1. MGB VII.1. An urn contains black and white balls. A sample of size n is drawn with replacement. What is the maximum-likelihood estimator of the ratio R of black to white balls in the urn? Suppose that one draws balls one by one with replacement until a black ball appears. Let X be the number of draws required (not counting the last draw). This operation is repeated n times to obtain a sample $X_1, X_2, \ldots, X_n$. What is the maximum-likelihood estimator of R on the basis of this sample?

Solution:

Since the urn contains both black and white balls, the ratio $R > 0$. Let Y be the random variable that distinguishes the color of the ball by

$$Y = \begin{cases} 1 & \text{if black,} \\ 0 & \text{if white.} \end{cases}$$

Then, if R is the proportion of black balls to white, $Y \sim \mathit{Bernoulli}\left(\frac{R}{R+1}\right)$. Let $R = \tau(p) = \frac{p}{1-p}$, then note that $\frac{R}{R+1} = p$. Since the MEL for a random sample from a Bernoulli distribution is known to be $\overline{Y}$, by the invariance of maximum likelihood, the MLE for R, $\widehat{R} = \frac{\overline{Y}}{1-\overline{Y}}$.

Now, let X be the number of white balls selected until a black is drawn. This is known to be a negative binomial random variable with one success (i.e. a geometric random variable). Hence the log-likelihood function for X is given by

$$\begin{aligned} \ell(R|_{X_1=x_1 \ldots X_n=x_n}) &= n \log\left(\frac{R}{R+1}\right) + \sum_{i=1}^{n} x_i \log\left(1 - \frac{R}{R+1}\right) \\ &= n \log\left(\frac{R}{R+1}\right) + \sum_{i=1}^{n} x_i \log\left(\frac{1}{R+1}\right), \end{aligned}$$

whose derivative with respect to R is given by

$$\begin{aligned} \frac{d\ell}{dR} &= n\frac{\cancel{(R+1)}}{R} \cdot \frac{(R+1) - R}{(R+1)^{\cancel{2}}} - \sum_{i=1}^{n} x_i \cdot \cancel{(R+1)}\frac{1}{(R+1)^{\cancel{2}}} \\ &= \frac{n - R\sum x_i}{R(R+1)}. \end{aligned}$$

Note that $\frac{d\ell}{dR} > 0$ for $0 < R < \overline{X}$, and $\frac{d\ell}{dR} < 0$ for $R > \overline{X}$ and $\frac{d\ell}{dR} = 0$ for $R = \overline{X}$. Hence, the MLE is $\widehat{R} = \overline{X}$.

$\square$

2. MGB VII.11 Let $X_1, \ldots, X_n$ be a random sample from some density which has mean μ and variance σ^2.

(a) Show that $\displaystyle\sum_{i=1}^{n} a_i X_i$ is an unbiased estimator of μ for any set of known constants $a_1, \ldots, a_n$ satisfying $\displaystyle\sum_{i=1}^{n} a_i = 1$

Solution:

Note that by linearity

$$E\left[\sum_{i=1}^{n} a_i X_i\right] = \sum_{i=1}^{n} a_i E[X_i] = \sum_{i=1}^{n} a_i \mu = \mu.$$

Hence $\sum_{i=1}^{n} a_i X_i$ is an unbiased estimator for μ.

$\square$ **(b)** If $\displaystyle\sum_{i=1}^{n} a_i = 1$, show that $Var\left[\displaystyle\sum_{i=1}^{n} a_i X_i\right]$ is minimized for $a_i = 1/n$, $i = 1, \ldots, n$. *Hint:* Prove that $\displaystyle\sum_{i=1}^{n} a_i^2 = \sum_{i=1}^{n} (a_i - 1/n)^2 + 1/n$ when $\displaystyle\sum_{i=1}^{n} a_i = 1$.

Solution:

Let us first prove the hint. That is,

$$\sum_{i=1}^{n} (a_i - 1/n)^2 = \sum_{i=1}^{n} a_i^2 - 2\sum_{i=1}^{n} \frac{a_i}{n} + n\frac{1}{n^2} = \sum_{i=1}^{n} a_i^2 - 1/n.$$

Now,

$$\begin{aligned}
Var\left[\sum_{i=1}^{n} a_i X_i\right] &= E\left[\left(\sum_{i=1}^{n} a_i X_i\right)^2\right] - \mu^2 \\
&= E\left[\sum_{i=j} a_i^2 X_i^2\right] + \cancel{E\left[\sum_{i \neq j} a_i a_j X_i X_j\right]} - \mu^2 \quad \text{by independence,} \\
&= \left(\sum_{i=1}^{n} (a_i - 1/n)^2 + 1/n\right) E\left[X_i^2\right] - \mu^2 \qquad \text{by the hint.}
\end{aligned}$$

The right hand side is minimized when the factor $\left(\cdot\right)$ is minimized, and this quadratic form is minimized when $\sum (a_i - 1/n)^2 = 0$ if and only if $a_i = 1/n$ for all i.

