Field Theoretic Perspective on Thrust in Accelerating Frames

Kevin Player*

January 21, 2025

The Unruh effect tells us that what we call particles is really just a matter of perspective.

Lee Smolin

Abstract

We analyze the quantum field theoretic framework where Unruh radiation was first described, interpreting acceleration as a driving source influencing the field. By conceptualizing this source as thrust, we provide a more intuitive mechanism for understanding radiation in accelerating frames. Using the equivalence principle, we extend this framework to Hawking radiation as well, demonstrating how a common source-response relationship manifests in accelerating frames.

1 Introduction

TODO: Fix up the INTRO

We first introduce preliminaries, ideas and notations for discussing quantum field theory in an accelerating frame. Then we review the positive frequency modes in Rindler spacetime. Next we introduce the idea of field theoretic driving sources. We then extend this formalism using the Fourier transform, studying the sources as a combination of emission and absorption terms. Finally, we work out an interpretation of thrust as Unruh radiation, and use the equivalence principle to study thrust's role in Hawking radiation as well.

Much of this note including notation and conventions are drawn directly from the overview [1], from the original source [2], and from [3]. As in these references, we only consider the scalar free field which is enough for the Unruh effect to become manifest.

2 The Unruh Effect from Field Theoretic Thrust

In this section, we introduce a field-theoretic interpretation of thrust as a driving source for particle production in non-inertial frames. By examining the dynamics of these driving sources, we derive the Unruh vacuum expectation value, highlighting its connection to acceleration and quantum field theory. This approach establishes a conceptual link between thrust, external sources, and the emergence of thermal radiation in an accelerating frame.

*kplayer@andrew.cmu.edu

2.1 Rindler and Minkowski Modes Review

Let $\hbar = c = 1$ and consider a uniform linear acceleration in $1+1$ dimensional spacetime. The full $1+3$ dimensional case does not add anything to the discussion, so without loss of generality we stick to the dimensions time t and space z where the boost is taking place. The massless Klein-Gordon equation

$$\Box\phi = \frac{\partial^2\phi}{\partial t^2} - \frac{\partial^2\phi}{\partial z^2} = 0, \tag{1}$$

has solutions which we describe using a basis $u = -t + z$ and $v = -t - z$, the null coordinates along the light cone. Any function of purely u, $f(u)$, is made up of modes that are constant on v. That is they are given by a 1-dimensional Fourier expansion

$$f(u) = \frac{1}{2\pi}\int e^{ip_u u}\hat{f}(p_u)dp_u. \tag{2}$$

supported on $p_v = 0$. The same is true for v, $g(v)$, p_v, and $\hat{g}(v)$ supported on $p_u = 0$. The full space of solutions to equation (1) are combinations of f and g, linear combinations of $\hat{f}(p_u)\delta(p_v)$ and $\hat{g}(p_v)\delta(p_u)$. These are supported on the massless shell $E = \pm p$. A basis of solutions are the "Minkowski modes"

$$\frac{1}{\sqrt{2\omega_k(2\pi)^{n-1}}}e^{i(kx-\omega_k t)} \tag{3}$$

where $\omega^2 = k^2$, covering the positive and negative frequency cases.

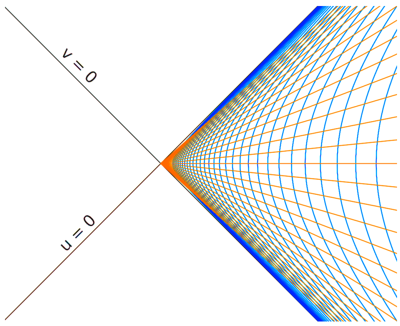

Figure 1: Rindler wedge W on the right.

