Chapter 2 notes

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- A linear combination of a list $v_1, v_2, ..., v_n$ of vectors in V is a single vector of the form

$$a_1v_1 + a_2v_2 + ... + a_nv_n$$

  with $a_1, ..., a_n \in F$

- The set of all linear combinations of a list of vectors $v_1, ..., v_n$ is denoted by $\text{span}(v_1, ..., v_n)$

- $\text{span}\ () = \{0\}$

- The span of a list of vectors in a vector space V is the smallest subspace of V containing these vectors.

  First we note that $\text{span}(v_1, ..., v_n)$ is in fact a subspace. The identity element exists in this set (as can be seen by setting $a_j = 0$) and it is in fact closed under scalar multiplication and addition, which is shown by the following.

  Note that if $v_1, ..., v_n$ is a list then the sum of two linear combinations is also in $\text{span}(v_1, ..., v_n)$ because

$$(a_1v_1 + ... + a_nv_n) + (c_1v_1 + ... + c_nv_n)$$

  And due to commutativity and associativity and distributivity we can arrange this to

$$(a_1 + c_1)v_1 + ... + (a_n + c_n)v_n$$

  $(a_j + c_j) \in F$ because of closure under addition in a field.

  which is indeed in $\text{span}(v_1, ..., v_n)$

  To see closure under scalar multiplication we note that for $\lambda \in F$ we can say

$$\lambda(a_1v_1 + ... + a_nv_n)$$

applying distributivity and associativity we find that

$$(\lambda a_1)v_1 + ... + (\lambda a_n)v_1$$

$\lambda a_j \in F$ by closure under multiplication in a field.

Now for the actual proof. Let U denote the smallest subspace such that each $v_j \in U$. Since $\text{span}(v_1, ..., v_n)$ is a subspace that contains each v_j we must have that

$$U \subseteq \text{span}(v_1, ..., v_n)$$

But $\text{span}(v_1, ..., v_n)$ contains all linear combinations of each v_j. Since subspaces are closed under linear combinations every subspace containing each v_j must contain $\text{span}(v_1, ..., v_n)$. Thus we must also have

$$\text{span}(v_1, ..., v_n) \subseteq U$$

and we must have $\text{span}(v_1, ..., v_n) = U$

- For a vector space V, if we have that $\text{span}(v_1, ..., v_n) = V$ we say that $\text{span}(v_1, ..., v_n)$ spans V.

- A vector space is a finite dimensional vector space is some list of vectors in it spans the vector space.

  F^n is always a finite dimensional vector space for all n as can be seen by considering the fundemental element of F^n which is $(x_1, ..., x_n)$.

$$(x_1, ..., x_n) = \text{span}((1, ..., 0), ..., (0, ..., 1, ..., 0), ..., (0, ..., 1))$$

- $p : F \to F$ is a polynomial with coefficients in F such that $p(x) = a_0 + a_1x + ... + a_mx^m$ for all $x \in F$.

- $P(F)$ is the set of all polynomials with coefficients in F. $P(F)$ is a vector space over F, meaning it is a subspace of F^F. It contains $0 : F \to F$, with $0(x) = 0$ for all $x \in F$. Furthermore it is closed under addition. Let $f, g \in P(F)$. $(f + g)(x) = f(x) + g(x)$ for all $x \in F$. And by defn

$$(f + g)(x) = \sum_{k=1}^{\infty} a_kx^k + \sum_{k=1}^{\infty} c_kx^k = \sum_{k=1}^{\infty}(a_k + c_k)x^k$$

  This is closed under multiplication as well, as can be easily checked.

- Coefficients are uniquely determined by a polynomial

- if $p \in P(F)$, then $\deg p = m$ if there exists $a_m \neq 0$ such that $p(x) = a_0 + a_1x + ... + a_mx^m$.

