1. Show that if an operator A is bounded and B is compact then AB and BA are compact operators.

Solution:

Let $\{x_n\}$ be a bounded sequence in $\mathcal{D}(A)$. Since A is bounded, $\{Ax_n\}$ is bounded, and thus $\{BAx_n\}$ is compact since B is a compact operator. Hence BA is a compact operator.

Now, let $\{x_n\}$ be a bounded sequence in $\mathcal{D}(B)$. For $\{ABx_n\}$, consider any subsequence $\{ABx_{n_k}\}$. Since B is a compact operator, there is a point of accumulation of $\{Bx_{n_k}\}$, say x. Since A is bounded, it is also continuous, and thus, Ax is also a point of accumulation of $\{ABx_{n_k}\}$. We have show that $\{ABx_n\}$ is a compact sequence, and thus AB is a compact operator.

$\square$

2. Prove that the inverse A^{-1} of a compact operator A in $h[a, b]$ is not bounded.

Solution:

For A^{-1} to be defined, it must be that $\Lambda = 0$ is not an eigenvalue. Thus, we can construct eigenvalue-eigenvector pairs $\{(\Lambda_n, \phi_n)\}_{n=1}^{\infty}$ such that $A\phi_n = \Lambda_n\phi_n$, $\|\phi_n\| = 1$, and $\Lambda_n \neq 0$ via the same arguments presented in class. Moreover, $|\Lambda_n| \to 0$ monotonically. Now, observe

$$A\phi_n = \Lambda\phi_n \iff \phi_n = \lambda_n A^{-1}\phi_n \iff A^{-1}\phi_n = \left(\frac{1}{\Lambda_n}\right)\phi_n,$$

Hence, $|A^{-1}\phi_n| = |\Lambda_n^{-1}|$ diverges to ∞, yet $\|\phi_n\| = 1$, and thus A^{-1} is not bounded.

$\square$

3. Let A be a bounded operator in a normed space N. Show that its null-space $\mathcal{N}(A)$ is a closed linear subspace of N.

Solution:

For $y, z \in \mathcal{N}(A)$, note $A(y+\alpha z) = Ay+\alpha Az = 0$, hence $\mathcal{N}(A)$ is a linear subspace of N. Now, take a sequence $\{y_n\}$ in $\mathcal{N}(A)$, so that $y_n \to y$. Since A is bounded, it is continuous, hence $Ay_n \to Ay$. However, the sequence $\{Ay_n\}$ is the constant sequence with $Ay_n = 0$, hence $Ay = 0$, and thus $y \in \mathcal{N}(A)$.

$\square$

4. Let A be a self-adjoint operator. Prove that eigenvectors corresponding to distinct eigenvalues are orthogonal.

Solution:

Let $\{(\Lambda_n, \phi_n)\}$ be the set of eigenvalue-eigenvector pairs of A. Observe

$$\Lambda_i(\phi_i, \phi_j) = (A\phi_i, \phi_j) = (\phi_i, A\phi_j) = \Lambda_j(\phi_i, \phi_j) \iff (\Lambda_i - \Lambda_j)(\phi_i, \phi_j) = 0.$$

So if $\Lambda_i \neq \Lambda_j$, then it must be that $(\phi_i, \phi_j) = 0$.

$\square$

5. For
$$Ay = \int_0^\pi (\sin 2x \sin 2s + \sin 5x \sin 5s) y(s)\, ds \quad \text{in } h[0,\pi]:$$

(a) Show that $\Lambda_0 = 0$ is an eigenvalue and find its multiplicity;

Solution:

For any integers $0 < m \leq n$, observe

$$\begin{aligned}\int_0^\pi \sin(ns)\sin(ms)\,ds &= \frac{1}{2}\int_0^\pi (\cos(ns - ms) - \cos(ns + ms))ds \\ &= \begin{cases} \frac{1}{2}\left(\frac{\sin((n-m)s)}{n-m} - \frac{\sin((n+m)s)}{n+m}\right)\Big|_{s=0}^{\pi} & \text{if } m < n, \\ \frac{\pi}{2} & \text{if } m = n. \end{cases} \\ &= \begin{cases} 0 & \text{if } m < n \\ \frac{\pi}{2} & \text{if } m = n. \end{cases}\end{aligned}$$

Therefore, if $y_n(x) = \sin(nx)$ for $|n| \neq 2, 5$, then $Ay_n = 0 + 0 = 0$, and each are eigenvectors corresponding to $\Lambda = 0$. Hence the multiplicity is ∞.

