1. Can we define the norm in $C^1[a,b]$ using the following expressions?

(a) $\|y\| = |y(a)+y(b)| + \max\limits_{x\in[a,b]} |y'(x)|$;

Solution:

This *does* define a norm.

(i) Clearly $\|y\| \geq 0$ and $\|0\| = 0$. Suppose $\|y\| = 0$. Then both $|y(a)+y(b)| = 0$ and $\max_{x\in[a,b]} |y'(x)| = 0$ since both terms are non-negative. Thus $y'(x) \equiv 0$, which implies $y(x) = c$ for some constant. But, $|y(a)+y(b)| = 2|c| = 0$ implies $y(x) = c = 0$.

(ii) By linearity of the derivative, we have

$$\|cy\| = |cy(a)+cy(b)| + \max_{x\in[a,b]} |cy'(x)| = |c|(|y(a)+y(b)| + \max_{x\in[a,b]} |y'(x)|) = |c|\|y\|.$$

(iii) Let $y, z \in C^1[a,b]$, and by linearity of the derivative and the triangle inequality for both $\mathbb{R}$ and $C[a,b]$,

$$\begin{aligned}
\|y+z\| &= |y(a)+z(a)+y(b)+z(b)| + \max_{x\in[a,b]} |y'(x)+z'(x)| \\
&\leq |y(a)+y(b)| + |z(a)+z(b)| + \max_{x\in[a,b]} |y'(x)| + \max_{x\in[a,b]} |z'(x)| \\
&= \|y\| + \|z\|.
\end{aligned}$$

□

(b) $\|y\| = |y(a)-y(b)| + \max_{x\in[a,b]} |y'(x)|$.

Solution:

This *does not* define a norm. Consider $y(x) = 1 \neq 0$, but

$$\|y\| = |1-1| + \max_{x\in[a,b]} |0| = 0.$$

□

2. Find the norm of $y(x) = \sin x$ in the following normed spaces:

(a) $C[0,2\pi]$;

Solution:

$$\|y\| = \max_{x\in[0,2\pi]} |\sin x| = 1.$$

□

(b) $C^1[0,2\pi]$;

Solution:

$$\|y\| = \max_{x\in[0,2\pi]} |\sin x| + \max_{x\in[0,2\pi]} |\cos x| = 2.$$

□

(c) $h[0, 2\pi]$.

Solution:

Using linearity of integrals and 2π-periodicity,

$$\|y\| = \sqrt{\int_0^{2\pi} \sin^2 x \, dx} = \sqrt{\int_0^{2\pi} \frac{1 - \cos(2x)}{2} \, dx} = \sqrt{2\pi \frac{1}{2} - 0} = \sqrt{\pi}.$$

□

3. Prove that an operator A is bounded iff it takes any bounded sequence to a bounded sequence.

Solution:

Suppose A is bounded. Let $\{y_n\}$ be a bounded sequence, such that $\|y_n\| \leq M$ for each n. If $y_n = 0$, then $A(y_n) = A(0) = 0$, since A is linear. Otherwise,

$$Ay_n = |y_n| A \frac{y_n}{|y_n|} \leq M\|A\|.$$

On the other hand, suppose A is unbounded. Then $\sup_{\|z\|=1} Az = \infty$. That is, for each n, there is z_n such that $\|z_n\| = 1$, yet $\|Az_n\| \geq n$. Hence, A does not take all bounded sequences to unbounded sequences.

□

4. Show that the Fredholm operator $Ay = \int_0^1 xs \, y(s)ds$ is bounded and find some estimate from the above for its norm $\|A\|$ if:

(a) $A : C[0, 1] \to C[0, 1]$;

$$\begin{aligned}
\|A\| &= \sup_{\|y\|=1} \|Ay\| \\
&= \sup_{\|y\|=1} \max_{x\in[0,1]} \left| \int_0^1 xs \, y(s)ds \right| \\
&= \sup_{\|y\|=1} \left| \int_0^1 s \, y(s)ds \right| \\
&\overset{\dagger}{\leq} \sup_{\|y\|=1} \sqrt{\int_0^1 s^2 ds} \cdot \sqrt{\int_0^1 y(s)^2 ds} \\
&= \sqrt{\int_0^1 s^2 ds} \cdot \sup_{\|y\|=1} \sqrt{\int_0^1 y(s)^2 ds} \\
&\leq \sqrt{\int_0^1 s^2 ds} \cdot \sqrt{(1-0) \cdot 1^2} \\
&= \frac{1}{\sqrt{3}}
\end{aligned}$$

