Biometrika (2014), **101**, 3, *pp.* 741–747
Printed in Great Britain
doi: 10.1093/biomet/asu021
Advance Access publication 16 July 2014

Construction of orthogonal and nearly orthogonal designs for computer experiments

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Summary

This paper presents new infinite families of orthogonal designs for computer experiments. In cases where orthogonal designs cannot exist, we construct alternative, nearly orthogonal designs. Our designs can accommodate many factors and a large set of levels. No iterative computer search is required. To build up the desired orthogonal designs we develop and use new infinite classes of periodic Golay pairs.

Some key words: Column-orthogonal design; Design of experiment; Golay sequence; Orthogonal design; Periodic Golay pair; U-type design.

1. Introduction

Computer experiments have become very common (e.g., Fang et al., 2006; Sacks et al., 1989). Latin hypercube designs, which have n equally spaced levels on n runs, form a major class of designs for computer experiments and have been widely studied (Georgiou, 2009; Lin et al., 2009; Sun et al., 2009; Yang & Liu, 2012; Georgiou & Efthimiou, 2014). Bingham et al. (2009) relaxed the condition that the number of levels for each factor must be identical to the run size, and constructed an extended and useful class of orthogonal and nearly orthogonal designs for computer experiments. Lin et al. (2010) proposed a method for constructing new designs for computer experiments by using the Kronecker product on small designs with specific properties. Under some circumstances these designs are orthogonal designs.

An orthogonal design, denoted by $\text{OD}(n; q^m)$, is an $n \times m$ matrix with entries from a set of q levels such that the m columns are pairwise orthogonal. The columns and rows can be identified with factors and experimental runs, respectively. Orthogonal designs have been referred to in the literature also as column-orthogonal designs (e.g., Sun et al., 2011). If the columns are only approximately orthogonal, then the design is called nearly orthogonal and denoted by $\text{NOD}(n; q^m)$. We consider orthogonal and nearly orthogonal designs such that the q levels are symmetrically placed about 0 and for every nonzero l, the levels $+l$ and $-l$ appear equally often in each column. As a consequence, these designs are mean orthogonal. Indeed, quite a few of the designs obtained here are of the U-type in the sense of having all q levels equally replicated in every column. Clearly, if the levels are equally spaced and $q = n$, then an $\text{OD}(n; q^m)$ as considered here reduces to an orthogonal Latin hypercube.

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The main result of this paper is the construction of new orthogonal and nearly orthogonal designs that can be used for computer experiments or for screening. These constructions rely heavily on circulant matrices, Golay sequences, and new classes of periodic Golay pairs, which are also developed in this paper. Our results fill gaps in the list of known periodic Golay pairs, orthogonal and nearly orthogonal designs. The methodology and new results equip practitioners with directly applicable tools for experimentation.

2. NEW RESULTS ON PERIODIC GOLAY PAIRS

Circulant matrices will be extensively used for the proposed constructions. A matrix is said to be circulant if each row vector is rotated one element to the right relative to the preceding row vector. Circulant matrices with an opposite direction of shift are called back-circulant. We will use $B = \mathrm{circ}(A)$ to denote that the circulant matrix B is generated by the row vector A. More details on circulant and back-circulant matrices can be found in Geramita & Seberry (1979, pp. 81–82).

Let $A = \{A_j : A_j = (a_{j,0}, \ldots, a_{j,n-1}),\ j = 1, \ldots, \ell\}$, be a set of ℓ row vectors of length n. The periodic autocorrelation function $P_A(s)$ and the non-periodic autocorrelation function $N_A(s)$ are defined, reducing $i + s$ modulo n, as

$$P_A(s) = \sum_{j=1}^{\ell} \sum_{i=0}^{n-1} a_{j,i} a_{j,i+s}, \quad N_A(s) = \sum_{j=1}^{\ell} \sum_{i=0}^{n-s-1} a_{j,i} a_{j,i+s} \quad (s = 0, \ldots, n-1).$$

The set of row vectors A is said to have zero periodic autocorrelation function if $P_A(s) = 0$ and zero non-periodic autocorrelation function if $N_A(s) = 0$ $(s = 1, \ldots, n-1)$. We again refer to Geramita & Seberry (1979, p. 130), for more details.

