1. MGB III.2

(a) Find the mode of the beta distribution

Solution:

The density function is given by

$$f(x|a,b) = \frac{1}{B(a,b)} x^{a-1}(1-x)^{b-1} I_{(0,1)}(x),$$

whose derivative is

$$\begin{aligned}
f'(x|a,b) &= \frac{1}{B(a,b)} \Big[(a-1)x^{a-2}(1-x)^{b-1} - (b-1)x^{a-1}(1-x)^{b-2}\Big] I_{(0,1)}(x) \\
&= \frac{1}{B(a,b)} x^{a-2}(1-x)^{b-2}\big[(a-1)(1-x) - (b-1)x\big] I_{(0,1)}(x) \\
&= \frac{1}{B(a,b)} x^{a-2}(1-x)^{b-2}\big[x(2-a-b) + (a-1)\big] I_{(0,1)}(x),
\end{aligned}$$

so long as $a \neq 1$ and $b \neq 1$ in which case $f(x) = I_{(0,1)}$ and the mode is clearly 1. Otherwise, $x = 0, 1$ are critical points, as well as $(a-1)/(a+b-2)$ when both $a < 1$ and $b < 1$ or when $a > 1$ and $b > 1$. When $a < 1$ or when $b < 1$, then both x^{1-a} and x^{1-b} are unbounded and, thus, the mode does not exist. When both $a, b > 1$ then $x(2-a-b) + (a-1)$ is a line with negative slope $(2-a-b)$, hence at the critical point $(a-1)/(a+b-2)$ f attains a maximum on $(0,1)$ by the first derivative test.

□

(b) Find the mode of the gamma distribution

Solution:

The density is given by

$$f(x|r,\lambda) = \frac{\lambda}{\Gamma(r)}(\lambda x)^{r-1} e^{-\lambda x} I_{(0,\infty)}(x) \quad \text{for } \lambda, r > 0,$$

whose derivative is given by

$$\begin{aligned}
f'(x|r,\lambda) &= \frac{\lambda}{\Gamma(r)}\Big[(r-1)\lambda(\lambda x)^{r-2}e^{-\lambda x} - \lambda(\lambda x)^{r-1}e^{-\lambda x}\Big] \\
&= \frac{\lambda^2}{\Gamma(r)}(\lambda x)^{r-2}e^{-\lambda x}\Big[(r-1) - \lambda x\Big].
\end{aligned}$$

Note that when $r < 1$, f is unbounded and thus no mode exists. When $r = 1$, $f'(x) < 0$ for all $x \in (0, \infty)$ so f is maximized as $x \to 0$, hence there is no mode since $0 \notin (0, 1)$. Otherwise, $x = \frac{r-1}{\lambda}$ is a critical point. Moreover, $(r-1) - \lambda x$ is a line with negative slope $-\lambda$ and by the first derivative test, f attains its maximum at $\frac{r-1}{\lambda}$ and, thus, is a mode.

□

2. MGB III.15 Let X be normally distributed with mean μ and variance σ^2. Truncate the density of X on the left at a and the right at b, and then calculate the mean of the truncated distribution. (Note that the mean of the truncated distribution should fall between a and b. Furthermore, if $a = \mu - c$ and $b = \mu + c$, then the mean of the truncated distribution is μ.

Solution:

The truncation, Y, is distributed with the p.d.f.

$$\frac{\phi_{\mu,\sigma^2}(y) I_{(a,b)}(y)}{\Phi_{\mu,\sigma^2}(b) - \Phi_{\mu,\sigma^2}(a)}.$$

So the mean is

$$\begin{aligned} E(Y) &= \frac{1}{\Phi_{\mu,\sigma^2}(b) - \Phi_{\mu,\sigma^2}(a)} \int_a^b y \phi_{\mu,\sigma^2}(y) dy \\ &= \frac{1}{\Phi_{\mu,\sigma^2}(b) - \Phi_{\mu,\sigma^2}(a)} \int_a^b \frac{y}{\sigma} \phi \left(\frac{y-\mu}{\sigma} \right) dy. \end{aligned}$$

