Trinomial Tree Models of Option Pricing

1. Introduction

Option pricing is essential for trading, hedging and risk management. Common tools for option pricing include the Black-Scholes-Merton model and binomial tree models. Since these methods are basic tools, of which the application and efficiency are often limited, the paper examines trinomial tree models, a more complex method of option pricing.

1.1 Black-Scholes-Merton Model

The Black-Scholes-Merton (BSM) model is a mathematical model developed by Fischer Black and Myron Scholes in 1973. Robert C. Merton also made important contributions to this theory. The generalized BSM model prices options using six parameters: the underlying asset price, strike price, time to maturity, implied volatility, risk free rate, and the cost of carry in a single formula.

$$c = Se^{(b-r)T}\Phi(\mathrm{d}_1) - \mathrm{X}\mathrm{e}^{-\mathrm{rT}}\Phi(\mathrm{d}_2)$$

$$p = Xe^{-rT}\Phi(-d_2) - Se^{(b-r)T}\Phi(-d_1)$$

$$d_1 = \frac{ln\left(\frac{S}{X}\right) + \left(b + \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}$$

$$d_2 = d_1 - \sigma\sqrt{T}$$

Where:

c is the price of call

p is the price of put

S is underlying price

X is strike price

T is the time to maturity

σ is the implied volatility

r is the risk-free rate

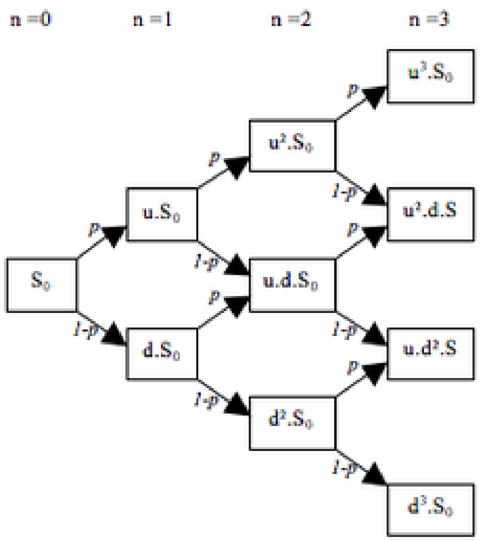
b is the cost of carry

$\Phi()$ is the normal CDF function

The BSM model assumes that stock price follows a geometric Brownian motion, and the risk-free rate and volatility are constant.

1.2 Binomial Tree Models

Unlike the BSM model, binomial tree model is a numeric method for option pricing. It assumes that the stock price either goes up (u) or goes down (d) in each step. Option value in each state of the last step can be calculated with the stock price. Then, the value of each node in the previous step can be calculated using the risk neutral probability p.



$$p = \frac{e^{rt/n} - d}{u - d}$$

$$u = e^{\sigma \sqrt{t/n}}$$

$$d = e^{-\sigma \sqrt{t/n}}$$

Figure 1: Binomial Tree

Binomial tree model is more flexible than BSM model since it can handle non-constant volatility and risk-free rate. It is also applicable to more complex options such as American options. When the number of steps increases, the result converges to the real value of options.

1.3 Trinomial Tree Models

Trinomial tree model is like the binomial model, with 3 paths for the stock price change. Typically, the stock price in each step goes up (u), keeps stable (m), or goes down (d).

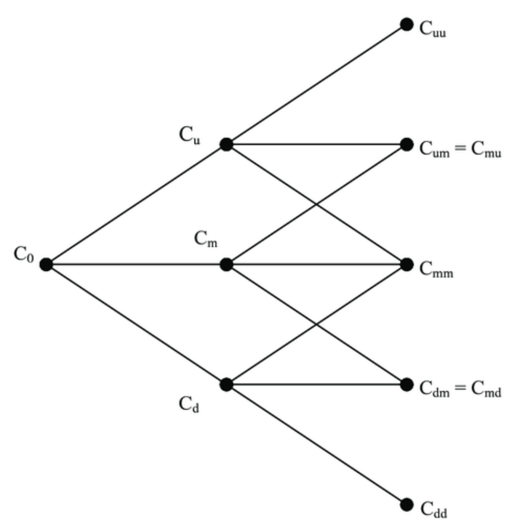


Figure 2: Trinomial Tree

Parameters for trinomial tree models are followed. Since tree models simulate geometric Brownian motion of stock price, u, d, and risk neutral probabilities are adjusted to match it. λ is the step size. In trinomial tree model, a popular and efficient choice of λ is $\sqrt{\frac{3}{2}}$, of which $p_m = \frac{1}{3}$. We will use this parameter to test the trinomial tree model.

$$u = e^{\lambda\sigma\sqrt{\Delta t}}$$

$$d = \frac{1}{u} = e^{-\lambda\sigma\sqrt{\Delta t}}$$

$$p_u = \frac{1}{2\lambda^2} + \frac{\left(r - \frac{\sigma^2}{2}\right)\sqrt{\Delta t}}{2\lambda\sigma}$$

$$p_d = \frac{1}{2\lambda^2} - \frac{\left(r - \frac{\sigma^2}{2}\right)\sqrt{\Delta t}}{2\lambda\sigma}$$

$$p_m = 1 - (p_u + p_d)$$

Trinomial Tree Models are more efficient since it can reach the same accuracy with fewer steps. We will test these models with different options to see if they are faster than binomial tree models.