Throughout this paper we use $\bar{a}$ to denote $-a$. Two sequences of variables are directed if the sequences have zero periodic autocorrelation function without relying on the commutativity of the variables. For example (a, b) and $(a, \bar{b})$ are two directed sequences while (a, b) and $(b, \bar{a})$ are not. Let A_j $(j = 1, \ldots, m)$ be m sequences of length n. These sequences are of type $(u_1, \ldots, u_t)$ if the sequences are composed of t variables, say x_i $(i = 1, \ldots, t)$ with x_i and $-x_i$ occurring a total of u_i times $(i = 1, \ldots, t)$. A set $\{A_1, \ldots, A_{2k}\}$ of square real matrices of order n is additive if $\sum_{i=1}^{2k} A_i A_i^{\mathrm{T}} = f I_n$ for some real number f and it is amicable if $\sum_{i=1}^{k} \left(A_{2i-1} A_{2i}^{\mathrm{T}} - A_{2i} A_{2i-1}^{\mathrm{T}} \right) = 0$ (Kharaghani, 2000). Two square matrices are disjoint if their elementwise product is the zero matrix; see Geramita & Seberry (1979, p. 2).

Golay sequences, also known as Golay pairs, are two row vectors (A_1, A_2) of length n, with elements from the set $\{1, -1\}$, that have zero non-periodic autocorrelation function. These sequences are known to exist for lengths $g = 2^a 10^b 26^c$, where a, b, c are nonnegative integers. More details can be found in Borwein & Ferguson (2003). Periodic Golay pairs (e.g., Bomer & Antweiler, 1990) are the natural generalization of Golay sequences with zero periodic autocorrelation function. Any positive integer g for which Golay sequences exist is called a Golay number. Similarly, any positive integer p for which a periodic Golay pair exists is called a periodic Golay number. It is obvious that any Golay number is also a periodic Golay number but the converse is not generally true. Both periodic Golay pairs and Golay sequences can be used to generate orthogonal block-circulant matrices even though the latter can provide some useful multiplication methods and give more general results. The existence of periodic Golay pairs is important especially when Golay sequences do not exist, cannot be constructed or are unknown.

Periodic Golay pairs of length n are also hard to construct. Periodic Golay pairs that are not Golay sequences are known only for lengths $n = 34, 50, 58, 74, 82, 122, 136, 164, 202, 226$ and all of them have been constructed by intensive computer search; see a 2013 unpublished technical report by D. Z. Djokovic and I. S. Kotsireas (arXiv: 1310.5773). With the use of Lemma 1 and Theorem 1 we construct new periodic Golay pairs, of an infinite variety of different lengths, without any computer search.

LEMMA 1. *There exist directed sequences with zero periodic autocorrelation function of lengths p and type (p, p), for $p \in P = \{34, 50, 58, 74, 82, 122, 136, 202, 226\}$.*

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Table 1. *Two-variable pairs of directed sequences of length p and type (p, p) with zero periodic autocorrelation function*

