Chapter 2: Vectors and Matrices

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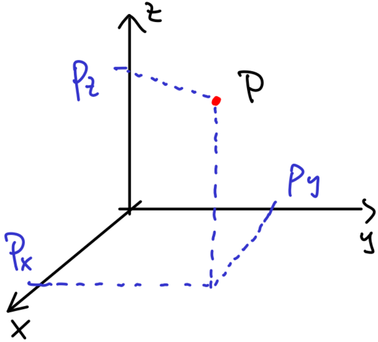
Content

Points and vectors

- $\mathbb{R}^2$: plane
- $\mathbb{R}^3$: space
- $\mathbb{R}^N$: N-dimensional space

Points in $\mathbb{R}^N$:

- A point P in $\mathbb{R}^N$ is specified by N coordinates: $P = (p_1, p_2, \ldots, p_N)$
- Different points have different coordinates

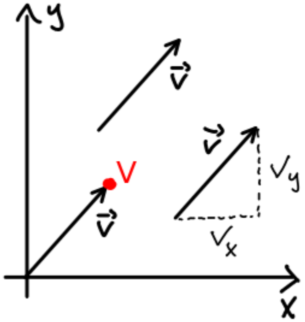


Points and vectors

Vectors in $\mathbb{R}^N$:

- A vector is a set of parallel arrows with
  - same lengths
  - same directions
- It is specified by N components

$$\vec{v} = (v_1, v_2, \ldots, v_N)$$



- Each point V defines a vector $\vec{v} = \overline{OV}$: arrow between origin and V

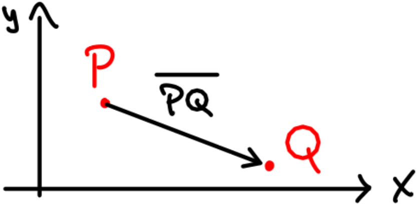
$$\overline{OV} = (v_1, v_2, \ldots v_N).$$

- $\overline{OV}$ is called *position vector* of V.

Points and vectors

Relation between points and vectors:

- Difference between two points P and Q: vector $\overline{PQ}$ with coordinates: $(q_1 - p_1, \ldots, q_N - p_N)$



- Rules:
  - point P + vector $\vec{v}$ = point Q
    coordinates of Q: $(p_1 + v_1, p_2 + v_2, \ldots p_N + v_N)$
  - vector + vector = vector
  - point + point: undefined

Exercise 1

A triangle (face) in $\mathbb{R}^3$ consists of

- A vertex (point) $A = (1, 0, 1)$
- Two edges (sides) $\vec{b} = (1, 0, 1)$ and $\vec{c} = (-1, 2, 1)$ emanating from A

Find

- the two other vertices B and C
- the edge $\vec{a}$ between B and C

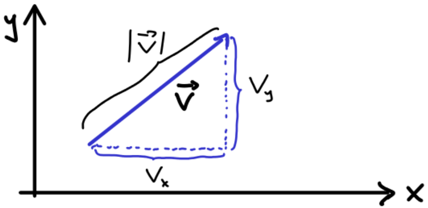
Elementary vector operations

1.) *Transposition*: Exchanging rows and columns

$$\begin{pmatrix} v_1 \\ \vdots \\ v_N \end{pmatrix}^T = (v_1, \ldots, v_N)\,, \qquad (v_1, \ldots, v_N)^T = \begin{pmatrix} v_1 \\ \vdots \\ v_N \end{pmatrix}$$

