1. MGB VI.18[c,d] On the F distribution:

(a) If X has an F distribution with m and n degrees of freedom, show that

$$W = \frac{mX/n}{1 + mX/n}$$

has a beta distribution.

Solution:

$\square$

(b) Use the result of part (a) and the beta function to find the mean and variance of the F distribution. [Find the first two moments of $mX/n = W/(1-W)$].

Solution:

$\square$

2. MGB VI.19[c,d] On the t distribution:

(a) If X is t-distributed, show that X^2 is F-distributed.

Solution:

$\square$

(b) If X is t-distributed with k degrees of freedom, show that $1/(1 + X^2/k)$ has a beta distribution.

Solution:

$\square$

3. Let $X_1, ..., X_n$ be a random sample from $N(\mu, \sigma^2)$. Define

$$\overline{X}_k = \frac{1}{k}\sum_{i=1}^{k} X_i, \qquad \mathcal{S}_k^2 = \frac{1}{k-1}\sum_{i=1}^{k}(X_i - \overline{X}_k)^2,$$

$$\overline{X}_{n-k} = \frac{1}{n-k}\sum_{i=k+1}^{n} X_i, \qquad \mathcal{S}_{n-k}^2 = \frac{1}{n-k-1}\sum_{i=k+1}^{n}(X_i - \overline{X}_{n-k})^2,$$

$$\overline{X} = \frac{1}{n}\sum_{i=1}^{n} X_i, \quad \text{and} \qquad \mathcal{S}^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \overline{X})^2.$$

Use known results about sampling from the normal distribution to answer the following:

(a) What is the distribution of $\sigma^{-2}\left[(k-1)\mathcal{S}_k^2 + (n-k-1)\mathcal{S}_{n-k}^2\right]$?

(b) What is the distribution of $(\frac{1}{2})(\overline{X}_k + \overline{X}_{n-k})$?

(c) What is the distribution of $\sigma^{-2}(X_i - \mu)$?

(d) What is the distribution of $\mathcal{S}_k^2/\mathcal{S}_{n-k}^2$?

(e) What is the distribution of $(\overline{X}0\mu)/(\mathcal{S}/\sqrt{n})$?

4. Let Z_1, Z_2 be a random sample of size 2 from $N(0,1)$ and X_1, X_2 be a random sample of size 2 from $N(1,1)$. Suppose the $Z_i's$ are independent of the $X_j's$. Use known results about sampling from the normal distribution to answer the following:

(a) What is the distribution of $\overline{X} - \overline{Z}$?

(b) What is the distribution of $(Z_1 + Z_2)/\sqrt{[X_2 - X_1)^2 + (Z_2 - Z_1)^2]/2}$?

(c) What is the distribution of $[(X_1 - X_2)^2 + (Z_1 - Z_2)^2 + (Z_1 + Z_2)^2]/2$?

(d) What is the distribution of $(X_2 + X_1 - 2)^2/(X_2 - X_1)^2$?

5. MGB VI.27 If $X_1, X_2, ..., X_n$ are indepently and normally distributed with the same mean but different variances $\sigma_1^2, \sigma_2^2, ..., \sigma_n^2$ and assuming that

$$U = \frac{\sum_{i=1}^{n} X_i/\sigma_i^2}{\sum_{i=j}^{n} 1/\sigma_j^2} \quad \text{and} \quad \frac{1}{\sigma_i^2}\sum_{i=1}^{n}(X_i - U)^2$$

are independently distributed, show that U is normal and V has the chi-square distribution with $n-1$ degrees of feedom.

6. MGB VI.29 Let a sample of size n_1 from a normal population (with variance σ_1^2) have sample variance S_1^2, and let a second sample of size n_2 from a second normal population (with mean μ_2 and variance σ_2^2) have mean $\overline{X}$ and variance $\mathcal{S}_2^2$. Find the joint density of

$$U = \frac{\sqrt{n_2}(\overline{X} - \mu)}{\mathcal{S}_2} \quad \text{and} \quad V = \frac{\mathcal{S}_1^2}{\mathcal{S}_2^2}$$

(Assume that samples are independent.)

7. Supplement 1. Let $Z_1, Z_2, ...$ be a sequence of random variables; and suppose that, for $n = 1, 2, ...$, the distribution of Z_n is as follows:

$$P(Z_n = n^2) = \frac{1}{n} \quad \text{and} \quad P(Z_n = 0) = 1 - \frac{1}{n}.$$

Show that

$$\lim_{n\to\infty} E(Z_n) = \infty \text{ but } Z_n \xrightarrow{p} 0.$$

8. Supplement 2. A sequence of random variables $Y_1, Y_2, ...$ is said to converge in the rth mean if $\lim_{n\to\infty} E(|Y_n - b|^r) = 0$. Prove that if a sequence of random variables converge to b in the quadratic mean, then the sequence also converges to b in probability.

9. Supplement 3. Let $X_1, X_2, ...$ be a sequence of random variables. By the Weak Law of Large Numbers (provided that $E(X^4) < \infty$) we have

$$\overline{X}_n \xrightarrow{p} \mu \quad \text{and} \quad \frac{1}{n}\sum_{i=1}^{n}(X_i - \overline{X}_n)^2.$$

Use these to prove that $\hat{\sigma}_n^2 \xrightarrow{p} \sigma^2$ where the sample variance $\hat{\sigma}_n^2$ is defined by

$$\hat{\sigma}_n^2 = \frac{1}{n}\sum_{i=1}^{n}(X_1 - \overline{X}_n)^2.$$

(Hint: Define $g : \mathbb{R}^2 \to \mathbb{R}$ as $g(y, z) = y - z^2$ which is a continuous function.)

10. Suppose that $X_1, \ldots, X_n$ form a random sample from a normal distribution with mean 0 and unknown variance σ^2.

(a) Determine the asymptotic distribution of the statistic $\left(\frac{1}{n}\sum_{i=1}^n X_i^2\right)^{-1}$.

(b) Find a variance stabilizing transformation for the statistic $\frac{1}{n}\sum_{i=1}^n X_i^2$.