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Powers of componentwise linear ideals: the Herzog–Hibi–Ohsugi conjecture and related problems

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Dedicated to Jürgen Herzog on
the occasion of his **80th** birthday

Abstract

In 1999, Herzog and Hibi introduced componentwise linear ideals. A homogeneous ideal I is componentwise linear if for all nonnegative integers d, the ideal generated by the homogeneous elements of degree d in I has a linear resolution. For square-free monomial ideals, componentwise linearity is related via Alexander duality to the property of being sequentially Cohen–Macaulay for the corresponding simplicial complexes. In general, the property of being componentwise linear is not preserved by taking powers. In 2011, Herzog, Hibi, and Ohsugi conjectured that if I is the cover ideal of a chordal graph, then I^s is componentwise linear for all $s \geq 1$. We survey some of the basic properties of componentwise linear ideals and then specialize to the progress on the Herzog–Hibi–Ohsugi conjecture during the last decade. We also survey the related problem of determining when the symbolic powers of a cover ideal are componentwise linear

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1 Introduction

Let R be a polynomial ring over a field k and let $I \subseteq R$ be a homogeneous ideal. The ideal I is often considered computationally simple if it has a *linear resolution*. In 1999, Herzog and Hibi [32] defined the "next best" class of ideals, called the componentwise linear ideals. Specifically, by letting $\langle I_d \rangle$ be the ideal generated by homogeneous polynomials of degree d in I, the ideal I is called a *componentwise linear* ideal if $\langle I_d \rangle$ has a linear resolution for all $d \geq 0$.

Componentwise linear ideals and their properties have become ubiquitous in commutative algebra, especially in combinatorial commutative algebra. Indeed, as shown in Herzog and Hibi's original paper, the square-free monomial ideals with the componentwise linear property provide an algebraic characterization of sequentially Cohen–Macaulay simplicial complexes. Componentwise linear ideals are also closely related to another class of ideals that have been extensively examined. A finitely generated R-module M is called

a *Koszul* module if its associated graded module $\operatorname{gr}_{\mathfrak{m}}(M) = \bigoplus_{i \geq 0} \mathfrak{m}^i M / \mathfrak{m}^{i+1} M$, with respect to the maximal homogeneous ideal $\mathfrak{m}$ of R, has a linear resolution over the associated graded ring $\operatorname{gr}_{\mathfrak{m}}(R) = \bigoplus_{i \geq 0} \mathfrak{m}^i / \mathfrak{m}^{i+1}$. Koszul algebras (first introduced by Priddy [47]) and Koszul modules (first considered by Şega [50] and by Herzog and Iyengar [38]) have enjoyed much attention from various areas of mathematics. Over a polynomial ring, a homogeneous ideal I is Koszul if and only if it is componentwise linear (see [49, Theorem 3.2.8] and [62, Proposition 4.9]), thus cementing the importance of componentwise linearity.

In the two decades since their introduction, one research theme in commutative algebra is to find new classes of componentwise linear ideals. At the same time, a broader research theme is to understand how properties of an ideal are preserved when taking powers. It is natural to ask the following question: If I is an ideal that is componentwise linear, does the ideal I^s also have this property? The answer to this question will be, in general, no. In fact, there are examples of ideals such that I has a linear resolution (and so is componentwise linear), but I^2 does not have a linear resolution. For a specific example, see Example 4.2.

One is then led to ask what extra hypotheses are required on I to ensure that I^s is also componentwise linear. The first investigation into this question was carried out by Herzog et al. [36]. While we will define necessary terminology in later sections, for now, it is enough to know that the ideal I in the statement below is constructed from the properties of a finite simple graph. Based upon their results and experiments, Herzog, Hibi, and Ohsugi posited the following conjecture:

Conjecture 1.1 (Herzog–Hibi–Ohsugi) *Let I be the cover ideal of a chordal graph. Then, I^s is componentwise linear for all $s \geq 1$.*

One of the main goals of this paper is to provide an up-to-date survey on what is known about Conjecture 1.1, and what is known about the more general question of powers of ideals that are componentwise linear. Besides describing which families of chordal graphs satisfy the above conjecture, we will also sketch out the broad strategies that have been used to verify the conjecture.

We will also survey a variation of Conjecture 1.1, which was initiated by Seyed Fakhari [53]. In this variation, it is asked whether or not $I^{(s)}$, the s-th symbolic power of the cover ideal, is componentwise linear. The two problems dovetail when the regular powers of a cover ideal are the same as its symbolic powers. This approach is typified by Kumar and Kumar's recent proof that Conjecture 1.1 holds for all trees (see [41]); in the case of trees, regular powers and symbolic powers of cover ideals agree, which allow Kumar and Kumar to exploit the properties of symbolic powers of ideals.

The majority of the results in this paper have appeared in the literature. We have, however, included one new result about edge ideals. In particular, we show that the edge ideals of complete m-partite graphs have the property that all of their symbolic powers are componentwise linear (see Corollary 5.22). To encourage further work on Conjecture 1.1, we have included some research questions in the final section.

We end this introduction with some final comments on our intended audience. When writing this survey, we assumed that our readers are familiar with minimal graded free resolutions, and possibly the Stanley–Reisner correspondence between simplicial complexes and square-free monomial ideals. Our goal in Sects. 2 and 3 is to provide a quick

summary of some of the basic facts surrounding componentwise linear ideals, plus pointers to the literature for proofs of these facts. We expect graduate students and researchers new to componentwise linear ideals will appreciate this approach. In the second half of the paper, we have been attempted to be more encyclopedic in our approach to the Herzog–Hibi–Ohsugi Conjecture, and the related questions concerning the symbolic powers. We hope experts will appreciate this snapshot of the current state-of-the-art regarding these problems.

2 Basics of componentwise linear ideals

In this section, we introduce componentwise linear ideals, and describe some of their basic properties. We emphasize techniques to determine if an ideal is componentwise linear (a theme that will be stressed throughout this survey). Let k be a field, and let $R = k[x_1, \ldots, x_n]$ denote a polynomial ring over k.

We begin by recalling the notion of a minimal graded free resolution. Given a homogeneous ideal I of R (an ideal generated by homogeneous elements of R), we can associate to I a *minimal graded free resolution*, that is, a long exact sequence of length $p \leq n$ of graded R-modules of the form

$$0 \to \bigoplus_{j \in \mathbb{N}} R(-j)^{\beta_{p,j}(I)} \to \bigoplus_{j \in \mathbb{N}} R(-j)^{\beta_{p-1,j}(I)} \to \cdots \to \bigoplus_{j \in \mathbb{N}} R(-j)^{\beta_{0,j}(I)} \to I \to 0.$$

Here, $R(-j)$ denotes the twisted graded R-module formed by setting $R(-j)_d = R_{d-j}$. The invariants $\beta_{i,j}(I)$ are the (i,j)-*th graded Betti numbers of* I, and they count the number of degree j generators of the i-th syzygy module of I.

We write $I = \langle f_1, \ldots, f_t \rangle$ if I is generated by $\{f_1, \ldots, f_t\}$. A homogeneous ideal $I = \langle f_1, \ldots, f_t \rangle$ has a *linear resolution* if $\deg f_1 = \cdots = \deg f_t = d$ for some integer $d \geq 0$ and, for all $i \geq 1$,

$$\beta_{i,i+j}(I) = 0 \text{ for all } j \neq d.$$

This is equivalent to the fact that the minimal graded free resolution of I has the form

$$0 \to R(-d-p)^{\beta_{p,p+d}(I)} \to \cdots \to R(-d-1)^{\beta_{1,1+d}(I)} \to R(-d)^{\beta_{0,d}(I)} \to I \to 0.$$

Example 2.1 Let $I = \langle x_1 x_2, x_2 x_3, x_3 x_4 \rangle$ in $R = k[x_1, x_2, x_3, x_4]$. Then, the minimal graded free resolution of I has the form

$$0 \to R(-3)^2 \to R(-2)^3 \to I \to 0.$$

So, the ideal I has a linear resolution.

On the other hand, consider the ideal $J = \langle x_1 x_3, x_2 x_3, x_1 x_2 x_4 x_5 \rangle$ in $R = k[x_1, \ldots, x_5]$. Note that J is not generated by forms of the same degree, so it cannot have a linear resolution. In particular, the graded free resolution of J has the form

$$0 \to R(-3)^1 \oplus R(-5)^1 \to R(-2)^2 \oplus R(-4)^1 \to J \to 0.$$

We will return to these ideals below.

We are now in a position to define the main objects of this survey. For any homogeneous ideal $I \subseteq R$ and integer $d \geq 0$, let

$$\langle I_d \rangle = \langle \{ F \in I \mid F \text{ is homogeneous of degree } d \} \rangle$$

denote the ideal generated by all the homogeneous elements of degree d in I. The following definition is then due to Herzog and Hibi [32]:

Definition 2.2 (*Herzog–Hibi*) A homogeneous ideal I of R is *componentwise linear* if for all integers $d \geq 0$, the ideal $\langle I_d \rangle$ has a linear resolution.

Based only upon the above definition, verifying whether or not a homogeneous ideal I is componentwise linear would involve checking an infinite number of conditions. Fortunately, it is possible to verify if I is componentwise linear by only checking a finite number of $d \in \mathbb{N}$ by using the Castelnuovo–Mumford regularity of I.

The *(Castelnuovo–Mumford) regularity of I* is defined to be

$$\mathrm{reg}(I) = \max\{j - i \mid \beta_{i,j}(I) \neq 0\}.$$

Roughly speaking, $\mathrm{reg}(I)$ measures the largest degree of a generator of a syzygy of I. This invariant can also be viewed as a measure of the complexity of I. We then require the following well-known fact (see, for example, [22, Proposition 1.1]).

Theorem 2.3 *Let I be a homogeneous ideal with $r = \mathrm{reg}(I)$. Then for all $d \geq r$, the ideal $\langle I_d \rangle$ has a linear resolution.*

Consequently, determining whether I is componentwise linear can be reduced to checking a finite number of cases.

