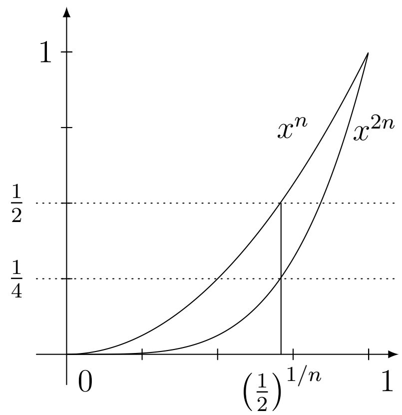
1. Is the sequence $y_n = x^n$ compact

(a) in $C[0,1]$?

Solution:

This sequence is not compact in this space. For $\{y_n\}$ we have

$$\|y_n - y_{2n}\|_{C[0,1]} = \sup_{x\in[0,1]} |x^n - x^{2n}| \geq \Big(|x^n - x^{2n}|\Big)\Big|_{x=\left(\frac{1}{2}\right)^{1/n}} = \frac{1}{4}$$



Hence $\{y_n\}$ has no points of accumulation, and thus cannot be compact.

□

(b) in $h[0,1]$?

Solution:

The sequence is compact in this space. In fact, it is Cauchy, since

$$\begin{aligned}
\|y_n - y_m\|^2_{h[0,1]} &= \|y_n\|^2_{h[0,1]} - 2\Big(y_n, y_m\Big) + \|y_m\|^2_{h[0,1]} \\
&= \int_0^1 x^{2n}\,dx - 2\int_0^1 x^{n+m}\,dx + \int_0^2 x^{2m}\,dx \\
&= \frac{1}{2n+1} - 2\frac{1}{n+m+1} + \frac{1}{2m+1} \to 0 \quad \text{as } m,n\to\infty.
\end{aligned}$$

So *every* subsequence has a point of accumulation.

□

2. Find characteristic values and eigenfunctions:

$$y(x) = \lambda \int_{-1}^{1} (xs + x^2 s^2)\, y(s)\, ds$$

Solution:

Let $f_1(x) = x$ and $f_2(x) = x^2$. Note that these are linearly independent. An eigenvector y satisfies

$$y = \lambda \left(x \int_{-1}^{1} s y(s)\, ds + x^2 \int_{-1}^{1} s^2 y(s)\, ds \right) = \lambda \left\{ \Big(f_1, y\Big) f_1(x) + \Big(f_2, y\Big) f_2(x) \right\}.$$

Taking inner products on both sides with both f_1 and f_2, we arrive at the linear system

$$\begin{bmatrix} \Big(f_1, y\Big) \\ \Big(f_2, y\Big) \end{bmatrix} = \lambda \begin{bmatrix} \Big(f_1, f_1\Big) & \Big(f_1, f_2\Big) \\ \Big(f_2, f_1\Big) & \Big(f_2, f_2\Big) \end{bmatrix} \begin{bmatrix} \Big(f_1, y\Big) \\ \Big(f_2, y\Big) \end{bmatrix} \iff Y = \lambda M Y.$$

The entries of M are given by

$$\begin{aligned} (f_1, f_1) &= \int_{-1}^{1} x^2\, dx = \frac{2}{3}, \\ (f_2, f_1) = (f_1, f_2) &= \int_{-1}^{1} x^3\, dx = 0, \qquad \text{So } M = \begin{bmatrix} \frac{2}{3} & 0 \\ 0 & \frac{2}{5} \end{bmatrix}. \\ (f_2, f_2) &= \int_{-1}^{1} x^4\, dx = \frac{2}{5}. \end{aligned}$$

Hence, the characteristic values satisfy

$$0 = |\lambda M - I| = \begin{vmatrix} \frac{2}{3}\lambda - 1 & 0 \\ 0 & \frac{2}{5}\lambda - 1 \end{vmatrix} \iff \lambda_1 = \frac{3}{2} \text{ or } \lambda_2 = \frac{5}{2}.$$

Since M is diagonal, the corresponding eigenvectors of coeffcients are given by $\binom{c_1}{0}$ and $\binom{0}{c_2}$, and, thus $y_1 = \frac{3}{2} c_1 x$ and $y_2 = \frac{5}{2} c_2 x^2$.

$\square$

3. Construct the Neumann series for the Volterra equation of the second kind

$$y(x) = \lambda \int_0^x s\, y(s)\, ds + 1$$

and find the solution.

