Miscellanious

Hamad Khan

October 2020

- Diagonalization follows from the following simple fact. Assume we have a 2x2 matrix and it has two linearly independent eigenvectors. The system of equations is

$$Ax_1 = \lambda_1 x_1$$
$$Ax_2 = \lambda_2 x_2$$

  This can be equivalently represented by

$$A[x_1\ x_2] = [Ax_1\ Ax_2] = [\lambda_1 x_1\ \lambda_2 x_2] = [x_1\ x_2]\begin{bmatrix}\lambda_1 & 0 \\ 0 & \lambda_2\end{bmatrix}$$

  Notice that we now have

$$A[x_1\ x_2] = [x_1\ x_2]\begin{bmatrix}\lambda_1 & 0 \\ 0 & \lambda_2\end{bmatrix}$$

  If we let $D = \begin{bmatrix}\lambda_1 & 0 \\ 0 & \lambda_2\end{bmatrix}$ (i.e. the diagonal matrix of eigenvalues) and we let P be the matrix of linearly independent eigenvectors. We have

$$AP = PD$$

  and by multiplying by P^{-1} on the right we get that

$$A = PDP^{-1}$$

  This of course generalizes to n-dimensions.

- Least squares solutions and more: `https://www.youtube.com/watch?v=Z0wELiinNVQ`

- Lagranges theoerem proofs/prerequisites

  `https://www.youtube.com/watch?v=5C54XzldHb8`

  `https://community.plu.edu/~sklarjk/fsaa/section-22.html`

  `https://www.wikiwand.com/en/Lagrange's_theorem_(group_theory)`

- You can use the following reasoning:

  Let H be a subgroup of G

  Left cosets of H form a partition of G. This can be seen by defining the equivalence relation $x \sim y$ iff $x = yh$ for $h \in H$. Thus we have $[a] = aH$. By the video linked above there is a one to one correspondence between equivalence relations and partitions. Hence for each $a_j \in G$ we have $[a_1] \cup [a_2] \cup ... \cup [a_n] = G$. Since these define a partition each $[a_j]$ is disjoint. Since if A and B are disjoint $n(A \cup B) = n(A) + n(B)$, we have $n([a_1]) + n([a_2]) + ... + n([a_n]) = n(G)$. This is the same as $n(a_1H) + n(a_2H) + ... + n(a_nH) = n(G)$. Since there exists a bijection $x \mapsto ax$ that is $H \mapsto aH$, we find that $n(a_jH) = n(a_kH) = n(a_1H)$. Then it is easy to see that $n|a_1H| = |G| = n|H|$ if we set $a = e$ where e is the identity under the group operation.

- The scalar triple product is $a \cdot (b \times c)$. This can be understood geometrically by the following diagram.

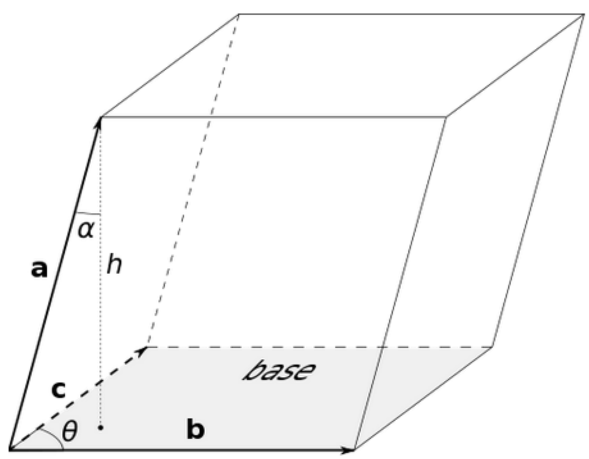

Figure 1: Parallelogram defined by three vectors

The magnitude of $b \times c$ can be seen as the area of the base of the parallelogram since $bcsin(\theta)$. Since $p = b \times c$ is a normal vector to the base we can denote $a \cdot p$ which says $apcos(\alpha)$. p is the area of the base and $acos(\alpha)$ is the height. Hence we get the area of the parallelogram.