p	Directed sequences D_1, D_2.
34	*aaaāāāb̄āb̄bb̄bab̄b̄bbabb̄abb̄bbaaābb̄abba*, *bāābaāb̄bbāāaābaābāāaabāaāab̄ab̄b̄b̄bbb*.
50	*āāāābbāb̄bb̄abbb̄āab̄bb̄aāaāb̄aab̄b̄abāabaab̄aābbab̄baab̄b̄aaa*, *bbbaabbāabāāb̄babbāb̄ābaabbab̄bb̄baāabb̄aāābaāab̄āāb̄b̄b̄b̄*.
58	*āāāāb̄aaāāaāāāāaaābb̄ab̄bbaāab̄bab̄b̄b̄b̄bab̄ab̄aabbāabb̄āabb̄aab̄aaaa*, *bbbbabbaābb̄aābb̄āābbababāaaaābāabb̄āāabaāb̄bb̄b̄b̄bb̄b̄b̄b̄bbab̄b̄b̄b̄*.
74	*aābaāaāb̄āāaaāaaaaābaabbāb̄abaab̄b̄āb̄ab̄b̄babaaāab̄b̄b̄āāaāb̄b̄bāabb̄ab̄bāb̄b̄āabaaab̄b̄b̄āa*, *bb̄aaābbbābb̄aāb̄āabaāb̄b̄āaab̄b̄b̄aaaabb̄bbābāaabab̄aabbābab̄āābbāb̄bbbb̄bbbb̄ab̄bb̄ābb*.
82	*aāb̄abaaaaabb̄āaab̄bābāb̄aāāāāab̄āab̄āb̄aāb̄b̄āāabbbab̄āaāb̄aaāāb̄b̄āāāaāaaab̄b̄bāāb̄b̄babab̄* *b̄āabb̄aā*, *b̄baābb̄aabābaab̄b̄āaāabbbb̄b̄b̄b̄b̄aab̄b̄bbab̄b̄babāāab̄b̄aab̄bab̄abb̄abb̄b̄b̄b̄bab̄āb̄āabbb̄aāb* *bbbbābab̄b*.
122	*ab̄aābaābb̄b̄b̄aaābāb̄aabab̄b̄b̄bāaāb̄aaaab̄aabab̄aaāb̄aabābbbbāb̄ābb̄aāb̄āb̄abābbb̄aaaab̄ābb̄b̄āb̄bbb* *āābb̄abāab̄b̄bbb̄b̄bb̄abbāāābbaaabb̄aābb̄aāab̄āā*, *b̄b̄abb̄baāb̄baābbbāāb̄b̄b̄āābaāaaāāaāabb̄ābaāb̄b̄aāāab̄aāab̄abbbbaāāb̄ābab̄abb̄aāb̄ab̄āāāāb̄ābbabb̄b* *babābbabbbbab̄bb̄āaaabābbab̄āb̄bbaaaāb̄bāb̄bab*.
202	*abb̄b̄āabbbāaāāb̄bābbb̄aāāb̄b̄b̄āb̄aāab̄aaaāāaab̄b̄āb̄aaab̄ābāb̄b̄b̄b̄b̄b̄āb̄bab̄aaābbāb̄bb̄ab̄āb̄b̄āaaaba* *ābaabbaābbbāb̄bb̄ab̄abbaāb̄abbb̄b̄āābaābbabb̄ab̄āaāabab̄aāabāāabbab̄aaab̄ābāaāāābb̄ābāb̄aaab̄bb̄ā* *b̄b̄b̄b̄aaabāb̄b̄b̄āāabāab̄b̄b̄b̄aab̄b̄aāb̄baab̄b̄āb*, *āb̄aabbaāb̄baabbaaāaabb̄āb̄b̄aaab̄ābbbaaāaab̄aāabbbab̄āb̄aāb̄b̄bb̄āb̄abbbabāab̄b̄āb̄b̄babāb̄b̄b̄ab* *aābāāb̄bāb̄b̄aaāābab̄bāābabaāab̄āāāb̄bāābbāb̄bābbb̄aab̄abaāab̄āāb̄bbabāab̄aaāāaab̄āb̄abbbab̄aāa* *bbb̄b̄bbbabb̄bab̄aaab̄b̄baāāb̄āab̄b̄bb̄āāābb̄āaāb*.
226	*ab̄ab̄b̄āb̄abb̄b̄abbbāb̄aaāabbbbbbāb̄bb̄b̄abbbb̄āb̄aāb̄bābabāb̄bbbaāb̄aaābaabaab̄b̄āab̄aāabāb̄aab̄ābāā* *b̄abaaaaabb̄āabb̄aaāabb̄ab̄āb̄aab̄bāb̄b̄b̄ābaāab̄ābāb̄bbābb̄b̄āb̄bbbabb̄āb̄b̄b̄āab̄aaāāāaaab̄b̄bab̄b̄b̄b* *bb̄bāb̄ābaāābb̄b̄ababab̄abab̄b̄āābbābābab̄b̄b̄b̄b̄b̄b̄babābbab̄āabb̄b̄āaab̄b̄b̄b̄bba*, *bāāaāāabbb̄aāabb̄abāāb̄abaaaāaaababāb̄āb̄āāb̄b̄aāababābaaāb̄b̄bāb̄abāb̄āaāaaabāaabbb̄b̄b̄b̄bbabb̄ā* *aāab̄aābāāāāb̄aāaāb̄āāab̄āb̄abb̄bāb̄aaab̄āabbab̄abāabb̄bbaāb̄b̄āaābbbbbābab̄b̄āb̄abbab̄āb̄b̄babb̄āabb* *ābbāb̄bbab̄b̄āāaāb̄ābāb̄āab̄bab̄aāāābaaāab̄āāāāāāb̄b̄bbab̄āāābaaābab̄aabab*.

