

1. MGB VI.18[c,d] On the F distribution:

(a) If X has an F distribution with m and n degrees of freedom, show that

$$W = \frac{mX/n}{1 + mX/n}$$

has a beta distribution.

Solution:

Note that W is a transformation of X by $g : (0, \infty) \rightarrow (0, 1)$ where

$$g(x) = \frac{mx/n}{1 + mx/n}.$$

Moreover, the inverse of g is given by

$$g^{-1}(w) = \frac{n}{m} \cdot \frac{w}{1-w}.$$

so the transformation is one-to-one, and the transformation theorem applies. The Jacobian is

$$\frac{n}{m} \cdot (1-w)^{-2},$$

so W has density

$$\begin{aligned} f_W(w) &= \frac{n}{m} (1-w)^{-2} f_{F(m,n)}(g^{-1}(w)) \\ &= \frac{n}{m} (1-w)^{-2} \frac{\left(\frac{m}{n}\right)^{m/2}}{B(m/2, n/2)} \left(\frac{m}{n} \frac{w}{1-w}\right)^{m/2-1} \left(1 + \frac{m}{n} \frac{w}{1-w}\right)^{-(m+n)/2} I_{(0,1)}(w) \\ &= \frac{1}{B(m/2, n/2)} \frac{\left(\frac{m}{n}\right)^{-1+m/2-m/2+1}}{1} w^{m/2-1} (1-w)^{-2+(-m/2+1)+(m+n)/2} \end{aligned}$$

since $1 + w/(1-w) = 1/(1-w)$,

$$= \frac{1}{B(m/2, n/2)} w^{m/2-1} (1-w)^{n/2-1}.$$

This is the p.d.f. of a beta distribution with parameters $a = m/2$ and $b = n/2$.

□

(b) Use the result of part (a) and the beta function to find the mean and variance of the F distribution. [Find the first two moments of $mX/n = W/(1-W)$].

Solution:

The r th moment of X is given by

$$\begin{aligned} E[X^r] &= \left(\frac{n}{m}\right)^r E[(mX/n)^r] = \left(\frac{n}{m}\right)^r E\left[\frac{W^r}{(1-W)^r}\right] \\ &= \left(\frac{n}{m}\right)^r \frac{1}{B(m/2, n/2)} \int_0^1 w^{m/2-1+r} (1-w)^{n/2-1-r} \\ &= \left(\frac{n}{m}\right)^r \frac{B(m/2+r, n/2-r)}{B(m/2, n/2)} \\ &= \left(\frac{n}{m}\right)^r \frac{\Gamma(m/2+r)\Gamma(n/2-r)}{\Gamma(m/2)\Gamma(n/2)}. \end{aligned}$$

So

$$\mu = \frac{n}{m} \cdot \frac{m/2}{n/2 - 1} = \frac{n}{n - 2},$$

and

$$\begin{aligned} \sigma^2 &= \left(\frac{n}{m}\right)^2 \frac{(m/2 + 1) \cdot m/2}{(n/2 - 1)(n/2 - 2)} - \left(\frac{n}{n - 2}\right)^2 \\ &= \frac{n^2(m + 2)}{m(n - 2)(n - 4)} - \frac{n^2}{(n - 2)^2} \\ &= \frac{n^2(m + 2)(n - 2) - n^2m(n - 4)}{m(n - 2)^2(n - 4)} \\ &= \frac{2n^2 \cdot (m + n - 4)}{m(n - 2)^2(n - 4)}. \end{aligned}$$

□

2. MGB VI.19[c,d] On the t distribution:

(a) If X is t -distributed, show that X^2 is F -distributed.

