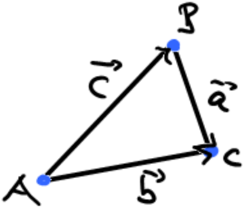
Exercise 1

A triangle (face) in $\mathbb{R}^3$ consists of

- A vertex (point) $A = (1, 0, 1)$
- Two edges (sides) $\vec{b} = (1, 0, 1)$ and $\vec{c} = (-1, 2, 1)$ emanating from A

Find

- the two other vertices B and C
- the edge $\vec{a}$ between B and C



$$B = A + \vec{c} = (0, 2, 2)$$

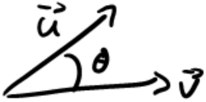
$$C = A + \vec{b} = (2, 0, 2)$$

$$\vec{a} = \overline{BC} = (2, -2, 0)$$

Exercise 2

a) What's the angle between the vectors $\vec{u} = (2, -1, 1)$ and $\vec{v} = (1, 1, 2)$?

b) What's the angle between the diagonal and an edge of a cube in $\mathbb{R}^3$?

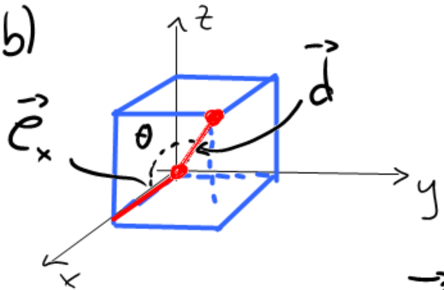


a) $\vec{u} \cdot \vec{v} = |\vec{u}||\vec{v}| \cos(\theta)$

$$\Rightarrow \cos(\theta) = \frac{\vec{u} \cdot \vec{v}}{|\vec{u}||\vec{v}|} = \frac{3}{\sqrt{6} \cdot \sqrt{6}} = 1/2$$

$$\Rightarrow \theta = \pi/3 \mathrel{\hat{=}} 60^\circ$$

b)



Unit vector pointing along diagonal

$$\vec{d} = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

Angle θ between $\vec{d}$ and edge $\vec{e}_x$:

$$\vec{d} \cdot \vec{e}_x = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{3}} = |\vec{d}||\vec{e}_x| \cos(\theta)$$

$$\Rightarrow \theta = 0{,}577 \mathrel{\hat{=}} 54{,}7^\circ$$

Exercise 3

Decompose $\vec{w} = (1, 2, 3)$ as $\vec{w}_\perp + \vec{w}_\|$, where $\vec{w}_\perp$ is perpendicular and $\vec{w}_\|$ along $\vec{v} = (2, -1, -2)$.
Check your result using the scalar product.

$$|\vec{v}| = 3 \Rightarrow \vec{e}_v = \frac{1}{3}(2, -1, -2), \quad \vec{e}_v \cdot \vec{w} = -2$$

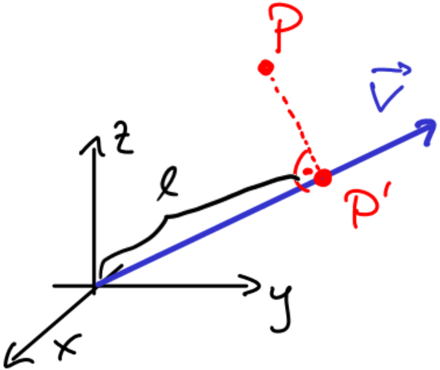
$$\Rightarrow \vec{w}_\| = (\vec{e}_v \cdot \vec{w})\vec{e}_v = \frac{2}{3}(-2, 1, 2)$$

$$\vec{w}_\perp = \vec{w} - \vec{w}_\| = \left(\frac{7}{3}, \frac{4}{3}, \frac{5}{3}\right)$$

check: $\vec{v} \cdot \vec{w}_\perp = \frac{1}{3}(2 \cdot 7 - 1 \cdot 4 - 2 \cdot 5) = 0$ ✓

Exercise 4

What's the orthogonal projection P' of the point $P = (2, -1, 3)$ onto the vector $\vec{v} = (4, -1, 2)$?



