ESO204A, Fluid Mechanics and rate Processes

Momentum Conservation principle differential formulation

Derivation of Navier-Stokes Equation

Chapter 4 of F M White
Chapter 5 of Fox McDonald

Continuity: $\frac{\partial \rho}{\partial t}+\nabla.(\rho\vec{u})=0$

RTT: $\frac{dB_{sys}}{dt}=\int_{CV}\left[\frac{\partial(\rho\beta)}{\partial t}+\nabla.(\rho\beta\vec{u})\right]d\forall$

$$\frac{\partial(\rho\beta)}{\partial t}+\nabla.(\rho\beta\vec{u})=\left(\beta\frac{\partial\rho}{\partial t}+\rho\frac{\partial\beta}{\partial t}\right)+\left[\beta\nabla.(\rho\vec{u})+\rho(\vec{u}.\nabla)\beta\right]$$

$$=\beta\left[\frac{\partial\rho}{\partial t}+\nabla.(\rho\vec{u})\right]+\rho\left[\frac{\partial\beta}{\partial t}+(\vec{u}.\nabla)\beta\right]$$

RTT: $\frac{dB_{sys}}{dt}=\int_{CV}\rho\left[\frac{\partial\beta}{\partial t}+(\vec{u}.\nabla)\beta\right]d\forall$

Momentum conservation: $\dfrac{d\left(m\vec{u}\right)}{dt}=\vec{F}\;=\vec{F}_{\mathrm{B}}+\vec{F}_{\mathrm{S}}$

RTT: $\dfrac{dB_{\mathrm{sys}}}{dt}=\int\limits_{\mathrm{CV}}\rho\left[\dfrac{\partial\beta}{\partial t}+\left(\vec{u}.\nabla\right)\beta\right]d\mathcal{V}$

using $B_{\mathrm{sys}}=m\vec{u}$ we have $\beta=\vec{u}$ and $\dfrac{dB_{\mathrm{sys}}}{dt}=\vec{F}_{\mathrm{B}}+\vec{F}_{\mathrm{S}}$

$$\vec{F}_{\mathrm{B}}+\vec{F}_{\mathrm{S}}=\int\limits_{\mathrm{CV}}\rho\left[\frac{\partial\vec{u}}{\partial t}+\left(\vec{u}.\nabla\right)\vec{u}\right]d\mathcal{V}$$

$$\int\limits_{\mathrm{CV}}\rho\vec{g}d\mathcal{V}+\vec{F}_{\mathrm{S}}=\int\limits_{\mathrm{CV}}\rho\left[\frac{\partial\vec{u}}{\partial t}+\left(\vec{u}.\nabla\right)\vec{u}\right]d\mathcal{V}$$

Surface forces are related to normal and shear stresses

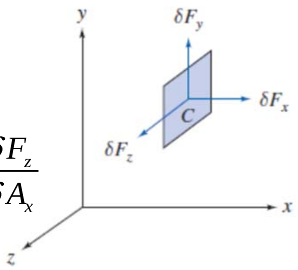
Stress in a fluid:

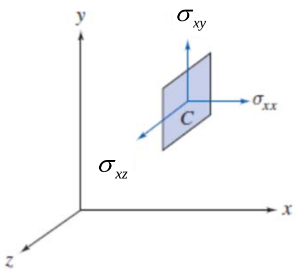
Normal stress $\sigma_{xx}=\lim\limits_{\delta A_x\to 0}\dfrac{\delta F_x}{\delta A_x}$

Shear $\sigma_{xy}=\lim\limits_{\delta A_x\to 0}\dfrac{\delta F_y}{\delta A_x}$ $\quad\sigma_{xz}=\lim\limits_{\delta A_x\to 0}\dfrac{\delta F_z}{\delta A_x}$

Surface normal direction

Force direction





σ_{ij} : Surface normal direction: *i*
Force direction: *j* **or**
Surface normal direction: -*i*
Force direction: -*j*

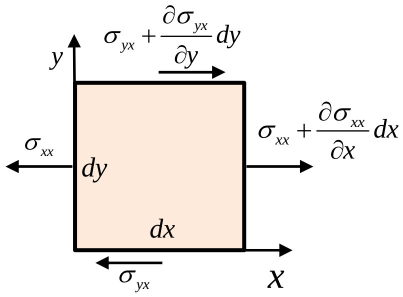
Surface force component in x-direction:

$$dF_{sx} = \left(\sigma_{xx} + \frac{\partial \sigma_{xx}}{\partial x}dx\right)dy - \sigma_{xx}dy$$

$$+\left(\sigma_{yx} + \frac{\partial \sigma_{yx}}{\partial y}dy\right)dx - \sigma_{yx}dx$$

$$= \left(\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{yx}}{\partial y}\right)d\cancel{V}$$

$$F_{sx} = \int_{CV}\left(\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{yx}}{\partial y}\right)d\cancel{V}$$



Recall: $$\int_{CV}\rho\vec{g}d\cancel{V} + \vec{F}_S = \int_{CV}\rho\left[\frac{\partial \vec{u}}{\partial t} + (\vec{u}.\nabla)\vec{u}\right]d\cancel{V}$$

x-mom Eq. $$\int_{CV}\rho g_x d\cancel{V} + F_{sx} = \int_{CV}\rho\left[\frac{\partial \vec{u}}{\partial t} + (\vec{u}.\nabla)\vec{u}\right]_x d\cancel{V}$$

$$F_{sx} = \int_{CV}\left(\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{yx}}{\partial y}\right)d\cancel{V}$$

$$\int_{CV}\rho g_x d\cancel{V} + \int_{CV}\left(\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{yx}}{\partial y}\right)d\cancel{V} = \int_{CV}\rho\left[\frac{\partial \vec{u}}{\partial t} + (\vec{u}.\nabla)\vec{u}\right]_x d\cancel{V}$$

