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Word Representability of the Graph k-cube

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Introduction

- Word representability of graphs is a fascinating field of study that seeks to understand how to represent a graph using words. The idea is to find a way to encode the structure of a graph into a sequence of symbols, such as letters or numbers, so that the resulting sequence captures the essential features of the graph.
- In this project, we aim to explore the idea of word representability of graphs through the cartesian product of graphs with the help of occurrence-based functions.

Definition

For a word w over an alphabet A, two letters x and y are said to alternate in w if between every two x's in w a y occurs and between every two y's in w an x occurs.

Stated otherwise: removing all letters but x and y from w results in a word $xyxy...$ or $yxyx...$ of even or odd length.

Definition

A word w over an alphabet A is called *k-uniform* if every $x \in A$ occurs exactly k times in w. A 1-uniform word over A is called a *permutation* of A.

We want to use a word over the alphabet V to represent a graph $G = (V, E)$. We will only talk about representing undirected graphs.

Definition

A graph $G = (V, E)$ is *word-representable* if there is a word w over the alphabet V such that:

- $\{x, y\} \in E$ if and only if x and y alternate in w;
- For all $x \in V$: $w_{\{x\}} \neq \epsilon$.

The word w is said to *represent*, or be a *representant* of G, and the graph that is represented by a word w is denoted by $G(w)$.

A word represents a unique graph, while a graph can have multiple words representing it. Also, a graph need not be word-representable.

Theorem

Let w be a k-uniform word that represents a graph G. Then any rotation of w represents G.

Lemma

(Kitaev and Lozin, 2015) Let w be a non-uniform word representing G. Then there exists a uniform word v that represents G.

Cartesian product of two graphs

The *Cartesian product* of two graphs $G = (V_G, E_G)$ and $H = (V_H, E_H)$ is defined as $G \square H = (V_{G\square H}, E_{G\square H})$, where $V_{G\square H} = V_G \times V_H$ and $E_{G\square H} = \{\{(x, x'), (y, y')\} \mid x = y \text{ and } \{x', y'\} \in E_H,$ or $x' = y'$ and $\{x, y\} \in E_G\}$.

The Cartesian product of a graph is interesting because it is easy to find copies of the original graphs in the product. The nodes in the Cartesian product have names (x, y) where x is a node in G and y is a node in H.

Occurrence-based functions

Definition

Let V and V' be (possibly different) alphabets, and let $N_k = \{1, \ldots, k\}$. The *labelling function* of finite words over V is defined by $H : V^* \to (V \times N_k)^*$, where the ith occurrence of each letter x is mapped to the pair (x, i), and k satisfies the property that every symbol occurs at most k times in w. Now $H(w)$ is called the *labelled version* of w.

An occurrence-based function is the composition $(h \circ H)$ of a homomorphism $h : (V \times N_k)^* \to (V')^*$ and the labelling function H. As a shorthand we will write $h(w)$ instead of $h(H(w))$.

Lemma

(Bas Broere and Zantema, 2019) Let w be a k-uniform word representing a graph G. For some $m > 1$ let $A_1, \ldots, A_m$ be non-empty subsets of $N_k = \{1, \ldots, k\}$ such that for all $j = 1, \ldots, k-1$ there exists an $i \in \{1, \ldots, m\}$ for which $\{j, j+1\} \subseteq A_i$. Then the $(\Sigma_{i=1}^{m} \# A_i)$-uniform word $w' = p_{A_1}(w) p_{A_2}(w) \cdots p_{A_m}(w)$ also represents the graph G.

The following algorithm gives a 2-representation w for a tree $G = (V, E)$:

1. $w = 11, C = \emptyset$;
2. While $C \neq V$ do the following:
   2.1 $A = \{x \in V \setminus C \mid w \text{ contains } x\}$;
   2.2 $B = \{\{a, y\} \in E \mid a \in A, y \in V \setminus (A \cup C)\}$;
   2.3 For all $a, y \in B$ replace w by $w = h_{a,y}(w)$, where

$$h_{a,y}(x, i) = \begin{cases} yxy & \text{if } a = x \text{ and } i = 2 \\ x & \text{otherwise} \end{cases}$$