2.) *Length* of a vector (Pythagorean theorem):

$$|\vec{v}| = \sqrt{\sum_{k=1}^{N} v_k^2} = \sqrt{v_1^2 + \ldots + v_N^2}$$

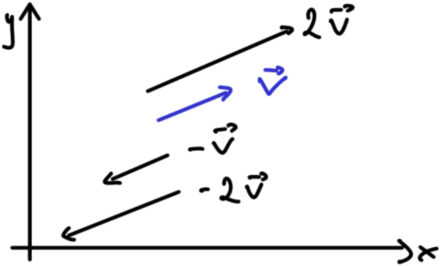


Elementary vector operations

3.) *Multiplication* with a scalar λ:

$$\lambda\vec{v} = (\lambda v_1, \dots, \lambda v_N)$$

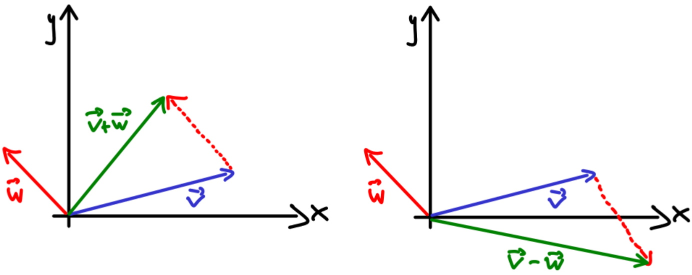
- changes length of $\vec{v}$
- leaves direction unchanged (for $\lambda > 0$)
- flips direction (for $\lambda < 0$)



Elementary vector operations

4.) *Vector addition* :

$$\vec{v} + \vec{w} = (v_1 + w_1, \ldots, v_N + w_N)$$

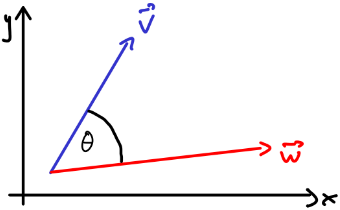


Scalar product

Two important versions:

$$\vec{v} \cdot \vec{w} = v_1 w_1 + \ldots + v_N w_N = \sum_{k=1}^{N} v_k w_k$$

$$\vec{v} \cdot \vec{w} = |\vec{v}||\vec{w}| \cos(\theta)$$



Special cases:

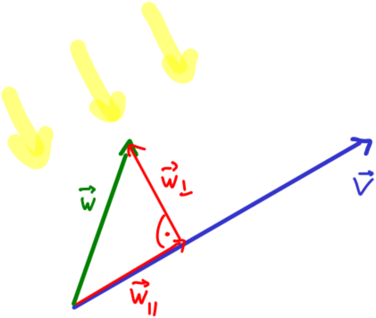
- $\vec{v}$ and $\vec{w}$ perpendicular $\iff$ $\vec{v} \cdot \vec{w} = 0$
- $\vec{v} \cdot \vec{v} = |\vec{v}|^2$ (because $\theta = 0$)

Exercise 2

- What's the angle between the vectors $\vec{u} = (2, -1, 1)$ and $\vec{v} = (1, 1, 2)$?
- What's the angle between the diagonal and an edge of a cube in $\mathbb{R}^3$?

Scalar product

Scalar product as projection:



- $\vec{w} = \vec{w}_{||} + \vec{w}_\perp$
- $|\vec{w}_{||}|$: length of orthogonal projection of $\vec{w}$ onto direction of $\vec{v}$.

$$\vec{v} \cdot \vec{w} = \vec{v} \cdot \left(\vec{w}_{||} + \vec{w}_\perp\right) = \underbrace{\vec{v} \cdot \vec{w}_{||}}_{|\vec{v}||\vec{w}_{||}|} + \underbrace{\vec{v} \cdot \vec{w}_\perp}_{0} = |\vec{v}||\vec{w}_{||}|$$

$$\boxed{\vec{v} \cdot \vec{w} = |\vec{v}| \text{ times length of projection of } \vec{w} \text{ on } \vec{v}}$$

Scalar product

Unit vector: vector with length 1

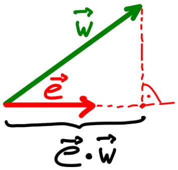
- Given some vector $\vec{v}$

$$\vec{e}_v := \frac{\vec{v}}{|\vec{v}|}$$

  is called unit vector along $\vec{v}$

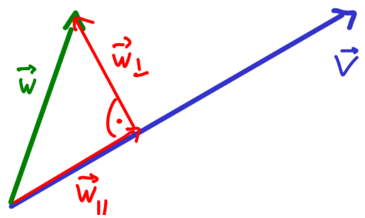
  - Check: $\left|\vec{e}_v\right| := \left|\frac{\vec{v}}{|\vec{v}|}\right| = \frac{|\vec{v}|}{|\vec{v}|} = 1$
  - Unit vectors just carry directional information

- $\vec{e} \cdot \vec{w}$: projection of some vector $\vec{w}$ into the direction of a unit vector $\vec{e}$.