Corollary 2.4 *Let I be a homogeneous ideal with $r = \mathrm{reg}(I)$. Then, I is componentwise linear if and only if $\langle I_d \rangle$ has a linear resolution for all $0 \leq d \leq r$. In particular, if I has a linear resolution, then I is also componentwise linear.*

Proof The if-and-only if statement follows directly from Theorem 2.3 and the definition. If I has a linear resolution, then all the homogeneous generators of I have the same degree, which coincides with its regularity r. Thus, $\langle I_d \rangle = \langle 0 \rangle$ for all $0 \leq d < r$, and $\langle I_r \rangle = I$. The conclusion now follows from the first part of the statement. □

Example 2.5 Let $J = \langle x_1 x_3, x_2 x_3, x_1 x_2 x_4 x_5 \rangle$ in $R = k[x_1, \ldots, x_5]$ be the ideal of Example 2.1. From the minimal graded free resolution of J given in Example 2.1, we have

$$\mathrm{reg}(J) = \max\{3 - 1, 5 - 1, 2 - 0, 4 - 0\} = 4.$$

So, by Corollary 2.4, we need to compute the minimal graded free resolutions of $\langle J_d \rangle$ for $d = 0, \ldots, 4$. Since $\langle J_d \rangle = \langle 0 \rangle$ for $d = 0, 1$, we focus on the remaining cases. For the ideal $\langle J_2 \rangle = \langle x_1 x_3, x_2 x_3 \rangle$, we have

$$0 \to R(-3) \to R(-2)^2 \to \langle J_2 \rangle \to 0;$$

for the ideal

$$\langle J_3 \rangle = \langle x_1^2 x_3, x_1 x_2 x_3, x_1 x_3 x_4, x_1 x_3 x_5, x_2^2 x_3, x_2 x_3^2, x_2 x_3 x_4, x_2 x_3 x_5 \rangle$$

the resolution is

$$0 \to R(-7)^2 \to R(-6)^{10} \to R(-5)^{20} \to R(-4)^{20} \to R(-3)^9 \to \langle J_3 \rangle \to 0;$$

and for the ideal $\langle J_4 \rangle$ (we suppress the 26 minimal generators) the resolution has the form

$$0 \to R(-8)^9 \to R(-7)^{43} \to R(-6)^{80} \to R(-5)^{71} \to R(-4)^{26} \to \langle J_4 \rangle \to 0.$$

Thus, the ideal J is componentwise linear.

We give some alternative ways to characterize when an ideal is componentwise linear. One such characterization is in terms of the generic initial ideal. We review the relevant terminology; for more on generic initial ideals, see [29].

Let $GL_n(k)$ denote the general linear group of order n over k, i.e., all the $n \times n$ invertible matrices with entries in k. Any matrix $g \in GL_n(k)$ acts on the variables $x = \{x_1, \ldots, x_n\}$ by a linear change of variables, i.e., x_i is sent to $g_{i,1}x_1 + g_{i,2}x_2 + \cdots + g_{i,n}x_n$ for $i = 1, \ldots, n$, where $(g_{i,1}, \ldots, g_{i,n})$ is the i-th row of g. Given $g \in GL_n(k)$ and a polynomial $f = f(x_1, \ldots, x_n) \in R$, then g acts on f by $g \cdot f := f(g \cdot x)$. Now, fix an ideal I of R and monomial order $>$. Every matrix $g \in GL_n(k)$ results in an initial ideal $\mathrm{in}_>(g \cdot I)$ where $g \cdot I = \langle g \cdot f \mid f \in I \rangle$. We say that two matrices g and g' are *equivalent* if

$$\mathrm{in}_>(g \cdot I) = \mathrm{in}_>(g' \cdot I).$$

This definition induces an equivalence relation on $GL_n(k)$, and thus, the equivalence classes partition the group $GL_n(k)$.

One of these partitions is quite "large" in the following sense:

Lemma 2.6 ([34, Theorem 4.1.2]) *For a fixed I and term order $>$, one of the equivalence classes is a non-empty Zariski open subset U inside $GL_n(k)$.*

Although we do not go into the details here, the Zariski open set inside of $GL_n(k)$ refers to the zero set of some ideal in $k[y_{i,j} \mid 1 \leq i, j \leq n]$. Note that for all $g \in U$, the Zariski open subset in Lemma 2.6, the initial ideal $\mathrm{in}_>(g \cdot I)$ is the same ideal.

Definition 2.7 Fix a term order $>$ on R, let I be an ideal of R, and let g be any element of the open Zariski subset of Lemma 2.6. The initial ideal $\mathrm{in}_>(g \cdot I)$ is called the *generic initial ideal* of I for the term order $>$. It is denoted $\mathrm{gin}_>(I) = \mathrm{in}_>(g \cdot I)$.

Roughly speaking, the generic initial ideal is the ideal we should expect if we pick a "random" matrix $g \in GL_n(k)$ and form $\mathrm{in}_>(g \cdot I)$.

With this terminology, we then have the following equivalent statements. In particular, an ideal I is componentwise linear if I and the generic initial ideal I with respect to the reverse lexicographical order have the same number of generators in each degree. We want to highlight that this statement also requires the hypothesis that the field k has characteristic zero.

Theorem 2.8 *Let I be a homogeneous ideal of R. Assume* $\mathrm{char}(k) = 0$. *Then, the following are equivalent:*

(1) *I is componentwise linear*
(2) *$\beta_{i,j}(I) = \beta_{i,j}(\mathrm{gin}_>(I))$ for all $i, j \geq 0$ where $>$ is the reverse lexicographical order.*
(3) *$\beta_{0,j}(I) = \beta_{0,j}(\mathrm{gin}_>(I))$ for all $j \geq 0$ where $>$ is the reverse lexicographical order.*

Proof The equivalence of (1) and (2) is [2, Theorem 1.1]. The equivalence of (1) and (3) first appears in the paper of Conca [11, Theorem 1.2], although Conca points out that this equivalence is implicit in the proof of [2, Theorem 1.1]. □

Example 2.9 Theorem 2.8 gives an alternative way to determine if an ideal is component-wise linear. The computer algebra package Macaulay2 [28] is able to compute the generic

initial ideal (although the algorithm is only probabilistic in the sense that it computes the initial ideal $g \cdot I$ for a random $g \in GL_n(k)$).

Here is a sample session applied to the componentwise linear ideal J of Example 2.1.

```
i1 : R = QQ[x_1..x_5]

i2 : j = monomialIdeal(x_1*x_3,x_2*x_3,x_1*x_2*x_4*x_5)

i3 : loadPackage ''GenericInitialIdeal''

i4 : gin j

o4 = ideal (x_1^2, x_1*x_2, x_2^4)
```

Since the ideal and its generic initial ideal have the same number of generators in each degree, the ideal is componentwise linear.

As mentioned in the introduction, componentwise linear ideals are related to the notion of Koszulness. We record this equivalence.

Theorem 2.10 *Let I be a homogeneous ideal of R. Assume* $\mathrm{char}(k) = 0$*. Then, I is componentwise linear if and only if the R-module I is a Koszul module.*

Proof See Römer [49, Theorem 3.2.8] or Yanagawa [62, Proposition 4.9].   □

Remark 2.11 Because of the equivalence of Theorem 2.10, componentwise linear ideals are sometimes called *Koszul ideals*; for example, see [16].

We end this section by introducing linear quotients, an extremely useful technique to show that an ideal is componentwise linear. Ideals with linear quotients were first defined by Herzog and Takayama [39] for monomial ideals; the more general definition is given below.

Definition 2.12 A homogeneous ideal I has *linear quotients* if the minimal generators of I can be ordered as $f_1, \ldots, f_s$ such that for each $i = 2, \ldots, s$, the ideal $\langle f_1, \ldots, f_{i-1} \rangle : \langle f_i \rangle$ is generated by linear forms.

Linear quotients can then be used to verify that an ideal is componentwise linear:

Theorem 2.13 ([34, Theorem 8.2.15]) *Let I be a homogeneous ideal, and suppose that I has linear quotients with respect to a minimal set of generators of I. Then, I is componentwise linear.*

Example 2.14 If we consider the ideal $J = \langle x_1 x_3, x_2 x_3, x_1 x_2 x_4 x_5 \rangle$ of Example 2.1, we have

$$\langle x_1 x_3 \rangle : \langle x_2 x_3 \rangle = \langle x_1 \rangle \quad \text{and} \quad \langle x_1 x_3, x_2 x_3 \rangle : \langle x_1 x_2 x_4 x_5 \rangle = \langle x_1, x_2 \rangle.$$

The ideal J has linear quotients, thus giving another way of seeing that J is componentwise linear.

Remark 2.15 We want to stress that the property of being componentwise linear (or having a linear resolution) depends upon the characteristic of the field k. A well-known example of this phenomenon, due to Reisner [48], is the square-free monomial ideal

$$I = \langle x_1 x_2 x_3, x_1 x_2 x_6, x_1 x_3 x_5, x_1 x_4 x_5, x_1 x_4 x_6, x_2 x_3 x_4, x_2 x_4 x_5, x_2 x_5 x_6, x_3 x_4 x_6, x_3 x_5 x_6 \rangle$$

in $R = k[x_1, \ldots, x_6]$. The ideal I has a linear resolution if and only if char$(k) \neq 2$. So I is componentwise linear if and only if char$(k) \neq 2$. This example also shows that the converse of Theorem 2.13 cannot hold, since linear quotients are a property that does not "see" the characteristic of the field.

3 Componentwise linearity of (square-free) monomial ideals

In this section, we recall properties of (square-free) monomial ideals that are componentwise linear. A highlight of this section is Herzog and Hibi's classification of square-free monomial ideals that are componentwise linear, which was one of main results in the paper that introduced componentwise linearity [32].

Let $V = \{x_1, \ldots, x_n\}$ be a collection of vertices. A *simplicial complex* Δ on V is a subset of the power set of V that satisfies the following two properties: (1) If $F \in \Delta$ and if $G \subseteq F$, then $G \in \Delta$; and (2) $\{x_i\} \in \Delta$ for all $i = 1, \ldots, n$. The maximal elements of Δ ordered with respect to inclusion are called the *facets* of Δ. If $F_1, \ldots, F_s$ is a complete list of the facets of Δ, then we usually write $\Delta = \langle F_1, \ldots, F_s \rangle$. In this case, we say Δ is generated by $F_1, \ldots, F_s$. An element $F \in \Delta$ is called a *face* of Δ. The *dimension* of F is dim $F = |F| - 1$. (We use the convention that dim $\emptyset = -1$.) The *dimension* of a simplicial complex is dim $\Delta = \max\{\dim F \mid F \in \Delta\}$. A simplicial complex is *pure* if all of its facets have the same dimension.

