A Constant-Factor Approximation Algorithm for the Asymmetric Traveling Salesman Problem

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Introduction to ATSP

- **What is ATSP?**
  - The **Traveling Salesman Problem (TSP)** is a well-known problem in which a salesman must visit a set of cities once and return to the starting city, minimizing total travel cost.
  - The **Asymmetric TSP (ATSP)** variation accounts for **directional distances**, meaning the cost from city A to B may differ from B to A.
- **Challenges in ATSP**:
  - **NP-hard**: Like other combinatorial optimization problems, ATSP is NP-hard, making exact solutions infeasible for large inputs.
  - **Approximation difficulty**: Without symmetry, prior algorithms could only achieve logarithmic approximation ratios.

Problem Statement and Goal

- **Objective**: Find a polynomial-time algorithm for ATSP that provides a constant-factor approximation.
- **Traditional Approach**:
  - The **Held-Karp relaxation** is often used as an LP relaxation to find a lower bound on TSP solutions.
  - Known integrality gap of **at least 2** for ATSP, making constant-factor approximations difficult.
- **Contribution of the Paper**:
- This paper achieves a constant-factor approximation for ATSP using a novel approach, specifically through a **reduction to Local-Connectivity ATSP** and **structured instance simplification**.

- **Definition**: **ATSP Instance**
- Formally, an ATSP instance is defined as a directed graph G=(V,E) where V represents cities and EE contains weighted edges with distances $w(e) \geq 0$ for all $e \in E$.
- **Definition**: **Held-Karp Relaxation**
- The **Held-Karp relaxation** (HK) is an LP relaxation for TSP, setting the basis for many approximations by relaxing the requirement for an integer solution:

$$\min \sum_{e \in E} w(e) \cdot x(e)$$

- Subject to:
- $x(\delta+(v)) = x(\delta-(v)) = 1$ for each $v \in V$.
- $x(\delta(S)) \geq 2$ for all subsets $S \subset V$.
- $x(e) \geq 0$ for all $e \in E$.
- This LP provides a **lower bound on tour costs** but cannot guarantee integer solutions. Its integrality gap is critical to approximations in ATSP..

Previous Work and Challenges

- **Previous Work and Challenges**
- **Previous Algorithms**:
  - Early solutions for ATSP achieved an approximation factor of (O(log(n))).
  - **Asadpour et al. (2010)** introduced **thin spanning trees** for approximations, achieving a (O(\log(n)/log(log(n)))) factor.
- **Challenges of ATSP**:
  - Unlike symmetric TSP, where Christofides' algorithm achieves a 1.5-approximation, **ATSP lacks such straightforward approximations**.
  - The **integrality gap** of the Held-Karp relaxation is bounded below by 2, making constant approximations elusive.

Main Contributions of the Paper

- This paper's main contributions are:
- **Constant-Factor Approximation**: The authors provide the first constant-factor approximation for ATSP.
- **Structured Reductions**:
  - Reduce ATSP to simpler cases, like **laminarly-weighted instances**.
- **Local-Connectivity ATSP**:
  - Simplifies ATSP by introducing a framework to handle local subtours across defined partitions of vertices.

Core Methodology Overview

- **Methodology**:
- **Held-Karp LP Relaxation**: Solves ATSP by obtaining a lower bound.
- **Laminarly-Weighted Instances**: Reduces complex graphs into simpler structures.
- **Recursive Reduction**:
  - The reduction is recursive, starting from general cases to **irreducible instances** and finally to **vertebrate pairs**.

Core Definitions and Reductions

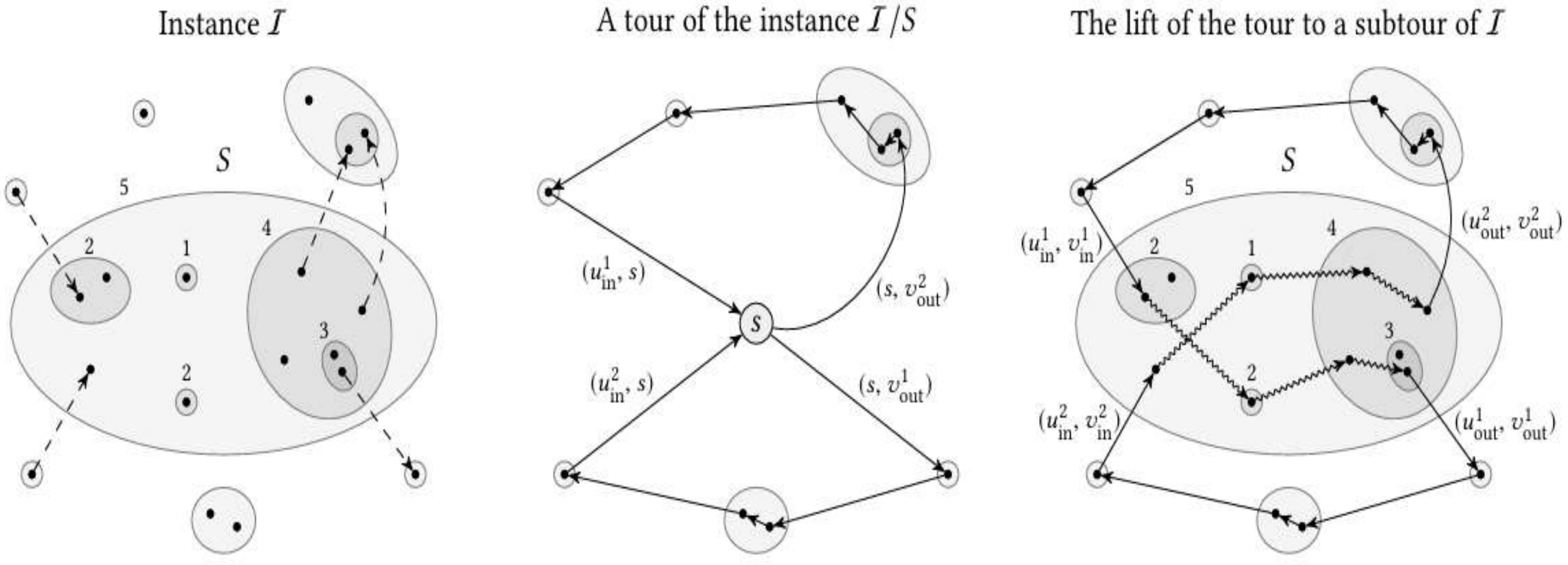
- **Definition**: **Laminar Family of Sets**
- A **laminar family** ({L}) is a collection of subsets such that for any two sets $A,B \in L$, either $A \cap B = \emptyset$, $A \subseteq B$, or $B \subseteq A$.
- **Reduction to Laminarly-Weighted Instances**:
- Instances where weights on edges can be represented using a laminar family of sets, making the problem more tractable.
- **Reduction Process**:
- **Held-Karp LP → Laminarly-Weighted Instances → Irreducible Instances → Vertebrate Pairs**.