- $P_m(F) = \{p \in P(F) \mid \deg p = m\}$. We will take $\deg 0 < -\infty$
- $P_m(F)$ is finite dimensional a finite dimensional vector space. Try proving it is a vector space on your own. To see that it is finite dimensional consider $f_j : F \to F$ with $f_j(x) = x^j$ for all $x \in F$. Then we have $P_m(F) = \text{span } (1, f_1, ..., f_m)$.
- $P(F)$ is infinite dimensional (no list of vectors spans it). Assume that a list in $P(F)$ spans it and let m denote the highest degree of polynomial in it. The conclusion follows.
- A list of vectors $v_1, ..., v_m \in V$ is linearly independent when the only way you can have $0 \in \text{span } (v_1, ..., v_m)$ such that $0 = a_1v_1 + ... + a_mv_m$ is if there exists $a_1, ..., a_m \in F$ such that $a_1 = ... = a_m = 0$.

  Notice that this implies that each vector $v \in \text{span } (v_1, ..., v_m)$ has a unique representation. We will prove that a list of vector's span is linearly independent if and only if each vectors' representation is unique.

  Assume each vector $v \in \text{span } (v_1, ..., v_m)$ with $v_1, ..., v_m \in F$ has a unique representation. Then since 0 is a vector

  $$0 = a_1v_1 + ... + a_mv_m$$

  It must be that $a_1 = ... = a_m = 0$ is one way to go about this. Since the vector's representation is unique it must be the only way.

  We now prove the other direction. Assume the list is linearly independent. Assume further that v has two representations.

  $$v = a_1v_1 + ... + a_mv_m$$

  $$v = c_1v_1 + ... + c_mv_m$$

  And thus $0 = (a_1 - c_1)v_1 + ... + (a_m - c_m)v_m$. Since the only way to write 0 is by setting each coefficient to 0, then $a_1 = c_1$, and so on until $a_m = c_m$. Thus the representation is unique.
- Equivalently, a list of vectors is linearly independent if every element in their span can only be written in one way. Reversing the above argument gives a proof for the converse, making this a necessary and sufficient condition.
- (0) is not linearly independent. And nor is any list of vectors containing 0.
- (v, kv) for $k \in F$ is not either. Let $0 = a_1v + a_2kv$. Now set $a_2 = -n$ and $a_1 = nk$ for $n \in F$ Multiple representations for 0 exist.

- Linear dependence is when a list of vectors is not linearly independent. This means for a list of vectors $v_1, ..., v_m \in V$ for $0 \in$ span $(v_1, ..., v_m)$ there exists $a_1, ..., a_m \in F$ for $0 = a_1v_1 + ... + a_mv_m$ such that at least one coefficient is not 0.

- Removing vectors from a linearly independent list doesn't stop it from being independent (every sublist of a linearly independent list is linearly independent- - - Check/prove if it's an if and only if statement)

- If one element of a list of vectors is a linear combination of the others, then the entire list is linearly dependent. Assume WLOG that $v_1 = a_2v_2 + ... + a_mv_m$ then $0 = a_2v_2 + ... + a_mv_m - v_1$ and $0 = -a_2v_2 - ... - a_mv_m + v_1$. Thus 0 can be represented in two different ways. (Try to check if this is an if and only if relation)

- Linear dependence lemma: If $v_1, ..., v_m$ is a linearly dependent list then there exists $j \in \{1, ..., m\}$ such that $v_j \in$ span $(v_1, ..., v_m)$ and further you can remove this v_j and still have the same span.

  Note that intuitively speaking if we have one element that is a linear combination of the others, you can rearrange the equation to make any vector a linear combination of the others.

  Hence, since $v_1, ..., v_m$ is a linearly dependent list we have $0 = a_1v_1 + ... + a_mv_m$ with at least one element not 0, assume that j is the largest element such that $a_j \neq 0$ in $0 = a_1v_1 + ... + a_mv_m$. Then everything after that must be 0. Hence we have $0 = a_1v_1 + ... + a_jv_j$. Now we simply rearrange to get

$$v_j = -\frac{a_1}{a_j}v_1 - ... - \frac{a_{j-1}}{a_j}v_{j-1}$$

  And thus $v_j \in$ span $(v_1, ..., v_{j-1})$.