We associate with Δ a square-free monomial ideal I_Δ in the polynomial ring $R = k[x_1, \ldots, x_n]$ as follows:

$$I_\Delta = \langle x_{i_1} \cdots x_{i_s} \mid \{x_{i_1}, \ldots, x_{i_s}\} \notin \Delta \rangle.$$

The ideal I_Δ is the *Stanley–Reisner ideal* of Δ; it captures many of the combinatorial properties of Δ. This construction can be reversed; that is, given any square-free monomial ideal I, we can construct its *Stanley–Reisner simplicial complex*

$$\Delta_I = \{\{x_{i_1}, \ldots, x_{i_s}\} \mid x_{i_1} \cdots x_{i_s} \text{ is a square-free monomial not in } I\}.$$

Example 3.1 Consider our running example (see Example 2.1)

$$J = \langle x_1 x_3, x_2 x_3, x_1 x_2 x_4 x_5 \rangle \subseteq k[x_1, \ldots, x_5].$$

On the vertex set $V = \{x_1, \ldots, x_5\}$, we then have

$$\Delta_J = \langle \{x_3, x_4, x_5\}, \{x_2, x_4, x_5\}, \{x_1, x_4, x_5\}, \{x_1, x_2, x_5\}, \{x_1, x_2, x_4\} \rangle.$$

We say that Δ is a *(sequentially) Cohen–Macaulay* simplicial complex if the quotient ring R/I_Δ is (sequentially) Cohen–Macaulay. Particularly, sequentially Cohen–Macaulay simplicial complexes are described as follows:

Definition 3.2 Let Δ be a simplicial complex with dim $\Delta = d$. For each $i = -1, \ldots, d$, let $\Delta(i) = \langle F \in \Delta \mid \dim F = i \rangle$, i.e., the simplicial complex generated by all the faces of dimension i in Δ. Then, Δ is *sequentially Cohen–Macaulay* if $\Delta(i)$ is Cohen–Macaulay for all $i = -1, \ldots, d$.

Given a simplicial complex Δ, the *Alexander dual* of Δ is the simplicial complex

$$\Delta^\vee = \langle V \setminus F \mid F \notin \Delta \rangle.$$

Using the above terminology, square-free monomial ideals that are componentwise linear can be classified. This result, which is due to Herzog and Hibi, is one of the first major results about componentwise linear ideals.

Theorem 3.3 ([32, Theorem 2.1]) *Let I be a square-free monomial ideal, and let $\Delta = \Delta_I$ be its Stanley–Reisner simplicial complex. Then, I is componentwise linear if and only if $\Delta^\vee$ is sequentially Cohen–Macaulay.*

Example 3.4 As noted multiple times, the ideal $J = \langle x_1 x_3, x_2 x_3, x_1 x_2 x_4 x_5 \rangle$ is componentwise linear. So the simplicial complex

$$\Delta_J^\vee = \langle \{x_3\}, \{x_2, x_4, x_5\}, \{x_1, x_4, x_5\} \rangle$$

is sequentially Cohen–Macaulay.

The above theorem generalizes an important result of Eagon and Reiner. We also record this result.

Theorem 3.5 ([17, Theorem 3]) *Let I be a square-free monomial ideal, and let $\Delta = \Delta_I$ be its Stanley–Reisner simplicial complex. Then, I has a linear resolution if and only if $\Delta^\vee$ is Cohen–Macaulay.*

Example 3.6 If I is the square-free monomial ideal of Remark 2.15, then the ideal I has a linear resolution if and only if $\mathrm{char}(k) \neq 2$. So $\Delta_I^\vee$ is Cohen–Macaulay if and only if $\mathrm{char}(k) \neq 2$.

Theorem 3.3 provides a new strategy to prove that an ideal is componentwise linear. In particular, instead of showing that I is componentwise linear, it is enough to show that $\Delta_I^\vee$ is sequentially Cohen–Macaulay. There are two combinatorial ways to determine if a simplicial complex is sequentially Cohen–Macaulay; we first give some relevant definitions.

For a vertex x in Δ, the *deletion* of x in Δ, denoted by $\mathrm{del}_\Delta(x)$, is the simplicial complex obtained by removing x and all faces containing x from Δ. Also, the *link* of x in Δ, denoted by $\mathrm{link}_\Delta(x)$, is the simplicial complex whose faces are

$$\{F \in \Delta \mid x \notin F \text{ and } F \cup \{x\} \in \Delta\}.$$

Definition 3.7 Let $\Delta = \langle F_1, \ldots, F_s \rangle$ be a simplicial complex.

(1) The complex Δ is *shellable* if there exists a linear order of its facets $F_1, \ldots, F_s$ such that for all $i = 2, \ldots, s$, the subcomplex $\langle F_1, \ldots, F_{i-1} \rangle \cap \langle F_i \rangle$ is pure and of dimension $(\dim F_i - 1)$.

(2) The complex Δ is *vertex decomposable* if either:

    (a) Δ is a *simplex* (i.e., it has a unique facet); or the empty complex; or

    (b) there exists a vertex x in Δ such that all facets of $\mathrm{del}_\Delta(x)$ are facets of Δ (i.e., x is a *shedding vertex*), and both $\mathrm{link}_\Delta(x)$ and $\mathrm{del}_\Delta(x)$ are vertex decomposable.

Remark 3.8 The notions of shellability and vertex decomposability were first given for *pure* simplicial complexes, that is, when all the facets have the same dimension. These notions were generalized by Björner and Wachs [5,6] to include non-pure simplicial complexes.

One approach in the literature (which we will return to later in the paper) to show a square-free monomial ideal is componentwise linear is to show that the associated simplicial complex has one of the above properties.

Theorem 3.9 *Let I be a square-free monomial ideal with associated simplicial complex $\Delta = \Delta_I$. If $\Delta^{\vee}$, the Alexander dual of Δ, is either vertex decomposable or shellable, then I is componentwise linear.*

Proof By [6, Theorem 11.3], a vertex decomposable simplicial complex Γ is also shellable. Stanley (see page 87 of [56]) first observed that if Γ is a shellable simplicial complex, then Γ is sequentially Cohen–Macaulay. The conclusion then follows from Theorem 3.3 since both hypotheses imply that $\Delta^{\vee}$ is sequentially Cohen–Macaulay.　　　□

For the case of square-free monomial ideals, there is an alternative way to verify that the ideal is componentwise linear. Given a square-free monomial ideal I, let $I_{[d]}$ be the ideal generated by all the square-free monomials of degree d in I. We then say I is *square-free componentwise linear* if $I_{[d]}$ has a linear resolution for all d. Checking whether a square-free monomial ideal is componentwise linear then reduces to checking whether or not it is square-free componentwise linear.

Theorem 3.10 ([32, Proposition 1.5]) *Let I be a square-free monomial ideal. Then, I is componentwise linear if and only if I is square-free componentwise linear.*

Remark 3.11 If I is a square-free monomial ideal in $R = k[x_1, \ldots, x_n]$, then I cannot have any square-free monomials of degree $> n$. So, to check if I is square-free componentwise linear, we only need to check that $I_{[d]}$ has a linear resolution for $0 \le d \le n$. In fact, since there is only one square-free monomial of degree n, namely $x_1 \cdots x_n$, either $I_{[n]} = \langle 0 \rangle$ or $I_{[n]} = \langle x_1 \cdots x_n \rangle$, and so $I_{[n]}$ always has a linear resolution. So we only need to check for $0 \le d < n$.

We expand our scope to now say a few words about monomial ideals more generally, and not just the square-free case. One common approach to studying monomial ideals is to use the process of polarization to turn the monomial ideal into a square-free monomial ideal in a much larger polynomial ring. In many instances, properties of the original ideal are preserved in the new ideal, and vice versa. It turns out that the linear quotient property is preserved under this operation.

We formally define the polarization procedure. Let $I = \langle x_1^{a_{1,1}} \cdots x_n^{a_{1,n}}, \ldots, x_1^{a_{t,1}} \cdots x_n^{a_{t,n}} \rangle$ be a monomial ideal in $R = k[x_1, \ldots, x_n]$. For $j = 1, \ldots, n$, set $b_j = \max\{a_{i,j} \mid 1 \le i \le t\}$; that is, b_j is the highest power of x_j that appears among the generators of I. In a polynomial ring

$$S = k[x_{1,1}, \ldots, x_{1,b_1}, x_{2,1}, \ldots, x_{2,b_2}, \ldots, x_{n,1}, \ldots, x_{n,b_n}]$$

we define the *polarization* of I to be the ideal

$$I^{\mathrm{pol}} = \langle x_{1,1} \cdots x_{1,a_{1,1}} x_{2,1} \cdots x_{2,a_{1,2}} \cdots x_{n,1} \cdots x_{n,a_{1,n}}, \ldots, x_{1,1} \cdots x_{1,a_{t,1}} x_{2,1} \cdots x_{2,a_{t,2}} \cdots x_{n,1} \cdots x_{n,a_{t,n}} \rangle.$$

That is, we replace $x_j^{a_{k,j}}$ with $x_{j,1} \cdots x_{j,a_{k,j}}$ in each generator of I.

As we saw earlier, linear quotients are a technique that can be used to check if an ideal is componentwise linear. The following result of Seyed Fakhari shows that for an arbitrary monomial ideal, we can use the polarization of the ideal to check if the original ideal has linear quotients.

Theorem 3.12 ([53], Lemma 3.5]) *Let I be a monomial ideal. Then, I has linear quotients if and only if I^{pol}, the polarization of I, has linear quotients.*

Example 3.13 Let $I = \langle y_1 y_3, y_2 y_3, y_1 y_2 y_4^2 \rangle$ in $R = k[y_1, y_2, y_3, y_4]$. We then have $b_1 = 1$, $b_2 = 1$, $b_3 = 1$, and $b_4 = 2$. In the polynomial ring $S = k[y_{1,1}, y_{2,1}, y_{3,1}, y_{4,1}, y_{4,2}]$, the polarization of I is

$$I^{\mathrm{pol}} = \langle y_{1,1} y_{3,1}, y_{2,1} y_{3,1}, y_{1,1} y_{2,1} y_{4,1} y_{4,2} \rangle.$$

If we relabel the variables so that $x_1 = y_{1,1}$, $x_2 = y_{2,1}$, $x_3 = y_{3,1}$, $x_4 = y_{4,1}$ and $x_5 = y_{4,2}$, then I^{pol} is the ideal J of our running example. So I has linear quotients (which can also be checked directly).

There are many classes of monomial ideals which have been identified as componentwise linear. While we do not survey all of this literature (since we wish to focus on powers of ideals), we highlight some families that are relevant for our future discussions.