Scalar product

Decomposition of vectors:



Explicit formulas for decomposing $\vec{w}$ along and perpendicular to $\vec{v}$:

$$|\vec{w}_{||}| = \vec{e}_v \cdot \vec{w} \implies \vec{w}_{||} = (\vec{e}_v \cdot \vec{w})\vec{e}_v = \frac{\vec{v} \cdot \vec{w}}{\vec{v}^2}\vec{v}$$

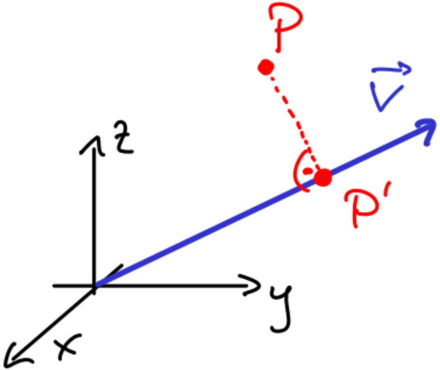
$$\vec{w}_\perp = \vec{w} - \vec{w}_{||}$$

Exercise 3

Decompose $\vec{w} = (1, 2, 3)$ as $\vec{w}_\perp + \vec{w}_\|$, where $\vec{w}_\perp$ is perpendicular and $\vec{w}_\|$ along $\vec{v} = (2, -1, -2)$.
Check your result using the scalar product.

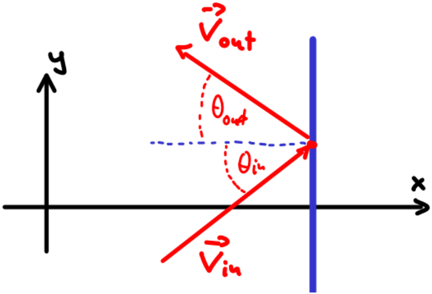
Exercise 4

What's the orthogonal projection P' of the point $P = (2, -1, 3)$ onto the vector $\vec{v} = (4, -1, 2)$?



Application: specular reflection

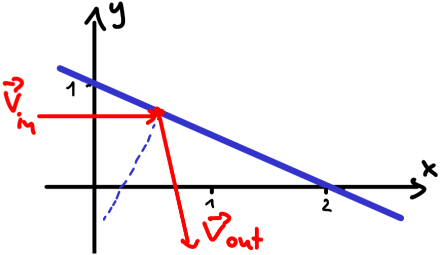
Specular reflection: *angle of incidence equals the angle of reflection*: $\theta_{\text{in}} = \theta_{\text{out}}$



Simple example:

- Reflection plane parallel to *y*-axis.
- What is $\vec{v}_{\text{out}}$ for $\vec{v}_{\text{in}} = (1.1, 1)$? $\vec{v}_{\text{out}} = (-1.1, 1)$

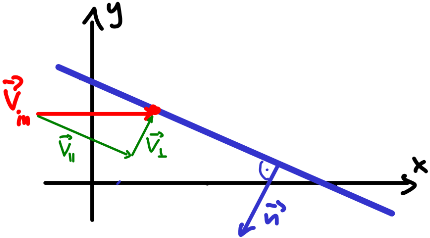
Exercise 5



What is $\vec{v}_{\text{out}}$ for $\vec{v}_{\text{in}} = (1, 0)$?

Application: specular reflection

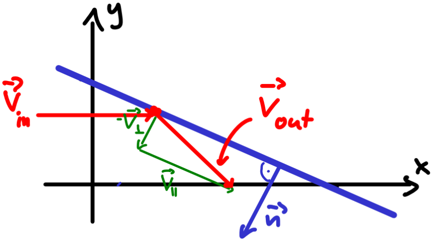
General case: reflection plane is characterized by unit normal vector $\vec{n}$.



