ESO204A, Fluid Mechanics and rate Processes

Dimensional Analysis and Similitude

Simple and powerful qualitative technique applicable to many fields of science and engineering

Chapter 5 of F M White
Chapter 7 of Fox McDonald

Dimensional Analysis

If certain physical phenomenon is governed by

$f\left(x_1, x_2, \ldots . x_n\right)=0$ where some/all of the variables (x) are dimensional

Then the above phenomena can be represented as

$\psi\left(\pi_1, \pi_2, \ldots . \pi_m\right)=0$ where all the variables (π) are non-dimensional

$m<n$

The nature of f and ψ may be obtained from experiments

Dimensional Analysis: Buckingham Pi Theorem

$$f(x_1, x_2, \ldots x_n) = 0 \Longrightarrow \psi(\pi_1, \pi_2, \ldots \pi_m) = 0$$

where some/all x are dimensional

where all π are non-dimensional

where $m < n,\ m = n - k$

Minimum number of **fundamental dimensions** involved: k

Example: $f(V, g, h) = 0$

$n = 3 \quad k = 2 \quad m = n - k = 1$

Pi Theorem: Repeating and non-repeating variables

$$(x_1, x_2, \ldots x_n) \Longrightarrow (x_{r1}, x_{r2}, \ldots x_{rk}; x_{nr1}, x_{nr2}, \ldots x_{nrm})$$

Construction of Pi-terms

$$\pi_1 = x_{nr1}(x_{r1})^{a_{11}}(x_{r2})^{a_{12}}(x_{r3})^{a_{13}}\ldots(x_{rk})^{a_{1k}}$$

$$\pi_2 = x_{nr2}(x_{r1})^{a_{21}}(x_{r2})^{a_{22}}(x_{r3})^{a_{23}}\ldots(x_{rk})^{a_{2k}}$$

.....

.....

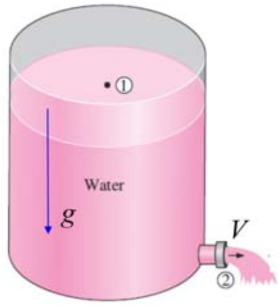
$$\pi_m = x_{nrm}(x_{r1})^{a_{m1}}(x_{r2})^{a_{m2}}(x_{r3})^{a_{m3}}\ldots(x_{rk})^{a_{mk}}$$

Selection of repeating variables:

- **They must be dimensionally independent**
- **Together, they must include all fundamental dimensions**

Experiment shows, for viscous flow $f\left(V,g,h,\nu\right)=0$

	M	L	T
V	0	1	-1
g	0	1	-2
h	0	1	0
ν	0	2	-1



$n=4 \quad k=2 \quad m=2$

We have to select two (02) repeating variables

Let's take the repeating variables: g,h

Non-repeating variables: V,ν

	L	T
V	1	-1
g	1	-2
h	1	0
ν	2	-1

$f\left(V,g,h,\nu\right)=0 \quad n=4 \quad k=2 \quad m=2$

Repeating variables: g,h

Non-repeating variables: V,ν

$$\pi_1 = x_{nr1}\left(x_{r1}\right)^{a_{11}}\left(x_{r2}\right)^{a_{12}}\left(x_{r3}\right)^{a_{13}}...\left(x_{rk}\right)^{a_{1k}}$$

$$\pi_1 = V\left(g\right)^a\left(h\right)^b \Rightarrow L^0T^0 = LT^{-1}\left(LT^{-2}\right)^a\left(L\right)^b = L^{1+a+b}T^{-1-2a}$$

$$\Rightarrow a=b=-1/2 \qquad \pi_1 = \frac{V}{\sqrt{gh}}$$

similarly $\pi_2 = \nu\left(g\right)^a\left(h\right)^b \Rightarrow L^0T^0 = L^2T^{-1}\left(LT^{-2}\right)^a\left(L\right)^b$

$$2+a+b=0=-1-2a \quad \Rightarrow a=-1/2, b=-3/2 \qquad \pi_2 = \frac{\nu}{\sqrt{gh^3}}$$

$$f(V,g,h,\nu)=0 \Rightarrow f_1\left(\frac{V}{\sqrt{gh}},\frac{\nu}{\sqrt{gh^3}}\right)=0$$

$$\frac{V}{\sqrt{gh}}=\mathrm{Fr}$$ **Froude number** $$\frac{\nu}{\sqrt{gh^3}}=\frac{V}{\sqrt{gh}}\frac{\nu}{Vh}=\frac{\mathrm{Fr}}{\mathrm{Re}}$$

