1. Construct the Neumann series for the Fredholm equation of the second kind

$$y(x) = \lambda \int_0^1 y(s)\, ds + f(x)$$

for "small" λ and $f(x) = 1$ and find the solution.

Solution:

Let A be the integral operator $\int_0^1 \cdot\, ds$. For small enough λ, the map $y \to \lambda Ay + f$ is a contraction whose fixed point is the solution and is given by repeated composition to an arbitrary function y_0. Successive applications of this map to $y_0 = 0$ yields the Neumann series

$$y_n = f + \lambda Af + \lambda^2 A^2 f + \cdots + \lambda^n A^n f. \tag{1}$$

Note that

$$A^n f = \int_0^1 \ldots \int_0^1 1 ds = 1. \tag{2}$$

So

$$y = \lim_{n\to\infty} y_n = \sum_{k=0}^{\infty} \lambda^k = \frac{1}{1-\lambda}$$

for $|\lambda| < 1$.

We can easily verify that y is a solution,

$$y = \frac{1}{1-\lambda} = \frac{\lambda + 1 - \lambda}{1-\lambda} = \lambda \frac{1}{1-\lambda} + 1 = \lambda Ay + f.$$

□

2. Construct the resolvent kernel for the above equation using various approaches:

(a) Assume that λ is "small". For what λ does the series for the resolvent kernel converge?

Solution:

We take $f \in C[0,1]$. Observe that $\|Ay\| \le \|y\|(1-0) = \|y\|$ and $\|A1\| = 1$, hence $\|A\| = 1$. Therefore, absolute and uniform convergence in (1) is guaranteed when $\|A\||\lambda| = |\lambda| < 1$. Taking the limit in (1), using the result that A^n is an integral operator with a given kernel k_n, and interchanging $\int$ with $\sum$, the resolvent operator is given by

$$y = f + \lambda \int_0^1 \sum_{n=0}^{\infty} \lambda^n k_{n+1}(x,s) f(s) ds \tag{3}$$

Using similar reasoning as in (2), terms in the series of the resolvent kernel are given by

$$\lambda^n k_{n+1}(x,s) = \lambda^n \int_0^1 k_n(x,s) k_n(s,x) = \lambda^n \cdot 1,$$

Continuing from (3), the resolvent is

$$y = f + \frac{\lambda}{1-\lambda} \int_0^1 f(s) ds. \tag{4}$$

□

(b) Use the fact that the kernel is symmetric and express the resolvent through the characteristic values and eigenfunctions.

Solution:

The characteristic values and eigenfunctions of A satisfy

$$\lambda \int_0^1 \phi_k(s)\, ds = \phi_k(x).$$

Hence all the eigenfunctions for A are constant functions, and the one orthonormal eigenfunction is $\phi_1(x) = 1$ with characteristic value $\lambda_1 = 1$.

Following the procedure outlined in Lecture 8, for $\lambda \neq \lambda_1 = 1$, the unique solution to

$$y = \lambda A y + f$$

is given by

$$y(x) = f(x) + \lambda \int_0^1 \sum_{k=1}^{\infty} \frac{\phi_k(s)\phi_k(x)}{\lambda_k - \lambda} f(s)\, ds = f(x) + \frac{\lambda}{1-\lambda} \int_0^1 f(s) ds.$$

Note that this agrees with (3) and extends it to $\lambda > 1$.

If $\lambda = 1$, then there are two cases to consider. First, if f is orthogonal to 1, i.e. $\int_0^1 f(s)\, ds = 0$, then $y = f$ since $Af = 0$. Note that these are not unique. If f is not orthogonal to 1, then there is no solution.

□

(c) Use the fact that the kernel is degenerate.

Solution:

Since the kernel is degenerate, we can reduce the problem by setting $c = \int_0^1 y(s) ds$, then integrating both sides of the problem. We get

$$c = \int_0^1 \lambda c + f(s) ds = \lambda c + f_1.$$

where $f_1 = \int_0^1 f(s) ds$. Solving this simple linear equation for c when $\lambda \neq 1$, we have $c = (1-\lambda)^{-1} f_1$ so

$$y = \lambda c + f = \frac{\lambda}{1-\lambda} \int_0^1 f(s) ds + f. \tag{5}$$

Note that this agrees with the previous results.