- Using the definition of the cross product it can be shown (by simply expanding them) that

$$a \cdot (b \times c) = a \cdot \begin{bmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix} = \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}$$

  This shows that the determinant of the matrix gives you the volume of the parallelotope in 3-dimensions.

- notice that $e^{inx} = cos(nx) + isin(nx)$. Hence for the n-th roots of unity we have $e^{i\frac{2\pi}{n}} = cos(\frac{2\pi}{n}) + isin(\frac{2\pi}{n})$ since $1 = e^{i2\pi}$

- If $a < b$ then the archimedian property garuntees that there exists $N \in$ such that $\frac{1}{Nt} < b - a$ and hence $a + \frac{1}{N} < b$

- Consider the differential equation $(1 + x)f'(x) = \alpha f(x)$. Show that the rhs and the lhs of the following satisfy this. For the lhs you will have to use the identities:

$$\binom{n}{k} = \frac{n}{k}\binom{n-1}{k-1}$$

  and pascal's identity:

$$\binom{n-1}{r} + \binom{n-1}{r-1} = \binom{n}{r}$$

  Plug in, use the first one, regroup the series according to powers of x, and then apply pascal's identity.

  Next show that the solutions are unique up to a multiplicative constant, which you need to show is equal to 1 next so that the lhs and rhs are shown to be equal. Next use the root test to determine the radius of convergence (which is 1).

$$(1 + x)^{\alpha} = \sum_{k=0}^{\infty} \binom{\alpha}{k} x^k$$

  and this is only valid for $|x| < 1$.

  Now we consider this:

$$\left(1 + \frac{x}{y}\right)^{\alpha} = \sum_{k=0}^{\infty} \binom{\alpha}{k} \left(\frac{x}{y}\right)^k$$

  This is only valid for $\left|\frac{x}{y}\right| < 1$ which is equivalent to $|x| < |y|$.

We now will multiply by y^α.

$$y^\alpha\left(1+\frac{x}{y}\right)^\alpha = y^\alpha\sum_{k=0}^{\infty}\binom{\alpha}{k}\left(\frac{x}{y}\right)^k$$

$$(y+x)^\alpha = \sum_{k=0}^{\infty}\binom{\alpha}{k}y^\alpha\left(\frac{x}{y}\right)^k = \sum_{k=0}^{\infty}\binom{\alpha}{k}y^{\alpha-k}x^k$$

Remember that we have generalized binomial coefficients to being

$$\binom{n}{k} = \frac{(n)_r}{k}$$

Google binomial series for more.

- Note that S_n for an arithmatic series is obtained by:

$$S_n = na + \frac{n(n-1)}{2}d$$

- Euler-mascheroni constant. Show it exists by using this:

  `https://math.stackexchange.com/questions/629630/simple-proof-euler-mascheroni-gamma-co`

  First step can be made easier by considering taylor series of $ln(1-x)$, and proving $\ln(1-x) < -x$

- A pdf about gamma and digamma functions.

  `https://fractional-calculus.com/gamma_digamma.pdf`

- Why matrix multiplication is defined the way it is:

  `https://math.stackexchange.com/questions/24456/matrix-multiplication-interpreting-and-`
  `24469#24469`

  Notice that this works out because

  $T \circ S(v) = Cv$ for some matrix C since $T \circ S$ is again a linear function and therefore has a matrix. Thus $T \circ S(f_j) = Cf_j$, and hence we can use vector multiplication to define each entry of the matrix Cf_j. Then we define $TS = C$, so that the nice properties from the composition of functions transfer over to the products of matrices/matrices.

  We can derive/motive matrix vector multiplication by the following, which also proves that every linear transformation has a matrix given two basis.

