Sheldon Axler notes

Hamad Khan

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1 $\mathbb{R}^n$ and $\mathbb{C}^n$

- Here, F will refer to either one of $\mathbb{R}$ and $\mathbb{C}$ (i.e. you can substitute in either and it will work out)
- If $x \in F$, and F is a field, x is a scalar, which emphasizes that x is not a vector.
- The complex numbers form a field.
- Ordinary space is $\mathbb{R}^3$, while a plane is $\mathbb{R}^2$
- A list of length n (a.k.a an n-tuple) is an ordered, finite collection of n elements of the form
$$(x_1, x_2, ..., x_n)$$
- () is a list of length 0 (a.k.a 0-tuple).
- We will use n-tuples to refer to lists of length n.
- F^n is the set of all n-tuples of elements from F.
$$F^n = \{(x_1, ..., x_2) \mid x_j \in F, j \in \mathbb{N}\}$$
- x_j is the j-th coordinate of $(x_1, ..., x_2, ..., x_n)$
- We can use a single letter to refer to an n-tuple. for example, we can say
$$x = (x_1, ..., x_n)$$
- We have defined addition in F^n as coordinatewise addition, and multiplication by a scalar by the coordinatewise multiplication of an element of F^n by an element of F.
- Through chapters 1 through 3 you can think of F as denoting an arbitrary field. Most theorems, definitions, and proofs work in arbitrary fields too.

2 Vector spaces

- A vector space is a set V, over a field F which has scalar multiplication and addition.
  - A scalar multiplication on a set V is a function that assigns the element $\lambda v \in V$ to each $\lambda \in F$, and each $v \in V$
  - An addition of vectors on a set V is a function that assigns the element $u + v \in V$ to some pair elements $u \in V$ and $v \in V$
- The following axioms are satisfied, for $u, v \in V$, and $a, b \in F$.
  - Abelian group under addition
  - Multiplicative associativity $a(bv) = (ab)v$
  - Multiplicative identity $1v = v$, $1 \in F$
  - Scalar multiplication and coordinatewise addition distribute over each other. $a(u + v) = au + av$, and $(a + b)(v) = av + bv$
- An abelian group satisfies the following properties (remember by using the acronym ASCII, but drop the S). (For all elements of the abelian group we have:)
  - Associativity
  - Commutativity
  - Identity
  - Inverse
- A ring is basically a field that is noncommutative under multiplication and that does not have garunteed inverses. Matrices form a ring.
- scalar multiplication depends on what field it is over, so we need to specify that. I.e. scalar multiplication by a complex number is different from one by the real numbers.
- F^n is a vector space over F.
- A field satisfies the following axioms:
  - An abelian group under addition
  - an abelian group under multiplication
  - Multiplication distributes over addition $a(v + u) = av + au$.
- If S is a set and F is a field, then F^S denotes the set of functions from S to F. If $f : S \to F$, then $f \in F^S$, while $f(x) \in F$.
- F^S forms a vector space with the following definitions, which have to hold for all $x \in S$, which means that S can not be empty:

- Addition between $f, g \in F^S$, is the function defined by:

$$(f+g)(x) = f(x) + g(x)$$

  for all $x \in S$

- For $\lambda \in F$, and $f \in F^S$, scalar multiplication is the function defined by

$$(\lambda f)(x) = \lambda f(x)$$

  for all $x \in S$

- Notice that here "+" and multiplication are whatever operations that F comes equipped with

- The additive identity is $f = 0$, where $0(x) = 0$ for all $x \in S$, where 0 denotes the additive identity in F. The proof of this is as follows.

$$(f+0)(x) = f(x) + 0(x) = f(x) + 0 = f(x)$$

  for all $x \in S$

- The additive inverse of $f \in F^S$ is defined by $-f(x) = (-f)(x)$ for all $x \in S$ and will be denoted by $-f \in F^S$. We will have $(f-f)(x) = f(x) - f(x) = 0 = 0(x)$ for all $x \in S$

- The rest of properties are easily verifiable.

* The elements of a vector space can be anything you want, it's an abstraction of the properties that F^n has.