(b) $A : h[0, 1] \to h[0, 1]$.

$$\begin{aligned}
\|A\| &= \sup_{\|y\|=1} \|Ay\| \\
&= \sup_{\|y\|=1} \sqrt{\int_0^1 \Big((Ay)(x)\Big)^2 dx} \\
&= \sup_{\|y\|=1} \sqrt{\int_0^1 \Big(\textstyle\int_0^1 xs \, y(s)ds\Big)^2 dx} \\
&= \sup_{\|y\|=1} \sqrt{\int_0^1 x^2 \Big(\textstyle\int_0^1 s \, y(s)ds\Big)^2 dx} \\
&= \sqrt{\int_0^1 x^2 dx} \sup_{\|y\|=1} \left| \int_0^1 s \, y(s)ds \right| \\
&= \frac{1}{\sqrt{3}} \sup_{\|y\|=1} \left| \int_0^1 s \, y(s)ds \right| \\
&\overset{\dagger}{\leq} \frac{1}{\sqrt{3}} \sup_{\|y\|=1} \sqrt{\int_0^1 s^2 ds} \cdot \sqrt{\int_0^1 y(s)ds} \\
&= \frac{1}{3}
\end{aligned}$$

Wherein †, we used the Cauchy-Schwartz estimate.

5. Show that the sequence $y_n(x) = \sin(nx)$ is bounded but not compact in the spaces:

(a) $h[0, \pi]$;

Solution:

Let $\{y_n\}$ be the sequence of functions defined by $y_n(x) = \sin(nx)$. We calculate similarly to problem 2,

$$\|y_n\|^2 = \int_0^{\pi} \sin^2(nx)\,dx = \int_0^{\pi} \frac{1-\cos(2nx)}{2}\,dx = \frac{\pi}{2} - 0 \overset{*}{=} \frac{\pi}{2}. \quad \text{(Note the norm is independent of } n.\text{)}$$

Hence $\{y_n\}$ is bounded. Observe for $n \neq m$

$$\begin{aligned}
(y_n, y_m) &= \int_0^{\pi} \sin(nx)\sin(mx)dx \\
&= \int_0^{\pi} \frac{1}{2}\Big(\cos((n-m)x) - \cos((n+m)x)\Big)dx \\
&= \frac{1}{2}\left(\frac{1}{n-m}\sin((n-m)x)\Big|_{x=0}^{\pi} - \frac{1}{n+m}\sin((n+m)x)\Big|_{x=0}^{\pi}\right)dx \\
&= 0.
\end{aligned}$$

Hence, $\{y_n\}$ form an orthogonal system in $h[0, \pi]$, and by a similar argument to the one presented in class,

$$\|y_n - y_m\|^2 = \|y_n\|^2 - 2(y_n, y_m) + \|y_m\|^2 = \|y_n\|^2 + \|y_m\|^2 \overset{*}{=} \pi$$

implies that any subsequence of $\{y_n\}$ cannot converge.

□

(b) $C[0, \pi]$.

Solution:

Clearly $\|y_n(x)\| = \max_{x\in[0,\pi]} |\sin(nx)| = 1$.

If $C[0, \pi]$ was compact then there would be a subsequence $\{y_{n_k}\}$ such that for sufficiently large k and j

$$\max_{x\in[0,\pi]} |\sin(n_k x) - \sin(n_j x)| < 1.$$

But, from part (a),

$$\pi \overset{*}{=} \int_0^{\pi} (\sin(n_k x) - \sin(n_j x))^2 dx \leq \pi \left(\max_{x\in[0,\pi]} |\sin(n_k x) - \sin(n_j x)|\right)^2 < \pi \cdot 1$$

a contradiction.

□