Solution:

Let A be the integral operator $\int_0^x \cdot\, sds$ and $f \equiv 1$. It can be shown that the map $y \mapsto \lambda Ay + f$ is a contraction operator whose fixed point is the solution and is given by repeated composition to an arbitrary function y_0. Successive applications of this map to $y_0 = 0$ yields the Neumann series

$$y_{n+1} = f + \lambda Af + \lambda^2 A^2 f + \cdots + \lambda^n A^n f. \tag{1}$$

Note that

$$\begin{aligned} Af &= \int_0^x sds = \frac{x^2}{2}, \\ A^2 f &= \int_0^x \frac{s^2}{2} sds = \frac{x^4}{2 \cdot 4}, \\ A^3 f &= \int_0^x \frac{s^4}{2 \cdot 4} sds = \frac{x^6}{2 \cdot 4 \cdot 6}, \\ &\vdots \\ A^n f &= \int_0^x \frac{s^{2n}}{2^n n!} sds = \frac{x^{2n+2}}{2^{n+1}(n+1)!}. \end{aligned} \tag{2}$$

So,

$$y = \lim_{n\to\infty} y_n = \sum_{k=0}^{\infty} \frac{\left(\frac{\lambda}{2}x^2\right)^k}{k!} = e^{\frac{\lambda}{2}x^2}.$$

We can easily verify that y is a solution,

$$\lambda \int_0^x s e^{\frac{\lambda}{2}s^2} + 1 = \lambda \int_0^{\frac{\lambda}{2}x^2} e^u\, du + 1 = e^{\frac{\lambda}{2}x^2} - 1 + 1 = e^{\frac{\lambda}{2}x^2}.$$

$\square$

4. Construct the resolvent kernel for the equation in the previous problem and use it to find the solution.

Solution:

Recall that for iterated Voltera-type integral operators,

$$\int_0^x \left(\int_0^t f(s)k_n(x,s)\,ds\right) k_1(x,s)dt = \int_0^x f(s)\left(\int_s^t k_n(x,t)k_1(t,s)\,dt\right) ds.$$

Let $k_1(x,s) = s$, then inductively define

$$\begin{aligned}
k_2(x,s) &= \int_s^x tsdt = \frac{s}{2}(x^2 - s^2),\\
k_3(x,s) &= \frac{s}{2}\int_s^x (x^2-t^2)tdt = \frac{s}{2}\int_0^{x^2-s^2} \frac{u}{2}\,du = \frac{s}{2^2\cdot 2}(x^2-s^2)^2,\\
k_4(x,s) &= \frac{s}{2^2\cdot 2}\int_s^x (x^2-t^2)^2 tdt = \frac{s}{2^2\cdot 2}\int_0^{x^2-s^2} \frac{u^2}{2}\,du = \frac{s}{2^3\cdot 2\cdot 3}(x^2-s^2)^3,\\
&\vdots
\end{aligned}$$

$$k_{n+1}(x,s) = \frac{s}{2^n\cdot n!}(x^2-s^2)^n. \tag{3}$$

Note that k_{n+1} converges absolutely and uniformly over $\mathbb{R}$. Hence, we can interchange the order of summation and integration in the Neumann series to express the resovent,

$$\begin{aligned}
y = Rf := f + \sum_{n=0}^{\infty}\lambda^{n+1}A^{n+1}f &= f(x) + \int_0^x \sum_{n=0}^{\infty}\lambda^{n+1}\frac{s}{2^n\cdot n!}(x^2-s^2)^n f(s)\,ds\\
&= f(x) + \lambda\int_0^x s\exp\left(\frac{x^2-s^2}{2\lambda^{-1}}\right) f(s)\,ds\\
&= 1 + \lambda\int_{\frac{\lambda x^2}{2}}^0 e^u\left(-\lambda^{-1}du\right)\\
&= e^{\frac{\lambda x^2}{2}}
\end{aligned}$$

Note that this solutions coincides with the previous problem.

□

5. Analyze the equation

$$y(x) = \lambda \int_{-1}^{1} (1 + xs)\, y(s)\, ds + \sin \pi x$$

and solve it for any λ.

