Spring Equations

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December 24, 2014

1 The differential equation

A mass m attached to a spring with spring constant k, friction constant r and external force $F(t)$ is described by the equation

$$m\ddot{y}(t) + r\dot{y}(t) + ky(t) = F(t).$$

$y(t)$ is the deviation of the pendulum mass from its rest position.
It is useful to introduce the eigen-frequency

$$\omega_0 = \sqrt{\frac{k}{m}}$$

and the dimensionless friction constant

$$d = \frac{r}{2\sqrt{km}}.$$

$d > 1$ means strong friction (no oscillations) and $d < 1$ means low friction. The equation then reads after dividing by m:

$$\ddot{y}(t) + 2d\omega_0\dot{y}(t) + \omega_0^2 y(t) = \frac{1}{m}F(t)$$

If the external force is exerted by moving the upper suspension point of the spring by some deviation $u(t)$ out of its rest position the force is $F(t) = k \cdot u(t)$ leading to the main equation

$$\ddot{y}(t) + 2d\omega_0\dot{y}(t) + \omega_0^2 y(t) = \omega_0^2 u(t) \tag{1}$$

2 Step- and Impulse response

This section assumes vanishing initial conditions, $y(0) = 0\,, \dot{y}(0) = 0$.
The impulse response is

- for $d > 1$:
$$h(t) = \sigma(t)\frac{\omega_0}{2\sqrt{d^2-1}}\mathrm{e}^{-d\,t\,\omega_0}\,\left(\mathrm{e}^{t\,\omega_0\,\sqrt{d^2-1}} - \mathrm{e}^{-t\,\omega_0\,\sqrt{d^2-1}}\right)$$
- for $d = 1$:
$$h(t) = \sigma(t)\cdot\omega_0^2 t e^{-\omega_0 t}$$
- for $0 \leq d < 1$:
$$h(t) = \sigma(t)\frac{\omega_0}{\sqrt{1-d^2}}\mathrm{e}^{-d\,t\,\omega_0}\,\sin\left(t\,\omega_0\,\sqrt{1-d^2}\right)$$

The step response is

- for $d > 1$:
$$g(t) = \sigma(t)\cdot\left(1 - \frac{1}{2}e^{-d\omega_0 t}\left(e^{t\omega_0\sqrt{d^2-1}}\left(1+\frac{d}{\sqrt{d^2-1}}\right) + e^{-t\omega_0\sqrt{d^2-1}}\left(1-\frac{d}{\sqrt{d^2-1}}\right)\right)\right)$$
- for $d = 1$:
$$g(t) = \sigma(t)\cdot\left(1 - e^{-\omega_0 t}(1+t\omega_0)\right)$$
- for $0 \leq d < 1$:
$$g(t) = \sigma(t)\cdot\left(1 - e^{-d\omega_0 t}\left(\cos(t\omega_0\sqrt{1-d^2}) + \frac{d}{\sqrt{1-d^2}}\sin(t\omega_0\sqrt{1-d^2})\right)\right)$$

3 Solution with initial conditions without external force

With no external force, i.e. $F(t) = 0$, but initial position y_0 and velocity v_0, i.e.

$$y(0) = y_0\,, \qquad \dot{y}(0) = v_0, \tag{2}$$

the solution of the equation is

- for $d > 1$:
$$\begin{aligned} y_{\text{init}}(t) = {} & \frac{v_0 + y_0\omega_0\sqrt{d^2-1} + d\,\omega_0 y_0}{2\omega_0\sqrt{d^2-1}}\mathrm{e}^{\omega_0(\sqrt{d^2-1}-d)t} \\ & - \frac{v_0 - y_0\omega_0\sqrt{d^2-1} + d\,\omega_0 y_0}{2\omega_0\sqrt{d^2-1}}\mathrm{e}^{-\omega_0(\sqrt{d^2-1}-d)t} \end{aligned}$$
- for $d = 1$:
$$y_{\text{init}}(t) = y_0\mathrm{e}^{-\omega_0 t} + (v_0 + y_0\omega_0)t\mathrm{e}^{-\omega_0 t}$$
- for $0 \leq d < 1$:
$$\begin{aligned} y_{\text{init}}(t) = {} & y_0\mathrm{e}^{-\omega_0 t}\cos(\omega_0\sqrt{1-d^2}\cdot t) \\ & + \frac{v_0 + d\,y_0\omega_0}{\omega_0\sqrt{1-d^2}}\sin(\omega_0\sqrt{1-d^2}\cdot t) \end{aligned}$$