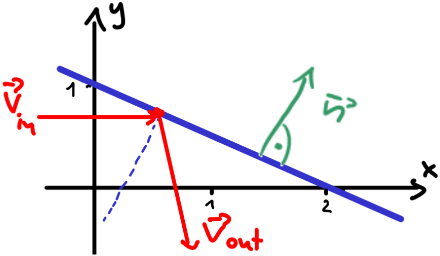
$$\overline{OP'} = l \cdot \vec{e}_v$$

$$\vec{e}_v = \frac{\vec{v}}{|\vec{v}|} = \frac{1}{\sqrt{21}} \begin{pmatrix} 4 \\ -1 \\ 2 \end{pmatrix}$$

$$l = \vec{e}_v \cdot \overline{OP} = \frac{1}{\sqrt{21}} (2, -1, 3) \begin{pmatrix} 4 \\ -1 \\ 2 \end{pmatrix} = \frac{15}{\sqrt{21}}$$

$$\Rightarrow \overline{OP'} = \frac{15}{\sqrt{21}} \cdot \frac{1}{\sqrt{21}} \begin{pmatrix} 4 \\ -1 \\ 2 \end{pmatrix} = \frac{15}{21} \begin{pmatrix} 4 \\ -1 \\ 2 \end{pmatrix}$$

Exercise 5



What is $\vec{v}_{\text{out}}$ for $\vec{v}_{\text{in}} = (1, 0)$?

Find out $\vec{n}$:

Unit vector parallel to blue line $\vec{t} = \frac{1}{\sqrt{5}}\begin{pmatrix} 2 \\ -1 \end{pmatrix}$

$\Rightarrow \vec{n} = \frac{1}{\sqrt{5}}\begin{pmatrix} 1 \\ 2 \end{pmatrix} \Leftrightarrow \vec{n} \cdot \vec{t} = 0$

$$\vec{v}_{in} \cdot \vec{n} = (1,0) \cdot \frac{1}{\sqrt{5}} \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \frac{1}{\sqrt{5}}$$

$$\Rightarrow \vec{v}_{out} = \vec{v}_{in} - 2(\vec{v}_{in} \cdot \vec{n})\,\vec{n}$$

$$= \begin{pmatrix} 1 \\ 0 \end{pmatrix} - 2 \cdot \frac{1}{\sqrt{5}} \cdot \frac{1}{\sqrt{5}} \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$$= \begin{pmatrix} 1 \\ 0 \end{pmatrix} - \frac{2}{5} \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 3/5 \\ -4/5 \end{pmatrix}$$

Exercise 6

- Find a unit vector $\vec{u}$ that is perpendicular to $\vec{v} = (2, -1, 1)$ and $\vec{w} = (1, 1, 2)$.
- Check your result by calculating $\vec{u} \cdot \vec{v}$ and $\vec{u} \cdot \vec{w}$.

$$\vec{v} \times \vec{w} = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} \times \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} -3 \\ -3 \\ 3 \end{pmatrix}$$

Normalize: $\vec{u} = \frac{1}{\sqrt{27}} \begin{pmatrix} -3 \\ -3 \\ 3 \end{pmatrix} = \frac{1}{\sqrt{3}} \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}$

check: $\vec{u} \cdot \vec{v} = \frac{1}{\sqrt{3}} (-1, -1, 1) \cdot \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} = 0$ ✓

$\vec{u} \cdot \vec{w} = \frac{1}{\sqrt{3}} (-1, -1, 1) \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} = 0$ ✓

Exercise 7

Calculate the following products:

$$\underbrace{\begin{pmatrix} 1 & 1 \\ 2 & 0 \end{pmatrix} \cdot \begin{pmatrix} 2 & 0 \\ 1 & 3 \end{pmatrix}}_{= \begin{pmatrix} 3 & 3 \\ 4 & 0 \end{pmatrix}} = ?\,, \qquad \underbrace{\begin{pmatrix} 2 & 0 \\ 1 & 3 \end{pmatrix} \cdot \begin{pmatrix} 1 & 1 \\ 2 & 0 \end{pmatrix}}_{= \begin{pmatrix} 2 & 2 \\ 7 & 1 \end{pmatrix}} = ?$$

$\Rightarrow$ The matrix product is non-commutative!

Exercise 8

Consider the matrix $\mathbf{A} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$

- Work out the inverse matrix $\mathbf{A}^{-1}$.
- Check the relation $\mathbf{A} \cdot \mathbf{A}^{-1} = \mathbf{E}$.

$$A^{-1} = \frac{1}{1 \cdot 4 - 2 \cdot 3} \begin{pmatrix} 4 & -2 \\ -3 & 1 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} -4 & 2 \\ 3 & -1 \end{pmatrix}$$

check: $A^{-1} \cdot A = \frac{1}{2} \begin{pmatrix} -4 & 2 \\ 3 & -1 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \checkmark$$