A monomial ideal I is *weakly polymatroidal* if for every pair of minimal generators $m_1 = x_1^{a_1} \cdots x_n^{a_n}$ and $m_2 = x_1^{b_1} \cdots x_n^{b_n}$ with $m_1 > m_2$ with respect to the the lexicographical ordering, and if $a_1 = b_1, \ldots, a_{t-1} = b_{t-1}$ but $a_t > b_t$, then there exists a $j > t$ such that $x_t(m_2/x_j) \in I$. The notion of weakly polymatroidal generalizes the notation of a stable ideal. An ideal I is *stable* if for any monomial $m = x_{i_1}^{a_{i_1}} \cdots x_{i_s}^{a_{i_s}}$ in I, if $j < i_s$, then $x_j(m/x_{i_s}) \in I$. As shown by Mohammadi and Moradi, these ideals all have linear quotients, thus providing us with a large class of componentwise linear monomial ideals.

Theorem 3.14 ([44, Theorem 1.3]) *If I is a weakly polymatroidal ideal, then I has linear quotients, and consequently, I is componentwise linear.*

We now introduce another class of ideals. For any $J = \{j_1, \ldots, j_s\} \subseteq \{1, \ldots, n\}$, we define $\mathfrak{m}_J = \langle x_{j_1}, \ldots, x_{j_s} \rangle$. For any integer $a \geq 1$, we call $\mathfrak{m}_J^a$ a *Veronese ideal* (see [33]). Given subsets $J_1, \ldots, J_s$ of $\{1, \ldots, n\}$ and positive integers $a_1, \ldots, a_s$, we call

$$I = \mathfrak{m}_{J_1}^{a_1} \cap \cdots \cap \mathfrak{m}_{J_s}^{a_s}$$

an *intersection of Veronese ideals*. Ideals of this type appear throughout the literature (for example, the primary decomposition of a square-free monomial ideal can viewed as the intersection of Veronese ideals). In some cases, we can determine if I is componentwise linear simply from the subsets $J_1, \ldots, J_s$. One such example is the following result of Francisco and Van Tuyl [26, Theorem 3.1], and generalized by Mohammadi and Moradi (whose result is presented below).

Theorem 3.15 ([44, Theorem 2.5]) *Let $J_1, \ldots, J_s, K$ be subsets of $[n] = \{1, \ldots, n\}$. Suppose that $J_i \cup J_j = [n]$ for all $i \neq j$ and $K \subseteq [n]$. Then,*

$$I = \mathfrak{m}_{J_1}^{a_1} \cap \cdots \cap \mathfrak{m}_{J_s}^{a_s} \cap \mathfrak{m}_K^b$$

is componentwise linear for any positive integers $a_1, \ldots, a_s, b$.

We end this section with a recent result of Dung, Hien, Nguyen, and Trung that allows one to build new componentwise linear ideals from old ones. We have only presented the monomial ideal version of this result, although the work of [16] is more general since the focus of their work is the linear defect of an ideal.

Theorem 3.16 ([16, Corollary 5.6]) *Let $R = k[x_1, \ldots, x_n]$. Let I' and T be non-trivial monomial ideals and x a variable such that*

(1) *I' is componentwise linear,*
(2) *$T \subseteq \langle x_1, \ldots, x_n \rangle I'$, and*
(3) *no generator of T is divisible by x.*

If $I = xI' + T$, then I is componentwise linear if and only if T is componentwise linear.

4 Componentwise linearity of regular powers

In this section, we survey the problem of determining when the regular powers of an ideal are componentwise linear. In particular, we will focus on the Herzog–Hibi–Ohsugi Conjecture on the behavior of cover ideals of chordal graphs.

One theme in commutative algebra is to understand how properties of an ideal are preserved when one takes powers of these ideals. This theme is encapsulated into the following broad question:

Question 4.1 *Let I be an ideal of a ring T. Suppose that the ideal I has some property $\mathcal{P}$. Does I^s also have property $\mathcal{P}$ for all integers $s \geq 1$?*

The monograph [9] looks at this question for a number of ideals that arise in either combinatorics or geometry. Given this theme, it is natural to ask if the property of being componentwise linear is preserved by taking powers. The answer turns out to be no in general as shown in the follow examples.

Example 4.2 Let

$$I = \langle x_1 x_2 x_3, x_1 x_2 x_6, x_1 x_3 x_5, x_1 x_4 x_5, x_1 x_4 x_6, x_2 x_3 x_4, x_2 x_4 x_5, x_2 x_5 x_6, x_3 x_4 x_6, x_3 x_5 x_6 \rangle$$

in $R = k[x_1, \ldots, x_6]$ be Reisner's example as given in Example 2.15. As already noted, this ideal has a linear resolution if $\mathrm{char}(k) \neq 2$. It was shown by Conca in [10, Remark 3] that when $\mathrm{char}(k) \neq 2$, the ideal I^2 does not have a linear resolution. Since $I^2 = \langle (I^2)_6 \rangle$, the square of I is not componentwise linear. Note that in [10], this example was attributed to Terai.

Sturmfels [57] gave another example, which does not depend upon the characteristic of the ground field. In particular, the ideal

$$I = \langle x_1 x_3 x_6, x_1 x_4 x_5, x_2 x_3 x_4, x_2 x_5 x_6, x_3 x_4 x_5, x_3 x_4 x_6, x_3 x_5 x_6, x_4 x_5 x_6 \rangle$$

has the property that I has a linear resolution in all characteristics, but I^2 does not have a linear resolution.

Conca's paper [12] also contains many other examples of (monomial) ideals with this behavior.

The previous example shows that we will require some extra hypotheses on I in order to guarantee that I^s is componentwise linear. Restricting to the case that I has a linear resolution is a natural starting point, and it turns out that we can say more in this case.

We say that a homogeneous ideal I has *linear powers* (following Bruns, Conca, and Varbaro [7]) if I has a linear resolution and I^s has a linear resolution for all $s \geq 2$. Ideals with linear powers can be classified in terms of their Rees algebras. If $I = \langle f_1, \ldots, f_s \rangle$ is

generated by homogeneous elements of degree d, the *Rees algebra* is

$$\text{Rees}(I) = \bigoplus_{s \in \mathbb{N}} I^s.$$

This ring has a bigraded structure given by $\text{Rees}(I)_{a,b} = (I^b)_a$ with $(a, b) \in \mathbb{N}^2$, that is, all the elements of degree a in the b-th power of I. We can give $\text{Rees}(I)$ a graded structure by setting the degree a part of $\text{Rees}(I)$ to be

$$\text{Rees}(I)_{(a,*)} = \bigoplus_{b \in \mathbb{N}} (I^b)_a.$$

With this grading $\text{Rees}(I)$ is a graded R-module. We denote its Castelnuovo–Mumford regularity with respect to this grading by $\text{reg}_{(1,0)}\text{Rees}(I)$. (There are other possible $\mathbb{N}$-gradings one can put on $\text{Rees}(I)$, so one wants to distinguish the grading used when taking the regularity.) We then have:

Theorem 4.3 ([7, Theorem 2.5]) *Let $I = \langle f_1, \ldots, f_s \rangle$ be a homogeneous ideal generated by forms of the same degree. Then, I has linear powers if and only if $\text{reg}_{(1,0)}\text{Rees}(I) = 0$.*

This result is generalized to the case of $I^s M$ for a module M in [8]. The "if" direction was first proved by Römer [49, Corollary 5.5].

While Example 4.2 shows that we should not expect arbitrary products of componentwise linear ideals to be componentwise linear, Conca, De Negri, and Rossi [13] gave a sufficient condition for this property.

Theorem 4.4 ([13, Theorem 2.20]) *Suppose that I and J are componentwise linear, and suppose d is the smallest degree of a generator of I. If $\dim R/\langle I_d \rangle \leq 1$, then IJ is componentwise linear. In particular, if $\dim R/\langle I_d \rangle \leq 1$, then I^s is componentwise linear for all $s \geq 1$.*

Another way to approach Question 4.1 is to restrict to families of ideals that are known to be componentwise linear, and check if the powers of ideals within this family continue to be componentwise linear. One such family of ideals is the cover ideals of graphs. We recall the relevant definitions and notation.

Let $G = (V, E)$ be a finite simple graph on the vertex set $V = \{x_1, \ldots, x_n\}$ and edge set E, which consists of unordered pairs of distinct elements of V. By identifying the vertex $x_i \in V$ with the variable in x_i in $R = k[x_1, \ldots, x_n]$, we can associate with G two square-free monomials ideals, the *edge ideal*

$$I(G) = \langle x_i x_j \mid \{x_i, x_j\} \in E \rangle,$$

and the *cover ideal*

$$J(G) = \bigcap_{\{x_i, x_j\} \in E} \langle x_i, x_j \rangle.$$

The terminology of edge ideal is used to highlight the fact that the minimal generators of $I(G)$ correspond to the edges of the graph. For the cover ideal, the minimal generators of $J(G)$ correspond to the minimal vertex covers of G. A subset $W \subseteq V$ is a *vertex cover* of G if $e \cap W \neq \emptyset$ for all $e \in E$. Edge and cover ideals give us an algebraic way to study graphs; for more on these ideals and their properties, see [31, 45, 54, 58]. By restricting to cover and edge ideals, one can exploit the extra combinatorial information.
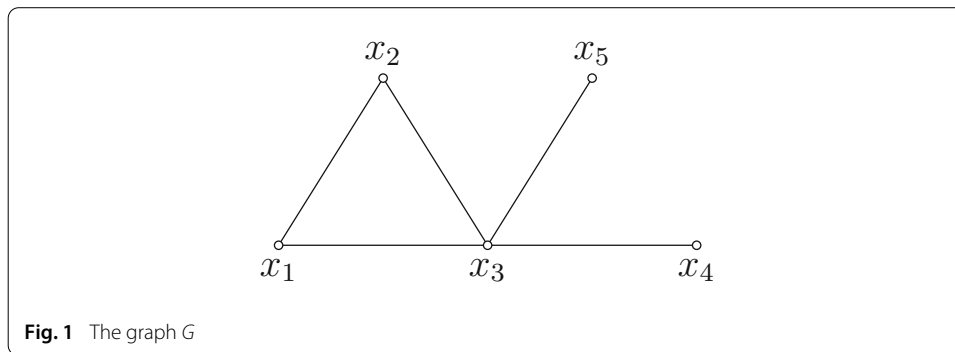
Fig. 1 The graph G

Example 4.5 Consider the graph G on five vertices as given in Fig. 1. For this graph, the edge ideal is

$$I(G) = \langle x_1 x_2, x_1 x_3, x_2 x_3, x_3 x_4, x_3 x_5 \rangle,$$

and the cover ideal is

$$J(G) = \langle x_1, x_2 \rangle \cap \langle x_1, x_3 \rangle \cap \langle x_2, x_3 \rangle \cap \langle x_3, x_4 \rangle \cap \langle x_3, x_5 \rangle = \langle x_1 x_3, x_2 x_3, x_1 x_2 x_4 x_5 \rangle.$$

Both ideals belong to the ring $R = k[x_1, \ldots, x_5]$. Observe that $J(G)$ is our running example. Also, each generator of $J(G)$ corresponds to a vertex cover of G. For example, if we look at the generator $x_1 x_3$, then every edge of G has either x_1 or x_3 as an endpoint.