* F^∞ is another example of a vector space. It is not composed of lists (since those are finite) but it is composed of all sequences of the form $(x_1, x_2, ...)$. $F = \{(x_1, x_2, ...) \mid x_j \in F\}$

* Notice that F^n is a special case of F^S, since we could think of a n-tuple as a function from $\{1, 2, ..., n\} \mapsto F$. Hence F^N is equivalent to $F^{\{1,2,3,...,n\}}$. F^∞ is too. It can be thought of $F^{\{1,2,3,...\}}$

* The following is in an arbitrary vector space

* The additive identity is unique, as can be seen by:

$$0' = 0' + 0 = 0 + 0' = 0$$

* The additive inverse is unique, as can be seen by: (first assume that $w \in V$ and $w' \in V$ are both additive inverses of $v \in V$

$$w = w + 0 = w + (v + w') = (w + v) + w' = 0 + w' = w'$$

- Due to uniqueness we can denote the additive inverse of v as $-v$. And $w - v$ is defined to be $w + (-v)$.
- When we say V without anything else, we're denoting a vector space over F.
- 0v = v, where $0 \in F$ and $v \in V$ this can be seen by:

$$0v = (0 + 0)v = 0v + 0v$$

Then by adding -0v to both sides we get

$$0v - 0v = 0 = 0v + 0v - 0v = 0v$$

Which gives us $0v = 0$

- a0 = 0, where $a \in F$, and $0 \in V$.

$$a0 = a(0 + 0) = a0 + a0$$

With the same shebang as above, we get that $a0 = 0$.

- (-1)v = -v, where $v \in V$, and $-1 \in F$:

$$v + (-1)v = 1v + (-1)v = (1 - 1)(v) = (0)(v) = 0$$

And hence (along with the uniqueness of inverses) $(-1)v = -v$.

- Do 1.1B

3 Subspaces

- If $U \subseteq V$, and U is a vector space with the same operations that are defined on V, U is a subspace of V.

  To check if U is a subspace of V, we need that:

  1. $0 \in U$
  2. U is closed under addition
  3. U is closed under scalar multiplication

  The last two make sure that addition and scalar multiplication work (remember that scalar multiplication and addition defined on a set V on a field F are functions that need to map to an element in the set, for all elements $u, v \in V$ to $u + v \in V$in the case of addition, and $\lambda \in F$ and $v \in V$, to $\lambda v \in V$).

We explicitly require property number 1 but not that there needs to be a multiplicative identity because the underlying field F remains the same, and the multiplicative identity comes from the field, not from the underlying set in the case of the additive identity.

The rest of the properties are garunteed by the fact that they hold for the larger vector space V (like commutativity, distributive law, associativity for both scalar multiplication and addition).

Additive inverses are garunteed by property number 3 because $-1(v) = -v$.

- A sum of m subsets is defined as

$$U_1 + ... + U_m = \{u_1 + ... + u_m \mid u_1 \in U_1, ..., U_m \in U_m\}$$

  We are actually interested in the sum of subspaces so we'll use this for that.

  In the following U_j denotes any of the possible $U_1, ...U_m$.

  Note that this is closed under addition (since we're considering all possible ways of adding the elements). It is also closed under scalar multiplication, since, as it is easy to show, $x \in U \iff ax \in U$. To show multiplicative closure we set $v = u_1 + ... + u_m$. Now we multiply by a scalar, say, $a \in F$. Thus we have $av = au_1 + ... + au_m$. Since, by definition we have $au_j \in U_j$, we have a sum of elements of U_j, and hence it is in $U_1 + ... + U_m$. We find that it is closed under scalar multiplication. We also have the 0 vector, meaning it is in fact a subspace.

  To further see that this is closed under addition, say we have three subspaces U_1, U_2, U_3, and say their sum is V. If there are elements $v, u \in U_1 + U_2 + U_3$. Then we say

$$v + u = v_1 + v_2 + v_3 + u_1 + u_2 + u_3$$

  (by definition $v = v_1 + v_2 + v_3$ and $u + u_1 + u_2 + u_3$). Since V is made up of all possible sums, this is included as well.

- Now we will prove that the sum of subspaces $U_1 + ... + U_m$ is in fact the smallest subspace that contains $U_1, ..., U_m$.

  To begin with, let S be the smallest subspace containing $U_1, ..., U_m$. Let U be the subspace sum, $U = U_1 + ... + U_m$. Clearly, $U_1, ..., U_m \subseteq S$, and $U_1, ..., U_m \subseteq U$. To see the second fact consider the subspace sum where all except one u are zero. The first one follows from the definition of S.

  Since S is a subspace, it must contain all the possible sums of $u_1 + ... + u_m$ because it must be closed under all forms of addition. In other words, $U \subseteq S$.