Solution:

Denote $Y = X^2$ and let $g : A_1 \cup A_2 \rightarrow (0, \infty)$ by $g(x) = x^2$ where $A_1 = (-\infty, 0)$, and $A_2 = (0, \infty)$. Note that g restricted to each A_i is one-to-one. Therefore, we can apply the transformation theorem:

$$\begin{aligned} f_Y(y) &= \left[\frac{1}{2} \frac{1}{\sqrt{y}} f_X(-\sqrt{y}) + \frac{1}{2} \frac{1}{\sqrt{y}} f_X(\sqrt{y}) \right] \\ &= \frac{1}{\sqrt{y}} f_X(\sqrt{y}) && \text{since } f_X \text{ is even} \\ &= \frac{1}{\sqrt{y}} \frac{\Gamma[(k+1)/2]}{\Gamma(k/2)} \frac{1}{\sqrt{k\pi}} \frac{1}{(1 + y/k)^{(k+1)/2}} \\ &= \frac{\Gamma[(1+k)/2]}{\Gamma(1/2)\Gamma(k/2)} \left(\frac{1}{k}\right)^{1/2} \frac{y^{(1-2)/2}}{(1 + \frac{1}{k}y)^{(k+1)/2}}. \end{aligned}$$

This is the density for a random variable distributed $F(1, k)$.

□

(b) If X is t -distributed with k degrees of freedom, show that $1/(1 + X^2/k)$ has a beta distribution.

Solution:

Using part (a), this is a corollary of MGB VI.18[d].

□

3. MGB VI.22 Let X_1, \dots, X_n be a random sample from $N(\mu, \sigma^2)$. Define

$$\begin{aligned}\bar{X}_k &= \frac{1}{k} \sum_{i=1}^k X_i, & \mathcal{S}_k^2 &= \frac{1}{k-1} \sum_{i=1}^k (X_i - \bar{X}_k)^2, \\ \bar{X}_{n-k} &= \frac{1}{n-k} \sum_{i=k+1}^n X_i, & \mathcal{S}_{n-k}^2 &= \frac{1}{n-k-1} \sum_{i=k+1}^n (X_i - \bar{X}_{n-k})^2, \\ \bar{X} &= \frac{1}{n} \sum_{i=1}^n X_i, \quad \text{and} & \mathcal{S}^2 &= \frac{1}{n-1} \sum_{i=1}^n (X_i - \bar{X})^2.\end{aligned}$$

Use known results about sampling from the normal distribution to answer the following:

(a) What is the distribution of $\sigma^{-2}[(k-1)\mathcal{S}_k^2 + (n-k-1)\mathcal{S}_{n-k}^2]$?

Distributing σ^{-2} , we have that this is the sum of independent Chi-squared distributions with degrees of freedom $k-1$ and $n-k-1$ respectively. By independence (moment generating function argument), the sum is distributed χ_{n-2}^2 .

(b) What is the distribution of $(\frac{1}{2})(\bar{X}_k + \bar{X}_{n-k})$?

We have that $\bar{X}_k \sim N(\mu, \sigma^2/k)$ and $\bar{X}_{n-k} \sim N(\mu, \frac{\sigma^2}{n-k})$. Hence, by independence, their sum is distributed $N\left(2\mu, \frac{\sigma^2 n}{k(n-k)}\right)$ since $\frac{1}{k} + \frac{1}{n-k} = \frac{n}{k(n-k)}$. Finally, multiplying by $1/2$ yields a distribution of $N\left(\mu, \frac{\sigma^2}{4} \left(\frac{\sigma^2 n}{k(n-k)}\right)\right)$.

(c) What is the distribution of $\sigma^{-2}(X_i - \mu)$?

Recall that $Z = (X_i - \mu)/\sigma \sim N(0, 1)$, hence $\sigma^{-2}(X_i - \mu) = \frac{1}{\sigma}Z \sim N(0, \frac{1}{\sigma^2})$.

(d) What is the distribution of $\mathcal{S}_k^2/\mathcal{S}_{n-k}^2$?

Note that $(k-1)\mathcal{S}_k^2/\sigma^2 \sim \chi_{k-1}^2$ and $(n-k-1)\mathcal{S}_{n-k}^2/\sigma^2 \sim \chi_{n-k-1}^2$. So

$$\frac{k-1}{n-k-1} \cdot \frac{\mathcal{S}_k^2}{\mathcal{S}_{n-k}^2} \sim F(k-1, n-k-1).$$

The distribution of $\mathcal{S}_k^2/\mathcal{S}_{n-k}^2$ can be obtained by transforming the above F distributed variable by multiplying by $(n-k-1)/(k-1)$. This gives a density

$$(k-1)/(n-k-1)f_Y\left(\frac{k-1}{n-k-1}\right),$$

where $Y \sim F(k-1, n-k-1)$.