$\vec{v}_{\text{in}}$ can be decomposed into $\vec{v}_{\text{in}} = \vec{v}_{\parallel} + \vec{v}_{\perp}$

- Perpendicular component: $\vec{v}_{\perp} = \left(\vec{v}_{\text{in}} \cdot \vec{n}\right) \vec{n}$
- Reflection turns $\vec{v}_{\perp}$ into $-\vec{v}_{\perp}$!
- $\vec{v}_{\parallel}$ remains unchanged!

Application: specular reflection



Reflected vector: $\vec{v}_{\text{out}} = \vec{v}_{\parallel} - \vec{v}_{\perp}$
Express in terms of $\vec{v}_{\text{in}}$ and $\vec{n}$:

$$\begin{aligned}
\vec{v}_{\text{out}} &= \underbrace{\vec{v}_{\text{in}} - \vec{v}_{\perp}}_{\vec{v}_{\parallel}} - \vec{v}_{\perp} \\
&= \vec{v}_{\text{in}} - 2\vec{v}_{\perp} = \vec{v}_{\text{in}} - 2\left(\vec{v}_{\text{in}} \cdot \vec{n}\right)\vec{n}
\end{aligned}$$

Consistency: $\vec{v}_{\text{out}}^2 = \vec{v}_{\text{in}}^2$ (check this!)

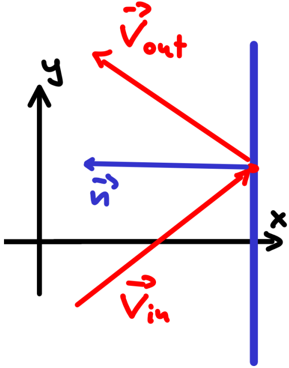
Application: specular reflection

Specular reflection law:

$$\vec{v}_{\text{out}} = \vec{v}_{\text{in}} - 2\left(\vec{v}_{\text{in}} \cdot \vec{n}\right)\vec{n}$$

with

- $\vec{v}_{\text{in}}$ incoming vector
- $\vec{v}_{\text{out}}$ outgoing vector
- $\vec{n}$: unit normal of reflection plane



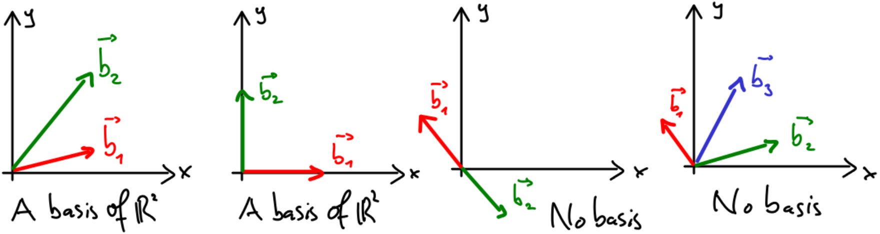
- Magnitude of speed does not change: $|\vec{v}_{\text{out}}| = |\vec{v}_{\text{in}}|$
- valid in any dimension (not just 2D)
- No need to calculate any angles
- See also example 3.13 in *Immersive Math*, Chapter 3.7 `http://immersivemath.com/ila/index.html`

Basis of $\mathbb{R}^N$

A set of N vectors $\vec{b}_1, \ldots, \vec{b}_N \in \mathbb{R}^N$ is called a *basis* if any vector $\vec{v} \in \mathbb{R}^N$ can be written as linear combination

$$\vec{v} = v_1 \vec{b}_1 + \ldots + v_N \vec{b}_N$$

$v_1, \ldots, v_N$ are components of $\vec{v}$ with respect to that basis.