2. Model Comparison

In this section, we will compare the efficiency, computational costs and errors for binomial and trinomial tree models. We use the following at-the-money European call option with the parameters (if not specified) to test them.

$$S_0 = 100$$

$$\mathrm{K} = 100$$

$$T = 1$$

$$r_f = 0.05$$

$$\sigma = 0.2$$

With Black-Scholes-Merton model, we can calculate the real value of the call, which is considered the benchmark price.

$$C = 10.45$$

2.1 Convergence Speed

Convergence speed is how fast the calculated option value of our model converge to the real value as the number of steps increases. The faster the result converges to the real value, the more efficient the model is.

We tested with the call option with previous parameters. We found that as the number steps increases, the trinomial model converges faster than binomial model.

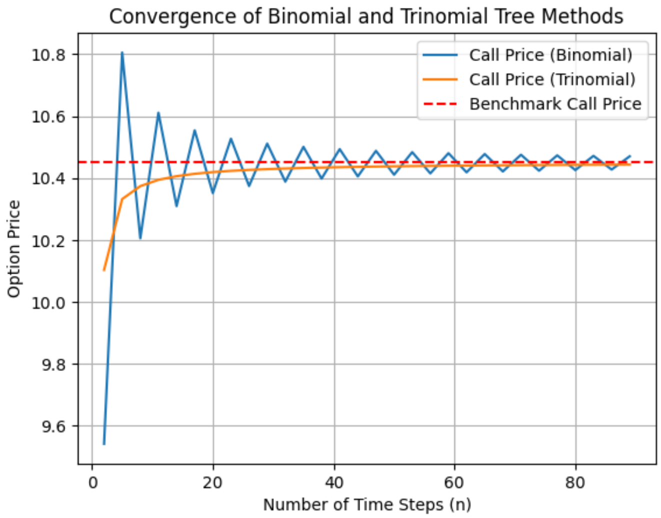


Figure 3: Convergence Speed of Binomial and Trinomial Tree Models

2.2 Computational Costs

Although trinomial model converges faster than binomial model, it costs more in computation for each step since each node has 3 states, while each node has 2 states in binomial model. To find out the total computational costs and efficiency for both models, we calculated the number of steps that are required to reach a given accuracy (error below 0.1%) and recorded the running time. The result indicates that although trinomial models have three nodes in each step, it still costs much less in computation compared to binomial models.

Table 1: Time of Computation

	Error	Number of Steps	Time (s)
Binomial	<0.1%	173	0.0109
Trinomial	<0.1%	62	0.0039

2.3 Test for Different Parameters

To find out whether trinomial models are still more efficient for in-the-money and out-of-the-money options, we tested with different strike prices (from \$80 to \$120). Trinomial models generally converge faster than binomial models as well.

Table 2: Error of Computation

Strike Price	Number of Steps	Error of Binomial Model	Error of Trinomial Model
80	10	1.04E-03	-6.64E-04
80	20	9.73E-04	-5.74E-04
80	30	-8.64E-04	-5.22E-04
80	40	4.96E-04	-1.90E-05
80	50	-4.59E-04	7.52E-05

90	10	1.36E-03	2.54E-03
90	20	6.96E-05	-5.58E-04
90	30	2.40E-03	8.49E-04
90	40	1.54E-03	5.16E-04
90	50	-1.15E-04	-4.05E-04
100	10	-1.89E-02	-5.92E-03
100	20	-9.50E-03	-3.00E-03
100	30	-6.35E-03	-2.01E-03
100	40	-4.77E-03	-1.51E-03
100	50	-3.82E-03	-1.21E-03
110	10	9.78E-03	2.66E-03
110	20	-1.22E-02	2.25E-03
110	30	5.03E-03	-1.17E-03
110	40	6.42E-03	2.45E-03
110	50	3.41E-03	7.75E-04
120	10	2.46E-02	1.36E-03
120	20	-3.25E-02	8.32E-05
120	30	8.94E-03	-7.76E-03
120	40	-8.22E-03	-1.91E-04
120	50	-1.64E-03	-7.89E-04

2.4 Test for American Option with Dividends

How does trinomial model work on the more complex American options? Here we tested the convergence speed for American options with parameters above. What is different in this test is that the stock now distributes dividends of \$2 on $T = 0.25$ and of \$1.5 on $T = 0.75$.

The result is below. The convergence of both methods seems unstable, but the trinomial model still converges more quickly than binomial model.

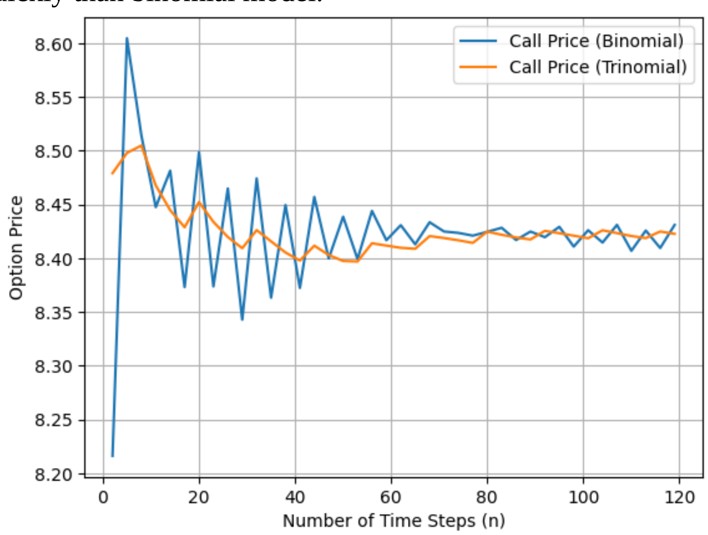


Figure 4: Convergence Speed of Binomial and Trinomial Tree Models for American Options

References

Kamrad, Bardia & Ritchken, Peter. (1991). Multinomial Approximating Models for Options with k State Variables. Management Science. 37. 1640-1652. 10.1287/mnsc.37.12.1640.