(e) What is the distribution of $(\bar{X} - \mu)/(\mathcal{S}/\sqrt{n})$?

If $Z = \frac{\bar{X} - \mu}{\sigma/\sqrt{n}}$, and $Y = \frac{n-1}{\sigma^2}\mathcal{S}^2$, then $Z \sim N(0, 1)$, $Y \sim \chi_{n-1}^2$ and

$$(\bar{X} - \mu)/(\mathcal{S}/\sqrt{n}) = (\sigma/\sqrt{n})Z \bigg/ \sqrt{\frac{Y\sigma^2}{n(n-1)}} = \sqrt{n-1}W,$$

where $W \sim t(n-1)$. Hence, the desired density is $(n-1)^{-1/2}f_t(w(n-1)^{-1/2})$, where f_t is the density of Student's t distribution.

4. MGB VI.23 Let Z_1, Z_2 be a random sample of size 2 from $N(0, 1)$ and X_1, X_2 be a random sample of size 2 from $N(1, 1)$. Suppose the Z_i 's are independent of the X_j 's. Use known results about sampling from the normal distribution to answer the following:

(a) What is the distribution of $\bar{X} - \bar{Z}$?

Since $\bar{X} \sim N(1, \frac{1}{2})$ and $\bar{Z} \sim N(0, \frac{1}{2})$, the linear combination $\bar{X} - \bar{Z}$ is distributed $N(1, 1)$.

(b) What is the distribution of $(Z_1 + Z_2)/\sqrt{[(X_2 - X_1)^2 + (Z_2 - Z_1)^2]/2}$?

The linear combination $Y = Z_1 + Z_2$ is distributed $N(0, 2)$. Hence, $Y/\sqrt{2} \sim N(0, 1)$.

Notice that the sample variance of X_1, X_2 is $S_X^2 = (X_1 - \bar{X})^2 + (X_2 - \bar{X})^2 = [(X_1 - X_2)/2]^2 + [(X_1 + X_2 - 2)/2]^2 = (X_2 - X_1)^2/2$. Similarly the sample variance for Z_1, Z_2 is $S_Z^2 = (Z_2 - Z_1)^2/2$. Since $\sigma^2 = 1$ and there's 1 degree of freedom S_Z^2 and S_X^2 are identically distributed χ_1^2 . They are independent as functions of independent samples. Thus, their sum, $U = S_X^2 + S_Z^2$ is distributed χ_2^2 .

Now, the random variable in question is

$$\frac{Y}{\sqrt{U}} = \frac{Y/\sqrt{2}}{\sqrt{U/2}},$$

which is a Student's t distribution with 2 degrees of freedom.

(c) What is the distribution of $[(X_1 - X_2)^2 + (Z_1 - Z_2)^2 + (Z_1 + Z_2)^2]/2$?

As in (b), we have that the random variable in question is

$$S_X^2 + S_Z^2 + (Y/\sqrt{2})^2.$$

Note that $(Y/\sqrt{2})^2 \sim \chi_1^2$. We have that S_X^2 is independent of S_Z^2 and $(Y/\sqrt{2})^2$ since samples are taken independently. Moreover, S_Z^2 and $(Y/\sqrt{2})^2$ are independent since S_Z^2 is independent of $\bar{Z} = 2Y$. Thus, the distribution is given by the sum of independent chi-squares, which is known to be distributed chi-squared with degrees of freedom equal to the sum of the degrees of freedom of each variable. I.e χ_3^2 .

(d) What is the distribution of $(X_2 + X_1 - 2)^2/(X_2 - X_1)^2$?