The following notations and results, (4) through (12), are taken from [1]. Let W be the $z > |t|$ Rindler "wedge" with coordinates

$$t = \frac{1}{a}e^{a\xi}\sinh(a\eta) \tag{4}$$

$$z = \frac{1}{a}e^{a\xi}\cosh(a\eta) \tag{5}$$

where $a > 0$ is an acceleration parameter (see Figure 1). The massless Klein-Gordon equation in Rindler coordinates is

$$\Box\phi = e^{-2a\xi}(-\partial_\eta^2 + \partial_\xi^2)\phi = 0. \tag{6}$$

which has a basis of solutions

$$r_k = \frac{1}{\sqrt{4\pi\omega_k}}e^{-i(\omega_k\eta - k\xi)} \tag{7}$$

for each wave number k and positive frequency $\omega_k = |k|$. These "Rindler modes" do not immediately extend to the entire (x,t)-plane but are confined to the Rindler wedge W.

Consider the analytic functions on the entire (x,t)-plane

$$h_k^{(u)} = \frac{e^{\frac{\pi\omega_k}{2a}}(au)^{\frac{i\omega_k}{a}}}{\sqrt{2\sinh\left(\frac{\pi\omega_k}{a}\right)}} \tag{8}$$

$$h_k^{(v)} = \frac{e^{\frac{\pi\omega_k}{2a}}(av)^{\frac{i\omega_k}{a}}}{\sqrt{2\sinh\left(\frac{\pi\omega_k}{a}\right)}} \tag{9}$$

These two functions are functions of u and v respectively, and so are solutions to equation (1). Each r_k can be extended to the entire plane as

$$e^{\frac{\pi\omega_k}{2a}}h_k^{(u)} - e^{-\frac{\pi\omega_k}{2a}}h_k^{(v)} = \sqrt{2\sinh\left(\frac{\pi\omega_k}{a}\right)}r_k \tag{10}$$

Let $c_k^{(1)}$ and $c_k^{(2)}$ be annihilation and creation operators for the positive and negative frequency Minkowski modes respectively. Define

$$b_k = \frac{1}{\sqrt{2\sinh\left(\frac{\pi\omega_k}{a}\right)}}\left(e^{\frac{\pi\omega_k}{2a}}c_k^{(1)} + e^{-\frac{\pi\omega_k}{2a}}c_{-k}^{(2)\dagger}\right). \tag{11}$$

Then the number of particles that an accelerating observer will see in the Minkowski vacuum $|0_M\rangle$ is seen to be

$$\langle n\rangle = \langle 0_M|\, b_k^\dagger b_k\, |0_M\rangle = \frac{1}{e^{\frac{2\pi\omega_k}{a}} - 1} \tag{12}$$

which is usually interpreted as (Unruh) radiation [2].

2.2 Fourier Transform of the Sources

We will take several Fourier transforms to elucidate the various modes and set up for some integrals. Let ϕ be a scalar free field in the flat $1+1$ dimensional Minkowski spacetime. We will consider $h_k^{(u)}$ and $h_k^{(v)}$ as driving W-event horizon sources

$$\rho = \alpha h_k^{(u)} + \beta h_k^{(v)}. \tag{13}$$

with positive real convex combination $\alpha + \beta = 1$. The drivers $h_k^{(u)}$ and $h_k^{(v)}$ are functions $f(u)$ of the past W-horizon and $g(v)$ of the future W-horizon respectively. Both generate excitations, which we identify with absorption and emission thrusts respectively.

The sources can originate from a coupling term, $\rho\phi$, added to the free scalar free Lagrangian.

$$\mathcal{L}_{driven} = \mathcal{L}_{original} + \rho\phi \tag{14}$$

where

$$\mathcal{L}_{original} = -\frac{1}{2}\partial^\mu\phi\partial_\mu\phi - \frac{1}{2}\phi^2. \tag{15}$$

This leads to an inhomogeneous Klein-Gordon equation

$$\Box\phi = \rho \tag{16}$$

as presented in [3][1]

We want to integrate ρ on shell in momentum space, which for a massless source is the positive energy part of the massless shell. The two positive energy "horizons" border

[1]In [3] it is assumed that the source is only active for a finite amount of time. We let ρ be active for all time. The argument in [3] seem to be adaptable to ρ.