Theorem on Constant-Factor Approximation

- **Theorem 1**: There exists a polynomial-time algorithm for ATSP that produces a solution with a weight within a constant factor of the Held-Karp lower bound.
- **Proof Outline**:
  - By recursively reducing the ATSP instance to vertebrate pairs, each sub-instance maintains a bounded approximation ratio.
  - Finally, the solution to the vertebrate pair approximates the original ATSP instance within a constant factor.

Reduction to Irreducible Instances

- **Irreducible Instances**: Instances where no further reduction is possible without losing essential connectivity.
- **Key Property**: Irreducible instances provide a bounded cost reduction, simplifying approximations.



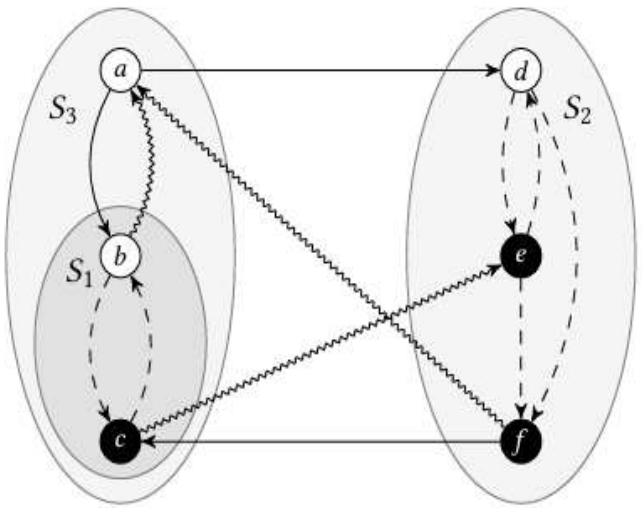
Vertebrate Pairs and Backbone Structure

- **Definition**: **Vertebrate Pair**:
  - A structured instance with a **backbone path** crossing all critical subsets.
- **Use in Approximation**:
  - Allows the algorithm to maintain connectivity across key points with manageable subtours.

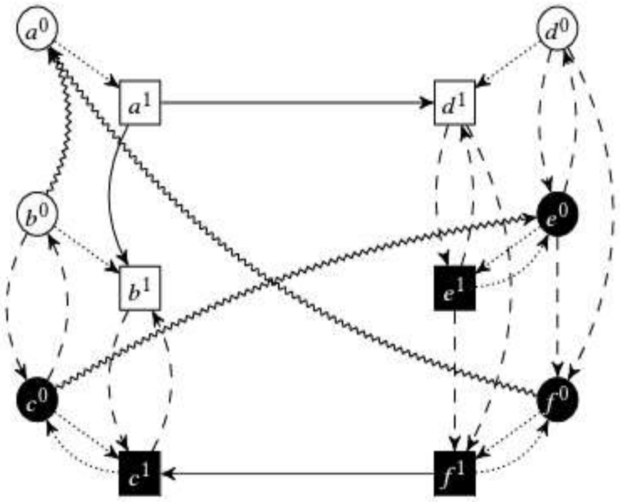
Local-Connectivity ATSP

- **Definition**: **Local-Connectivity ATSP**
- Finds subtours that only need to connect specific vertex sets.
- **Algorithm**:
- Ensures each subtour is an **Eulerian edge set**, which collectively approximate the full tour in ATSP.

G and $\mathcal{L}_{\geq 2} = \{S_1, S_2, S_3\}$



G_{sp}



Results and Implications

- **Results**:
  - Constant-factor approximation achieved for ATSP.
  - Bounded the integrality gap of Held-Karp to a constant for ATSP.
- **Implications**:
  - This technique can apply to other directed graph problems, paving the way for constant approximations in asymmetric routing.

Conclusion and Future Work

- **Conclusion**: This work demonstrates the first constant-factor approximation algorithm for ATSP, addressing a major open problem in approximation algorithms.
- **Future Directions**:
  - Optimizing the constant factor further.
  - Applying similar reductions to related NP-hard problems, such as **bottleneck ATSP**.

References

- **Primary Paper**: Svensson, Tarnawski, Végh. *A Constant-Factor Approximation Algorithm for the Asymmetric Traveling Salesman Problem*. STOC 2018.
- Additional sources on ATSP approximations.