1. MGB VII.29[a,b,c,d,e] Let $X_1, \ldots, X_n$ be a random sample from $N(\theta, 1)$.

(a) Find the Cramér-Rao lower bound for the variance of unbiased estimators of θ, θ^2 and $P[X > 0]$.

Solution:

Let us first calculate

$$\begin{aligned}\left[\frac{\partial}{\partial\theta}\log f(x|\theta)\right]^2 &= \left[\frac{\partial}{\partial\theta}\log\left(\frac{1}{\sqrt{2\pi}}e^{-(x-\theta)^2/2}\right)\right]^2 \\ &= \left[\frac{\partial}{\partial\theta}\left(\frac{(x-\theta)^2}{2} - \log(\sqrt{2\pi})\right)\right]^2 \\ &= (x-\theta)^2\end{aligned}$$

So for a given $\tau(\theta)$ and any estimator T, the Cramér-Rao lower bound is given by

$$\begin{aligned}Var_\theta(T) &\geq \frac{[\tau(\theta)]^2}{nE_\theta\left[\frac{\partial}{\partial\theta}\log f(x|\theta)\right]^2} \\ &= \frac{[\tau(\theta)]^2}{nE_\theta[x-\theta]^2} \\ &= \frac{[\tau(\theta)]^2}{n}\end{aligned}$$

So, in the case of $\tau(\theta) = \theta$, the lower bound for any given estimator is $1/n$. When $\tau(\theta) = \theta^2$, the lower bound is $4\theta^2/n$. Finally, note that $P[X > 0] = P[X - \theta > -\theta] = 1 - \Phi(-\theta) = \Phi(\theta)$, where Φ is the c.d.f. for the standard normal distribution. So, the lower bound for an estimate for this parameter is $\dfrac{\phi(\theta)^2}{n} = \dfrac{e^{-x^2}}{n\sqrt{2\pi}}$

□

(b) Is there an unbiased estimator of θ^2 for $n = 1$? If so, find it.

Solution:

Consider $\widehat{\Theta} = X_1^2 - 1$, and note $E_\theta[X_1^2 - 1] = E_\theta[X_1^2] - 1 = (1 - \theta^2) - 1 = \theta^2$. Hence, $\widehat{\Theta}$ is unbiased.

□

(c) Is there an unbiased estimator of $P[X > 0]$? If so, find it.

Solution:

Consider $\widehat{\Theta} = (1/n)\sum I_{[0,\infty)}(X_i)$, and note $E_\theta\left[(1/n)\sum I_{[0,\infty)}(X_i)\right] = 1/n\sum E_\theta[I_{[0,\infty)}(X_i)] = (1/n)\sum P[X_i > 0] = P[X > 0]$.

□

(d) What is the maximum-likelihood estimator of $P[X > 0]$?

Recall that $\overline{X}$ is a maximum likelihood estimator for θ, so by invariance $\Phi(\overline{X})$ is a maximum likelihood estimator for $P[X > 0] = \Phi(\theta)$.

(e) Is there a UMVUE of θ^2? If so, find it.

Solution:

Note that

$$f(x|\theta) = \frac{1}{\sqrt{2\pi}} e^{-(x-\theta)^2/2} = \left(\frac{e^{-\theta^2/2}}{\sqrt{2\pi}}\right)\left(e^{-x^2/2}\right)\exp(\theta \cdot x)$$

Hence $\sum X_i$ is a complete sufficient statistic. By the Lehmann-Scheffé theorem, the UMVUE is given by an unbiased estimate of θ^2 which is a function of $\sum X_i$. Since $X_1^2 - 1$ is an unbiased estimator of θ^2, such a function is given by (the argument † is due to a generous hint from Jack Lelko)

$$\begin{aligned} E_\theta\left[X_1^2 - 1 \middle| \sum X_i = s\right] &\overset{\dagger}{=} E_\theta\left[\left(\left(X_1 - \overline{X}\right) + \overline{X}\right)^2 \middle| \sum X_i = s\right] - 1 \\ &= E_\theta\left[\left(X_1 - \overline{X}\right)^2 \middle| \sum X_i = s\right] - \cancelto{0}{2E_\theta\left[(X_1 - \overline{X})\overline{X} \middle| \sum X_i = s\right]} + \ldots \\ &\qquad E_\theta\left[\overline{X}^2 \middle| \sum X_i = s\right] - 1 \\ &= \frac{n-1}{n} - (s/n)^2 - 1, \end{aligned}$$

where $E_\theta\left[\left(X_1 - \overline{X}\right)^2 \middle| \sum X_i = s\right]$ since $1 \cdot (n-1) = E_\theta\left[\sum (X_i - \overline{X})^2\right] = \sum E_\theta[X_1 - \overline{X}]^2 = nE_\theta[X_1 - \overline{X}]^2$.

So the UMVUE is

$$\widehat{\Theta} = \frac{n-1}{n} - \overline{X}^2 - 1.$$

$\square$

2. MGB VII.30 For a random sample from the Poisson distribution, find an unbiased estimator of $\tau(\lambda) = (1+\lambda)e^{-\lambda}$. Find a maximum-likelihood estimator of $\tau(\lambda)$. Find the UMVUE of $\tau(\lambda)$.

Solution:

Consider the estimator $t(X_1, \ldots, X_n) = I_{\{0,1\}}(X_1)$, and note $E[I_{\{0,1\}}(X_1) = e^{-\lambda} + \lambda e^{-\lambda} = (1+\lambda)e^{-\lambda}$. Hence, the estimator is unbiased.