It turns out that the cover ideals of chordal graphs provide a large family of componentwise ideals. Given a graph $G = (V, E)$, the *induced subgraph* on the set $W \subseteq V$ is the graph $G_W = (W, E_W)$ where $E_W = \{e \in E \mid e \subseteq W\}$; that is, an edge e of G also belongs to G_W if and only if both endpoints of e belong to W. A *cycle of length n* is the graph

$$C_n = (\{x_1, \ldots, x_n\}, \{\{x_1, x_2\}, \{x_2, x_3\}, \ldots, \{x_{n-1}, x_n\}, \{x_n, x_1\}\}).$$

A graph G is a *chordal graph* if G has no induced subgraphs isomorphic to a C_n with $n \geq 4$. The graph in Example 4.5 is an example of chordal graph.

The following result is due to Francisco and Van Tuyl:

Theorem 4.6 ([25], Theorem 3.2]) *If G is a chordal graph, then the cover ideal $J(G)$ is componentwise linear.*

Note that the statement in [25] uses the equivalence of Theorem 3.3; that is, the conclusion is that $\Delta_{J(G)}^{\vee}$ is a sequentially Cohen–Macaulay simplicial complex. To prove Theorem 4.6, Francisco and Van Tuyl use Theorem 3.10 to show that $J(G)$ is square-free componentwise linear. Other proofs of Theorem 4.6 exist making use of Theorem 3.9: (1) Van Tuyl and Villarreal [59] showed that $\Delta_{J(G)}^{\vee}$ is shellable, from which we can deduce that $J(G)$ is componentwise linear; (2) Erey [18] gave an alternative ordering of the generators of $J(G)$ to prove that $J(G)$ has the linear quotients; and (3) Dochtermann and Engström [15], and independently Woodroofe [61], proved that $\Delta_{J(G)}^{\vee}$ was vertex decomposable when G is chordal, which implies $\Delta_{J(G)}^{\vee}$ is shellable.

Herzog et al. [36] were the first to consider the problem of when powers of componentwise linear ideals are also componentwise linear. Using Theorem 4.6 as their starting point, they proposed the following conjecture [36, Conjecture 2.5].

Conjecture 4.7 (The Herzog–Hibi–Ohsugi conjecture) *If G is a chordal graph with cover ideal $J(G)$, then $J(G)^s$ is componentwise linear for all $s \geq 1$.*

Part of the difficulty of this conjecture lies in the fact that $J(G)^s$ is no longer a square-free monomial ideal if $s \geq 2$. The majority of the proofs for Theorem 4.6 as described above rely heavily on the fact that $J(G)$ is a square-free monomial ideal. Answering the Herzog–Hibi–Ohsugi Conjecture will rely on new techniques for proving a monomial ideal is componentwise linear.

There is a growing body of families of chordal graphs that satisfy Conjecture 4.7, thus pointing toward the validity of the conjecture. The proof strategies broadly fall into two categories. One strategy is to find a description or ordering of the generators of $J(G)^s$ and then show that the ideal has linear quotients to apply Theorem 2.13, or $J(G)^s$ satisfies a property like being weakly polymatroidal and apply a result like Theorem 3.14. The second strategy is to use properties of the Rees algebra, as in Theorem 4.3. Current attacks on the conjecture have also exploited the structure of chordal graphs. For example, some approaches exploit chordal graphs with "lots" of edges, e.g., a graph with a large complete graph (defined below) as a subgraph. At the other extreme, the conjecture has been investigated for chordal graphs with "few" edges, e.g., trees or graphs with very rigid structure.

Herzog, Hibi, and Ohsugi provided the initial evidence for the validity of Conjecture 4.7. Under the extra assumption that $J(G)$ has a linear resolution (which is equivalent to the fact that $\Delta_{J(G)}^\vee$ is a Cohen–Macaulay simplicial complex by Theorem 3.5), they show that $\mathrm{reg}_{(1,0)} R(J(G)) = 0$ and use the approach of Theorem 4.3 to verify Conjecture 4.7 for all cover ideals of chordal graphs with a linear resolution.

Theorem 4.8 ([36, Theorem 2.7]) *If G is a chordal graph such that cover ideal $J(G)$ has a linear resolution, then $J(G)^s$ is componentwise linear for all $s \geq 1$.*

We need to introduce some special classes of chordal graphs. The *complete graph K_n* on n vertices is the graph with vertex set $\{x_1, \ldots, x_n\}$ and edge set $\{\{x_i, x_j\} \mid 1 \leq i < j \leq n\}$. We call a graph G on $n + m$ vertices a *star graph* based on K_n if the vertices of G can be relabeled so that the induced graph on $\{x_1, \ldots, x_n\}$ is the complete graph K_n, and for any $n \leq i < j \leq n + m$, the edge $\{x_i, x_j\} \notin E$. The graph in Fig. 4.5 is an example of a star graph based on K_3 since the induced graph on $\{x_1, x_2, x_3\}$ is a K_3.

Mohammadi [42] introduced a wider class of ideals that generalized this construction that were called *generalized star graphs*. While we will not recall this construction, the idea is similar in that one glues together a collection of complete graphs in a prescribed way to form a "core," and then one is allowed to attach some extra edges. We then have the following result.

Theorem 4.9 *Conjecture 4.7 is true for the cover ideals of the following chordal graphs:*

(1) [36, Theorem 2.3] *Star graphs based on K_n.*
(2) [42, Theorem 1.5] *Generalized star graphs.*

Example 4.10 For any complete graph K_n, the cover ideal $J(K_n)$ has a linear resolution. This can be checked directly from the fact that $J(K_n) = \langle x_1 \cdots \widehat{x_i} \cdots x_n \mid 1 \leq i \leq n \rangle$, and that this ideal has linear quotients. So all powers of $J(K_n)$ are componentwise linear.

As a second example, the graph in Fig. 4.5 is a star graph based on K_3. Consequently, the ideal of our running example, that is, $J(G) = \langle x_1 x_3, x_2 x_3, x_1 x_2 x_4 x_5 \rangle$, has the property that all of its powers has the componentwise linear property.

Herzog, Hibi, Ohsugi's proof of Theorem 4.9 uses properties of the Rees algebra $R(J(G))$. Mohammadi gives a different proof for star graphs based on K_n that shows for each graph G in this family, the ideal $J(G)^s$ is weakly polymatroidal. In fact, the following theorem gives a combinatorial way to check if all powers of $J(G)$ are componentwise linear. Given a graph $G = (V, E)$, the *clique complex* of G is the simplicial complex

$$\mathrm{Cliq}(G) = \langle W \subseteq V \mid G_W = K_{|W|} \rangle;$$

in other words, the clique complex consists of all the subsets of V such that induced graph on that subset is a complete graph. For any simplicial complex Δ, a facet $F \in \Delta$ has a *free vertex* if there is some vertex $x_i \in F$ that only appears in F, but no other facet. We denote the set of all facets of Δ with a free vertex by $\mathcal{FV}(\Delta)$. We then have the following tool:

Theorem 4.11 ([43, Corollary 2.4]) *Let $G = (V, E)$ be a chordal graph with clique complex* $\mathrm{Cliq}(G)$*. If $|V| - 1 \le |\bigcup_{F \in \mathcal{FV}(\mathrm{Cliq}(G))} F|$, then the ideal $J(G)^s$ is componentwise linear for all $s \ge 1$.*

Example 4.12 The graph G in Fig. 1 has clique complex

$$\mathrm{Cliq}(G) = \langle \{x_1, x_2, x_3\}, \{x_3, x_4\}, \{x_3, x_5\} \rangle.$$

Note that each facet has a free vertex: x_1, x_2 only appear in the first facet, while x_4 only appears in the second, and x_5 only in the third. Since

$$|V| - 1 = 4 \le |\{x_1, x_2, x_3\} \cup \{x_3, x_4\} \cup \{x_3, x_5\}| = 5,$$

all powers of the cover ideal $J(G)$ are componentwise linear by Theorem 4.11.

Erey (see [19,20]) approached Conjecture 4.7 by finding an order of the minimal generators of $J(G)^s$ that gives linear quotients. In the statement below, the *path graph P_n* is the graph with vertex set $V = \{x_1, \ldots, x_n\}$ and edge set $E = \{\{x_1, x_2\}, \{x_2, x_3\}, \ldots, \{x_{n-1}, x_n\}\}$. A graph is *$(C_4, 2K_2)$-free* if it has no induced graph isomorphic to C_4 or two copies of K_2.

Theorem 4.13 *Conjecture 4.7 is true for the cover ideals of the following chordal graphs:*

(1) [19, Theorem 3.7] *Chordal graphs that are also $(C_4, 2K_2)$-free graphs.*
(2) [20, Theorem 4.3] *The path graphs P_n.*

Remark 4.14 Erey showed a stronger result in [19, Theorem 3.7], namely the cover ideal $J(G)$ of *any* $(C_4, 2K_2)$-free graph G satisfies the property that $J(G)^s$ is componentwise linear for all $s \ge 1$. Note that the cycle C_5 is a $(C_4, 2K_2)$-free graph that is not chordal.

Remark 4.15 As an intermediate step, Erey and Qureshi first proved that $J(P_n)^2$ was componentwise linear in [21, Theorem 5.1]. Erey was later able to extend this result to all powers, as noted above.

