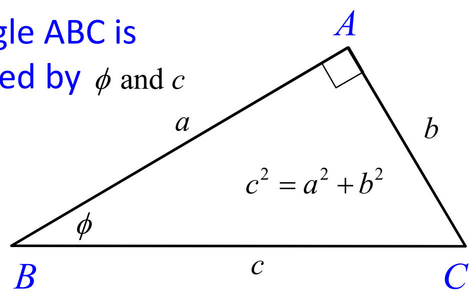
ESO204A, Fluid Mechanics and rate Processes

Dimensional Analysis and Similitude

Simple and powerful qualitative technique applicable to many fields of science and engineering

Chapter 5 of F M White
Chapter 7 of Fox McDonald

Right-angled triangle ABC is uniquely determined by ϕ and c



Area of triangle ABC: $A_c = f_1(\phi, c)$

Rearranging:

$\phi = f_2(A_c, c)$

dimensionless $\quad L^2 \quad L$

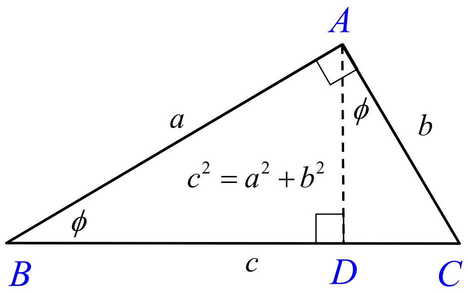
$$\Rightarrow \phi = f_3\left(\frac{A_c}{c^2}\right) \quad \Rightarrow \frac{A_c}{c^2} = f(\phi)$$

$$\boxed{\Rightarrow A_c = c^2 f(\phi)}$$

Area of triangle ABC: $A_c = c^2 f(\phi)$

Similarly area of triangle ABD: $A_a = a^2 f(\phi)$

Similarly area of triangle ACD: $A_b = b^2 f(\phi)$



ϕ : smaller of two acute angles

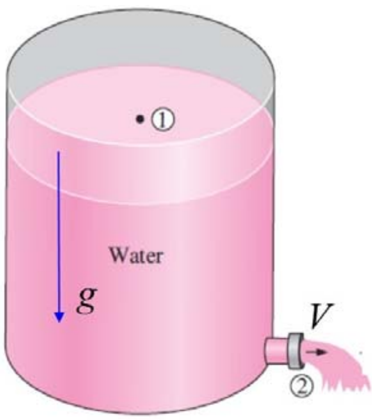
$$A_c = A_a + A_b$$

$$c^2 f(\phi) = a^2 f(\phi) + b^2 f(\phi) \quad \Rightarrow c^2 = a^2 + b^2$$

The theorem is proved using dimension-related arguments only

For quasi-steady, incompressible, frictionless flow, we have derived (Bernoulli Eq.)

$$V = \sqrt{2gh}$$



Alternate approach

Experiment (or thought experiment) suggests

$$V = f_1(g,h) \quad \Rightarrow V = f_2(u)$$

$$u = g^m h^n$$

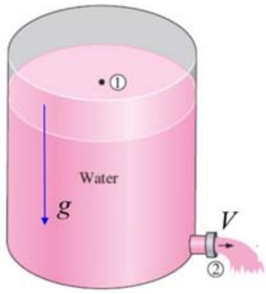
$$u = \sqrt{gh}$$

where u is made of g and h
and has the same dimension as that of V

$$\underbrace{u}_{LT^{-1}} = \underbrace{g^m}_{LT^{-2}} \underbrace{h^n}_{L}$$

$$u = \sqrt{gh}$$

$$LT^{-1} = \left(LT^{-2}\right)^m \left(L\right)^n$$
$$= L^{m+n} T^{-2m}$$
$$m + n = 1; -2m = -1$$
$$m = n = 1/2$$



$V = f_2\left(\sqrt{gh}\right)$ Think about the nature of f_2

$V = c\sqrt{gh};\ c$ is a constant

The constant can be evaluated from experiments

The solution may also be written as

$$\pi = \frac{V}{\sqrt{gh}} = c;\ \pi \text{ is nondiemsnional}$$

Summary

Given $f\left(V, g, h\right) = 0$ Where V, g, and h are dimensional, as shown before

The above system is equivalent to

$\psi\left(\pi\right) = 0;\ \pi$ is nondimensional $\quad \pi = \dfrac{V}{\sqrt{gh}}$

The technique described above is known as **Dimensional Analysis**

Dimensional Analysis

If certain physical phenomenon is governed by

$$f\left(x_1, x_2, \ldots. x_n\right) = 0$$

where some/all of the variables $\left(x\right)$ are dimensional

Then the above phenomena can be represented as

$$\psi\left(\pi_1, \pi_2, \ldots. \pi_m\right) = 0$$

$m < n$

where all the variables $\left(\pi\right)$ are non-dimensional

The nature of f and ψ may be obtained from experiments

Dimensional Analysis: Buckingham Pi Theorem

$$f\left(x_1, x_2, \ldots. x_n\right) = 0 \Rightarrow \psi\left(\pi_1, \pi_2, \ldots. \pi_m\right) = 0$$

