Intro to Computer Graphics
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Problem sheet for chapter 7: Linear maps and transformation matrices

Problem 1:

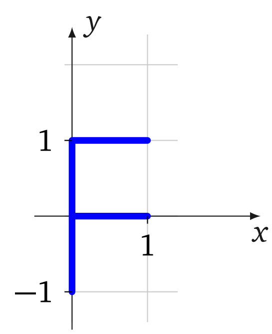
Consider the following maps $\mathbb{R}^2 \to \mathbb{R}^2$:

$$A_1 : \begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} -y \\ x \end{pmatrix} \qquad A_2 : \begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} x+1 \\ y \end{pmatrix} \qquad A_3 : \begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} x+y \\ y \end{pmatrix}$$

(a) Which of the maps are linear?

(b) For the linear maps: Write down the representation matrices.

(c) For the linear maps: Draw pictures how they transform the letter F:

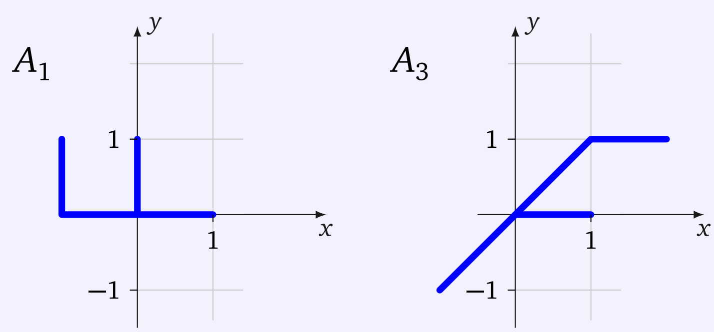


Lösung:

(a) All maps except A_2 are linear.

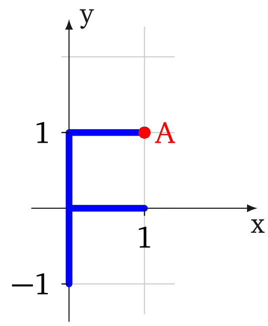
(b) $A_1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, $\quad A_3 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$

(c) Images of F under the linear maps:



Problem 2:

Consider the letter F and the two transforms **R** and **S** shown below:



$$\mathbf{S} = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & 2 \end{pmatrix}, \qquad \mathbf{R} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$$

(a) Draw the letter F after the transform $\mathbf{R} \cdot \mathbf{S}$ has been applied to it. What are the coordinates of the transformed point A?

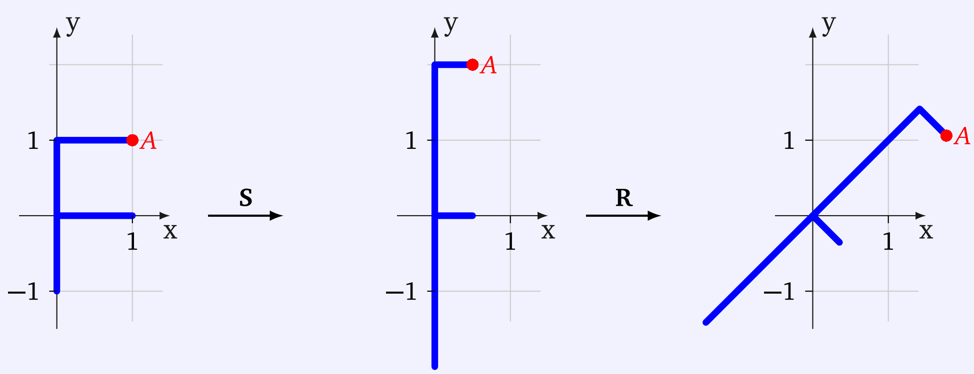
(b) Draw the letter F after the transform $\mathbf{S} \cdot \mathbf{R}$ has been applied to it. What are the coordinates of the transformed point A?

Lösung:

$\mathbf{S}$ is a scaling transform, $\mathbf{R}$ is a rotation by 45°.

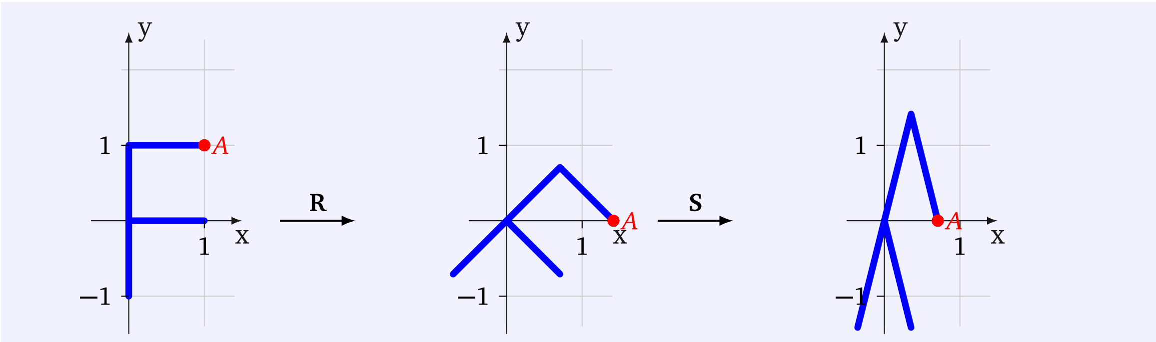
(a) The point $A = (1, 1)$ gets transformed as follows:

$$R \cdot S \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} \frac{1}{2} & 2 \\ -\frac{1}{2} & 2 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{1}{2\sqrt{2}} \begin{pmatrix} 5 \\ 3 \end{pmatrix} \approx \begin{pmatrix} 1.7678 \\ 1.0607 \end{pmatrix}$$



(b) The point $A = (1, 1)$ gets transformed as follows:

$$S \cdot R \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ -2 & 2 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \approx \begin{pmatrix} 0.7071 \\ 0 \end{pmatrix}$$



Problem 3:

Consider the rotation matrix $\mathbf{R}(\varphi) = \begin{pmatrix} \cos(\varphi) & -\sin(\varphi) \\ \sin(\varphi) & \cos(\varphi) \end{pmatrix}$.