□

(b) Find all nonzero eigenvalues Λ_k and corresponding unit eigenvectors $\phi_k(x)$, $\|\phi_k\| = 1$.

Solution:

Let $f_1(x) = \sin(2x)$ and $f_2(x) = \sin(5x)$. Any eigenvector y satisfies

$$\Lambda y = Ay = \big((f_1, y)\, f_1 + (f_2, y)\, f_2\big)$$

implies

$$\Lambda(f_i, y) = \big((f_i, f_1)(f_1, y) + (f_i, f_2)(f_2, y)\big).$$

Denote $c = \Big((y, f_1), (y, f_2)\Big)^T$ and M the 2×2 matrix with coefficients (f_i, f_j), so

$$\Lambda c = Mc.$$

From the discussion in **(a)**, $(f_i, f_j) = \frac{\pi}{2}\delta_{ij}$, so M is a diagonal matrix with $\frac{\pi}{2}$ on the diagonal. Hence $\lambda = \frac{\pi}{2}$ is the only eigenvalue, and it has multiplicity 2. The corresponding orthonormal eigenvectors are $\phi_1(x) = \frac{2}{\pi}\sin(2x)$ and $\phi_2(x) = \frac{2}{\pi}\sin(5x)$.

□

6. Find characteristic values and eigenfunctions satisfying

$$y(x) = \lambda \int_{-1}^{1} (xs^2 + x^2 s) y(s)\, ds.$$

Solution:

Let $f_1(x) = x$ and $f_2(x) = x^2$. Note that these are linearly independent. An eigenvector y satisfies

$$y = \lambda \left(x \int_{-1}^{1} s^2 y(s)\, ds + x^2 \int_{-1}^{1} s y(s)\, ds \right) = \lambda\big(f_1(f_2, y) + f_2(f_1, y)\big)$$

$$\iff (f_i, y) = \lambda\big((f_i, f_1)(f_2, y) + (f_i, f_2)(f_1, y)\big) = \lambda\big((f_i, f_2)(f_1, y) + (f_i, f_1)(f_2, y)\big).$$

In matrix vector form,

$$\begin{pmatrix} (f_1, y) \\ (f_2, y) \end{pmatrix} = \lambda \begin{pmatrix} (f_1, f_2) & (f_1, f_1) \\ (f_2, f_2) & (f_2, f_1) \end{pmatrix} \begin{pmatrix} (f_1, y) \\ (f_2, y) \end{pmatrix} \iff c = \lambda M c.$$

We need only calculate three inner products. That is

$$\begin{aligned} (f_1, f_1) &= \int_{-1}^{1} x^2\, dx = \frac{2}{3}, \\ (f_2, f_1) = (f_1, f_2) &= \int_{-1}^{1} x^3\, dx = 0, \\ (f_2, f_2) &= \int_{-1}^{1} x^4\, dx = \frac{2}{5}. \end{aligned}$$

The characteristic values satisfy

$$\left| M - \frac{1}{\lambda} I \right| = \begin{vmatrix} -\frac{1}{\lambda} & \frac{2}{3} \\ \frac{2}{5} & -\frac{1}{\lambda} \end{vmatrix} = 0 \iff \frac{1}{\lambda^2} - \frac{4}{15} = 0 \iff \lambda_{1,2} = \pm\frac{\sqrt{15}}{2}.$$

For $\lambda_1 = \frac{\sqrt{15}}{2}$, a corresponding eigenvector has coordinates satisfying

$$\frac{2}{\sqrt{15}}(f_1, y) = \frac{2}{3}(f_2, y),$$

so if $(f_1, y) = \sqrt{15}$ and $(f_2, y) = 3$, then $y_1 = 3x + \sqrt{15}x^2$.

For $\lambda_1 = -\frac{\sqrt{15}}{2}$, a corresponding eigenvector has coordinates satisfying

$$\frac{2}{\sqrt{15}}(f_1, y) = -\frac{2}{3}(f_2, y),$$

so if $(f_1, y) = -\sqrt{15}$ and $(f_2, y) = 3$, then $y_1 = 3x - \sqrt{15}x^2$.

$\square$