Herzog, Hibi, and Moradi were also able to prove that the same result for P_n as a consequence of a more general result, which again uses the Rees algebra. We recall how one can construct the Rees algebra of an ideal $I = \langle f_1, \ldots, f_s \rangle \subseteq R = k[x_1, \ldots, x_n]$ when

I is not necessarily generated by terms of the same degree. In the ring $R[t]$, consider the subring

$$R[It] = k[f_1 t, f_2 t, \ldots, f_s t] \subseteq R[t].$$

Let $S = k[y_1, \ldots, y_s, x_1, \ldots, x_n]$. We define a k-algebra homomorphism $\varphi : S \to R$ by

$$y_i \mapsto f_i t \ \text{ and } \ x_j \mapsto x_j$$

for $i = 1, \ldots, s$ and $j = 1, \ldots, n$. Let $J = \ker \varphi$. The ideal J is called the *defining ideal of the Rees ring* Rees(I). A new criterion for when an ideal is componentwise linear is then given in terms of J.

Theorem 4.16 ([35, Theorem 2.6]) *Let I be a monomial ideal, and let J be the defining ideal of the Rees ring* Rees(I). *There exists a monomial order $>$ such that if the initial ideal* $\text{in}_{>}(J)$ *is generated by quadratic monomials with respect to this order, then I^s is componentwise linear for all $s \geq 1$.*

The definition of the required monomial order can be found in [35]. Using Theorem 4.16, Herzog, Hibi, and Moradi were able to verify Conjecture 4.7 for more families of chordal families. A *biclique graphs* is a graph on the vertex set $\{x_1, \ldots, x_p, y_1, \ldots, y_q, z_1, \ldots, z_r\}$ such that the induced graphs on $\{x_1, \ldots, x_p, y_1, \ldots, y_q\}$ and on $\{y_1, \ldots, y_q, z_1, \ldots, z_r\}$ are complete graphs. *Cameron–Walker graphs* are graphs whose induced matching number equals its matching number; we do not formally define this family here, but point the reader to [40]. Note that Cameron–Walker graphs are not chordal in general.

Theorem 4.17 ([35, Corollary 4.7]) *Conjecture 4.7 is true for the cover ideals of the following chordal graphs:*

(1) *Biclique graphs.*
(2) *The path graphs P_n.*

In addition, the cover ideals of Cameron–Walker graphs whose bipartite graph is a complete bipartite graph also satisfy the property that all of their powers are componentwise linear.

Kumar and Kumar have recently shown that Conjecture 4.7 holds for all trees. *Trees* are graphs which have no induced cycles, and thus, they are examples of chordal graphs. Kumar and Kumar's proof uses a different strategy then the above results. In the case that G is a tree, $J(G)^s$ equals its s-symbolic power (to be defined in the next section). It can then be shown that $(J(G)^s)^{\text{pol}}$, the polarization of $J(G)^s$, is the cover ideal of another graph. The proof for trees then shows that the cover ideal of this new graph is also componentwise linear. This strategy will be expanded upon in more detail when we look at symbolic powers in the next section.

Theorem 4.18 ([41, Corollary 3.5]) *Conjecture 4.7 is true for the cover ideals of all trees.*

Remark 4.19 The above result is slightly stronger since it is shown that $J(G)^s$ has linear quotients for all $s \geq 0$ when G is a tree.

We round out this section by describing three results not directly related to Conjecture 4.7, but related to the more general theme of Question 4.1. The first result concerns the cover ideals of graphs that may not be chordal. The second result concerns edge ideals,

not cover ideals, and the third is for quadratic monomial ideals (which includes all edge ideals).

A graph G is *bipartite* if the vertex set of G can be partitioned as $V = V_1 \cup V_2$ so that for every edge $e \in E$, one has $e \cap V_1 \neq \emptyset$ and $e \cap V_2 \neq \emptyset$. That is, every edge has one endpoint in V_1 and the other in V_2. We then have the following result for the cover ideals of bipartite graphs; an earlier version of this result appeared in [44, Theorem 2.2].

Theorem 4.20 ([53, Corollary 3.7]) *Let G be a bipartite graph. Then, $J(G)$ has linear powers if and only if $J(G)$ has a linear resolution.*

As is evident from this section, the majority of work on powers of componentwise linear ideals has focused on cover ideals. Of course, similar questions could be asked about edge ideals. To date, the primary focus has been on the linear property, as demonstrated in the next two results.

Theorem 4.21 ([19, Theorem 2.12]) *Let G be a $(C_4, 2K_2)$-free graph with edge ideal $I(G)$. Then, $I(G)^s$ is componentwise linear for all $s \geq 2$; in fact, $I(G)^s$ has a linear resolution for all $s \geq 2$.*

Note that in the above result the ideal $I(G)$ may or may not be componentwise linear, but its powers are. Our final result looks at quadratic monomial ideals that need not be square-free.

Theorem 4.22 ([37, Theorem 3.2]) *Let I be a quadratic monomial ideal. Then, I has linear powers if and only if I has a linear resolution.*

5 Componentwise linearity of symbolic powers

In this section, we move beyond the Herzog–Hibi–Ohsugi conjecture to address the question of when *symbolic* powers of an ideal are componentwise linear. For any arbitrary ideal $I \subseteq R$, the *s-th symbolic power* of I is the ideal

$$I^{(s)} = \bigcap_{P \in \mathrm{Ass}(I)} (I^s R_P \cap R)$$

where $\mathrm{Ass}(I)$ is the set of associated primes of I and R_P is the ring R localized at the prime ideal P. In the case that I is a square-free monomial ideal with primary decomposition $I = P_1 \cap \cdots \cap P_r$, its s-th symbolic power is given by

$$I^{(s)} = P_1^s \cap \cdots \cap P_r^s.$$

In particular, the s-th symbolic power of the cover ideal of a graph G satisfies

$$J(G)^{(s)} = \bigcap_{\{x_i, x_j\} \in E} \langle x_i, x_j \rangle^s.$$

We survey a number of recent articles focusing on the class of cover ideals of graphs that have addressed the following umbrella question:

Question 5.1 *For which graphs G is $J(G)^{(s)}$ componentwise linear for all $s \geq 1$?*

Going forward, we will employ the following terminology. For a graph G, its *independent complex*, denoted by $\mathrm{Ind}(G)$, is the simplicial complex whose faces are independent sets

in G. A set $A \subseteq V$ is an *independent set* if for all $e \in E$, $e \not\subseteq A$. Equivalently, $V \setminus A$ is a vertex cover. It is not hard to show that $\mathrm{Ind}(G) = \Delta_{J(G)}^{\vee}$.

Definition 5.2 A graph G is *vertex decomposable*, respectively, *shellable*, if its independent complex $\mathrm{Ind}(G)$ is vertex decomposable, respectively, shellable.

Note that by Theorem 3.9, if a graph G is vertex decomposable or shellable, then $J(G)$ is componentwise linear.

When G is a bipartite graph, it is known (cf. [27]) that $J(G)^s = J(G)^{(s)}$ for all $s \geq 1$, and so Question 5.1 reduces to the question of when regular powers of the cover ideal of a graph are componentwise linear—this question has been discussed in the previous section and is closely related to the Herzog–Hibi–Ohsugi conjecture. In this case, by combining previous work of Seyed Fakhari [53] and of Selvaraja and Skelton [52], one obtains the following result.

Theorem 5.3 ([53, Theorem 3.6 and Corollary 3.7] and [52, Theorem 5.3])
Let G be a bipartite graph, and thus, $J(G)^s = J(G)^{(s)}$.

(1) *The following are equivalent:*

    (a) *$J(G)^s$ is componentwise linear for all $s \geq 1$,*
    (b) *$J(G)^s$ is componentwise linear for some $s > 1$,*
    (c) *$J(G)^s$ has linear quotients for all $s \geq 1$,*
    (d) *G is a vertex decomposable graph.*

(2) *The following are equivalent:*

    (a) *$J(G)^s$ has a linear resolution for all $s \geq 1$,*
    (b) *$J(G)$ has a linear resolution,*
    (c) *G is a pure vertex decomposable graph (i.e., $\mathrm{Ind}(\Delta) = \Delta_{J(G)}^{\vee}$ is also a pure simplicial complex).*

For an arbitrary graph G, the general approach to investigate symbolic powers of the cover ideal $J(G)$ is to view the polarization of these symbolic powers as the cover ideals of other graphs constructed from G. Particularly, the following constructions, due to Seyed Fakhari [53] and Kumar and Kumar [41], have proved to be essential in this line of work.

Construction 5.4 (*Duplicating vertices*) Let G be a graph over the vertex set $V_G = \{x_1, \ldots, x_n\}$, and let $s \geq 1$ be an integer. We construct a new graph, denoted by G_s, as follows:

$$V_{G_s} = \{x_{i,p} \mid 1 \leq i \leq n, \ 1 \leq p \leq s\}, \text{ and}$$
$$E_{G_s} = \left\{ \{x_{i,p}, x_{j,q}\} \mid \{x_i, x_j\} \in E_G \text{ and } p + q \leq s + 1 \right\}.$$

Construction 5.5 (*Duplicating edges*) Let G be a graph with vertex set $V_G = \{x_1, \ldots, x_n\}$ and edge set $E_G = \{e_1, \ldots, e_m\}$.

(1) Let $r \in \mathbb{Z}_{\geq 0}$ and $e = \{x_i, x_j\} \in E_G$. Set

$$V(e(r)) = \{x_{l,p} \mid l \in \{i, j\} \text{ and } 1 \leq p \leq r\}, \text{ and}$$
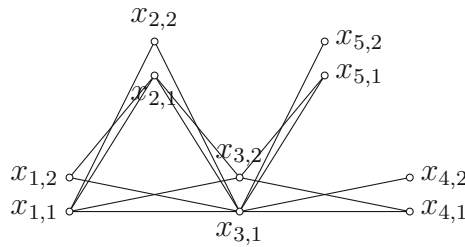$$E(e(r)) = \left\{ \{x_{i,p}, x_{j,q}\} \mid p + q \leq r + 1 \right\}.$$

Fig. 2 The graph G_2 constructed from the graph G in Fig. 1

(2) For an ordered tuple $(s_1, \ldots, s_m) \in \mathbb{Z}_{\geq 0}^m$, we construct a new graph, denoted by $G(s_1, \ldots, s_m)$, as follows:

$$V_{G(s_1,\ldots,s_m)} = \bigcup_{i=1}^{m} V(e_i(s_i)), \text{ and}$$

$$E_{G(s_1,\ldots,s_m)} = \bigcup_{i=1}^{m} E(e_i(s_i)).$$

Obviously, for $s_1 = \cdots = s_m = s$, we have $G(s_1, \ldots, s_m) = G_s$. The use of Constructions 5.4 and 5.5 is reflected in the following lemma.

