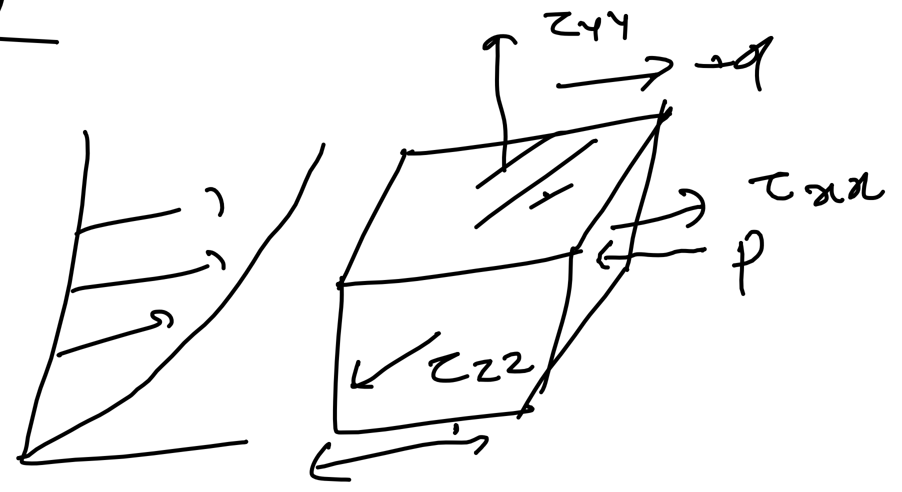
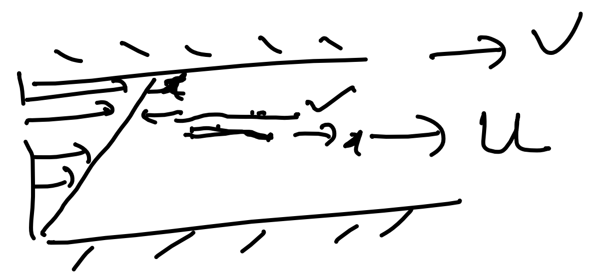
# Differential balances: Momentum balance (Continued)




$$\frac{dF}{d\forall} = \rho \frac{dV}{dt}$$ (we derived yesterday)

$$dF = (dF)_{body} + (dF)_{surface}$$

$$\left(\frac{dF}{d\forall}\right)_{body} = \rho \vec{g}$$

$$\left(\frac{dF}{d\forall}\right)_{surface} = \underbrace{-\nabla p}_{\text{Pressure}} + \underbrace{\nabla \cdot \tau_{ij}}_{\text{viscous}}$$

$$\nabla \cdot \tau_{ij} = \left\{\left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)\right\} \cdot \left\{(ii\,\tau_{xx} + ji\,\tau_{yx} + ki\,\tau_{zx}) + (ij\,\tau_{xy} + jj\,\tau_{yy} + kj\,\tau_{zy}) + (ik\,\tau_{xz} + jk\,\tau_{yz} + kk\,\tau_{zz})\right\}$$

$$= \left(\frac{\partial \tau_{xx}}{\partial x} i + \frac{\partial \tau_{yx}}{\partial y} i + \frac{\partial \tau_{zx}}{\partial z} i\right) + \left(\frac{\partial \tau_{xy}}{\partial x} j + \frac{\partial \tau_{yy}}{\partial y} \hat{j} + \frac{\partial \tau_{zy}}{\partial z} \hat{j}\right) + \left(\frac{\partial \tau_{xz}}{\partial x} k + \frac{\partial \tau_{yz}}{\partial y} \hat{k} + \frac{\partial \tau_{zz}}{\partial z} \hat{k}\right)$$

$\nabla \cdot$ $\tau_{ij}$ tensor

$$\boxed{\rho \frac{dV}{dt} = -\nabla p + \rho g + \nabla \cdot \tau_{ij}}$$

Expansion in $x, y, z$

$$x: \quad \rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right) = \rho\frac{du}{dt} = -\frac{\partial p}{\partial x} + \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} + \rho g_x$$

$$y: \quad \rho\left(\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z}\right) = \rho\frac{dv}{dt} = -\frac{\partial p}{\partial y} + \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \tau_{yy}}{\partial y} + \frac{\partial \tau_{zy}}{\partial z} + \rho g_y$$

$$z: \quad \rho\left(\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z}\right) = \rho\frac{dw}{dt} = -\frac{\partial p}{\partial z} + \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \tau_{zz}}{\partial z} + \rho g_z$$

Laminar

$\longrightarrow u$

– This is differential momentum balance, valid for any general motion of fluid, Particular fluid being characterized by particular viscous stress

→ ⟶ Newtonian ⟶ (✓)
  ⟶ Nonnewtonia (X)

Newton's law of viscosity → Generalization – (Bird, stewart, Lightfoot) Transport Phenomena