When $\lambda = 1$, we have the condition that $f_1 = 0$. So there is no solution when $f_1 \neq 0$, and non-unique solutions $y = c + f$ otherwise. This coincides with the symmetric kernel analysis.

□

3. Analyze the equation

$$y(x) = \lambda \int_{-1}^{1} (2xs^3 + 5x^2 s^2) y(s)\, ds + 7x^4 + 3$$

and solve it for any λ.

Solution:

Let A denote the integral operator above and $f(x) = 7x^4 + 3$. Note that the kernel is degenerate; i.e. it is of the form $\sum_{i=1}^{2} a_i(x) b_i(s)$ where $a_1(x) = 2x$, $b_1(s) = s^3$, $a_2(x) = 5x^2$, and $b_2(s) = s^2$. Hence, solutions are of the form

$$y(x) = \lambda a_1(x)\Big(y, b_1\Big) + \lambda a_2(x)\Big(y, b_2\Big) + f(x), \tag{6}$$

where $\Big(\cdot, \cdot\Big)$ denotes the integral inner product.

Taking integral inner products on both sides of (6) by b_1 and b_2, we have,

$$Y = \lambda K Y + F, \tag{7}$$

where Y, K and F are the obvious vectors/matrices of inner products. The entries of K and F are

$$K = \begin{bmatrix} \Big(a_1, b_1\Big) & \Big(a_2, b_1\Big) \\ \Big(a_1, b_2\Big) & \Big(a_2, b_2\Big) \end{bmatrix} = \begin{bmatrix} \int_{-1}^{1} 2\,x^4 dx & \int_{-1}^{1} 5\,x^5 dx \\ \int_{-1}^{1} 2\,x^3 dx & \int_{-1}^{1} 5\,x^4 dx \end{bmatrix} = \begin{bmatrix} \frac{4}{5} & 0 \\ 0 & 2 \end{bmatrix},$$

and

$$F = \begin{bmatrix} \Big(f, b_1\Big) \\ \Big(f, b_2\Big) \end{bmatrix} = \begin{bmatrix} \int_{-1}^{1} 7\,s^7 + 3\,s^3 ds \\ \int_{-1}^{1} 7\,s^6 + 3\,s^2 ds \end{bmatrix} = \begin{bmatrix} 0 \\ 4 \end{bmatrix}.$$

Thus, (7) reduces to

$$\left. \begin{aligned} \Big(y, b_1\Big) &= \tfrac{4}{5}\lambda \Big(y, b_1\Big) \\ \Big(y, b_2\Big) &= 2\lambda \Big(y, b_2\Big) + 4 \end{aligned} \right\}. \tag{8}$$

When $\lambda = \frac{5}{4}$, the value of $\Big(y, b_1\Big) = c$ is arbitrary and $\Big(y, b_2\Big) = -\frac{8}{3}$, and from (6) we can write the non-unique solutions as

$$y(x) = c\frac{5}{2}x - \frac{50}{3}x^2 + 7x^4 + 3.$$

When $\lambda = \frac{1}{2}$, then (8) gives rise the contradiction $0 = 4$, and there is no solution.

When $\lambda \neq \frac{5}{4}$ and $\lambda \neq \frac{1}{2}$, the coefficients in Y are given by

$$Y = (I - \lambda K)^{-1} F = \begin{bmatrix} \dfrac{1}{1 - \frac{4}{5}\lambda} & 0 \\ 0 & \dfrac{1}{1 - 2\lambda} \end{bmatrix} \begin{bmatrix} 0 \\ 4 \end{bmatrix} = \begin{bmatrix} 0 \\ \dfrac{4}{1 - 2\lambda} \end{bmatrix}$$

and finally substituting into (6) gives

$$y(x) = \frac{4\lambda}{1-2\lambda}5x^2 + 7x^4 + 3.$$

$\square$