where some/all x are dimensional

where all π are non-dimensional

where $m < n, \ m = n - k$

Minimum number of **fundamental dimensions** involved: k

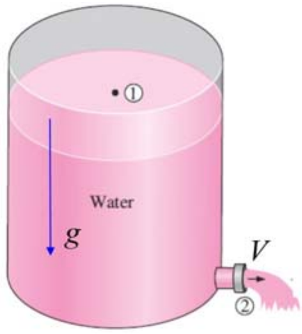
Example: $f\left(V, g, h\right) = 0$

$$n = 3 \quad k = 2 \quad m = n - k = 1$$

Experiment shows

$f(V,g,h)=0 \;\Rightarrow \psi(\pi)=0$

where π is made of V, g and h and
π is dimensionless



Above Equation suggests

$$\pi = V^a g^b h^c \qquad L^0T^0 = \left(LT^{-1}\right)^a \left(LT^{-2}\right)^b \left(L\right)^c = L^{a+b+c}T^{-a-2b}$$

$$a+b+c=-a-2b=0 \quad \Rightarrow b=c=-a/2$$

$$\pi = \left(\frac{V}{\sqrt{gh}}\right)^a$$

Any arbitrary value of a should be ok

$$\psi\left(\frac{V}{\sqrt{gh}}\right)=0$$

Pi Theorem: Repeating and non-repeating variables

$$\left(x_1, x_2, \ldots x_n\right) \Longrightarrow \left(x_{r1}, x_{r2}, \ldots x_{rk}; x_{nr1}, x_{nr2}, \ldots x_{nrm}\right)$$

Construction of Pi-terms

$$\pi_1 = x_{nr1}\left(x_{r1}\right)^{a_{11}}\left(x_{r2}\right)^{a_{12}}\left(x_{r3}\right)^{a_{13}}\ldots\left(x_{rk}\right)^{a_{1k}}$$

$$\pi_2 = x_{nr2}\left(x_{r1}\right)^{a_{21}}\left(x_{r2}\right)^{a_{22}}\left(x_{r3}\right)^{a_{23}}\ldots\left(x_{rk}\right)^{a_{2k}}$$

.....

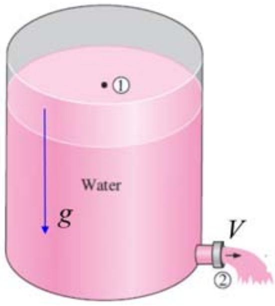
.....

$$\pi_m = x_{nrm}\left(x_{r1}\right)^{a_{m1}}\left(x_{r2}\right)^{a_{m2}}\left(x_{r3}\right)^{a_{m3}}\ldots\left(x_{rk}\right)^{a_{mk}}$$

Selection of repeating variables:

- **They must be dimensionally independent**
- **Together, they must include all fundamental dimensions**

Experiment shows, for viscous flow $f(V, g, h, \nu) = 0$



	M	L	T
V			
g			
h			
ν			

$n = 4 \quad k = 2 \quad m = 2$

We have to select two (02) repeating variables

Let's take the repeating variables: g, h

Non-repeating variables: V, ν

	L	T
V	1	-1
g	1	-2
h	1	0
ν	2	-1

$f(V, g, h, \nu) = 0 \quad n = 4 \quad k = 2 \quad m = 2$

Repeating variables: g, h

Non-repeating variables: V, ν

$$\pi_1 = x_{nr1}(x_{r1})^{a_{11}}(x_{r2})^{a_{12}}(x_{r3})^{a_{13}}\ldots(x_{rk})^{a_{1k}}$$

$$\pi_1 = V(g)^a(h)^b \Rightarrow L^0T^0 = LT^{-1}\left(LT^{-2}\right)^a(L)^b = L^{1+a+b}T^{-1-2a}$$

$$\Rightarrow a = b = -1/2 \qquad \boxed{\pi_1 = \frac{V}{\sqrt{gh}}}$$

similarly $\pi_2 = \nu(g)^a(h)^b \Rightarrow L^0T^0 = L^2T^{-1}\left(LT^{-2}\right)^a(L)^b$

$$2 + a + b = 0 = -1 - 2a \quad \Rightarrow a = -1/2, b = -3/2 \qquad \boxed{\pi_2 = \frac{\nu}{\sqrt{gh^3}}}$$

$$f(V, g, h, \nu) = 0 \quad \Rightarrow \quad f_1\left(\frac{V}{\sqrt{gh}}, \frac{\nu}{\sqrt{gh^3}}\right) = 0$$

$$\frac{V}{\sqrt{gh}} = \mathrm{Fr}$$ **Froude number**

$$\frac{\nu}{\sqrt{gh^3}} = \frac{V}{\sqrt{gh}}\frac{\nu}{Vh} = \frac{\mathrm{Fr}}{\mathrm{Re}}$$

We may also write $f_2(\mathrm{Fr}, \mathrm{Fr}/\mathrm{Re}) = 0 \quad \mathrm{Fr} = \psi(\mathrm{Fr}/\mathrm{Re})$

Frictionless flow: $\mathrm{Fr} = \text{constant}$

Viscous flow: $\mathrm{Fr} = \psi(\mathrm{Fr}/\mathrm{Re})$ Experiments are necessary to find the nature of function