  Let $f : V \to W$, for some finite dimensional vector spaces V and W with basis $\{v_1, .., v_n\}$ and $\{w_1, ..., w_m\}$. Consider a vector $v \in V$. Since there is a basis we can uniquely represent v as $v = c_1v_1 + ... + c_nv_n$.

$$v = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$$

Now we are ready to consider $f(v)$

$$f(v) = f(c_1v_1 + ... + c_nv_n) = c_1f(v_1) + ... + c_nf(v_n)$$

Thus the value of the function is completely determined by $f(v_1), ..., f(v_n)$. Now since $f(v_j) \in W$, and W has a basis, we can write $f(v_j) = a_{1j}w_1 + ... + a_{mj}w_m$. Now substitute.

$$f(v) = c_1(a_{11}w_1 + ... + a_{m1}w_m) + ... + c_n(a_{1n}w_1 + ... + a_{mn}w_m)$$

And we distribute, and then rearrange according to the basis vector w_j. We get:

$$f(v) = (c_1a_{11} + ... + c_na_{1n})w_1 + ... + (c_1a_{m1} + ... + c_na_{mn})w_m$$

But this is precisely

$$f(v) = \begin{bmatrix} c_1a_{11} + ... + c_na_{1n} \\ \vdots \\ c_1a_{m1} + ... + c_na_{mn} \end{bmatrix}$$

Now we define matrix vector multiplication such that

$$f(v) = \begin{bmatrix} c_1a_{11} + ... + c_na_{1n} \\ \vdots \\ c_1a_{m1} + ... + c_na_{mn} \end{bmatrix} = \begin{bmatrix} a_{11} & ... & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & ... & a_{mn} \end{bmatrix} \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$$

We define

$$\begin{bmatrix} a_{11} & ... & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & ... & a_{mn} \end{bmatrix}$$

As the matrix M of f. Thus $f(v) = Mv$

This gives us matrix multiplication (almost) for free! First, consider $T \circ S$. Now this is another linear transformation, which means this can be represented by a matrix C. Thus $T \circ S(v) = Cv$. Consider the column vector f_j that is 0 everywhere except for the j-th entry, where it is 1. Then consider $T \circ S(f_j) = Cf_j$. Sf_j is simply the j-th column of S, and Cf_j is simply the j-th column of C. Using the vector multiplication formula from above, we can find the (i, j)-th entry of the matrix of C.

- `https://math.stackexchange.com/questions/154099/remembering-taylor-series` How to derive/remember taylor series.

- Consider this integral.

$$\int_a^\infty \ln\left(1+\frac{a^2}{x^2}\right)dx$$

with $a > 0$ We may approximate this with the taylor series of $\ln\left(1+\frac{a^2}{x^2}\right)$. Thus

$$\ln\left(1+\frac{a^2}{x^2}\right) \approx \sum_{n=1}^{k}(-1)^{n+1}\frac{a^{2n}}{nx^{2n}}$$

Integrating both sides and interchanging the sum and integral (this is valid since we have a finite sum), we achieve (for k sufficiently large)

$$\int_{a+\epsilon}^\infty \ln\left(1+\frac{a^2}{x^2}\right)dx \approx \sum_{n=1}^{k}\frac{(-1)^{n+1}a^{2n}}{n}\int_{a+\epsilon}^\infty \frac{1}{x^{2n}}dx$$

(we're using $a+\epsilon$ bcs we want convergence since this maclaurin series converges iff $a < x$).

$$\int_{a+\epsilon}^\infty \ln\left(1+\frac{a^2}{x^2}\right)dx \approx \sum_{n=1}^{k}\frac{(-1)^{n+1}a^{2n}}{n(1-2n)}(-1)(a+\epsilon)^{-2n+1}$$

$$\lim_{\epsilon\to 0}\int_{a+\epsilon}^\infty \ln\left(1+\frac{a^2}{x^2}\right)dx \approx -\lim_{\epsilon\to 0}\sum_{n=1}^{k}\frac{(-1)^{n+1}a^{2n}}{n(1-2n)}(a+\epsilon)^{-2n+1}$$

$$\lim_{\epsilon\to 0}\int_{a+\epsilon}^\infty \ln\left(1+\frac{a^2}{x^2}\right)dx \approx -\sum_{n=1}^{k}\frac{(-1)^{n+1}a}{n(1-2n)}$$

$$\lim_{\epsilon\to 0}\int_{a+\epsilon}^\infty \ln\left(1+\frac{a^2}{x^2}\right)dx \approx -a\sum_{n=1}^{k}\frac{(-1)^{n+1}}{n(1-2n)}$$