Since the above relation is true for arbitrary CV

$$\rho\left[\frac{\partial u}{\partial t} + (\vec{u}.\nabla)u\right] = \rho g_x + \frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{yx}}{\partial y}$$

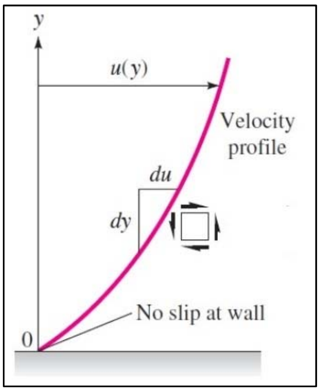
Newton's law of viscosity

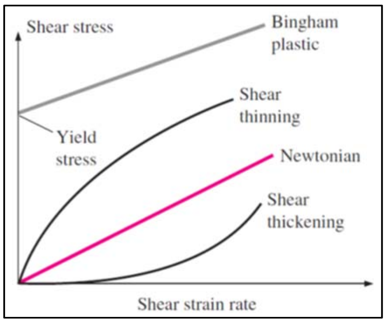
stress $\propto$ strain-rate $\qquad \sigma_{yx} = \mu \dfrac{\partial u}{\partial y}$

μ: viscosity

Fluid following above relation: Newtonian fluid

Ex: air (most gases), water





Non-Newtonian fluid examples: ketchup, blood, toothpaste

Generalized form of Newton's law for incompressible flow (Stokes)

$$\sigma_{yx} = \mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) \qquad \sigma_{xx} = -p + 2\mu\frac{\partial u}{\partial x}$$

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{yx}}{\partial y} = \frac{\partial}{\partial x}\left(-p + 2\mu\frac{\partial u}{\partial x}\right) + \mu\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)$$

$$= -\frac{\partial p}{\partial x} + \mu\frac{\partial^2 u}{\partial x^2} + \mu\frac{\partial^2 u}{\partial y^2} + \mu\frac{\partial}{\partial x}\underbrace{\cancel{\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)}}_{=0,\ \text{incompressible}}$$

$$= -\frac{\partial p}{\partial x} + \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$$

x-mom. Eq. for Newtonian fluid, 2-D incompressible flow

$$\rho\left[\frac{\partial u}{\partial t}+(\vec{u}.\nabla)u\right]=\rho g_x-\frac{\partial p}{\partial x}+\mu\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)$$

$$\frac{\partial u}{\partial t}+(\vec{u}.\nabla)u=g_x-\frac{1}{\rho}\frac{\partial p}{\partial x}+\nu\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right);\nu=\mu/\rho$$

μ : viscosity (dynamic viscosity), Pa-s

ν : kinematic viscosity, $\mathrm{m^2/s}$

In 3-D
$$\frac{\partial u}{\partial t}+(\vec{u}.\nabla)u=g_x-\frac{1}{\rho}\frac{\partial p}{\partial x}+\nu\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right)$$
$$=g_x-\frac{1}{\rho}\frac{\partial p}{\partial x}+\nu\nabla^2 u$$

Similarly *y*- and *z*-direction mom. Eq.

$$\frac{\partial v}{\partial t}+(\vec{u}.\nabla)v=g_y-\frac{1}{\rho}\frac{\partial p}{\partial y}+\nu\nabla^2 v \qquad \frac{\partial w}{\partial t}+(\vec{u}.\nabla)w=g_z-\frac{1}{\rho}\frac{\partial p}{\partial z}+\nu\nabla^2 w$$

Momentum Eq. in vector form (incompressible, Newtonian)

$$\frac{\partial \vec{u}}{\partial t}+(\vec{u}.\nabla)\vec{u}=\vec{g}-\frac{1}{\rho}\nabla p+\nu\nabla^2\vec{u}$$

Above Eq. is known as **Navier-Stokes Equation** (Navier 1825, Stokes 1850)

Inviscid form: $$\frac{\partial \vec{u}}{\partial t} + (\vec{u}.\nabla)\vec{u} = \vec{g} - \frac{1}{\rho}\nabla p$$

Known as **Euler Equation** (Euler, 1757)

Applications: Geophysical flows, aerodynamics, flows far away from walls

Bernoulli
1700-1782

Euler
1707-1783

Navier
1785-1836

Stokes
1819-1903

$$\frac{\partial \vec{u}}{\partial t} + (\vec{u}.\nabla)\vec{u} = \vec{g} - \frac{1}{\rho}\nabla p + \nu\nabla^2\vec{u} \qquad \vec{u}.\nabla \equiv u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z}$$

3-D, N-S eq., scalar form, Cartesian coordinate:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} = g_x - \frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right)$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} = g_y - \frac{1}{\rho}\frac{\partial p}{\partial y} + \nu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2}\right)$$

$$\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} = g_z - \frac{1}{\rho}\frac{\partial p}{\partial z} + \nu\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2}\right)$$

Incompressible continuity: $$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$