Proof. To obtain the required directed sequences we apply the construction techniques from Koukouvinos & Seberry (1999) on periodic Golay pairs. The derived directed sequences are provided in Table 1. □

THEOREM 1. *If g is a Golay number and p a periodic Golay number then t = gp is a periodic Golay number.*

Proof. Construct the Golay sequences A and B of length g using the known multiplication techniques for Golay sequences. These two sequences have zero non-periodic autocorrelation function. Use the directed sequences D_1 and D_2 of length p given in Table 1, and replace their variable a with the Golay sequence A and their variable b with the Golay sequence B. This will give the desired periodic Golay pairs of length gp. So, gp is a periodic Golay number. □

Example 1. Suppose we wish to construct a periodic Golay pair (T_1, T_2) of length 68. We have $68 = 2 \times 34$, so we have to use the directed sequences D_1, D_2 of length 34 and type (34, 34) given in Table 1 with the Golay sequences $A = (+, +)$ and $B = (+, -)$, where $+$ stands for $+1$ and $-$ stands for -1. The resulting periodic Golay pair is

$$T_1 = (+ + + + + + - - - - - - - + - - - + + - - + + - + + - + - + + - + - \\ + + + - - + + + + - - + + - + - + + + + - - + - - + + + + - + - + +),$$

$$T_2 = (+ - - - - - + - + + - - - + + - + - - - - - + + - - + - + + - - + - \\ - - - - + + + + + - - - + + - - + + - + + + - + - + - + + - + - + -).$$

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Corollary 1. *Periodic Golay pairs can be constructed for all lengths* $t = 2^{a_1} 10^{a_2} 26^{a_3} p^{a_4}$*, where* a_1, a_2, a_3 *are nonnegative integers,* $p \in \{34, 50, 58, 74, 82, 122, 202, 226\}$ *and* $a_4 = 0, 1$.

3. Construction of orthogonal designs

Let A_i $(i = 1, \ldots, 8)$ be circulant matrices of order n and R_n the back-diagonal identity matrix of order n. The following constructions are known:

$$C_2 = \begin{pmatrix} A_1 & A_2 R_n \\ -A_2 R_n & A_1 \end{pmatrix}, \quad C_4 = \begin{pmatrix} A_1 & A_2 R_n & A_3 R_n & A_4 R_n \\ -A_2 R_n & A_1 & A_4^{\mathrm{T}} R_n & -A_3^{\mathrm{T}} R_n \\ -A_3 R_n & -A_4^{\mathrm{T}} R_n & A_1 & A_2^{\mathrm{T}} R_n \\ -A_4 R_n & A_3^{\mathrm{T}} R_n & -A_2^{\mathrm{T}} R_n & A_1 \end{pmatrix},$$

