Real Time Signal Processing with Symmetric and Asymmetric Support Intervals

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Outline

Importance of Real Time Signal Processing

What is real time signal processing?

- Applications
  - Speech recognition
  - Audio signal processing
  - Video compression
  - Weather forecasting
  - Economic forecasting
  - Medical imagining (e.g., CAT, MRI)
  - And more...

Outline

Introduction and Motivation

Problem Outline

Problem Approach and Steps

Code Comments

Results and Conclusions

What is the problem?

Goal: We wish to reconstruct some generated signal $\hat{x}$ that has been distorted by some error and convolution processes.

Solution: Find the “convolution inverse” of a. We call this the reconstruction operator R.

Reconstruction Operator R

- Following Lecture 13, we seek a linear reconstruction operator R that we assume is given by convolution with some r supported on a specified interval Δ, so that $\widehat{x} = r * x$.
- It was shown that such an operator satisfies

$$H(r) = E(\widehat{x} - x)^2 = \left\langle P(r - P^{-1}q), r - P^{-1}q \right\rangle_{\Delta} + f_0 - \left\langle q, P^{-1}q \right\rangle_{\Delta}$$

  where P is the operator associated with convolution by $p = a * \phi * a^* + \sigma^2 \delta$ and $q = a * \phi$.
- So, for a given Δ support for r, the reconstruction kernel is uniquely determined by $r = P^{-1}q|_{\Delta}$, and

$$\mathrm{Var}\,\widehat{x} = H_{min} = f_0 - \left\langle q, P^{-1}q \right\rangle_{\Delta}.$$

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Problem Approach Overview

1. Specify problem settings
   - $y = a * x + \nu, \ \ \nu \ (0, \sigma^2 \delta).$
   - $\hat{x} = r * y$
     - $r = P^{-1} q$
   - Choose some $\Delta = [-d, d]$ $\rightarrow$ compute $r = P^{-1} q|_\Delta$
   - Choose some $\Delta = [T, \tau]$ $\rightarrow$ compute $r = P^{-1} q|_\Delta$
2. Compute "optimal" Δ for r by
   - $H(\Delta) = E(\hat{x}_i - x_i)^2 = f_0 - \langle q, P^{-1} q \rangle_\Delta$
   - Plot $H(\Delta)$ vs either d in the symmetric case, or (T, τ) in the asymmetric case.
3. Choose "optimal" Δ $\rightarrow$ compute $\widehat{x} = r * y$.
4. Illustrate the result of estimation with uncertainty given by
   - $\sqrt{E(\hat{x}_i - x_i})^2 = \sqrt{H}$

Problem Strategy: Step 1

Specify the main ingredients of simulated measurement system:

- Specify point spread function (influence function) a,
  - Symmetric
  - Asymmetric
- Covariance function ϕ for the signal x:

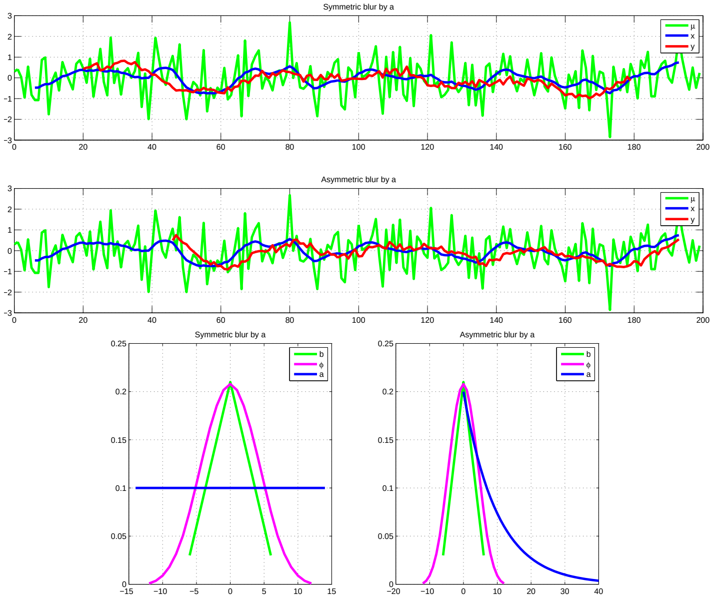
$$\phi = \mathrm{Cov}\,(x) = b * b^* \tag{1}$$

- Variance σ^2 of the independent components of the additive random noise ν

$$y = a * x + \nu, \;\; \nu \; (0, \sigma^2 \delta). \tag{2}$$

- Choose support interval $\Delta = [-d, d]$ or $\Delta = [-T, \tau]$

Signal and Covariance Setup



Problem Simulation

Specify the main ingredients of simulated measurement system:

- Finitely supported point spread function (influence function),
  - Symmetric case: $a_i = \frac{1}{10}$ for $|i| < 15$.
  - Asymmetric case: $a_i = \frac{2}{10}e^{-i/40}$ for $0 \leq i \leq 40$.
- The covariance function ϕ for the signal x is given by

$$\phi = \mathrm{Cov}\,(x) = b * b^*$$

  where $b_i = \frac{21}{100}(1 - |i|)$ for $|i| \leq 7$.
- Measurement noise is modeled with a zero mean Gaussian ν with a specified $\sigma^2 = \frac{1}{100}$.
- Finally, the data is given by

$$y = a * x + \nu$$

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Finitely Supported Function Data Type

- The fundamental data type in `Matlab` is the “matvec” whose operations are not “natural” for finitely supported discrete functions.
- `Matlab` supports object oriented programming which allows for the creation of a custom data type for which we can implement convolution as the natural multiplication.

```
classdef FinSupFun % Finite Support Function
    properties (SetAccess = private)
...
    end

    methods
...
    end
end
```

Finitely Supported Function Data Type

- From the demo codes provided, convolution and addition were "overrided" to allow statements like

```
mu = FinSupFun(randn(1,N),0); % Construct finitely
    supported white noise
x = b .* mu; % Inner convolution
y0 = a .* x;
y = y0 + FinSupFun(s*randn(size(y0.f)),y0.l);
phi = b*b'; % Outer convolution
p = a*phi*a' + FinSupFun(s^2);
q = phi*a';
```

- Note that the $*$ and $.*$ operations now represent convolution between `FinSupFun` objects.
- This makes code easier to read and debug.

Finitely Supported Function Data Type

- We added the following methods

```
function c = restricted_to(a,l,r) % Restrict support to [l,r], if [l,r] is
      bigger than [a.l,a.r], then pad with zeroes.
    L = max(l,a.l); % Left endpoint of restricted interval
    R = min(r,a.r); % Right endpoint of restricted interval
    ...
end

function c = mldivide(a,b) % \ De-convolution by constructing toeplitz matrix. a
      must be symmetric
  n = length(b.f);
  toeplitz_row = [a.f((a.r+1):end), zeros(1, n-(a.r))];% This needs to be from
       the center of p and padded with zeros
  ...
end
```

- so computing $P^{-1}q$ on a restriced interval are easily implemented as follows

```
q_delta1 = q.restricted_to(-d,d); % Restrict (or zero pad) to [-d,d].
q_delta2 = q.restricted_to(-T,tau); % Restrict (or zero pad) to [-T,tau].
r1 = p \ q_delta1;
r2 = p \ q_delta2;
```

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References

[1] Golubtsov, P. (2015). Theoretical Big Data Analytics course notes.