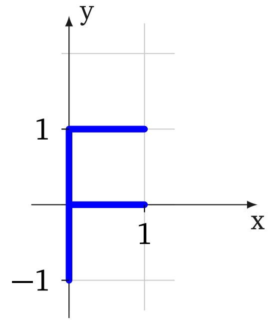
Intro to Computer Graphics
K. Jünemann

Problem sheet for chapter 8: Affine maps and homogeneous coordinates

Problem 1:

Consider the letter F and the two transforms **S** and **T** described below:



$$\mathbf{S} = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & 2 \end{pmatrix}$$

$\mathbf{T} =$ translation by 1 unit into x-direction

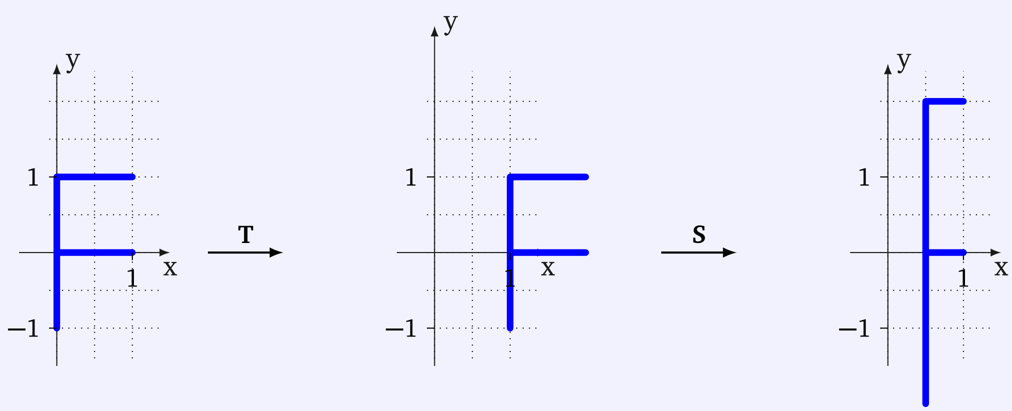
(a) Draw the letter F after the transform $\mathbf{S} \cdot \mathbf{T}$ has been applied to it.

(b) Draw the letter F after the transform $\mathbf{T} \cdot \mathbf{S}$ has been applied to it.

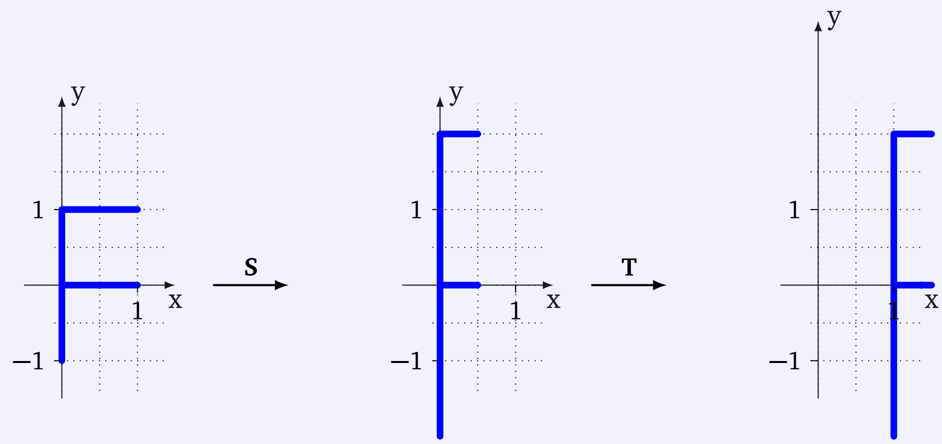
(c) Write down the 3×3-matrix representations of $\mathbf{S} \cdot \mathbf{T}$ and $\mathbf{T} \cdot \mathbf{S}$.

Lösung:

(a) $\mathbf{S} \cdot \mathbf{T}$ is



(b) $\mathbf{T} \cdot \mathbf{S}$ is



(c) Because of the presence of translations everything has to be described as 3×3 matrices with homogeneous coordinates. The matrix representations of $\mathbf{S}$ and $\mathbf{T}$ are

$$\mathbf{S} = \begin{pmatrix} \frac{1}{2} & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad \mathbf{T} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

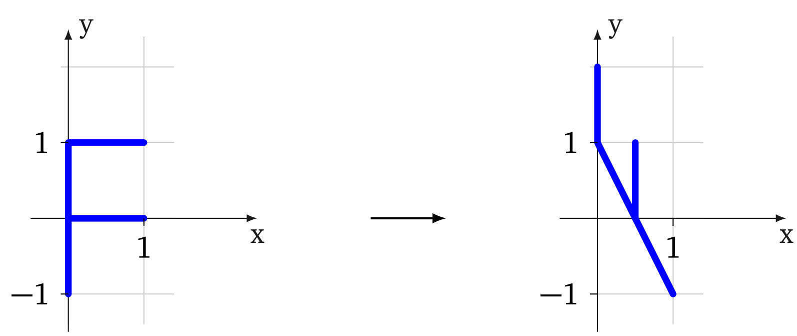
Matrix multiplication gives

$$\mathbf{T} \cdot \mathbf{S} = \begin{pmatrix} \frac{1}{2} & 0 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad \mathbf{S} \cdot \mathbf{T} = \begin{pmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

For $\mathbf{S} \cdot \mathbf{T}$ the translation part also gets scaled. $\mathbf{T} \cdot \mathbf{S}$ feels more natural. That's why it is the default order in `three.js`.

Problem 2:

Consider an affine map that transforms the left F to the right F:



(a) Is the map linear? Why or why not?

(b) Determine a 3×3 matrix representing this map in terms of homogeneous coordinates. Hint: Split the map into a linear part and a translation. What does the linear part do to the unit vectors $\vec{e}_x$ and $\vec{e}_y$?

Lösung:

(a) The map is non-linear because it moves the origin.

(b) The translation is by $(1/2, 0)$ because this is where the origin is moved to. After undoing the translation, one sees that the basis vectors transform as

$$\vec{e}_x = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \to \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \qquad \vec{e}_y = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \to \begin{pmatrix} -1/2 \\ 1 \end{pmatrix}$$

The 2×2 linear transform A and the 3×3 affine transform B are therefore:

$$A = \begin{pmatrix} 0 & -1/2 \\ 1 & 1 \end{pmatrix}, \qquad B = \begin{pmatrix} 0 & -1/2 & 1/2 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Problem 3:

Construct a 3×3 matrix that implements a (generalized) rotation of $90°$ in $\mathbb{R}^2$ around the point $(2, 1)$ in terms of homogeneous coordinates. As a test apply your matrix to the points $(3, 1)$ and $(3, 2)$. Also, construct the 3×3 matrix implementing the inverse map.

Lösung:

The pivot is $\vec{p} = (2, 1)^T$. The pure rotation matrix around $90°$ is

$$R_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

With $\vec{p} - R \cdot \vec{p} = (3, -1)^T$ we get for the 3×3 matrix

$$R_3 = \begin{pmatrix} 0 & -1 & 3 \\ 1 & 0 & -1 \\ 0 & 0 & 1 \end{pmatrix}.$$

Testing

- $R_3(3, 1, 1)^T = (2, 2, 1)^T$
- $R_3(3, 2, 1)^T = (1, 2, 1)^T$,

both of which make sense. The inverse matrix rotates around $-90°$:

$$R_3^{-1} = \begin{pmatrix} 0 & 1 & 1 \\ -1 & 0 & 3 \\ 0 & 0 & 1 \end{pmatrix}.$$