$p_u <= 0$ and $p_v <= 0$, see Figure 2. Proceeding to take the Fourier transform of the function $f(u)^2$, we drop p_u to just p for the time being to increase legibility. We will continue to assume that ω_k and a are positive. Define the kernel

$$A = e^{-ipu}(au)^{\frac{i\omega_k}{a}} du \tag{17}$$

and then we have

$$\hat{f}(\_u) = \frac{e^{\frac{\pi\omega_k}{2a}}}{\sqrt{2\sinh\left(\frac{\pi\omega_k}{a}\right)}} \int_{-\infty}^{\infty} A \tag{18}$$

where $L = \int_{-\infty}^{0} A$ and $R = \int_{0}^{\infty} A$ are the left and right sides of the total integral $L + R$.

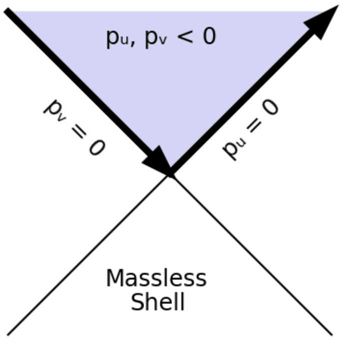

Figure 2: Massless shell is when $p_u = p_v = 0$

We rewrite the integrals using a complex changes of variables, $s = ipu$, and contour integrals.

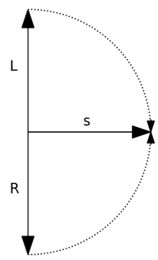

Figure 3: Using contours with large radius we convert the L integral that goes to $i\infty$, and the R integral that goes to $-i\infty$, to integrals with s going to real ∞.

The L integral for real $p < 0$ is

$$\begin{aligned}
L(p) &= -\int_0^{i\infty} \left(\frac{ias}{-p}\right)^{\frac{i\omega_k}{a}} \left(\frac{i}{-p}\right) ds \\
&= \frac{-i}{a}\left(\frac{a}{-p}\right)^{\frac{i\omega_k}{a}+1} \int_0^{\infty} (is)^{\frac{i\omega_k}{a}} e^{-s} ds \\
&= \frac{-i}{a}\left(\frac{a}{-p}\right)^{\frac{i\omega_k}{a}+1} \Gamma\left(\frac{i\omega_k}{a}+1\right) e^{-\frac{\pi\omega_k}{2a}} \\
&= \frac{1}{2}\Gamma\left(\frac{i\omega_k}{a}+1\right) e^{-\frac{\pi\omega_k}{2a}} B(p)
\end{aligned} \tag{19}$$

[2] WLOG since $g(v)$ is of the same form.

where

$$B(p) = \frac{-2i}{a}\left(\frac{a}{-p}\right)^{\frac{i\omega_k}{a}+1} \tag{20}$$

is as shown. This is using a large radius contour which rotates the endpoint 90 degrees clockwise.

The same calculation for R is done using a counter-clockwise contour this time.

$$R(p) = \frac{1}{2}\Gamma\left(\frac{i\omega_k}{a}+1\right)e^{-\frac{\pi\omega_k}{2a}}B(p) \tag{21}$$

We get back to $h_k^{(u)}$ and apply the normalization from equation (18)

$$\hat{h_k}^{(u)}(p_u) = \frac{e^{\frac{\pi\omega_k}{2a}}}{\sqrt{2\sinh\left(\frac{\pi\omega_k}{a}\right)}}(L(p_u)+R(p_u)) \tag{22}$$

So

$$\begin{aligned}
\hat{h_k}^{(u)}(p_u) &= \frac{\Gamma\left(\frac{i\omega_k}{a}+1\right)}{\sqrt{2\sinh\left(\frac{\pi\omega_k}{a}\right)}}B(p_u) \\
\hat{h_k}^{(v)}(p_v) &= \frac{\Gamma\left(\frac{i\omega_k}{a}+1\right)}{\sqrt{2\sinh\left(\frac{\pi\omega_k}{a}\right)}}B(p_u)
\end{aligned} \tag{23}$$