Now, recall that $\overline{X}$ is a maximul likelihood estimator for λ. So, by invariance, $(1+\overline{X})e^{\overline{X}}$ is a maximum liklihood estimator for $(1+\lambda)e^{-\lambda}$.

Note that the p.d.f. of a Poisson distribution is $f(x|\lambda) = e^{-\lambda}\lambda^x/x! = e^{-\lambda}\big(\frac{1}{x!}I_{\mathbb{Z}^+}(x)\big)e^{-\ln(\lambda)x}$, so it is of the exponential family. Hence $S := \sum X_i$ is a complete sufficent statistic. By the Lehmann-Scheffé theorem, an unbiased estimator that is a function of this statistic is the UMVUE for $\tau(\lambda)$. Such a statistic is given by

$$E\left[I_{\{0,1\}}(X_1)\middle|\sum X_i = s\right]. \tag{1}$$

We evaluate this expectation by first finding the conditional distribution of $I_{\{0,1\}}(X_1)|\sum X_1 = s$. It can take on only 0 or 1, hence we evaluate the probabilities

$$\begin{aligned}
P\left[X_1 = 0\middle|\sum_{i=1}^n X_i = s\right] &= \frac{P\left[X_1 = 0; \sum_{i=1}^n X_i = s\right]}{P\left[\sum_{i=1}^n X_i = s\right]} \\
&= \frac{P\left[X_1 = 0\right]\cdot P\left[\sum_{i=2}^n X_i = s\right]}{P\left[\sum_{i=1}^n X_i = s\right]} \\
&= \frac{e^{-\lambda}\cdot\left[e^{-(n-1)\lambda}\big((n-1)\lambda\big)^s/s!\right]}{e^{-n\lambda}(n\lambda)^s/s!} \\
&= \left(\frac{n-1}{n}\right)^s,
\end{aligned}$$

and

$$\begin{aligned}
P\left[X_1 = 1\middle|\sum_{i=1}^n X_i = s\right] &= \frac{P\left[X_1 = 1\right]\cdot P\left[\sum_{i=2}^n X_i = s-1\right]}{P\left[\sum_{i=1}^n X_i = s\right]} \\
&= \frac{e^{-\lambda}\lambda\cdot\left[e^{-(n-1)\lambda}\big((n-1)\lambda\big)^{s-1}/(s-1)!\right]}{e^{-n\lambda}(n\lambda)^s/s!} = s\frac{(n-1)^{s-1}}{n^s}
\end{aligned}$$

Hence the statistic (1) is

$$E\left[I_{\{0,1\}}(X_1)\middle|\sum X_i = S\right] = \left(\frac{n-1}{n}\right)^S + S\frac{(n-1)^{S-1}}{n^S}, \quad \text{where } S = \sum X_i.$$

$\square$

3. MGB VII.36 Show that

$$E_\theta\left[\left(\frac{\partial}{\partial\theta}\log f(X|\theta)\right)^2\right] = -E_\theta\left[\frac{\partial^2}{\partial\theta^2}\log f(X|\theta)\right],$$

assuming that $f(X|\theta)$ has sufficiently bounded derivatives to allow the interchange of the operators E_θ and $\frac{\partial^2}{\partial\theta^2}$

Solution:

We begin by evaluating the right hand side,

$$\begin{aligned}
E_\theta\left[\frac{\partial^2}{\partial\theta^2}\log f(X|\theta)\right] &= -E_\theta\left[\frac{\partial}{\partial\theta}\left(\frac{1}{f(X|\theta)}\cdot\frac{\partial}{\partial\theta}f(X|\theta)\right)\right] \\
&= -E_\theta\left[\frac{-1}{f(X|\theta)^2}\left(\frac{\partial}{\partial\theta}f(X|\theta)\right)^2 + \frac{\partial^2}{\partial\theta^2}f(X|\theta)\right] \\
&= E_\theta\left[\left(\frac{1}{f(X|\theta)}\frac{\partial}{\partial\theta}f(X|\theta)\right)^2\right] + \cancelto{0}{\frac{\partial^2}{\partial\theta^2}E_\theta\left[f(X|\theta)\right]} \\
&= E_\theta\left[\left(\frac{\partial}{\partial\theta}\log f(X|\theta)\right)^2\right]
\end{aligned}$$

$\square$

4. MGB VII.43[a, b, c] Let $Z_1, \ldots, Z_n$ be a random sample from $N(0, \theta^2), \theta > 0$. Define $X_i = |Z_i|$, and consider estimation of θ and θ^2 on the basis of the random sample $X_1, \ldots, X_n$.

(a) Find the UMVUE of θ^2 if such exists.

(b) Find an estimator of θ^2 that has uniformly smaller mean-squared error than the estimator that you found in part (a)

(c) Find the UMVUE of θ if such exists.

5. MGB VII.52 Let θ be the true I.Q. of a certain student. To measuer his I.Q., the student takes a test, and it is known that his test scores are normally distributed with mean μ and standard deviation 5.

(a) The student takes the I.Q. test and gets a score of 130. What is the maximum-likelihood estimate of θ?

(b) Suppose that it is known that I.Q.'s of students of a certain age are distributed normally with mean 100 and variance 225; that is, $\Theta \sim N(100, 225)$. Let X denote a student's test score [X is distributed $N(\theta, 25)$]. Find the posterior distribution of Θ given $X = x$. What is the posterior Bayes estimate of the student's I.Q. if $X = 130$?