1. Show that any contraction operator A is:

(a) Continuous, i.e. for any convergent sequence $y_n \to y$ the sequence $Ay_n \to Ay$;

(b) "Bounded" in the following sense: for any bounded sequence y_n the sequence Ay_n is bounded.

Solution: to part (a)

Let $0 \leq c < 1$ such that $\|Ay_n - Ay\| \leq c\|y_n - y\|$ given by A a contraction. Hence $\|Ay_n - Ay\| \to 0$ at least as fast as $\|y_n - y\| \to 0$.

□

Solution: to part (b)

Let $\|y_n\| \leq B$ for all n and c as above. Then

$$\|Ay_n\| = \|Ay_n - A0 + A0\| \leq \|Ay_n - A0\| + \|A0\| \leq cB + \|A0\|.$$

□

2. Construct the Neumann series for the Volterra equation of the second kind

$$y(x) = \lambda \int_0^x y(s)ds + 1$$

and find the solution.

Solution:

Let A be the integral operator given by $Ay = \int_0^x y(s)ds$. The Neumann series is given by

$$y_n = \sum_{k=0}^{n-1} \frac{(x\lambda)^k}{k!}.$$

To see this, the inductive step is given by

$$y_{n+1} = \lambda \int_0^x y_n ds + 1 = \lambda \int_0^x \sum_{k=0}^{n-1} \frac{(x\lambda)^k}{k!} ds + 1 = \lambda \sum_{k=0}^{n-1} \frac{\lambda^k s^{k+1}}{(k+1)!} + 1 = \sum_{k=0}^{n} \frac{(x\lambda)^k}{k!}.$$

This is precisely the series for $e^{\lambda x}$.

□

3. Construct the resolvent kernel for the equation in Problem 2 and use it to find the solution.

Solution:

The nth term for resolvent kernel is given by

$$K_n(x, s) = \frac{(x-s)^{n-1}}{(n-1)!}.$$

To see this, the inductive step is

$$K_{n+1}(x,s) = \int_s^x K_n(x,t)1\,dt = \int_s^x \frac{(x-t)^{n-1}}{(n-1)!}dt = -\frac{(x-t)^n}{n!}\bigg|_{t=s}^{x} = \frac{(x-s)^n}{n!}.$$

Note that $\sum \lambda^{n-1}K_n$ converges absolutely and uniformly to $e^{\lambda(x-s)}$, hence the solution is given by

$$y = 1 + \lambda R_\lambda 1 = 1 + \lambda \int_0^x e^{\lambda(x-s)}\,1\,ds = 1 + \lambda e^{\lambda x}\left(\frac{-1}{\lambda}e^{-\lambda s}\bigg|_{s=0}^{x}\right) = e^{\lambda x}.$$

□

4. Reduce the equation in Problem 2 to a Cauchy problem and find the solution.

Solution:

By differentiating both sides of the equation and noting $y(0) = 1$, we see that the problem is equivalent to the ordinary differential equation

$$y' = \lambda y, \quad y(0) = 1.$$

By inspection, this simple equation has the solution $y(x) = e^{\lambda x}$.

□