2.3 Interpretation

The driving source ρ, with mixed absorption and emission thrusts $\alpha+\beta=1$, contribute excitations to the scalar field ϕ. Equations (13) and (23) let us write down the expected change of energy

$$\begin{aligned}
\mathbb{E}[\Delta E] &= \frac{1}{4\pi}\int|\rho(p)|^2dp \\
&= \frac{\alpha}{4\pi}\int\left|\hat{h_k}^{(u)}(p_u)\right|^2dp_u + \frac{\beta}{4\pi}\int\left|\hat{h_k}^{(v)}(p_v)\right|^2dp_v \\
&= \frac{\left|\Gamma\left(\frac{i\omega_k}{a}+1\right)\right|^2}{2\sinh\left(\frac{\pi\omega_k}{a}\right)}\frac{1}{4\pi}\int|B(p)|^2dp \\
&= \frac{\left|\Gamma\left(\frac{i\omega_k}{a}+1\right)\right|^2}{2\pi a\sinh\left(\frac{\pi\omega_k}{a}\right)}\int a/|p|^2dp \\
&= I(\omega_k)P
\end{aligned} \tag{24}$$

where the integrals are on the positive energy massless shell with contributions from p_u on the left piece and p_v on the right piece. We factored out $P=\int a/|p|^2$ with a remaining p independent positive real coefficient $I(\omega_k)$.

Without being more careful we end up with inferred problems — The integrals do not converge at zero, where P explodes. But this infinity cancels when we compare the spectral radiances to each other, $I(\omega_{k_1})/I(\omega_{k_2})$.

The magnitude of our Gamma function has known asymptotics [5, Eq. 5.11.9]

$$\left|\Gamma\left(\frac{i\omega_k}{a}+1\right)\right|^2 \sim \left(\frac{2\pi\omega_k}{a}\right)e^{-\frac{\pi\omega_k}{a}} \tag{25}$$

Plugging this into equation (24) we find the average energy of the mode, the $1+1$ dimensional Planck distribution function, and thus recover the Unruh's radiation spectrum from a thrust driven field.

$$\frac{1}{P}\mathbb{E}[\Delta E] = I(\omega_k) \sim \frac{\omega_k}{e^{\frac{2\pi\omega_k}{a}} - 1} \tag{26}$$

Compare this to equation (12) and references [2] and [1].

3 Conclusion and Prediction

If we do not account for the thrust required to accelerate a detector, then we recover it instead as a thermal unknown in the vacuum, Unruh radiation. But, it would seem that we can explain Unruh radiation directly using thrust in the illustrative ways outlined in this paper. The prediction of this note is that Unruh radiation and Hawking-Bekenstein radiation, separate from thrust, should not appear.

4 Acknowledgments

Thanks to Ben Commeau and Daniel Justice for useful discussions.

References

[1] E. Frodden and N. Valdés, "Unruh effect: Introductory notes to quantum effects for accelerated observers," *International Journal of Modern Physics A*, vol. 33, p. 1830026, sep 2018.

[2] W. G. Unruh, "Notes on black-hole evaporation," *Phys. Rev. D*, vol. 14, pp. 870–892, Aug 1976.

[3] E. Zurich, "Chapter 3 quantum field theory," 2012.

[4] J.-L. Thiffeault, M. Purcell, and Y. Correll, "What a rindler observer sees in a minkowski vacuum," *Physics Letters A*, 05 1998.

[5] "*NIST Digital Library of Mathematical Functions.*" http://dlmf.nist.gov/, Release 1.1.8 of 2022-12-15. F. W. J. Olver, A. B. Olde Daalhuis, D. W. Lozier, B. I. Schneider, R. F. Boisvert, C. W. Clark, B. R. Miller, B. V. Saunders, H. S. Cohl, and M. A. McClain, eds.