  Since S is the smallest subspace containing $U_1, ..., U_m$ we must have $S \subseteq U$. Thus we can conclude that $U = S$.

- A subspace sum is a direct sum if and only if each element of $U_1+...+U_m = \{u_1+u_2+...+u_m \mid u_j \in U_j\}$ can be written in only one way as $u_1+...+u_m$. Any reordering of this sum is still a direct sum as long as each u_j stays in U_j.

- The sum of m subspaces is a direct sum if and only if the way to write 0 is by taking each $u_j \in U_j$ as 0.

  First assume that the subspace sum is a direct sum. Therefore, we have to be able to write 0 in one way. The only way this is possible for any possible subspace is if we set each part of the vector sum as 0 since if we could write $0 = u_1 + u_2 + ... + u_m$, with at least one $u_j \neq 0$ we could write $(-1)0 = 0 = -u_1 - u_2 - ... - u_m$, with at least one $u_j \neq 0$ and hence the representation won't be unique.

  Now we assume that we can uniquely write $0 = 0 + 0 + ... + 0$ (m times). We will now show that this implies all elements are unique. We take $v = v_1+...+v_m$ and $v = u_1+...+u_m$. Now $v-v = 0 = (v_1-u_1)+...+(v_m-u_m)$. Since $0 = 0 + 0 + ... + 0$. We must now have.

  $$\begin{aligned} 0 &= v_1 - u_1 \\ 0 &= v_2 - u_2 \\ 0 &= v_3 - u_3 \\ &\vdots \\ 0 &= v_m - u_m \end{aligned}$$

  Thus $v_j = u_j$ and the representation is unique.

- We will now show that, if U and W are subspaces, then $U + W$ is a direct sum if and only if $U \cap W = \{0\}$.

  To begin with, assume that $U + W$ is a direct sum. Next, consider $s \in U \cap W$, which implies $s \in U + W$. We need to look at the equation $0 = s + (-s)$. Since s is in both U and W, we can put the condition $s \in U$ and $-s \in W$. Then, by unique representation in a direct sum, we must have $s = 0 = -s$. Hence $U \cap W = \{0\}$

  To prove the other direction, we consider $U \cap W = \{0\}$, $u \in U$, $w \in W$. Suppose that $0 = u + w$, we next find that $-w = u$. Since this implies that $u \in W$ since $-w \in W$, and $-w \in U$ because $u \in U$, we find that $u, w \in U \cap W = \{0\}$. Hence the only way to write 0 is by the sum of 0s from the parent vector subspaces. By the earlier result we must have that $U + W$ is a direct sum.

- For intuition on this consider two subspaces U, W of V. Let $U \cap W = \{..., -a, a, 0, b, -b, ...\}$. Assume they have a direct sum. Notice that this means $a, b \in U + W$. Thus we must have $-a, -b \in U + W$, indeed this was implied by the inverses being in the intersection as well. Since $U + W$ is a direct sum, we must have that $0 = 0 + 0$ by the earlier result. However by definition $a - a = 0$ and $b - b = 0$ so this representation is not unique. Therefore it isn't a direct sum. This is a contradiction, so our assumption that we could have $U \cap W \neq \{0\}$ and $U + W$ being a direct sum at the same time was wrong.

- This result is the closest thing to having two subspaces be disjoint. Hence direct sums are subspace sums between "disjoint" subspaces.
- Remember the earlier result is only between two vector subspaces.
- In the complex numbers, there are n complex roots for $\sqrt[n]{a}$
- The intersection of two subspaces of V is also a subspace of V. This generalizes to any number of subspaces of V. (as can be shown by induction).
- for induction to work you need a way to get from one element to the other, hence it won't work with uncountably infinite sets.
- countable infinity is when you can stop at any time and assign a natural number. Uncountable infinities occur when you can't assign a natural number at any time. Thus the real numbers are an uncountable infinity since there's no way to get to the "next real number", since for each m, n we have $m < x < n$.
- The following are from exercises $1.C$
- If U_1 and U_2 are subspaces of V. The intersection $U_1 \cap U_2$ is a subspace too.
- The union of two subspaces is a subspace if and only if one is contained in the other
- $U + U = U$
- $U + W = W + U$
- $(A + B) + C = A + (B + C)$
- The operation of addition on subspaces of V has the additive identity $\{0\}$. Only $\{0\}$ has an additive inverse.
- If U_1, U_2, W are subspaces of V and $U_1 + W = U_2 + W$, then it is not necessary that $U_1 = U_2$. The same holds for direct sums.