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Problem sheet for chapter 2: Vectors and Matrices

Problem 1:

A point A in three-dimensional space has coordinates $(4,3,8)$, another point B has coordinates $(2,10,5)$. What is the distance between the points?

Lösung:

The vector between the two points is

$$\vec{v} = \overline{AB} = (2-4, 10-3, 5-8) = (-2, 7, -3)$$

The distance between the points is the length of this vector:

$$|\vec{v}| = \sqrt{4+49+9} = \sqrt{62} \approx 8$$

Problem 2:

(a) Which of the following three vectors are perpendicular?

$$\vec{a} = (-8,1,2) \qquad \vec{b} = (4,-3,1) \qquad \vec{c} = (-1,-2,-2).$$

(b) Calculate the angle between the vectors $\vec{a} = (1,-1,1)$ and $\vec{b} = (-1,1,-1)$.

Lösung:

(a) $\vec{a}\cdot\vec{b} = -33$, $\vec{a}\cdot\vec{c} = 2$, $\vec{b}\cdot\vec{c} = 0$. Therefore $\vec{b}$ and $\vec{c}$ are perpendicular to each other.

(b) $|\vec{a}| = |\vec{b}| = \sqrt{3}$, $\vec{a}\cdot\vec{b} = -3$. Hence

$$\cos(\gamma) = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|} = -1 \quad\Rightarrow\quad \gamma = \pi \quad\Rightarrow\quad \vec{a} = -\vec{b}$$

Problem 3:

A straight line in $\mathbb{R}^3$ passes through the origin and the point $Q = (-4,4,2)$. Which point P' on this line is closest to point $P = (1,2,3)$?

Lösung:

$\vec{e}_Q = \frac{\overline{OQ}}{|\overline{OQ}|} = \frac{1}{3}\begin{pmatrix}-2\\2\\1\end{pmatrix}$ is the unit vector pointing from the origin to Q.

Since P' sits on the line through the origin and Q, we have $\overline{OP'} = |\overline{OP'}|\cdot\vec{e}_Q$. The length $|\overline{OP'}|$ is the length of the projection of $\overline{OP}$ onto the unit vector $\vec{e}_Q$, i.e.

$$|\overline{OP'}| = \vec{e}_Q\cdot\overline{OP} = \frac{1}{3}\begin{pmatrix}-2\\2\\1\end{pmatrix}\cdot\begin{pmatrix}1\\2\\3\end{pmatrix} = \frac{5}{3}.$$

Therefore:

$$\overline{OP'} = \frac{5}{3} \cdot \frac{1}{3} \begin{pmatrix} -2 \\ 2 \\ 1 \end{pmatrix} = \frac{5}{9} \begin{pmatrix} -2 \\ 2 \\ 1 \end{pmatrix}$$

In other words: $P' = (-10/9, 10/9, 5/9)$.

Problem 4:

What is the area of the triangle defined by the following points?

$$A = (1, 2)\,, \qquad B = (7, 2)\,, \qquad C = (3, 6)$$

Hint: Think of the points as points in the x-y-plane of $\mathbb{R}^3$ and use the cross-product.

Lösung:

Two sides of the triangle in $\mathbb{R}^3$ are

$$\overline{AB} = \begin{pmatrix} 6 \\ 0 \\ 0 \end{pmatrix}, \qquad \overline{AC} = \begin{pmatrix} 2 \\ 4 \\ 0 \end{pmatrix}$$

The cross product is

$$\overline{AB} \times \overline{AC} = \begin{pmatrix} 0 \\ 0 \\ 24 \end{pmatrix}$$

The area of the triangle is half the magnitude of the cross product:

$$\text{Area} = \frac{1}{2} \left| \overline{AB} \times \overline{AC} \right| = 12$$

Problem 5:

Consider a plane in $\mathbb{R}^3$ that is positioned and oriented such that the three points

$$A = (1, 2, 3)\,, \qquad B = (-1, 1, 5)\,, \qquad C = (2, 2, 2)$$

are located within the plane. A particle with velocity $\vec{v}_{\text{in}} = (-1, 2, 3)$ hits the plane and is reflected specularly (i.e. incoming angle equals outgoing angle). Determine the velocity of the particle after reflection.

Lösung:

Determine unnormalized normal vector $\vec{u}$ to plane:

$$\vec{u} = \overline{AB} \times \overline{AC} = \begin{pmatrix} -2 \\ -1 \\ 2 \end{pmatrix} \times \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

With $\vec{u}^2 = 2$ and $\vec{v}_{\text{in}} \cdot \vec{u} = 2$ the result is

$$\vec{v}_{\text{out}} = \vec{v}_{\text{in}} - 2\frac{\vec{v}_{\text{in}} \cdot \vec{u}}{\vec{u}^2}\vec{u} = \begin{pmatrix} -1 \\ 2 \\ 3 \end{pmatrix} - 2\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -3 \\ 2 \\ 1 \end{pmatrix}$$

Problem 6:

Calculate the products $A \cdot B$, $A^T \cdot B^T$, $B \cdot A$ and $B^T \cdot A^T$ for the matrices A and B given below. What is the relation between $(A \cdot B)^T$ and the individual matrices A^T und B^T?

$$A = \begin{pmatrix} 2 & -1 \\ 0 & 3 \end{pmatrix}, \qquad B = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

Lösung:

$$A \cdot B = \begin{pmatrix} 2 & -1 \\ 0 & 3 \end{pmatrix} \cdot \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 0 & 3 \end{pmatrix}$$

$$B \cdot A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 2 & -1 \\ 0 & 3 \end{pmatrix} = \begin{pmatrix} 2 & 2 \\ 0 & 3 \end{pmatrix}$$

$$A^T \cdot B^T = \begin{pmatrix} 2 & 0 \\ -1 & 3 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 2 & 3 \end{pmatrix} = (B \cdot A)^T$$

$$B^T \cdot A^T = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} 2 & 0 \\ -1 & 3 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 1 & 3 \end{pmatrix} = (A \cdot B)^T$$

Problem 7:

Which matrices X_1 and X_2 satisfy the matrix equations $X_1 \cdot A = B$ as well as $A \cdot X_2 = B$ with

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 2 & -1 \\ 1 & 3 \end{pmatrix}?$$

Lösung:

$X_1 = B \cdot A^{-1}$ and $X_2 = A^{-1} \cdot B$. With

$$A^{-1} = \begin{pmatrix} 0 & 1 \\ 1 & -2 \end{pmatrix} \qquad \text{and} \qquad B^{-1} = \frac{1}{7}\begin{pmatrix} 3 & 1 \\ -1 & 2 \end{pmatrix}$$

this becomes:

$$X_1 = \begin{pmatrix} -1 & 4 \\ 3 & -5 \end{pmatrix} \qquad \text{and} \qquad X_2 = \begin{pmatrix} 1 & 3 \\ 0 & -7 \end{pmatrix}.$$