.$$

The kernel $G(x, s)e^{2s}$ is not symmetric. However, using the same "trick" used in class, (2) is equivalent to the symmetric Fredholm problem

$$\phi(x) = -\lambda \int_0^1 e^x G(x, s) e^s \, \phi(s)\, ds.$$

where y can be recovered from $\phi(x) = y(x)e^x$.

□

2. Find characteristic values and eigenfunctrions for the Fredholm equation

$$y(x) = \lambda \int_0^1 K(x,s)y(s)\,ds,$$

where

$$K(x,s) = \begin{cases} 1-s, & 0 \le x \le s \le 1, \\ 1-x, & 0 \le s \le x \le 1 \end{cases}$$

by reducing it to a Sturm-Liouville problem.

Solution:

Note that y satisfies

$$\begin{aligned} y(x) &= \lambda \int_x^1 (1-s)y(s)\,ds + \lambda(1-x)\int_0^x y(s)\,ds \\ y'(x) &= \cancel{-\lambda(1-x)y(x) + \lambda(1-x)y(x)} - \lambda \int_0^x y(s)\,ds \\ y''(x) &= -\lambda y(x). \end{aligned}$$

Moreover,

$$\begin{aligned} y(0) &= \lambda \int_0^1 (1-s)y(s)\,ds \quad \text{and} \quad y(1) = 0 + 0 = 0. \\ y'(0) &= 0 \quad \text{and} \quad y'(1) = -\lambda \int_0^1 y(s)\,ds. \end{aligned}$$

At least, y must satisfy the system

$$\begin{cases} y'' = -\lambda y & 0 < x < 1 \\ y(1) = 0 & y'(0) = 0. \end{cases}$$

If $\lambda > 0$, solutions to this system are given by

$$\begin{cases} y(x) = c_1 \cos(\sqrt{\lambda}x) + c_2 \sin(\sqrt{\lambda}x) \\ y(1) = c_1 \cos\sqrt{\lambda} + c_2 \sin\sqrt{\lambda} = 0 \quad y'(0) = \sqrt{\lambda}c_2 = 0 \end{cases}$$

implies that $\lambda_m = \left(\frac{m\pi}{2}\right)^2$ for odd m, with corresponding eigenfuctions $y_m(x) = c\cos\left(\sqrt{\lambda_m}x\right)$.

If $\lambda < 0$, then set $\omega^2 = -\lambda$, and solutions satisfy

$$\begin{cases} y(x) = c_1 e^{\omega x} + c_2 e^{-\omega x} \\ y(1) = c_1 e^{\omega} + c_2 e^{\omega} = 0 \quad y'(0) = c_1\omega - c_2\omega = 0 \end{cases}$$

which only holds when $c_1 = c_2 = 0$. Thus, there are no negative characteristic values.

Note that these solutions are, in fact, sufficient since

$$y_m'(1) = -c\lambda_m \sin\left(\sqrt{\lambda_m}x\right) = -\lambda_m \int_0^1 y_m(s)ds$$

and

$$\begin{aligned}
c\lambda_m \int_0^1 (1-s)\cos\left(\sqrt{\lambda_m}x\right)\, ds &= c\lambda_m \cancel{\left.\frac{(1-s)\sin(\sqrt{\lambda_m}s)}{\sqrt{\lambda_m}}\right|_0^1} + \frac{c\lambda_m}{\sqrt{\lambda_m}} \int_0^1 \sin\left(\sqrt{\lambda_m}s\right)\, ds \\
&= -\frac{c\lambda_m}{\lambda_m} \cos\left(\sqrt{\lambda_m}s\right)\Big|_0^1 \\
&= c.
\end{aligned}$$

$\square$