The linear combination $W = X_2 + X_1 - 2 \sim N(0, 2)$. So $(W/\sqrt{2})^2 \sim \chi_1^2$. Thus, the random variable in question is

$$\frac{(W/\sqrt{2})^2}{S_X^2} \sim F(1, 1).$$

5. MGB VI.27 If X_1, X_2, \dots, X_n are independently and normally distributed with the same mean but different variances $\sigma_1^2, \sigma_2^2, \dots, \sigma_n^2$ and assuming that

$$U = \frac{\sum_{i=1}^n X_i / \sigma_i^2}{\sum_{j=1}^n 1 / \sigma_j^2} \quad \text{and} \quad V = \sum_{i=1}^n \frac{(X_i - U)^2}{\sigma_i^2}$$

are independently distributed, show that U is normal and V has the chi-square distribution with $n - 1$ degrees of freedom.

Solution:

Note that U is a linear combination of normal random variables, hence its distribution is given by

$$\begin{aligned} U &\sim N \left(\mu \frac{\sum_{i=1}^n 1/\sigma_i^2}{\sum_{j=1}^n 1/\sigma_j^2}, \left[\sum_{i=1}^n \frac{\sigma_i^2}{\sigma_i^4} / \left(\sum_{j=1}^n 1/\sigma_j^2 \right)^2 \right] \right) \\ &= N \left(\mu, \left(\sum_{j=1}^n 1/\sigma_j^2 \right)^{-1} \right). \end{aligned} \quad (1)$$

To evaluate the distribution of V , note

$$\begin{aligned} \sum_{i=1}^n \frac{(X_i - \mu)^2}{\sigma_i^2} &= \sum_{i=1}^n \frac{(X_i - U + U - \mu)^2}{\sigma_i^2} \\ &= \sum_{i=1}^n \frac{(X_i - U)^2}{\sigma_i^2} + 2(U - \mu) \sum_{i=1}^n \frac{(X_i - U)}{\sigma_i^2} + (U - \mu)^2 \sum_{i=1}^n \frac{1}{\sigma_i^2}. \\ &= \sum_{i=1}^n \frac{(X_i - U)^2}{\sigma_i^2} + 2(U - \mu) \left[\sum_{i=1}^n \frac{X_i}{\sigma_i^2} - U \sum_{i=1}^n \frac{1}{\sigma_i^2} \right] + (U - \mu)^2 \sum_{i=1}^n \frac{1}{\sigma_i^2}. \end{aligned}$$

since $U \sum 1/\sigma_i^2 = \sum X_i/\sigma_i^2$

$$\stackrel{\dagger}{=} V + (U - \mu)^2 \sum_{i=1}^n \frac{1}{\sigma_i^2}.$$

We know that $(X_i - \mu)/\sigma_i$ is a standard normal random variable so $\sum_{i=1}^n (X_i - \mu)^2/\sigma_i^2 \sim \chi_n^2$. Moreover, $(U - \mu)/\sqrt{\sum 1/\sigma_i}$ is also a standard normal random variable by (1). Hence, its square (which is the second term on the right hand side of \dagger) is distributed χ_1^2 . The equality of the random variables above implies that their moment generating functions are equal. Moreover, since V is assumed independent of U , it is independent of $(U - \mu)^2 \sum 1/\sigma_i$. Thus the moment generating function of the right hand side of \dagger factors. I.e.

$$\left(\frac{1}{1-2t} \right)^{n/2} = M_V(t) \cdot \left(\frac{1}{1-2t} \right)^{1/2} \iff M_V(t) = \left(\frac{1}{1-2t} \right)^{(n-1)/2}.$$

This is the moment generating function for a χ_{n-1}^2 distributed random variable.

□

6. MGB VI.29 Let a sample of size n_1 from a normal population (with variance σ_1^2) have sample variance S_1^2 , and let a second sample of size n_2 from a second normal population (with mean μ_2 and variance σ_2^2) have sample mean \bar{X} and sample variance S_2^2 . Find the joint density of

$$U = \frac{\sqrt{n_2}(\bar{X} - \mu_2)}{S_2} \quad \text{and} \quad V = \frac{S_1^2}{S_2^2}$$

(Assume that samples are independent.)

Solution:

Let us first establish a lemma.