⟶ Laminar ⟶
⟶ Turbulent → – Viscous stress + turbulent stresses $\rho\frac{dV}{dt} = -\nabla p + (\nabla\cdot\tau_{ij})_{laminar} + (\nabla\cdot\tau_{ij})_{turbulent}$

Assumption : Laminar, Newtonian, incompressible

$$\tau_{yx} = \tau_{xy} = \mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right), \quad \tau_{zx} = \tau_{xz} = \mu\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right), \quad \tau_{yz} = \tau_{zy} = \mu\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right)$$

$$\tau_{xx} = 2\mu\frac{\partial u}{\partial x}, \quad \tau_{yy} = 2\mu\frac{\partial v}{\partial y}, \quad \tau_{zz} = 2\mu\frac{\partial w}{\partial z}$$

$$V = u(x,y,z)\,\hat{i} + v(x,y,z)\,\hat{j} + w(x,y,z)\,\hat{k}$$

$$x: \quad \rho g_x - \frac{\partial p}{\partial x} + 2\mu\frac{\partial^2 u}{\partial x^2} + \mu\frac{\partial^2 u}{\partial y^2} + \mu\frac{\partial^2 v}{\partial x\partial y} + \mu\frac{\partial^2 w}{\partial x\partial z} + \mu\frac{\partial^2 u}{\partial z^2} = \rho\frac{du}{dt}$$

$$✓ \quad \rho g_x - \frac{\partial p}{\partial x} + \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) + \mu\frac{\partial}{\partial x}\underbrace{\left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial w}{\partial z}\right)}_{0} = \rho\frac{du}{dt}$$

$$y: \quad \begin{cases} \rho g_x - \frac{\partial p}{\partial x} + \mu\nabla^2 u = \rho\frac{du}{dt} \\ \rho g_y - \frac{\partial p}{\partial y} + \mu\nabla^2 v = \rho\frac{dv}{dt} \\ \rho g_z - \frac{\partial p}{\partial z} + \mu\nabla^2 w = \rho\frac{dw}{dt} \end{cases}$$

Navier-stokes' $eq^n$

CFD ←

The stream functions ($\psi$) → ← (2D)

variable → $u, v, w, p$ → (by solving Navier-stokes')

→ we find a $f^n$ by automatically statistfying the continuity

→ Then momentum $eq^n$'s are solved for a single variable $\psi$

→ $\psi$ → continuity

↓ Transfer momentum $eq^n$ in term of $\psi$ ⇒ solve → to get $\psi$

$\psi$ → $(u, v)$ or $(u, w)$ or $(v, w)$ → $p$ (momentum)

↳ two velocity component & $p$

For 2D incompressible flow

$$\cancel{\frac{\partial \rho}{\partial t}} + \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

$$u = \frac{\partial \psi}{\partial y}, \quad v = -\frac{\partial \psi}{\partial x}$$

→ $\rho \frac{dV}{dt} = \rho g - \nabla p + \mu \nabla^2 V$

Bird, Stewart, Lightfoot

kinematic viscosity $\frac{\mu}{\rho} = \nu$

→ by taking curl $\nabla \times (eq^n)$

$$\frac{\partial \psi}{\partial y} \frac{\partial}{\partial x} \nabla^2 \psi - \frac{\partial \psi}{\partial x} \frac{\partial}{\partial y} \nabla^2 \psi = \nu \nabla^4 \psi \Rightarrow \text{Solve for } \psi \rightarrow$$

Physical significance of $\psi$

## stream line

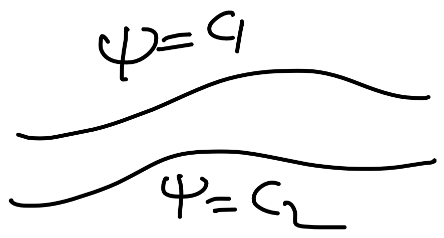
$$\frac{dx}{u} = \frac{dy}{v} = \cancel{\frac{dz}{w}}$$

$$v\,dx = u\,dy$$

$$-\frac{\partial \psi}{\partial x}\,dx = \frac{\partial \psi}{\partial y}\,dy$$

$$\frac{\partial \psi}{\partial x}\,dx + \frac{\partial \psi}{\partial y}\,dy = 0$$

$d\psi = 0 \Rightarrow \psi =$ constant along a streamline




$$V = \frac{\partial \psi}{\partial y}\hat{i} - \frac{\partial \psi}{\partial x}\hat{j}$$

$$\nabla \times \underline{V} = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j}\right) \times \left(\frac{\partial \psi}{\partial y}\hat{i} - \frac{\partial \psi}{\partial x}\hat{j}\right) = -\nabla^2\psi\,\hat{k} = \text{Vorticity } \nabla \times V$$

irrotational $\nabla \times V = 0 \Rightarrow \boxed{\nabla^2\psi = 0} \rightarrow \psi \rightarrow u, v$