(a) Verify that it has unit determinant, i.e. $\det(\mathbf{R}(\varphi)) = 1$,

(b) Verify that it is orthogonal: $\mathbf{R}(\varphi)^T = \mathbf{R}(\varphi)^{-1}$

(c) Verify that the composition of two rotations is a rotation by the sum of the angles, i.e.

$$\mathbf{R}(\varphi) \cdot \mathbf{R}(\theta) = \mathbf{R}(\varphi + \theta)$$

Hint: Use the addition theorems $\sin(\varphi+\theta) = \sin(\varphi)\cos(\theta) + \cos(\varphi)\sin(\theta)$ and $\cos(\varphi+\theta) = \cos(\varphi)\cos(\theta) - \sin(\varphi)\sin(\theta)$.

Lösung:

(a) $\det(\mathbf{R}(\varphi)) = \cos^2(\varphi) + \sin^2(\varphi) = 1$

(b) It is orthogonal because

$$\mathbf{R}^T(\varphi) \cdot \mathbf{R}(\varphi) = \begin{pmatrix} \cos(\varphi) & \sin(\varphi) \\ -\sin(\varphi) & \cos(\varphi) \end{pmatrix} \cdot \begin{pmatrix} \cos(\varphi) & -\sin(\varphi) \\ \sin(\varphi) & \cos(\varphi) \end{pmatrix} = \mathbf{E}$$

implying that $\mathbf{R}(\varphi)^T = \mathbf{R}(\varphi)^{-1}$.

(c) $\mathbf{R}(\varphi) \cdot \mathbf{R}(\theta) = \begin{pmatrix} \cos(\varphi) & -\sin(\varphi) \\ \sin(\varphi) & \cos(\varphi) \end{pmatrix} \cdot \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}$

$$= \begin{pmatrix} \cos(\varphi)\cos(\theta) - \sin(\varphi)\sin(\theta) & -\sin(\varphi)\cos(\theta) - \cos(\varphi)\sin(\theta) \\ \sin(\varphi)\cos(\theta) + \cos(\varphi)\sin(\theta) & \cos(\varphi)\cos(\theta) - \sin(\varphi)\sin(\theta) \end{pmatrix} = \begin{pmatrix} \cos(\varphi+\theta) & -\sin(\varphi+\theta) \\ \sin(\varphi+\theta) & \cos(\varphi+\theta) \end{pmatrix}$$

Problem 4:

Consider the following two vectors in $\mathbb{R}^3$:

$$\vec{v} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \qquad \vec{w} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

(a) What is the angle between $\vec{v}$ and $\vec{w}$.

(b) Find a 3×3 rotation matrix $\mathbf{R}$ that rotates $\vec{v}$ into $\vec{w}$, i.e. satisfies $\vec{w} = \mathbf{R} \cdot \vec{v}$.

(c) Check that $\mathbf{R}$ is orthogonal and has a determinant of 1.

(d) Find a matrix that rotates $\vec{w}$ into $\vec{v}$.

Lösung:

(a)

$$\vec{v} \cdot \vec{w} = 1 = |\vec{v}| \cdot |\vec{w}| \cos(\theta) = 2\cos(\theta) \implies \theta = \frac{\pi}{3}$$

(b) The axis of rotation $\vec{u}$ is perpendicular to both vectors, i.e. can be computed as cross product:

$$\vec{v} \times \vec{w} = \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix} \implies \vec{u} = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}$$

With $c := \cos(\theta) = \frac{1}{2}$ and $s := \sin(\theta) = \frac{\sqrt{3}}{2}$ we get

$$s u_k = \pm\frac{1}{2}, \qquad (1-c)u_k^2 + c = \frac{2}{3}, \qquad (1-c)u_k u_j = \pm\frac{1}{6}$$

The axis-angle formula then gives:

$$\mathbf{R} = \begin{pmatrix} \frac{2}{3} & -\frac{1}{6} + \frac{1}{2} & -\frac{1}{6} - \frac{1}{2} \\ -\frac{1}{6} - \frac{1}{2} & \frac{2}{3} & \frac{1}{6} - \frac{1}{2} \\ -\frac{1}{6} + \frac{1}{2} & \frac{1}{6} + \frac{1}{2} & \frac{2}{3} \end{pmatrix} = \frac{1}{3} \begin{pmatrix} 2 & 1 & -2 \\ -2 & 2 & -1 \\ 1 & 2 & 2 \end{pmatrix}$$

One can check that it satisyfies $\mathbf{R} \cdot \vec{v} = \vec{w}$.

(c) To check orthogonality one calculates $\mathbf{R} \cdot \mathbf{R}^T = \mathbf{E}$.

(d) Because of orthogonality we have $\mathbf{R}^T \cdot \vec{w} = \vec{v}$ which can easily be checked.