Lemma 1. If $Y \sim \chi_k^2$, then $aY \sim \text{Gamma}(\frac{1}{2}, \frac{k}{2a})$.

Proof. Consider the moment generating function of aY

$$\begin{aligned} M_{aY}(t) &= M_Y(at) \\ &= \left(\frac{\frac{k}{2}}{\frac{k}{2} - at} \right)^{\frac{1}{2}} \\ &= \left(\frac{\frac{k}{2a}}{\frac{k}{2a} - t} \right)^{\frac{1}{2}} \\ &= M_W(t) \quad \text{where } W \sim \text{Gamma}\left(\frac{1}{2}, \frac{k}{2a}\right). \end{aligned}$$

□

Suppose $S_2^2 = s$ is given. Then

$$U|_{S_2^2=s} \sim N(0, \sigma_2^2/s^2), \quad \frac{(n_1 - 1)}{\sigma_1^2} s^2 V|_{S_2^2=s} \sim \chi_{n_1-1}^2 \quad \text{and} \quad \frac{n_2 - 1}{\sigma_2^2} S_2^2 \sim \chi_{n_2-2}^2. \quad (2)$$

Moreover, these random variables are conditionally independent since the samples are taken independently (i.e. functions of the samples are independent). So, the joint conditional density factors

$$f_{U,V|S_2^2=s}(u, v|s) = f_{U|S_2^2=s}(u|s) f_{V|S_2^2=s}(v|s).$$

So, we can recover the joint distribution f_{U,V,S_2^2} and integrate out s to obtain the desired joint distribution. I.e.

$$\begin{aligned} f_{U,V}(u, v) &= \int_0^\infty f_{U,V,S_2^2}(u, v, s) ds \\ &= \int_0^\infty f_{U,V|S_2^2=s}(u, v|s) f_{S_2^2}(s) ds \\ &= \int_0^\infty f_{U|S_2^2=s}(u|s) f_{V|S_2^2=s}(v|s) f_{S_2^2}(s) ds. \end{aligned}$$

Using lemma 1 and (2), we have an integral expression for the joint density.

□

7. Supplement 1. Let Z_1, Z_2, \dots be a sequence of random variables; and suppose that, for $n = 1, 2, \dots$, the distribution of Z_n is as follows:

$$P(Z_n = n^2) = \frac{1}{n} \quad \text{and} \quad P(Z_n = 0) = 1 - \frac{1}{n}.$$

Show that

$$\lim_{n \rightarrow \infty} E(Z_n) = \infty \quad \text{but} \quad Z_n \xrightarrow{p} 0.$$

Solution:

Note that for each n , the events $Z_n = n^2$ and $Z_n = 0$ are disjoint and that the probability of their sum is 1, hence the support of each Z_n is $\{0, n^2\}$ and has the p.m.f.

$$f_{Z_n}(m) = \begin{cases} \frac{1}{n} & \text{if } m = n^2 \\ 1 - \frac{1}{n} & \text{if } m = 0 \\ 0 & \text{otherwise.} \end{cases}$$

Thus, we can calculate the expected value of each Z_n as

$$\begin{aligned} E(Z_n) &= \sum_{0, n^2} x f_{Z_n}(x) \\ &= 0 \left(1 - \frac{1}{n}\right) + n^2 \left(\frac{1}{n}\right) \\ &= n. \end{aligned}$$

Hence $E(Z_n) \rightarrow \infty$ exactly as $n \rightarrow \infty$.

However, suppose that $\varepsilon > 0$ is given. Then

$$\begin{aligned} \lim_{n \rightarrow \infty} P(|Z_n - 0| < \varepsilon) &= \lim_{n \rightarrow \infty} P(|Z_n| < \varepsilon) \\ &= \lim_{n \rightarrow \infty} P(Z_n = 0) \quad \text{for if } n \geq \sqrt{\varepsilon} \text{ then } 0 < \varepsilon \leq n^2, \\ &= \lim_{n \rightarrow \infty} \left(1 - \frac{1}{n}\right) \\ &= 1. \end{aligned}$$

Hence $Z_n \xrightarrow{p} 0$.