We may also write $f_2(\mathrm{Fr},\mathrm{Fr}/\mathrm{Re})=0$ $\quad \mathrm{Fr}=\psi(\mathrm{Fr}/\mathrm{Re})$

Frictionless flow: $\mathrm{Fr}=\text{constant}$

Viscous flow: $\mathrm{Fr}=\psi(\mathrm{Fr}/\mathrm{Re})$ Experiments are necessary to find the nature of function

Advantages of dimensional analysis

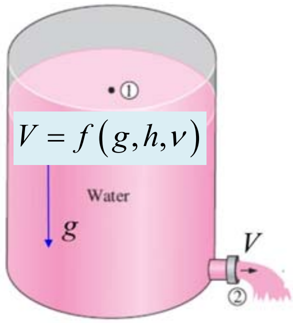
$$f(V,g,h,\nu)=0 \Rightarrow \mathrm{Fr}=\psi(\mathrm{Fr}/\mathrm{Re})$$

$$\mathrm{Fr}=\frac{V}{\sqrt{gh}} \qquad \frac{\mathrm{Fr}}{\mathrm{Re}}=\frac{\nu}{\sqrt{gh^3}}$$

- Less number of experiments are necessary, as opposed to the dimensional system
- Experiments become inexpensive
- Data reduction becomes easier, single plot is sufficient to show the results

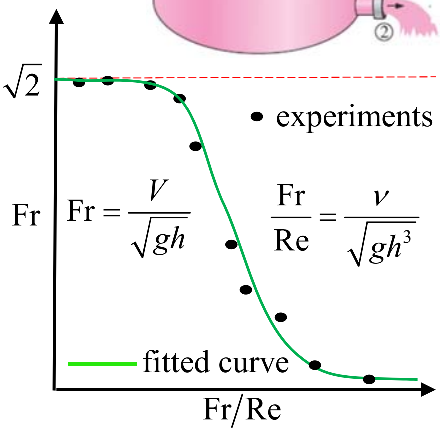
Dimensional analysis shows

$$\mathrm{Fr} = \psi\left(\mathrm{Fr}/\mathrm{Re}\right)$$



We would like to do a laboratory experiment now

We can vary h and measure V for few cases

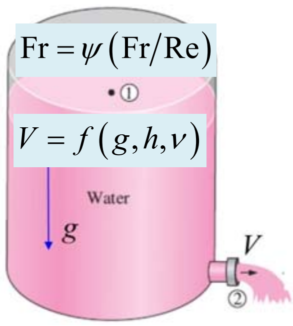


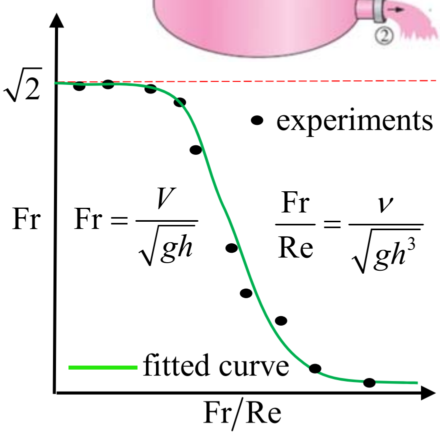
Finally we do a curve-fitting

*Nature of curve is arbitrary in this case, actual curve may look slightly different

Conclusions

- One Figure is enough, as opposed to many Figs of dimensional system
- Tank size is irrelevant as long as our assumptions aren't violated





The plot, developed from experiments in a smaller (model) tank may be used for similar bigger (prototype) tank

Similitude: Basic idea behind model testing

For the present case study $$\mathrm{Fr} = \psi\left(\frac{\nu}{\sqrt{gh^3}}\right)$$

Since the relation holds for similar model and prototype tanks

$$\text{if } \left(\frac{\nu}{\sqrt{gh^3}}\right)_{\text{model}} = \left(\frac{\nu}{\sqrt{gh^3}}\right)_{\text{prototype}}$$

$$\text{then } (\mathrm{Fr})_{\text{model}} = (\mathrm{Fr})_{\text{prototype}}$$

Model studies (similitude)