Basis of $\mathbb{R}^N$

Important special cases:

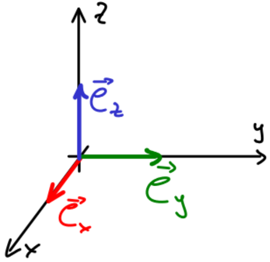
- *Orthogonal* basis: all $\vec{b}_k$ are pairwise perpendicular, i.e. $\vec{b}_k \cdot \vec{b}_l = 0$ for $k \neq l$.
- *Normal* basis: all $\vec{b}_k$ are unit vectors.
- *Orthonormal* basis: Basis is orthogonal and normal

Standard orthonormal basis of $\mathbb{R}^3$:

$$\vec{e}_x \,,\ \vec{e}_y \,,\ \vec{e}_z$$

Different notation:

$$\vec{e}_1 \,,\ \vec{e}_2 \,,\ \vec{e}_3$$



Basis of $\mathbb{R}^N$

Component notation (here in $\mathbb{R}^3$):

$$\vec{e}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} , \ \vec{e}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} , \ \vec{e}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

An arbitrary vector is a linear combination of basis vectors

$$\begin{aligned} \vec{v} &= \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = v_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + v_2 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + v_3 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \\ &= v_1 \vec{e}_1 + v_2 \vec{e}_2 + v_3 \vec{e}_3 \end{aligned}$$

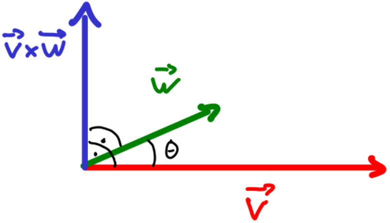
Calculation of components: $v_k = \vec{e_k} \cdot \vec{v}$
v_k: length of projection of $\vec{v}$ onto basis vector $\vec{e}_k$

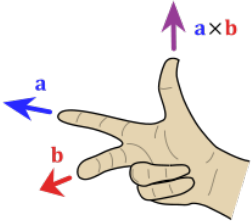
The vector product

Let $\vec{v}, \vec{w} \in \mathbb{R}^3$ be two vectors including an angle θ. The *vector product* $\vec{v} \times \vec{w}$ is a vector

- whose length is $|\vec{v}|\,|\vec{w}|\sin(\theta)$,
- which is perpendicular to $\vec{v}$ and $\vec{w}$,
- which forms a right-handed system with $\vec{v}$ and $\vec{w}$.

- The vector product is special to $\mathbb{R}^3$!
- The vector product is also known as *cross product*.



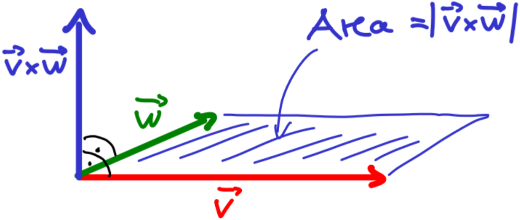


Right-Hand-Rule (From: Wikipedia)

The vector product

Properties of the vector product:

- The vector product is unique to $\mathbb{R}^3$.
- Due to orientation the vector product is anti-commutative: $\vec{v} \times \vec{w} = -\vec{w} \times \vec{v}$
- $\vec{v} \times \vec{w} = \vec{0} \iff \vec{v}$ and $\vec{w}$ are parallel.
- $\left|\vec{v} \times \vec{w}\right|$ is the area of the parallelogram spanned by $\vec{v}$ and $\vec{w}$.



The vector product

Calculating the vector product: The definition implies

$$\vec{e}_1 \times \vec{e}_2 = \vec{e}_3 \ , \ \vec{e}_1 \times \vec{e}_3 = -\vec{e}_2 \ , \ \vec{e}_2 \times \vec{e}_3 = \vec{e}_1 \, .$$

Applying this to

$$\vec{v} = v_1 \vec{e}_1 + v_2 \vec{e}_2 + v_3 \vec{e}_3 \ , \ \vec{w} = w_1 \vec{e}_1 + w_2 \vec{e}_2 + w_3 \vec{e}_3$$

gives

$$\boxed{\vec{v} \times \vec{w} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} \times \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} = \begin{pmatrix} v_2 w_3 - v_3 w_2 \\ v_3 w_1 - v_1 w_3 \\ v_1 w_2 - v_2 w_1 \end{pmatrix}}$$

Exercise 6

- Find a unit vector $\vec{u}$ that is perpendicular to $\vec{v} = (2, -1, 1)$ and $\vec{w} = (1, 1, 2)$.
- Check your result by calculating $\vec{u} \cdot \vec{v}$ and $\vec{u} \cdot \vec{w}$.