$$C_8 = \begin{pmatrix} A_1 & A_2 & A_4 R_n & A_3 R_n & A_6 R_n & A_5 R_n & A_8 R_n & A_7 R_n \\ -A_2 & A_1 & A_3 R_n & -A_4 R_n & A_5 R_n & -A_6 R_n & A_7 R_n & -A_8 R_n \\ -A_4 R_n & -A_3 R_n & A_1 & A_2 & -A_8^{\mathrm{T}} R_n & A_7^{\mathrm{T}} R_n & A_6^{\mathrm{T}} R_n & -A_5^{\mathrm{T}} R_n \\ -A_3 R_n & A_4 R_n & -A_2 & A_1 & A_7^{\mathrm{T}} R_n & A_8^{\mathrm{T}} R_n & -A_5^{\mathrm{T}} R_n & -A_6^{\mathrm{T}} R_n \\ -A_6 R_n & -A_5 R_n & A_8^{\mathrm{T}} R_n & -A_7^{\mathrm{T}} R_n & A_1 & A_2 & -A_4^{\mathrm{T}} R_n & A_3^{\mathrm{T}} R_n \\ -A_5 R_n & A_6 R_n & -A_7^{\mathrm{T}} R_n & -A_8^{\mathrm{T}} R_n & -A_2 & A_1 & A_3^{\mathrm{T}} R_n & A_4^{\mathrm{T}} R_n \\ -A_8 R_n & -A_7 R_n & -A_6^{\mathrm{T}} R_n & A_5^{\mathrm{T}} R_n & A_4^{\mathrm{T}} R_n & -A_3^{\mathrm{T}} R_n & A_1 & A_2 \\ -A_7 R_n & A_8 R_n & A_5^{\mathrm{T}} R_n & A_6^{\mathrm{T}} R_n & -A_3^{\mathrm{T}} R_n & -A_4^{\mathrm{T}} R_n & -A_2 & A_1 \end{pmatrix}.$$

Arrays C_2 and C_4 are orthogonal when the circulant matrices used are additive (Geramita & Seberry, 1979, pp. 97–9). Array C_8 is orthogonal when the circulant matrices used are simultaneously additive and amicable (Kharaghani, 2000).

From periodic Golay pairs T_1, T_2 of length t and their corresponding circulant matrices $Z_1 = \mathrm{circ}(T_1)$, $Z_2 = \mathrm{circ}(T_2)$ we can define

$$X = (Z_1 + Z_2)/2, \quad W = (Z_1 - Z_2)/2. \tag{1}$$

Lemma 2. *The matrices* X *and* W *in* (1) *are circulant, disjoint, additive, amicable matrices of order* t*, with entries from* $\{0, 1, -1\}$ *and* $X \pm W$ *is a* ± 1 *matrix.*

Proof. Follows by simple calculations. □

Let (T_1, T_2) be the periodic Golay pair of length $t = 2^{a_1} 10^{a_2} 26^{a_3} p^{a_4}$, where a_1, a_2, a_3 are nonnegative integers, $p \in \{34, 50, 58, 74, 82, 122, 202, 226\}$ and $a_4 = 0$ or 1. Using matrices X and W from (1), consider the constructions

$$A_i = (2i-1)X + (2i+1)W, \qquad A_{i+1} = (2i+1)X - (2i-1)W, \qquad (i = 1, 3, 5, 7), \tag{2}$$

$$A_{2i-1} = (2i-1)X + 2iW, \qquad A_{2i} = 2iX - (2i-1)W, \qquad (i = 1, 2, 3, 4), \tag{3}$$

$$A_i = (2i+1)X + (2i+3)W, \qquad A_{i+1} = (2i+3)X - (2i+1)W, \qquad (i = 1, 3, 5, 7), \tag{4}$$

$$A_{2i-1} = 2iX + (2i+1)W, \qquad A_{2i} = (2i+1)X - 2iW, \qquad (i = 1, 2, 3, 4), \tag{5}$$

$$A_i = (2i+3)X + (2i+5)W, \qquad A_{i+1} = (2i+5)X - (2i+3)W, \qquad (i = 1, 3, 5, 7), \tag{6}$$

$$A_{2i-1} = (2i+1)X + 2(i+1)W, \qquad A_{2i} = 2(i+1)X - (2i+1)W, \qquad (i = 1, 2, 3, 4). \tag{7}$$