$\square$

3. MGB VII.12[a,b,c] Let $X_i, \ldots, X_n$ be a random sample from the discrete density function $f(x;\theta) = \theta^x(1-\theta)^{1-x} I_{\{0,1\}}(x)$, where $0 \le \theta \le 1/2$. Note that $\underline{\overline{\Theta}} = \{\theta : 0 \le \theta \le \frac{1}{2}\}$.

(a) Find a method-of-moments estimator θ, and then find the mean and mean-squared error of your estimator.

Solution:

Note that f is the pdf for a Bernoulli distribution where θ is chosen among p and $1-p$ so that $0 < \theta < 1/2$. Hence $\mu = \theta$. Matching this with the first raw sample moment, we have that the MOM estimator is $\widehat{\Theta}_1 = \vartheta_1(X_1, \ldots, X_n) = \overline{X}$.

This statistic is clearly unbiased, hence the $MSE_{\vartheta_1}(\theta) = Var\left[\,\overline{X}\,\right] = \frac{\theta(1-\theta)}{n}$.

$\square$

(b) Find a maximum-likelihood estimator of θ, and then find the mean and mean-squared error of your estimator.

4. MGB VII.44[b,c] Let $X_1, \ldots, X_n$ be a random sample from

$$f(x;\theta) = e^{-(x-\theta)} I_{[\theta,\infty)}(x) \quad \text{for } -\infty < \theta < \infty.$$

(a) Find a maximum-likelihood estimator of θ.

Consider the liklihood function

$$\begin{aligned} L(\theta|_{x_1 \ldots x_n}) &= \prod_{i=1}^{n} e^{-x_i+\theta} I_{[\theta,\infty)}(x_i) \\ &= e^{n\theta} e^{-\sum x_i} \qquad \text{for } x_i \in [\theta,\infty) \text{ and } 0 \text{ otherwise.} \end{aligned}$$

Note that this is an exponentially increasing function in θ. Also, since $x_1 \ldots x_n$ are given values of $X_1, \ldots, X_n$, we have that $\theta \leq \min\{x_1 \ldots x_n\}$. Hence, $\widehat{\theta} = \min\{x_1, \ldots, x_n\}$ maximizes the likelihood.

(b) Find a method-of-moments estimator of θ.

The first raw moment is

$$\begin{aligned} \mu &= \int_{\theta}^{\infty} x e^{-(x-\theta)} dx \\ &= \int_{0}^{\infty} (u+\theta) e^{-u} du \\ &= \Gamma(2) + \theta\Gamma(1) \\ &= 1 + \theta. \end{aligned}$$

Matching moments, $\overline{X} = 1 + \widehat{\Theta}_2$, we estimate $\widehat{\Theta}_2 = \overline{X} - 1$. Note that this is different than the MOM estimate.

5. MGB VII.54[b] Let $X_1, \dots, X_n$ be a random sample from the density

$$f(x;\alpha,\theta) = (1-\theta)\theta^{x-\alpha} I_{\alpha+\mathbb{Z}^+}(x),$$

where $-\infty < \alpha < \infty$, and $0 < \theta < 1$, and $\alpha + \mathbb{Z}^+ := \{\alpha, \alpha+1, \dots\}$. Find the maximum-likelihood estimator of (α, θ).

Consider the likelihood function with θ fixed in $0 < \theta < 1$.

$$L(\alpha|_{\theta,x_1,\dots,x_n}) = (1-\theta)^n \theta^{\sum x_i} \theta^{-n\alpha} \prod_{i=1}^{n} I_{\alpha+\mathbb{Z}^+}(x_i).$$

Note that L non-zero only when it is a positive integer multiple of $\min\{x_1, \dots, x_n\}$. That is $I_{\alpha+\mathbb{Z}^+}(x_i) = I_{x^*+\mathbb{Z}^+}(\alpha)$ where $x^* = \min\{x_1, \dots, x_n\}$. Moreover, θ^{-na} decreases on the support of α. Hence $L|_\theta$ is maximized when $\widehat{\alpha} = \min\{x_1, \dots, x_n\}$ for all $0 < \theta < 1$.

We now fix $\widehat{\alpha}$ and and maximize with respect to θ. That is the the derivative of the likelihood function given $\widehat{\alpha}$ is

$$\begin{aligned}
\frac{dL|_{\widehat{\alpha}}}{d\theta} &= -n(1-\theta)^{n-1}\theta^{\sum x_i - n\alpha} + \left(\sum x_i - n\alpha\right)(1-\theta)^n \theta^{\sum x_i - n\alpha - 1} \\
&= \theta^{\sum x_i - n\alpha - 1}(1-\theta)^{n-1}\left[\theta n + \left(\sum x_i - n\alpha\right)(1-\theta)\right].
\end{aligned}$$

This quantity is 0 only when $\widehat{\theta} = \frac{1}{n}\sum x_i$. Moreover for $\frac{dL}{d\theta} > 0$ for $0 < \theta < \widehat{\theta}$ and $\frac{dL}{d\theta} < 0$ for $\widehat{\theta} < \theta < 1$, hence this is a maximum for $0 < \theta < 1$. Hence, the MLE is

$$\widehat{(\alpha,\theta)} = \left(\min\{x_1, \dots, x_n\}, \overline{X}\right).$$