Solution:

This is a Fredholm Equation of the 2nd kind with a degenerate kernel $k(x,s) = 1 + xs = k_1(s)k_1(x) + k_2(x)k_2(s)$ where $k_1(x) = 1$ and $k_2(x) = x$. Then solutions satisfy

$$y(x) = \lambda k_1(x)\Big(y, k_1\Big) + \lambda k_2(x)\Big(y, k_2\Big) + f(x), \tag{4}$$

where $f(x) = \sin \pi x$. Taking integral inner products on both sides of (4) by k_1 and k_2, we have,

$$Y = \lambda K Y + F, \tag{5}$$

where Y, K and F are the obvious vectors/matrices of inner products. The entries of K and F are

$$K = \begin{bmatrix} \Big(k_1, k_1\Big) & \Big(k_2, k_1\Big) \\ \Big(k_1, k_2\Big) & \Big(k_2, k_2\Big) \end{bmatrix} = \begin{bmatrix} \int_{-1}^{1} dx & \int_{-1}^{1} x dx \\ \int_{-1}^{1} x dx & \int_{-1}^{1} x^2 dx \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & \frac{2}{3} \end{bmatrix},$$

and

$$F = \begin{bmatrix} \Big(f, k_1\Big) \\ \Big(f, k_2\Big) \end{bmatrix} = \begin{bmatrix} \int_{-1}^{1} \sin \pi s ds \\ \int_{-1}^{1} s \sin \pi s ds \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{2}{\pi} \end{bmatrix}.$$

Thus, (5) implies

$$\left.\begin{aligned} \Big(y, b_1\Big)(1 - 2\lambda) &= 0 \\ \Big(y, b_2\Big)(1 - \tfrac{2}{3}\lambda) &= \tfrac{2}{\pi} \end{aligned}\right\}. \tag{6}$$

When $\lambda = \frac{1}{2}$, the value of $\Big(y, b_1\Big) = c$ is arbitrary and $\Big(y, b_2\Big) = \frac{3}{\pi}$, and from (4) we can write the non-unique solutions as

$$y(x) = \frac{c}{2} - \frac{3}{2\pi} x + \sin \pi x.$$

When $\lambda = \frac{3}{2}$, then (6) gives rise the contradiction $0 = \frac{2}{\pi}$, and there is no solution.

When $\lambda \neq \frac{1}{2}$ and $\lambda \neq \frac{2}{\pi}$, then $\Big(y, b_1\Big) = 0$ and $\Big(y, b_2\Big) = \frac{2}{\pi}(1 - \frac{2}{3}\lambda)^{-1}$ and substituting into (4) gives

$$y(x) = \frac{2\lambda}{\pi\left(1 - \frac{2}{3}\lambda\right)} x + \sin \pi x.$$

$\square$

6. Construct the resolvent kernel for the equation in the previous problem and use it to find the solution

Solution:

Using the notation from the last problem, and assuming $\lambda \neq \frac{1}{2}$ and $\lambda \neq \frac{2}{\pi}$. Let

$$Q = (I - \lambda K)^{-1} = \begin{bmatrix} (1-2\lambda)^{-1} & 0 \\ 0 & (1-\frac{2}{3}\lambda)^{-1} \end{bmatrix}, \tag{7}$$

then (5) implies $Y = QF$. Putting these coordinates into (4) and pulling the integral out of the linear combination gives

$$\begin{aligned} y(x) &= f(x) + \lambda\big(k_1(x)[QF]_1 + k_2(x)[QF]_2\big) \\ &= f(x) + \lambda \left(\begin{bmatrix} k_1(x) & k_2(x) \end{bmatrix} \left[\begin{pmatrix} f, k_1 \\ f, k_2 \end{pmatrix} \right] \right) \\ &= f(x) + \lambda \int_{-1}^{1} \left(\begin{bmatrix} k_1(x) & k_2(x) \end{bmatrix} Q \begin{bmatrix} k_1(s) \\ k_2(s) \end{bmatrix} \right) f(s)\, ds \qquad (8) \\ &= \sin \pi x + \lambda \int_{-1}^{1} \left(\cancel{(1-2\lambda)^{-1}} + \left(1 - \frac{2}{3}\lambda\right)^{-1} xs \right) \sin \pi s \, ds \\ &= \sin \pi x + \lambda x \left(1 - \frac{2}{3}\lambda\right)^{-1} \frac{2}{\pi}. \end{aligned}$$

Note that this coincides with the previous problem. The resolvent kernel is given in (8),

$$R_{\lambda,s,x}(x,s) := \begin{bmatrix} k_1(x) & k_2(x) \end{bmatrix} Q \begin{bmatrix} k_1(s) \\ k_2(s) \end{bmatrix} = (1-2\lambda)^{-1} + \left(1 - \frac{2}{3}\lambda\right)^{-1} xs.$$

$\square$