Theorem 2. *Let* $t = 2^{a_1} 10^{a_2} 26^{a_3} p^{a_4}$*, where* a_1, a_2, a_3 *are nonnegative integers,* $p \in \{34, 50, 58, 74, 82, 122, 202, 226\}$ *and* $a_4 = 0$ *or* 1*. The following orthogonal designs for computer experiments exist:* (i) $\mathrm{OD}(2^{b+1}t; q_1^{\ell})$ *and* (ii) $\mathrm{OD}(2^{b+1}t + s; q_2^{\ell})$*,* $q_1 = 2^{b+1}$*,* $q_2 = 2^{b+1} + 1$*,* $\ell \leqslant 2^b t$*, for* $b = 1, 2, 3$ *and for any positive integer* s*.*

Proof. To prove the existence of orthogonal designs (i) and (ii) of the theorem we use the constructions given in (2) and (3) respectively. From Lemma 2 we have that the matrices $\{A_i\}_{i=1}^{8}$ are circulant,

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Table 2. *Some orthogonal designs* $\mathrm{OD}(n; q^m)$ *constructed from two circulant matrices. All designs for* $q = 4, 8, 16$ *or* $q = n$ *are additionally of U-type. Designs marked with an asterisk are constructed using periodic Golay pairs that are developed as in* § 2

$q=4$		$q=5$		$q=8$		$q=9$		$q=16$		$q=17$	
n	m	n	m	n	m	n	m	n	m	n	m
4	2	5	2	8	4	9	4	16	8	17	8
8	4	9	4	16	8	17	8	32	16	33	16
16	8	17	8	32	16	33	16	64	32	65	32
32	16	33	16	64	32	65	32	128	64	129	64
40	20	41	20	80	40	81	40	160	80	161	80
64	32	65	32	128	64	129	64	256	128	257	128
80	40	81	40	160	80	161	80	320	160	321	160
104	52	105	52	208	104	209	104	416	208	417	208
128	64	129	64	256	128	257	128	512	256	513	256
136	68*	137	68*	272	136*	273	136*	544	272*	545	272*
160	80	161	80	320	160	321	160	640	320	641	320
200	100*	201	100*	400	200*	401	200*	800	400*	801	400*
208	104	209	104	416	208	417	208	832	416	833	416
232	116*	233	116*	464	232*	465	232*	928	464*	929	464*
256	128	257	128	512	256	513	256	1024	512	1025	512
272	136*	273	136*	544	272*	545	272*	1088	544*	1089	544*
296	148*	297	148*	592	296*	593	296*	1184	592*	1185	592*
320	160	321	160	640	320	641	320	1280	640	1281	640
328	164*	329	164*	656	328*	657	328*	1312	656*	1313	656*
400	200	401	200	800	400	801	400	1600	800	1601	800
416	208	417	208	832	416	833	416	1664	832	1665	832
464	232*	465	232*	928	464*	929	464*	1856	928*	1857	928*
488	244*	489	244*	976	488*	977	488*	1952	976*	1953	976*
512	256	513	256	1024	512	1025	512	2048	1024	2049	1024
544	272*	545	272*	1088	544*	1089	544*	2176	1088*	2177	1088*

additive and amicable. For $b = 1, 2, 3$ and a fixed $1 \leqslant \ell \leqslant 2^b t$, the desired orthogonal design is obtained by randomly selecting any ℓ columns from the matrix $(C_{2^b}^{\mathrm{T}}, -C_{2^b}^{\mathrm{T}})^{\mathrm{T}}$ or $(C_{2^b}^{\mathrm{T}}, 0_{2^b t\times s}, -C_{2^b}^{\mathrm{T}})^{\mathrm{T}}$ for (i) or (ii) respectively, where C_{2^b} is the square matrix of order $2^b t$ obtained using $\{A_1, \ldots, A_{2^b}\}$. □

To illustrate the construction described in Theorem 2, we provide a detailed example. Even though the desired design in this example is quite large, it is quite easy to construct and to present due to its block-circulant structure.