4 Solution with harmonic external force and zero initial conditions

Let's assumue the motion of the upper suspension is

$$u(t) = u_0 \cdot \sin(\omega_e t)$$

with some given amplitude u_0, and external frequency ω_e. Also, the pendulum motion starts with zero initial conditions, i.e.

$$y(0) = 0\,, \qquad \dot{y}(0) = 0.$$

Then the solution of the main equation (1) is

- for $0 \le d < 1$:

$$y_{\text{ext}}(t) = 2\,d\,\omega_0\,\omega_e A e^{-d\,t\,\omega_0}\left(\cos(t\,\omega_0\,D) + \frac{\sin(t\,\omega_0\,D)\left(2\,d^2\,\omega_0{}^2 - \omega_0{}^2 + \omega_e{}^2\right)}{2\,d\,\omega_0{}^2\,D}\right)$$

$$-A\left(\omega_e{}^2\,\sin(t\,\omega_e) - \omega_0{}^2\,\sin(t\,\omega_e) + 2\,d\,\omega_0\,\omega_e\,\cos(t\,\omega_e)\right)$$

  with

$$D = \sqrt{1-d^2}\,, \qquad A = \frac{u_0\,\omega_0{}^2}{4\,d^2\,\omega_0{}^2\,\omega_e{}^2 + \omega_0{}^4 - 2\,\omega_0{}^2\,\omega_e{}^2 + \omega_e{}^4}$$

- for $d = 1$:

$$y_{\text{ext}}(t) = 2A\omega_0\,\omega_e\,e^{-t\,\omega_0}$$
$$+A(\omega_0{}^2 - \omega_e^2)\sin(t\,\omega_e) - 2A\omega_0\,\omega_e\,\cos(t\,\omega_e)$$
$$+t\,B\omega_e\,e^{-t\,\omega_0}$$

  with

$$A = \frac{u_0\omega_0^2}{(\omega_0{}^2 + \omega_e{}^2)^2}\,, \qquad B = \frac{u_0\omega_0^2}{\omega_0{}^2 + \omega_e{}^2}$$

- for $d > 1$:

$$y_{\text{ext}}(t) = 2\,d\,A\omega_0\,\omega_e\,e^{-d\,t\,\omega_0}\left(\cosh(t\,\omega_0\,D) + \frac{\sinh(t\,\omega_0\,D)\left(2\,d^2\,\omega_0{}^2 - \omega_0{}^2 + \omega_e{}^2\right)}{2\,d\,\omega_0{}^2\,D}\right)$$

$$-A\left(\omega_e{}^2\,\sin(t\,\omega_e) - \omega_0{}^2\,\sin(t\,\omega_e) + 2\,d\,\omega_0\,\omega_e\,\cos(t\,\omega_e)\right)$$

  with

$$A = \frac{u_0\,\omega_0{}^2}{4\,d^2\,\omega_0{}^2\,\omega_e{}^2 + \omega_0{}^4 - 2\,\omega_0{}^2\,\omega_e{}^2 + \omega_e{}^4}$$
$$D = \sqrt{d^2 - 1}$$

5 Solution with harmonic external force and non-zero initial conditions

Let's assumue the motion of the upper suspension is

$$u(t) = u_0 \cdot \sin(\omega_e t)$$

with some given amplitude u_0, and external frequency ω_e. Also, the pendulum motion starts with non-zero initial conditions (2). Then the solution $y(t)$ of the main equation (1) splits into two pieces

$$y(t) = y_{\text{init}}(t) + y_{\text{ext}}(t)$$

where the first term is the motion due to the initial conditions and the second term the motion due to the external force.