  Next, if we consider $u \in$ span $(v_1, ..., v_m)$ we have $u = a_1v_1 + ... + a_mv_m$. Therefore we must simply replace v_j with the equation found above. Thus $u \in$ span $(v_1, ..., v_{j-1}, v_{j+1}, ..., v_m)$

  Notice that we need to modify the proof to avoid division by 0. We do this by noticing that $0 = a_1v_1$, and a_1 isn't 0, hence $v_1 = 0$. This is the same thing as saying $v_1 \in$ span $() = \{0\}$. Additionally, for the second part we can say $u \in$ span $(v_1, ..., v_m)$ is just $u = a_1v_1 + ... + a_mv_m = 0 + ... + a_mv_m = a_2v_2 + ... + a_mv_m$ which is equivalent to $u \in$ span $(v_2, ..., v_m)$

- Intuitively speaking the previous part says that if no element was in the span of the previous ones you'd have a linearly independent list.

- Furthermore it says that, given that the first vector is nonzero, you can always find a j-th vector where $j > 1$ is a linear combination of the previous ones.

- Here we are considering sublists of lists, for example we look at whether (v_1) is indepdendent, and then if (v_1, v_2) is. And so on.

- You can't remove a vector from a linearly independent list and keep the span the same. To see this, assume that you can. The vector that you removed will however be in the previous span and since the spans are the same it must be in the new span. This means that the vector was in fact a linear combination of the others and therefore the list was linearly dependent deriving a contradiction.

- Note that if one element is in the span of a list then the other elements don't necessarily have to be in the span of that list with them removed and that element added.

- We must now show that the length of a linearly independent list in V is always less than or equal to the length of a spanning list in V.

  To see this let $B = w_1, ..., w_n$ and be a spanning list for V and $A = u_1, ..., u_m$ and be a linearly independent list in V. Let us add u_1 to A.

  For step 1 We now have $u_1, w_1, ..., w_n$. Since B is a spanning list, $u_1 \in$ span (B). We know that a list is linearly dependent if one of the elements can be written as a linear combination of the other elements. By the previous lemma we must have for some j that $v_j \in$ span $(v_1, ..., v_{j-1})$. Note that if $j = 1$ then we have $u_1 \in \text{span } () = \{0\} \implies u_1 = 0$ hence A is not linearly independent which is a contradiction. Therefore $j > 1$. Hence we can safely remove some w_k and keep our span of V.

  In general we have the j-th step being the following. We notice that we have a spanning list for V, which is B. Now we append u_j in front of the other u's. This will be a new spanning list and therefore the u_j can be written as a linear combination of its elements and thus the new list is linearly dependent. By the earlier lemma, one vector is a linear combination of the previous ones, this can't be one of the u's since those are linearly independent. Thus it is one of the w's and therefore we can remove that one.

  Every step has a list B of length n. Hence $|A| \leq |B| = n$. There are at most as many w's as u's.

- This theorem can be used to check whether a list is linearly independent or is a spanning list.

- We will now use the earlier result to show that every subspace U of a finite dimensional vector space V is also a finite dimensional vector space. We do this by a clever construction.

  If $U = \{0\}$ we are done. However if this is not the case then we will take $v_1 \in U$ and see if $U = \text{span } (v_1)$.

  In general if $U \neq \text{span } (v_1, ..., v_{j-1})$ then we take $v_j \notin \text{span } (v_1, ..., v_{j-1})$. By the contrapositive of the linear dependence lemma, this shows that

$v_1, ..., v_j$ is linearly independent. Every linaerly independent list in V's length is always less than the length of a spanning list of V. Hence the construction is bounded above and therefore is finite.

- Do 14,15,16,17 of 2A
- A basis is a list of vectors that is both a spanning list and is linearly independent
- It is clear that a basis for a vector space V is a basis if and only if all $v \in V$ can be written as $v = a_1v_1 + ... + a_nv_n$ in a unique way. This follows from the if and only if relation established earlier (a list is linearly independent if and only if each representation of each vector in their span is unique).