Lemma 5.6 ([53, Lemma 3.4]) *Let G be a graph, and let $J(G)$ be its cover ideal. For any integer $s \geq 1$, the polarization $\left(J(G)^{(s)}\right)^{\mathrm{pol}}$ coincides with the cover ideal of G_s.*

Example 5.7 We illustrate the above ideas by using the graph of Example 4.5 for $s = 2$. The graph G_2 is then given in Fig. 2.

Note that $J(G) = \langle x_1 x_3, x_2 x_3, x_1 x_2 x_4 x_5 \rangle$, so the ideal $J(G)^{(2)}$ is given by

$$J(G)^{(2)} = \langle x_1^2 x_3^2, x_1 x_2 x_3^2, x_2^2 x_3^2, x_1 x_2 x_3 x_4 x_5, x_1^2 x_2^2 x_4^2 x_5^2 \rangle.$$

The polarization of $J(G)^{(2)}$ is then the ideal

$$(J(G)^{(2)})^{\mathrm{pol}} = \langle x_{1,1} x_{1,2} x_{3,1} x_{3,2}, x_{1,1} x_{2,1} x_{3,1} x_{3,2}, x_{2,1} x_{2,2} x_{3,1} x_{3,2},$$

$$x_{1,1} x_{2,1} x_{3,1} x_{4,1} x_{5,1}, x_{1,1} x_{1,2} x_{2,1} x_{2,2} x_{4,1} x_{4,2} x_{5,1} x_{5,2} \rangle.$$

This ideal then satisfies $(J(G)^{(2)})^{\mathrm{pol}} = J(G_2)$.

Lemma 5.6 fits into the context of studying the componentwise linearity of symbolic powers of the cover ideal $J(G)$ via the following result. It allows us to, instead of looking at the componentwise linearity of $J(G)^{(s)}$, consider when G_s is vertex decomposable, which is a combinatorial property and could be more natural to examine.

Lemma 5.8 *Let G be a graph, and let $s \in \mathbb{N}$. If G_s is vertex decomposable, then $J(G_s)$ has linear quotients. Particularly, if G_s is vertex decomposable, then $J(G)^{(s)}$ has linear quotients and is componentwise linear.*

Proof By Theorem 3.9, we know that if G_s is vertex decomposable, then $J(G_s)$ has linear quotients. Thus, Lemma 5.6 implies $(J(G^{(s)})^{\mathrm{pol}}$ has linear quotients. This together with Theorem 3.12 implies that $J(G)^{(s)}$ has linear quotients. The last statement follows from Theorem 2.13. $\qquad\square$

By applying Lemma 5.8, Seyed Fakhari [53], Selvaraja [57], and Kumar and Kumar [41] showed that the following special classes of graphs G have the property that $J(G)^{(s)}$ is componentwise linear for any $s \geq 1$.

Theorem 5.9 *Let G be a graph.*

(1) [53, Theorem 3.6] *If G is very well covered and $J(G)$ has a linear resolution, then $J(G)^{(s)}$ has linear quotients for all $s \geq 1$.*

(2) [51, Corollary 4.7] *If G is a Cameron–Walker graph, then $J(G)^{(s)}$ has linear quotients for all $s \geq 1$.*

(3) [41, Theorem 3.4 and Corollary 3.5] *If G is a tree on n vertices, then for any tuple $(s_1, \ldots, s_{n-1}) \in \mathbb{Z}_{\geq 0}^{n-1}$, $G(s_1, \ldots, s_{n-1})$ is a vertex decomposable graph. Particularly, $J(G)^s = J(G)^{(s)}$ has linear quotients for all $s \geq 1$.*

(4) [41, Corollary 4.5] *If G is a uni-cyclic vertex decomposable graph, then $J(G)^{(s)}$ is componentwise linear for all $s \geq 1$.*

In the above statement, a graph is *uni-cyclic* if the graph has only one induced cycle, and a graph G is *very well covered* if all of its maximal vertex covers have cardinality $\frac{1}{2}|V|$.

Seyed Fakhari [55] improved his previous result [53, Theorem 3.6] (see Theorem 5.9 (1)). In particular, he classified all the graphs whose symbolic powers have a linear resolution.

Theorem 5.10 ([55, Theorem 3.4]) *Let G be a graph with no isolated vertices. Then, the following are equivalent:*

(1) *$J(G)^{(s)}$ has a linear resolution for all $s \geq 1$,*

(2) *$J(G)^{(s)}$ has a linear resolution for some $s \geq 1$, and*

(3) *G is very well covered and $\mathrm{Ind}(G)$ is Cohen–Macaulay.*

In a different approach, Selvaraja and Skelton [52] gave the following sufficient condition for $J(G)^{(s)}$ to fail componentwise linearity for all $s \geq 1$.

Theorem 5.11 ([52, Theorem 3.]) *Let G be graph, and suppose that $J(G)^{(s)}$ is not componentwise linear for either $s = 1$ and 2, or $s = 2$ and 3. Then, $J(G)^{(s)}$ is not componentwise linear for all $s \geq 2$.*

The strategy to prove Theorem 5.11 is to use Seyed Fakhari's construction of G_s and then show that in the inductive hypothesis of G_s being vertex decomposable, a subgraph obtained from G_s by removing the neighbors of a shedding vertex is isomorphic to G_{s-2}.

In the same spirit, finding conditions so that $J(G)^{(s)}$ fails to be componentwise linear for all $s \geq 1$, Selvaraja and Skelton [52] gave the following result. Note that for a vertex decomposable graph G with a shedding sequence $x_{\alpha(1)}, \ldots, x_{\alpha(l)}$, let $\{x_{\gamma(1)}, \ldots, x_{\gamma(r)}\}$ be the collection of isolated vertices remaining in $G \setminus \{x_{\alpha(1)}, \ldots, x_{\alpha(l)}\}$. The *spanning bipartite graph* $\mathcal{B}_G$ is defined to be the bipartite graph with the bipartition of the vertices $\{x_{\alpha(1)}, \ldots, x_{\alpha(l)}\} \cup \{x_{\gamma(1)}, \ldots, x_{\gamma(r)}\}$ and edges

$$\{\{x_{\alpha(i)}, x_{\gamma(j)}\} \mid \{x_{\alpha(i)}, x_{\gamma(j)}\} \in E_G, 1 \leq i \leq l \text{ and } 1 \leq j \leq r\}.$$

Theorem 5.12 ([52, Theorem 3.6]) *Let G be a vertex decomposable graph. If there exists an independent set A such that $\mathcal{B}_{G \setminus N[A]}$ is not vertex decomposable, then $J(G)^{(s)}$ is not componentwise linear for all $s \geq 2$.*

The necessary condition in Theorem 5.12 is also sufficient to achieve the componentwise linearity of $J(G)^{(s)}$, for all $s \geq 1$, for a special class of vertex decomposable graphs, namely the class of W-graphs. Selvaraja and Skelton [52] defined a W-*graph* G to be graph such that $G \setminus N[A]$ has a simplicial vertex for any independent set A. A vertex x is a *simplicial vertex* if the induced graph on x and all of its neighbors is a complete graph.

Theorem 5.13 ([52, Theorem 4.2]) *Let G be a W-graph. Then, the following are equivalent:*

(1) $\mathcal{B}_{G \setminus N[A]}$ *is vertex decomposable for any independent set A in G.*
(2) $J(G)^{(s)}$ *is componentwise linear for all $s \geq 1$.*
(3) $J(G)^{(s)}$ *is componentwise linear for some $s \geq 2$.*

In addressing Question 5.1 and identifying new classes of graphs for which all symbolic powers of the cover ideal are componentwise linear, the following approach has been investigated: Combinatorially modify a given graph G to obtain a new graph G' with the required property that $J(G')^{(s)}$ is componentwise linear for any $s \geq 1$. Specifically, originating from Villarreal's work [60], the process of adding *whiskers* (or *whiskering*) to the vertices of a graph has been studied and developed by many authors and from various directions (cf. [3,4,14,16,24,30,41,51,52]).

Definition 5.14 By adding a *whisker* to a vertex x of a graph G, one adds a new vertex y and the edge $\{x, y\}$ to G. Let $S \subseteq V_G$ be a subset of the vertices in G. Then, we denote by $G \cup W(S)$ the graph obtained by adding a whisker to G at each vertex in S.

The first result in this approach to Question 5.1 is due to Dung, Hien, Nguyen, and Trung [16], which shows that by adding a whisker to every vertex of any given graph one obtains new graph with the desired property. The case $s = 1$ of the following theorem in fact implies Villarreal's result in [60].

Theorem 5.15 ([16, Corollary 5.9]) *Let G be a graph, and let $H = G \cup W(V_G)$ be the graph obtained by adding a whisker at every vertex in G. Then, $J(H)^{(s)}$ is componentwise linear for all $s \geq 1$.*

Dung, Hien, Nguyen, and Trung, in fact, proved a stronger statement than Theorem 5.15 in [16, Theorem 5.7], where they showed that the same conclusion holds if at least one whisker is added to every vertex of G.

Cook and Nagel [14] generalized the process of whiskering to that of *clique-whiskering* to extend Villarreal's previous work [60].

Definition 5.16 Let G be a graph. A *clique partition* π of G is a partition of the vertices in G into disjoint (possibly empty) subsets $W_1, \ldots, W_t$ such that the induced graphs G_{W_i} are a complete (or empty) graph in G for all $i = 1, \ldots, t$. A *clique-whiskering* of G associated with a clique partition π, denoted by G^π, is a graph over the vertices $V_{G^\pi} = V_G \cup \{y_1, \ldots, y_t\}$ and has edges

$$
E_{G^\pi} = E_G \cup \left(\bigcup_{i=1}^{t} \{\{v, y_i\} \mid v \in W_i\} \right).
$$

Selvaraja [51] proved the following theorem, of which the case where $s = 1$ was known in the previous work of Cook and Nagel [14].

Theorem 5.17 ([51, Theorem 4.9]) *Let G be a graph, and let π be a clique vertex partition of G. Then, $J(G^\pi)^{(s)}$ has linear quotients for any $s \geq 1$.*

Inspired by Theorem 5.15, the following question arises naturally: For which subset $S \subseteq V_G$ of the vertices in a graph G do we have that $J(G \cup W(S))^{(s)}$ is componentwise linear for all $s \in \mathbb{N}$? A number of special configurations of such subsets S have been identified. The case $s = 1$ in the following result of Selvaraja [51] was already known by Francisco and Hà in [24].