□

8. Supplement 2. A sequence of random variables Y_1, Y_2, \dots is said to converge in the r th mean if $\lim_{n \rightarrow \infty} E(|Y_n - b|^r) = 0$. Prove that if a sequence of random variables converge to b in the quadratic mean, then the sequence also converges to b in probability.

Solution:

Let $\varepsilon > 0$ be given. First note that because the function $f(x) = x^2$ is increasing for $x > 0$,

$$\begin{aligned} P[|Y_n - b| < \varepsilon] &= P[(Y_n - b)^2 < \varepsilon^2] \\ &\stackrel{\dagger}{=} 1 - P[(Y_n - b)^2 \geq \varepsilon^2]. \end{aligned}$$

Recall Markov's inequality (denoted Chebychev's Inequality in MGB) guarantees,

$$P[(Y_n - b)^2 \geq \varepsilon^2] \leq \frac{E[(Y_n - b)^2]}{\varepsilon^2},$$

Hence,

$$\begin{aligned} 1 - \frac{E[(Y_n - b)^2]}{\varepsilon^2} &\leq 1 - P[(Y_n - b)^2 \geq \varepsilon^2] \\ \iff 1 - \frac{E[(Y_n - b)^2]}{\varepsilon^2} &\stackrel{\dagger}{\leq} P[|Y_n - b| < \varepsilon] \\ \iff \lim_{n \rightarrow \infty} \left\{ 1 - \frac{E[(Y_n - b)^2]}{\varepsilon^2} \right\} &\leq \lim_{n \rightarrow \infty} P[|Y_n - b| < \varepsilon] \\ \iff 1 - 0 &\leq \lim_{n \rightarrow \infty} P[|Y_n - b| < \varepsilon]. \end{aligned}$$

Since probability is bounded above by 1, we have $\lim_{n \rightarrow \infty} P[|Y_n - b| < \varepsilon] = 1$ and, thus, convergence in probability.

□

9. Supplement 3. Let X_1, X_2, \dots be a sequence of random variables. By the Weak Law of Large Numbers (provided that $E(X^4) < \infty$) we have

$$\overline{X}_n \xrightarrow{p} \mu \quad \text{and} \quad \frac{1}{n} \sum_{i=1}^n X_i^2 \xrightarrow{p} E(X^2).$$

Use these to prove that $\hat{\sigma}_n^2 \xrightarrow{p} \sigma^2$ where the sample variance $\hat{\sigma}_n^2$ is defined by

$$\hat{\sigma}_n^2 = \frac{1}{n} \sum_{i=1}^n (X_i - \overline{X}_n)^2.$$

(Hint: Define $g : \mathbb{R}^2 \rightarrow \mathbb{R}$ as $g(y, z) = y - z^2$ which is a continuous function.)

Solution:

Consider the joint random variable $(\frac{1}{n} \sum_{i=1}^n X_i^2, \overline{X}_n)$ and the continuous transformation $g : \mathbb{R}^2 \rightarrow \mathbb{R}$, $g(y, z) = y - z^2$. Then, by the statements above and Theorem*,

$$g\left(\frac{1}{n} \sum_{i=1}^n X_i^2, \overline{X}_n\right) \xrightarrow{p} g(E(X^2), \mu) = E(X^2) - \mu^2 = \sigma^2.$$

We now calculate,

$$\begin{aligned} g\left(\frac{1}{n} \sum_{i=1}^n X_i^2, \overline{X}_n\right) &= \frac{1}{n} \sum_{i=1}^n X_i^2 - \overline{X}_n^2 \\ &= \frac{1}{n} \sum_{i=1}^n X_i^2 - 2\overline{X}_n + \overline{X}_n^2 \\ &= \frac{1}{n} \sum_{i=1}^n X_i^2 - 2\left(\frac{1}{n} \sum_{i=1}^n X_i\right) \overline{X}_n + \overline{X}_n^2 \\ &= \frac{1}{n} \sum_{i=1}^n (X_i - \overline{X}_n)^2. \end{aligned}$$

□

Theorem*: Suppose $\mathbf{Y}_n : \Omega \rightarrow \mathbb{R}^n$ is a random variable such that $\mathbf{Y}_n \xrightarrow{p} \mathbf{b}$. Then, if $g : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is a continuous transformation then $g(\mathbf{Y}_n) \xrightarrow{p} g(\mathbf{b})$.