Let $u = (y - \mu)/\sigma$ then $y/\sigma = u - \mu/\sigma$ and $dy = \sigma du$, so continuing from above,

$$\begin{aligned} E(Y) &= \frac{1}{\Phi_{\mu,\sigma^2}(b) - \Phi_{\mu,\sigma^2}(a)} \int_{(a-\mu)/\sigma}^{(b-\mu)/\sigma} \Big(\sigma u \phi(u) - \mu \phi(u) \Big) du \\ &= \frac{1}{\Phi_{\mu,\sigma^2}(b) - \Phi_{\mu,\sigma^2}(a)} \int_{(a-\mu)/\sigma}^{(b-\mu)/\sigma} \sigma u \phi(u) du - \mu \left[\Phi \left(\frac{b-\mu}{\sigma} \right) - \Phi \left(\frac{a-\mu}{\sigma} \right) \right] \\ &= \frac{\sigma}{\sqrt{2\pi} (\Phi_{\mu,\sigma^2}(b) - \Phi_{\mu,\sigma^2}(a))} \int_{(a-\mu)/\sigma}^{(b-\mu)/\sigma} u e^{-u^2/2} du + \mu. \end{aligned}$$

Let $w = u^2/2$, then $dw = u du$ and continuing from above

$$\begin{aligned} E(Y) &= \frac{\sigma^2}{\Phi_{\mu,\sigma^2}(b) - \Phi_{\mu,\sigma^2}(a)} \cdot \frac{1}{\sigma\sqrt{2\pi}} \int_{(a-\mu)^2/(2\sigma^2)}^{(b-\mu)^2/(2\sigma^2)} e^{-w} dw + \mu. \\ &= \sigma^2 \frac{\phi_{\mu,\sigma}(a) - \phi_{\mu,\sigma}(b)}{\Phi_{\mu,\sigma^2}(b) - \Phi_{\mu,\sigma^2}(a)} + \mu. \end{aligned}$$

□

3. MGB III.17 Let X be the life in hours of a radio tube. Assume that X is normally distributed with mean 200 and variance σ^2. If a purchaser of such radio tubes requires that at least 90 percent of the tubes have lives exceeding 150 hours, what is the largest value of σ can be and still have the purchaser satisfied?

Solution:

Since $X \sim N(200, \sigma^2)$, we have that

$$\begin{aligned} P(X \geq 150) &= 1 - \Phi_{200,\sigma^2}(150) \\ &= 1 - \Phi\left(\frac{150-200}{\sigma^2}\right) \\ &= \Phi\left(\frac{50}{\sigma^2}\right) \geq .9 \end{aligned}$$

Note that Φ is strictly increasing and bounded between 0,1, hence it has a unique value z^* such that $\Phi(z^*) = .9$ and $\Phi(z) \geq .9$ for all $z \geq z^*$. Thus, we require $\frac{50}{\sigma^2} \geq z*$ or equivalently $\sigma^2 \leq \frac{50}{z^*}$ since both z^* and σ^2 are greater than 0. We can obtain a numerical estimate for the upper bound on the variance with the program `R` using the command `50/qnorm(.9)` which yields `39.01521`.

□

4. MGB III.19a The distribution given by

$$f(x|\beta) = \frac{1}{\beta^2} x e^{-\frac{1}{2}(x/\beta)^2} I_{(0,\infty)}(x) \quad \text{for } \beta > 0$$

is called the *Raleigh* distribution. Show that the mean and variance exist and find them.

Solution:

The nth moment of X is given by

$$\begin{aligned} E(X^n) &= \int_0^\infty \frac{1}{\beta^2} x^{n+1} e^{-\frac{1}{2}(x/\beta)^2}\, dx \\ &= \int_0^\infty \beta^n (2u)^{\frac{n}{2}} e^{-u} du \qquad \text{where } u = \frac{1}{2}(x/\beta)^2 \text{ and } du = x/\beta^2 dx \\ &= \beta^n 2^{\frac{n}{2}} \Gamma\left(1 + \frac{n}{2}\right). \end{aligned}$$

Thus $\mu = E(X) = \frac{2\beta}{\sqrt{2}}\sqrt{\pi} = \beta\sqrt{\frac{\pi}{2}}$ and $\sigma^2 = E(X^2) - E(X)^2 = 4\beta^2 - \beta^2\frac{\pi}{2} = \beta^2\frac{4-\pi}{2}$.

□

5. MGB III.20a The distribution given by

$$F(x|\beta) = \frac{4}{\beta^3\sqrt{\pi}} x^2 e^{-x^2/\beta^2} I_{(0,\infty)}(x) \quad \text{for } \beta > 0$$

is called the *Maxwell* distribution. Show that the mean and variance exist and find them.