Certain fluid mechanical phenomenon is governed by

$$f(\pi_1, \pi_2, \ldots . \pi_n) = 0 \quad \text{where } \pi_i \text{ are non-dimensional}$$

When the model is **similar** to the prototype

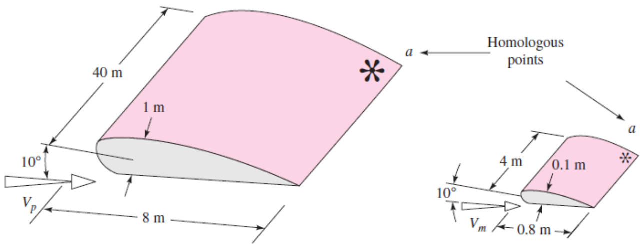
$$(\pi_i)_{\text{model}} = (\pi_i)_{\text{prototype}}, \quad i = 1, 2, \ldots n$$

Complete **similarity** requires

Geometric similarity + **Kinematic** similarity + **Dynamic** similarity

Geometric similarity: length-scale matching

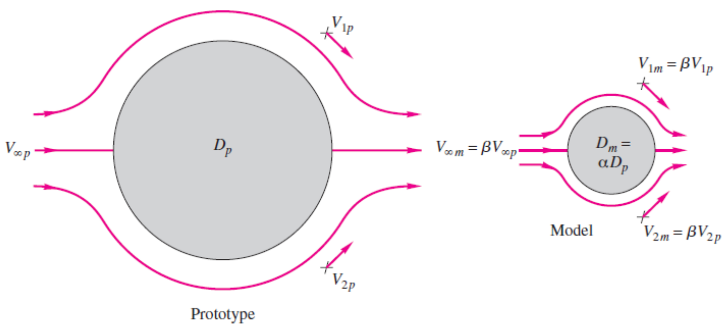
A model and prototype are geometrically similar if and only if all body dimension in all three coordinates have the same linear scale ratio



All angels, flow direction, orientation with the surroundings must be preserved

Kinematic similarity (velocity-scale matching)

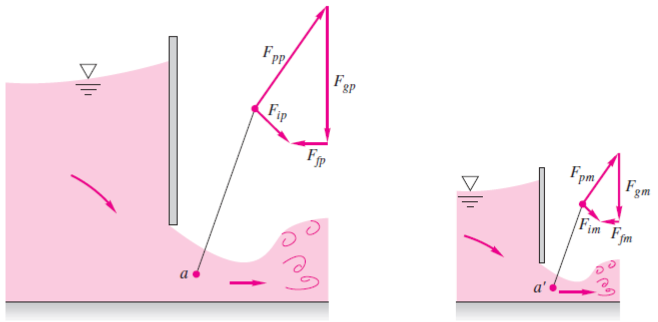
A model and prototype are kinematically similar if homologous particles lie at homologous points at homologous time



Kinematic similarity requires geometric similarity

Dynamic similarity (force-scale matching)

A model and prototype are dynamically similar if ratio of any two forces are same for model and prototype



Dynamic similarity requires geometric, kinematic similarities

Check before model testing

Geometric similarity + matching of **Pi-terms**

- Geometric similarity depends on proper design, manufacturing, material choice
- Proper choice of variables to include necessary fluid dynamical effects
- In reality, it is not always possible to attain complete similarity, and forced to work with partial similarity. May happens for more than three dominant forces

When fluid flows over an object, the object experiences fluid resistance known as 'drag force'

For incompressible flow with 'smooth' objects the drag force is given by

$$F = f\left(L, u, \rho, \mu\right)$$

Conduct dimensional analysis to identify the dimensionless numbers associated with the above phenomenon

A running car experiences fluid resistance known as 'drag force' $F = f\left(L, u, \rho, \mu\right)$

We are interested to conduct a model study in a wind tunnel to know the drag on the prototype

	M	***L***	***T***
$\boldsymbol{u}$			
$\boldsymbol{L}$			
$\boldsymbol{\rho}$			
$\boldsymbol{F}$			
$\boldsymbol{\mu}$			

$n = 5 \qquad k = 3 \qquad m = 2$

Repeating: L, u, ρ

$$\pi_1 = F\left(L\right)^a \left(u\right)^b \left(\rho\right)^c \qquad \pi_2 = \mu\left(L\right)^a \left(u\right)^b \left(\rho\right)^c$$