Example 2. Suppose we wish to construct an $\mathrm{OD}(272; 4^\ell)$, $\ell = 1, \ldots, 136$. From Theorem 1, we construct the periodic Golay pair (T_1, T_2) of length $t = 68$ as is illustrated in Example 1. Define the disjoint circulant matrices X and W as in Lemma 2. So,

$$X = \mathrm{circ}(+000000-00---+0-000--0+000000+0-+-000-0++++-\\ -0+00-++0+0000-+0++-+-+0),$$

$$W = \mathrm{circ}(0+++++-0--0000-0-++00+0-++-+-0+000+++0-\\ 000000+0-+000+0--+-00+000000+).$$

We have $b = 1$, so by construction (2) we have

$$A_1 = \mathrm{circ}(133333\bar{3}\bar{1}\bar{3}\bar{3}\bar{1}\bar{1}\bar{1}1\bar{3}\bar{1}\bar{3}33\bar{1}\bar{1}31\bar{3}33\bar{3}3\bar{3}13\bar{1}1\bar{1}333\bar{1}\bar{3}1111\bar{1}\bar{1}31\bar{3}3\bar{1}1131\bar{3}\bar{3}3\bar{3}\bar{1}1311\bar{1}1\bar{1}13),$$

$$A_2 = \mathrm{circ}(3\bar{1}\bar{1}\bar{1}\bar{1}\bar{1}1\bar{3}11\bar{3}\bar{3}\bar{3}31\bar{3}1\bar{1}\bar{1}\bar{3}\bar{3}\bar{1}31\bar{1}\bar{1}1\bar{1}13\bar{1}\bar{3}3\bar{3}\bar{1}\bar{1}\bar{1}\bar{3}13333\bar{3}\bar{3}\bar{1}31\bar{1}\bar{3}33\bar{1}311\bar{1}1\bar{3}3\bar{1}33\bar{3}3\bar{3}3\bar{1}).$$

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It is easy to verify that matrices A_1 and A_2 are circulant, additive and amicable. The desired $\text{OD}(272, 4^\ell)$ is obtained by randomly selecting any ℓ columns from the matrix $(C_{136}^{\mathrm{T}}, -C_{136}^{\mathrm{T}})^{\mathrm{T}}$, where C_{136} is the square matrix of order 136, obtained using the matrices A_1 and A_2 in structure C_2.

The orthogonal design presented in Example 2 is constructed by two circulant matrices. The parameters of some further orthogonal designs are given in Table 2.

The following corollaries are straightforward applications of Theorem 2 but they provide new infinite classes of orthogonal designs.

Corollary 2. *If t is a Golay or a periodic Golay number then the following orthogonal designs for computer experiments exist:* $\text{OD}(4t; 4^{2t})$, $\text{OD}(4t+1; 5^{2t})$, $\text{OD}(8t; 8^{4t})$, $\text{OD}(8t+1; 9^{4t})$, $\text{OD}(16t; 16^{8t})$ *and* $\text{OD}(16t+1; 17^{8t})$.