Solution:

The nth moment of X is given by

$$\begin{aligned}
E(X^n) &= \int_0^\infty \frac{4}{\beta^3\sqrt{\pi}} x^{n+2} e^{-\frac{1}{2}(x/\beta)^2}\, dx \\
&= \frac{4}{\sqrt{\pi}} \int_0^\infty \beta^n (2u)^{\frac{n+1}{2}} e^{-u} du \qquad \text{where } u = \frac{1}{2}(x/\beta)^2 \text{ and } du = x/\beta^2 dx \\
&= \frac{\beta^n}{\sqrt{\pi}} 2^{\frac{n+5}{2}} \Gamma\left(\frac{n+3}{2}\right).
\end{aligned}$$

Thus $\mu = E(X) = \frac{2^4\beta}{\sqrt{\pi}}$ and $\sigma^2 = E(X^2) - E(X)^2 = 4\beta^2 - \beta^2\frac{\pi}{2} = \beta^2\frac{4-\pi}{2}$.

□

6. MGB III.28 Show that

$$P(X \geq k) = \sum_{x=k}^{n} \binom{n}{x} p^x (1-p)^{n-x} = \frac{1}{B(k, n-k+1)} \int_0^p u^{k-1}(1-u)^{n-k} du$$

for X a binomially distributed random variable. That is, if X is binomially distributed with parameters n and p and Y is beta-distributed with parameters k and $n-k+1$, then $F_Y(p) = 1 - F_X(k-1)$.

Solution:

Let us evaluate the integral on the far right hand side via integration by parts $n-k$ times. That is,

$$\begin{aligned}
&\frac{1}{B(k, n-k+1)} \int_0^p u^{k-1}(1-u)^{n-k} du \\
&= \frac{1}{B(k, n-k+1)} \left((1-u)^{n-k} \frac{u^k}{k}\Big|_0^p + \int_0^p \frac{u^k}{k}(n-k)(1-u)^{n-(k+1)}\, du \right) \\
&\dots \\
&= \frac{1}{B(k, n-k+1)} \sum_{x=k}^{n} \left[\frac{(n-k)!}{(n-x)!}(1-p)^{n-k}\right] \cdot \left[\frac{(k-1)!}{x!} p^k\right] \\
&= \frac{\Gamma(n+1)}{\Gamma(k)\Gamma(n-k+1)} \sum_{x=k}^{n} \left[\frac{(n-k)!}{(n-x)!}(1-p)^{n-k}\right] \cdot \left[\frac{(k-1)!}{x!} p^k\right] \\
&= \frac{n!}{(k-1)!(n-k)!} \sum_{x=k}^{n} \left[\frac{(n-k)!}{(n-x)!}(1-p)^{n-k}\right] \cdot \left[\frac{(k-1)!}{x!} p^k\right] \\
&= \sum_{x=k}^{n} \binom{n}{x} p^x (1-p)^{n-x} = P(X \geq k)
\end{aligned}$$

□

7. MGB V.10 A certain explosive device will detonate if any one of n short-lived fuses lasts longer than .8 seconds. Let X_i represent the life of the ith fuse. It can be assumed that each X_i is uniformly distributed over the interval 0 to 1 second. Furthermore, it can be assumed that the X_i's are independent.

Solution:

□

8. MGB V.13 Let X_1 and X_2 be independent standard normal randiom variables. Let $Y_1 = X_1 + X_2$ and $Y_2 = X_1^2 X_2^2$.

(a) Show that the join moment generating function of Y_1 and Y_2 is

$$\frac{\exp\left[t_1^2/(1-2t_2)\right]}{1-2t_2} \quad \text{for } -\infty < t_1 < \infty \text{ and } -\infty < t_2 < \frac{1}{2}.$$

Solution:

□

(b) Find the correlation coefficient of Y_1 and Y_2.

Solution:

□

9. MGB V.22 Kitty Oil Co. has has decided to drill for oil in 10 different locations; the cost of drilling at each location is \$10,000. (Total cost is then \$100,000.) The probability of finding oil in a given location is only $\frac{1}{5}$, but if oil is found at a given location then the amount of money the company will get selling oil (excluding the initial \$10,000 drilling cost) from that location is an exponential random variable with mean \$50,000. Let Y be the random variable that denotes the number of locations where oil is found, and let Z denote the total amount of money received from selling oil from all the locations.

(a) Find $E(Z)$.

Solution:

□

(b) Find $P(Z > 100,000|Y = 1)$ and $P(Z > 100,000|Y = 2)$.

Solution:

□

(c) How would you find $P(Z > 100,000)$? Is $P(Z > 100,000) > \frac{1}{2}$.

Solution:

□

10. MGB V.54 Let X_1 and X_2 be independent random variables, each normally distributed with parameters $\mu = 0$ and $\sigma^2 = 1$. Find the joint distribution of $Y_1 = X_1^2 + X_2^2$ and $Y_2 = X_1/X_2$. Find the marginal distribution of Y_1 and of Y_2. Are Y_1 and Y_2 independent?

Solution:

□