Matrices

An $N \times N$ *matrix* is a collection of N column vectors $\vec{a}_1, \ldots, \vec{a}_N \in \mathbb{R}^N$ written as

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1N} \\ a_{21} & a_{22} & \cdots & a_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ a_{N1} & a_{N2} & \cdots & a_{NN} \end{pmatrix}$$

Matrix element: a_{kl}

- First index k: row index
- Second index l: column index

Unit matrix: $\mathbf{E} = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}$

Matrices

Matrix multiplication:

The *product* $\mathbf{C} = \mathbf{A} \cdot \mathbf{B}$ of two $N \times N$ matrices $\mathbf{A}$ and $\mathbf{B}$ is defined to be an $N \times N$ matrix with elements

$$c_{kl} = \sum_{n=1}^{N} a_{kn} b_{nl}$$

c_{kl}: scalar product of the k-th row of $\mathbf{A}$ with the l-th column of $\mathbf{B}$.

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \cdot \begin{pmatrix} 5 & 6 \\ 7 & 8 \end{pmatrix} = \begin{pmatrix} 19 & 22 \\ 43 & 50 \end{pmatrix}$$

Exercise 7

Calculate the following products:

$$\begin{pmatrix} 1 & 1 \\ 2 & 0 \end{pmatrix} \cdot \begin{pmatrix} 2 & 0 \\ 1 & 3 \end{pmatrix} =?\,, \qquad \begin{pmatrix} 2 & 0 \\ 1 & 3 \end{pmatrix} \cdot \begin{pmatrix} 1 & 1 \\ 2 & 0 \end{pmatrix} =?$$

Matrices

Special cases of matrix multiplication:

- Matrix $\times$ column vector $=$ column vector

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \cdot \begin{pmatrix} 5 \\ 6 \end{pmatrix} = \begin{pmatrix} 17 \\ 39 \end{pmatrix}$$

- Row vector $\times$ matrix $=$ row vector

$$\begin{pmatrix} 5 & 6 \end{pmatrix} \cdot \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} 23 & 34 \end{pmatrix}$$

- Other combinations don’t work

Matrices

The *inverse matrix* of an $N \times N$ matrix $\mathbf{A}$ is the $N \times N$ matrix $\mathbf{A}^{-1}$ satisfying

$$\mathbf{A} \cdot \mathbf{A}^{-1} = \mathbf{A}^{-1} \cdot \mathbf{A} = \mathbf{E}$$

- Only a square matrix can have an inverse matrix
- Even for a square matrix $\mathbf{A}^{-1}$ need not exist
- For 2×2 matrices:

$$\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \implies \mathbf{A}^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

Exercise 8

Consider the matrix $\mathbf{A} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$

- Work out the inverse matrix $\mathbf{A}^{-1}$.
- Check the relation $\mathbf{A} \cdot \mathbf{A}^{-1} = \mathbf{E}$.

Matrices

Properties of matrix multiplication:

- $\mathbf{A} \cdot \mathbf{E} = \mathbf{E} \cdot \mathbf{A} = \mathbf{A}$
- $\mathbf{A}_1 \cdot \mathbf{A}_2 \neq \mathbf{A}_2 \cdot \mathbf{A}_1$ (in general)
- $(\mathbf{A}_1 \cdot \mathbf{A}_2)^T = \mathbf{A}_2^T \cdot \mathbf{A}_1^T$
- $(\mathbf{A}_1 \cdot \mathbf{A}_2)^{-1} = \mathbf{A}_2^{-1} \cdot \mathbf{A}_1^{-1}$