Proof. Let $\varepsilon > 0$ and $0 < \delta < 1$. By continuity of g , there exists $\hat{\delta} > 0$ such that if $|\mathbf{y} - \mathbf{b}| < \hat{\delta}$ then $|g(\mathbf{y}) - g(\mathbf{b})| < \varepsilon$. In the context of the random variable \mathbf{Y}_n , $\{e \in \Omega : |\mathbf{Y}_n(e) - \mathbf{b}| < \hat{\delta}\} \stackrel{\dagger}{\subseteq} \{e \in \Omega : |g(\mathbf{Y}_n(e)) - g(\mathbf{b})| < \varepsilon\}$. By hypothesis, we can choose N so that if $n \geq N$ implies $P(|\mathbf{Y}_n - \mathbf{b}| < \hat{\delta}) \geq 1 - \delta$. Note that because of the containment[†] mentioned, $P(\{e \in \Omega : |g(\mathbf{Y}_n(e)) - g(\mathbf{b})| < \varepsilon\}) \geq P(\{e \in \Omega : |\mathbf{Y}_n(e) - \mathbf{b}| < \hat{\delta}\}) \geq 1 - \delta$, and, thus we have satisfied the desired convergence in probability. □

10. Supplement 4. Suppose that X_1, \dots, X_n form a random sample from a normal distribution with mean 0 and unknown variance σ^2 .

(a) Determine the asymptotic distribution of the statistic $(\frac{1}{n} \sum_{i=1}^n X_i^2)^{-1}$.

Solution:

Recall that the square of a standard normal random variable is distributed chi-squared with one degree of freedom. Hence, $X_i^2/\sigma \sim \chi_1^2$. This is a gamma distribution with $\lambda = 1/(2\sigma^2)$ and $r = 1/2$ (by MGB VI.29 Lemma 1).

Thus, we can think of $X_1^2, X_2^2, \dots, X_n^2$ as a random sample from gamma distribution with $\lambda = 1/(2\sigma^2)$ and $r = 1/2$. Denote them as W_i and their sample mean as \bar{W} . The mean and variance of this gamma distribution are given by $\mu_W = r/\lambda = \sigma^2$ and $\sigma_W^2 = r/\lambda^2 = 2\sigma^4$, respectively. If $g(w) = w^{-1}$, then note that $g'(\mu_W) = -1/(\sigma^4) \neq 0$. By the delta method theorem we have that

$$\frac{\sqrt{n} [(\bar{W})^{-1} - (\sigma^2)^{-1}]}{(-1/\sigma^4) \sqrt{2\sigma^4}} \xrightarrow{D} Z \sim N(0, 1).$$

Hence $(\frac{1}{n} \sum_{i=1}^n X_i^2)^{-1} = (\bar{W})^{-1}$ is asymptotically distributed $N(\sigma^{-2}, \frac{2}{n\sigma^4})$ (removing the negative by symmetry of the normal).

□

(b) Find a variance stabilizing transformation for the statistic $\frac{1}{n} \sum_{i=1}^n X_i^2$.

Solution:

Note that $\sigma_W = \sqrt{2\sigma^2} = \sqrt{2}\mu_W$. To stabilize the variance, we seek g such that

$$\begin{aligned} 1 = g'(\mu_W)\sigma_w &\iff g(\mu_W) = \int \sigma_w^{-1} d\mu_W = \frac{1}{\sqrt{2}} \int \mu_W^{-1} d\mu_W \\ &= \frac{1}{\sqrt{2}} (\ln |\mu_W| + c). \end{aligned}$$

Hence, if we transform the data by $g(W) = \ln(W)$, we can expect that the variance of the transformed asymptotic distribution will not depend on the mean.

□