Theorem 5.18 ([51, Corollary 4.5]) *Let G be graph and let $S \subset V_G$ be a vertex cover of G. Then, $J(G \cup W(S))^{(s)}$ has linear quotients for any $s \geq 1$.*

As a consequence to Theorem 5.18, Selvaraja and Skelton [52] obtained the following corollary; the case $s = 1$ was again known in [24].

Corollary 5.19 ([52, Corollary 4.6]) *Let G be a graph and $S \subseteq G$. If $|S| \geq |V(G)| - 3$, then $J(G \cup W(S))^{(s)}$ is componentwise linear for all $s \geq 1$.*

The condition that S is a vertex cover in Theorem 5.18 is improved by Gu, Hà, and Skelton [30]. We call a subset S of the vertices in G a *cycle cover* if every cycle in G contains at least a vertex in S. A vertex cover is necessarily a cycle cover, but contains a lot more vertices in general.

Theorem 5.20 ([30, Theorem 3.10]) *Let G be a graph, and let S be a cycle cover of G. Let H be the graph obtained by adding at least one whisker to each vertex in S. Then, $J(H)^{(s)}$ is componentwise linear for all $s \geq 1$.*

Theorem 5.20 is slightly generalized further in [30, Theorem 4.6], where it is shown that instead of adding just a whisker at each vertex in S, one can add a *non-pure star complete graph*, a graph constructed by adjoining complete graphs of different sizes at a single vertex, which has at least one whisker (see [30] for more details).

We round out this section by pointing out that there has been little work into the case of symbolic powers of edge ideals. We complete this paper with one result in this direction that makes use of Theorem 3.15.

Theorem 5.21 *Let G be a graph and suppose that $W_1, \ldots, W_t$ is a complete list of minimal vertex covers. Suppose that $W_i \cup W_j = V$ for all $i \neq j$. Then, $I(G)^{(s)}$ is componentwise linear for all $s \geq 1$.*

Proof Suppose that $W_1, \ldots, W_t$ is a complete list of minimal vertex covers of G. The edge ideal $I(G)$ then has the following primary decomposition

$$I(G) = \langle x \mid x \in W_1 \rangle \cap \cdots \cap \langle x \mid x \in W_t \rangle.$$

For a proof, see [58, Corollary 1.35]. For a variable x_i, let $\text{supp}(x_i) = \{i\}$. Note we can rewrite $\langle x \mid x \in W_i \rangle = \mathfrak{m}_{J_i}$ as a Veronese ideal with $J_i = \{\text{supp}(x) \mid x \in W_i\} \subseteq [n]$. Thus, since $I(G)$ is a square-free monomial ideal, we can write the s-th symbolic powers of $I(G)$

as

$$I(G)^{(s)} = \mathfrak{m}_{J_1}^s \cap \cdots \cap \mathfrak{m}_{J_t}^s.$$

The conclusion now follows by Theorem 3.15.   □

A graph G is a *complete m-partite graph* if the vertices V can be partitioned as $V = V_1 \cup V_2 \cup \cdots \cup V_m$ such that for all $i \neq j$, if $x \in V_i$ and $y \in V_j$, then $\{x, y\} \in E$. Note that the complete graph K_n is the complete n-partite graph with $V_i = \{x_i\}$ for all i. We then have the following result.

Corollary 5.22 *Let G be a complete m-partite graph (for any $m \geq 2$). Then, $I(G)^{(s)}$ is componentwise linear for any $s \geq 1$.*

Proof If $V = V_1 \cup V_2 \cup \cdots \cup V_m$ is the partition of V, then the minimal vertex covers of G have the form $W_i = V_1 \cup \cdots \cup \widehat{V_i} \cup \cdots \cup V_m$ for $i = 1, \ldots, m$, where we mean V_i is omitted. Now, apply Theorem 5.21.   □

6 Future research directions

We finish this paper with some problems which we hope will generate future work.

A natural way to generalize Herzog, Hibi, and Ohsugi's conjecture is consider objects more general than chordal graphs. Over the last decade, there has been interest in generalizing the property of chordality of graphs to simplicial complexes (see, for example, [1] and references therein). The notion of a cover ideal can be generalized to simplicial complexes as follows: Given a simplicial complex Δ, the cover ideal of Δ is

$$J(\Delta) = \langle x_{i_1} \cdots x_{i_s} \mid \{x_{i_1}, \ldots, x_{i_s}\} \cap F \neq \emptyset \ \ \text{for all facets } F \in \Delta \rangle.$$

The following question then generalizes the Herzog–Hibi–Ohsugi Conjecture:

Question 6.1 *Let Δ be a "chordal" simplicial complex (using an appropriate definition of chordal) with cover ideal $J(\Delta)$. Is $J(\Delta)$ componentwise linear? Is $J(\Delta)^s$ componentwise linear for all $s \geq 1$? We can also ask similar questions for $J(\Delta)^{(s)}$.*

We have put chordal in quotes since it is not clear which generalization of chordality one will want to use. A starting point to attack this question would be the case of simplicial trees, as defined by Faridi [23]. In fact, when Δ is a simplicial tree, Faridi has already shown that $J(\Delta)$ is componentwise linear (see [23, Corollary 5.5]).

Theorems 4.8 and 4.20 show that if G is a chordal or bipartite graph such that $J(G)$ has a linear resolution, then all powers of $J(G)$ have a linear resolution. This leads to the following question, which was first posed by Mohammadi [43, Question 4.1].

Question 6.2 *Let G be a graph such that the cover ideal $J(G)$ has a linear resolution. Does $J(G)^s$ have a linear resolution for all $s \geq 1$? If not, what hypotheses are needed on G to give this conclusion?*

Mohammadi has shown that the previous question is true for *cactus graphs* (see [43, Theorem 4.3]). A cactus graph is one where each edge belongs to at most one induced cycle in the graph. We are not aware of any other families of graphs for which there is a positive (or negative!) answer to Question 6.2.

As noted in Sect. 4, we do not know of many examples for edge ideals whose powers are componentwise linear. We formalize this as a question.

Question 6.3 *Let G be a graph with edge ideal I(G). What properties on G imply that* $I(G)^s$ *is componentwise linear? Similarly, what conditions imply* $I(G)^{(s)}$ *is componentwise linear?*

Observe that $I(G)^s$ is generated in a single degree. Thus, for $I(G)^s$ to be componentwise linear, it needs to have a linear resolution. Hence, Question 6.3 for $I(G)^s$ reduces to asking when a power of an edge ideal has a linear resolution. See Peeva and Nevo [46] for some work in this direction.

We also add a question from Selvaraja and Skelton's work (see [52, Question 5.8]).

Question 6.4 *Let G be a vertex decomposable graph.*

(1) *If* $J(G)^{(2)}$ *is not componentwise linear, is it true that* $J(G)^{(s)}$ *is not componentwise linear for all* $s \geq 3$?
(2) *If* $\mathcal{B}_{G \setminus N[A]}$ *is vertex decomposable for any independent set A, is* $J(G)^{(s)}$ *a componentwise linear ideal for all* $s \geq 2$?

We consider another question inspired by Selvaraja and Skelton's work, namely Theorem 5.11. As was shown in this theorem, one can determine if $J(G)^{(s)}$ is not componentwise linear by checking for small values of s. This leads to a much more general question:

Question 6.5 *Suppose that I is an ideal that is componentwise linear. Does there exists an integer* $t \geq 1$ *such that if* $I^{(i)}$ *is componentwise linear for all* $1 \leq i \leq t$, *then* $I^{(s)}$ *is componentwise linear for all* $s \geq t$?

An answer to the above question even in the case that $I = I(G)$ or $J(G)$ would be of great interest, especially if the value of t is related to a graph invariant. Note that the analogous question for regular powers has a negative answer. In particular, Conca [12, Theorem 3.1] showed that for any integer $d > 1$, there exists an ideal $I(d)$ such that $I(d)^k$ has a linear resolution (and hence, is componentwise linear) for all $1 \leq k < d$, but $I(d)^d$ does not have a linear resolution (and hence, is not componentwise linear).

As seen in Theorems 5.15, 5.17, and 5.20, the operation of whiskering can turn a graph G into a new graph H such that $J(H)^{(s)}$ is componentwise linear for all $s \geq 1$. We know of no similar results for the regular powers. We thus pose the following question:

Question 6.6 *Given a graph G, can we attach whiskers to G so that the resulting graph H has the property that* $J(H)^s$ *is componentwise linear for all* $s \geq 1$. *Can we classify all the ways to add whiskers to G to make H so that* $J(H)^s$, *respectively,* $J(H)^{(s)}$, *is componentwise linear.*

Moving beyond cover ideals, in [3,4], the authors together with Biermann and Francisco gave a generalization for the whiskering process that works also for simplicial complexes. We would like to understand if this process would produce more general monomial ideals with the property that all the symbolic powers of the Alexander dual of its Stanley–Reisner ideal are componentwise linear.

For a subset W of the vertices of a simplicial complex Δ, the *restriction* of Δ on W, denoted by $\Delta|_W$, is the simplicial complex whose faces are $\{F \in \Delta \mid F \subseteq W\}$.

Definition 6.7 Let Δ be a simplicial complex on the vertices V, let $W \subseteq V$, and let $t \in \mathbb{N}$.

(1) A *partial t-coloring* of $\Delta|_W$ is given by a partition $W = W_1 \cup \cdots \cup W_t$ (where the set W_i can be empty) such that no facet of $\Delta|_W$ contains more than one vertex in each W_i.

(2) Let χ be a partial t-coloring of $\Delta|_W$ given by $W = W_1 \cup \cdots \cup W_t$. We define a new simplicial complex Δ_χ on the vertex set $V \cup \{y_1, \ldots, y_t\}$, with faces

$$\{\sigma \cup \tau \mid \sigma \in \Delta, \tau \subseteq \{y_1, \ldots, y_t\} \text{ and } \sigma \cap W_j = \emptyset \text{ if } y_j \in \tau\}.$$

Question 6.8 *Let Δ be a simplicial complex, and let W be a subset of the vertices in Δ. For a partial t-coloring χ of $\Delta|_W$, let I_{Δ_χ} be the Stanley–Reisner ideal of Δ_χ and let J_{Δ_χ} be its Alexander dual. What conditions on W and χ would guarantee that $J_{\Delta_\chi}^{(s)}$ is componentwise linear for all $s \geq 1$?*

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