Corollary 3. *Let $t = 2^{a_1}10^{a_2}26^{a_3}p^{a_4}$, where a_1, a_2, a_3 are nonnegative integers, $p \in \{34, 50, 58, 74, 82, 122, 202, 226\}$ and $a_4 = 0$ or 1. The following orthogonal designs for computer experiments exist:* (i) $\text{OD}(2^{b+1}t; q_1^\ell)$ *and* (ii) $\text{OD}(2^{b+1}t + s; q_2^\ell)$, *$\ell \leqslant 2^b t$, for $b = 1, 2, 3$, for any positive integer s, $q_1 = 2, \ldots, 2^{b+1}$, and $q_2 = 3, \ldots, 2^{b+1} + 1$.*

Proof. The results follow from Theorem 2 by equating and/or killing levels, as needed, before foldover. Here equating means setting the largest level equal to the previous level, with repetition if necessary, and killing means setting the largest level equal to zero, with repetition if necessary. □

The proposed method can also be expanded and used for the construction of nearly orthogonal designs, as in Bingham et al. (2009). In such cases, the correlation of the derived designs is pre-calculated by the periodic function of the used row vectors and the design structure.

Theorem 3. *Let $t = 2^{a_1}10^{a_2}26^{a_3}p^{a_4}$, where a_1, a_2, a_3 are nonnegative integers, $p \in \{34, 50, 58, 74, 82, 122, 202, 226\}$ and $a_4 = 0$ or 1. The following nearly orthogonal designs for computer experiments exist:* (i) $\text{NOD}(2(2^b t + 1); [\{2(2^{b+1} + 1)\}^\ell])$, (ii) $\text{NOD}(2^{b+1}t + s + 2; (2^{b+1} + 3)^\ell)$, (iii) $\text{NOD}(2^2(2^{b-1}t + 1); [\{2^2(2^{b-1} + 1)\}^\ell])$ *and* (iv) $\text{NOD}(2^{b+1}t + s + 4; (2^{b+1} + 5)^\ell), \ell \leqslant 2^b t$, *for $b = 1, 2, 3$ and for any positive integer s.*

Proof. To prove the existence of the nearly orthogonal designs described in (i), (ii), (iii) and (iv), we use the construction given in (4), (5), (6), and (7) respectively. From Lemma 2 we have that matrices $\{A_i\}_{i=1}^8$ are circulant, additive and amicable. For $b = 1, 2, 3$ and a fixed $1 \leqslant \ell \leqslant 2^b t$, the desired nearly orthogonal design (i), (ii), (iii), or (iv) is obtained by randomly selecting any ℓ columns from the matrix $(C_{2^b}^{\mathrm{T}}, 1^{\mathrm{T}}, -1^{\mathrm{T}}, -C_{2^b}^{\mathrm{T}})^{\mathrm{T}}$, $(C_{2^b}^{\mathrm{T}}, 1^{\mathrm{T}}, 0_{2^b t\times s}^{\mathrm{T}}, -1^{\mathrm{T}}, -C_{2^b}^{\mathrm{T}})^{\mathrm{T}}$, $(C_{2^b}^{\mathrm{T}}, 3^{\mathrm{T}}, 1^{\mathrm{T}}, -1^{\mathrm{T}}, -3^{\mathrm{T}}, -C_{2^b}^{\mathrm{T}})^{\mathrm{T}}$, or $(C_{2^b}^{\mathrm{T}}, 2^{\mathrm{T}}, 1^{\mathrm{T}}, 0_{2^b t\times s}^{\mathrm{T}}, -1^{\mathrm{T}}, -2^{\mathrm{T}}, -C_{2^b}^{\mathrm{T}})^{\mathrm{T}}$, respectively, where C_{2^b} is the square matrix of order $2^b t$ obtained using $\{A_1, \ldots, A_{2^b}\}$. □

A design is called ℓ-orthogonal if the sum of the elementwise products of any ℓ columns is zero. Obviously, an ℓ-orthogonal design is mean orthogonal when $\ell = 1$ and its main effects are pairwise orthogonal when $\ell = 2$. From the foldover structure of the designs we have the following lemma.

Lemma 3. *All the designs constructed by Theorems 2 and 3 are $2\ell + 1$ orthogonal, ℓ any nonnegative integer.*

Acknowledgement

The authors thank the editor and the referees for their valuable comments and suggestions.

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[*Received December* 2013. *Revised April* 2014]

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