   2.4 Replace $C = C \cup A$.

An Example I

We will represent the tree in figure below

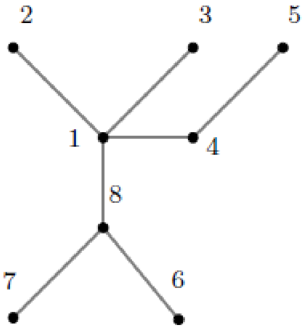

Figure: A tree

We can do steps 1, 2.1 and 2.2 all at once. This gives $w = 11$, $A = \{1\}$, $B = \{\{1,2\},\{1,3\},\{1,4\},\{1,8\}\}$ and $C = \emptyset$. So in 2.3 and 2.4 we get:

- $h_{1,2}(w) = 1212$;
- $h_{1,3}(1212) = 123132$;
- $h_{1,4}(123132) = 12341432$;
- $h_{1,8}(12341432) = 1234818432$;
- $C = \{1\}$.

An Example III

Since $C \neq V$ we have to go on and get $A = \{2, 3, 4, 8\}$, $B = \{\{4, 5\}, \{8, 6\}, \{8, 7\}\}$. After 2.4 this gives $w = h_{8,7}(h_{8,6}(h_{4,5}(w))) = 1234816787654532$ and $C = \{1, 2, 3, 4, 8\}$.
After this we still need to go on, but we have already processed all edges so we will see that $B = \emptyset$ in this step and it results in $C = V$. The word $w = 1234816787654532$ represents the graph.

Cartesian Product with K_2

Theorem

Let G be a k-representable graph for $k > 1$ and let w be a k-representant of G. Then the graph $G \square K_2$ is $(k+1)$-representable with representant $w' = f(w)g(w)$ for the occurrence based functions f, g defined by

$$f(x,i) = \begin{cases} x_1 & \text{if } i = 1 \\ x_2 x_1 & \text{if } 1 < i \leq k \end{cases} \qquad g(x,i) = \begin{cases} x_2 & \text{if } i = 1 \\ x_1 x_2 & \text{if } i = 2 \\ \epsilon & \text{if } 2 < i \leq k \end{cases}$$

The Graph Q_k

Theorem

For every $k \geq 1$, the k-cube Q_k is k-representable.

Proof.

The proof is by induction on k. For $k = 1$ we observe that $Q_1 = K_2$, being 1-representable by the word $w = 12$.
For $k = 2$ we observe that Q_2 is the 4-cycle being 2-representable by the word $w = 31421324$.
For the induction step for $k > 2$, we use Theorem 8 giving a k-uniform representant for Q_k from a $(k-1)$-uniform representant of Q_{k-1}. $\square$

Extension to K_n

Theorem

Let G be a k-representable graph for $k > 1$ and let w be a k-representant of G. Then the graph $G \square K_n$ is $(k+n-1)$-representable with representant $w' = f_n(w) f_{n-1}(w) \cdots f_1(w)$ for the occurrence based functions f_i defined by

$$f_1(x, i) = \begin{cases} x_1 & \text{if } i = 1 \\ x_n x_{n-1} \ldots x_1 & \text{if } 1 < i \leq k \end{cases}$$

and

$$f_j(x, i) = \begin{cases} x_j & \text{if } i = 1 \\ x_{j-1} \ldots x_1 x_n \ldots x_j & \text{if } i = 2 \\ \epsilon & \text{if } 2 < i \leq k \end{cases}$$

for $j = 2, \ldots, n$.

- In this study, we utilized occurrence-based functions as a fundamental tool to construct a word that represents a modified graph, based on a given word representing a graph.
- By applying the key lemma (Lemma 7), we were able to use this approach to construct a word that represents the Cartesian product of a graph with K_n.
- By restricting this approach to the Cartesian product with K_2, we established a construction for a k-uniform word that represents the k-cube graph, Q_k.
- As a result, we have shown that the representation number of Q_k is at most k.

However, it remains an open question whether the representation number of Q_k is exactly k. To prove this, further investigation is needed to determine whether there exists an l-uniform word that represents Q_k, where $l < k$.

References

Broere, B. [B.]. (2018). *Word representable graphs* (Master's thesis). at Radboud University, Nijmegen.

Broere, B. [Bas], & Zantema, H. (2019). The k-dimensional cube is k-representable. *J. Autom. Lang. Comb.*, *24*(1), 3–12. https://doi.org/10.25596/jalc-2019-003

Kitaev, S., & Lozin, V. (2015). *Words and graphs*. Springer.

Thank You

for your attention.

Do you have any question?