Now just use partial fractions to get

$$\lim_{\epsilon\to 0}\int_{a+\epsilon}^\infty \ln\left(1+\frac{a^2}{x^2}\right)dx \approx -a\sum_{n=1}^{k}(-1)^{n+1}\left(\frac{1}{n}+\frac{2}{(1-2n)}\right)$$

$$\lim_{\epsilon\to 0}\int_{a+\epsilon}^\infty \ln\left(1+\frac{a^2}{x^2}\right)dx \approx -a\left(\sum_{n=1}^{k}\frac{(-1)^{n+1}}{n}+2\sum_{n=1}^{k}\frac{(-1)^{n+1}}{(1-2n)}\right)$$

$$\lim_{\epsilon\to 0}\int_{a+\epsilon}^\infty \ln\left(1+\frac{a^2}{x^2}\right)dx \approx -a\left(\ln(2)+2\sum_{n=1}^{k}\frac{(-1)^{n+1}}{(1-2n)})\right)$$

use the substitution n = b+1 and the taylor series of (x) at $x = 1$ to obtain

$$\lim_{\epsilon \to 0} \int_{a+\epsilon}^{\infty} \ln\left(1 + \frac{a^2}{x^2}\right) dx \approx -a\left(\ln(2) + 2(-\arctan(1)\right)$$

$$\lim_{\epsilon \to 0} \int_{a+\epsilon}^{\infty} \ln\left(1 + \frac{a^2}{x^2}\right) dx \approx -a\left(\ln(2) - \frac{\pi}{2}\right)$$

$$\lim_{\epsilon \to 0} \int_{a+\epsilon}^{\infty} \ln\left(1 + \frac{a^2}{x^2}\right) dx \approx a\left(\frac{\pi}{2} - \ln(2)\right)$$

Making this more rigorous can be done using a limit on k.

- Note that
$$\pi = 4\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}$$
by the taylor series of $\arctan(x)$ evaluated at 1.

- general formula for sum of the first k, p-th powers of natural numbers. (Faulhaber's formula). `https://www.wikiwand.com/en/Faulhaber%27s_formula#/Proof_with_exponential_generating_function`

- note that
$$\phi = \frac{1}{\sqrt{2\pi}} \int_{\infty}^{x} e^{\frac{-t^2}{2}} dt$$
can be approximated using taylor series. We use the trick
$$\phi = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{0} e^{\frac{-t^2}{2}} dt + \frac{1}{\sqrt{2\pi}} \int_{0}^{x} e^{\frac{-t^2}{2}} dt$$
We do this in order to garuntee convergence of the taylor series of $e^{\frac{-t^2}{2}}$. Otherwise we have to evaluate $\lim_{t \to -\infty} t^{2k}$ which is a bad situation if we want to do anything usable. The left integral is easily seen to be $\frac{1}{2}$ by rewriting it in terms of the gaussian integral. The right one is approximated by
$$e^{\frac{-t^2}{2}} \approx \sum_{k=0}^{n} \frac{(-t^2)^k}{2^k k!}$$
interchanging the sum and integral we find that
$$\phi \approx \frac{1}{2} + \frac{1}{\sqrt{2\pi}} \sum_{k=0}^{n} \frac{(-1)^k x^{2k+1}}{2^k k!(2k+1)}$$

$$e^x = 1 + \int_0^x 1dt + \int_0^x \int 1dtdt + \int_0^x \int \int 1dtdtdt + ... = \sum_{k=0}^{\infty} \frac{x^n}{n!}$$

- https://math.stackexchange.com/questions/668/whats-an-intuitive-way-to-think-about-the