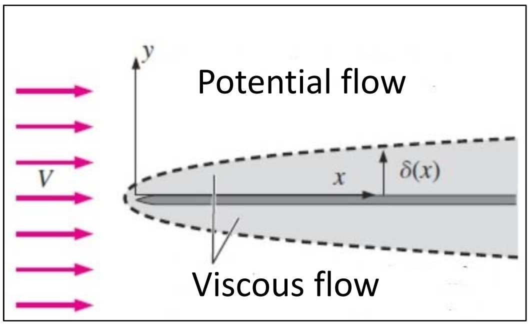
ESO204A, Fluid Mechanics and Rate Processes

# Incompressible flows over immersed bodies (External Flow)

Chapter 7 of F M White
Chapter 9 of Fox McDonald

**Boundary Layer** Very thin layer of fluid near solid surface where viscous effects are dominant

Outside boundary layer flow remains largely inviscid (amenable to potential flow solutions)

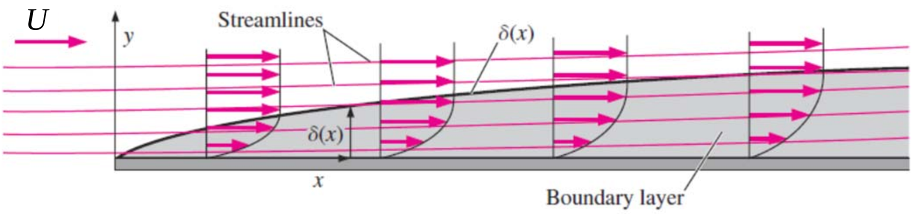



**Prandtl showed**

$\delta(x) \ll x$

Leads to meaningful approximation of N-S Eq.

## Boundary Layer over a flat plate



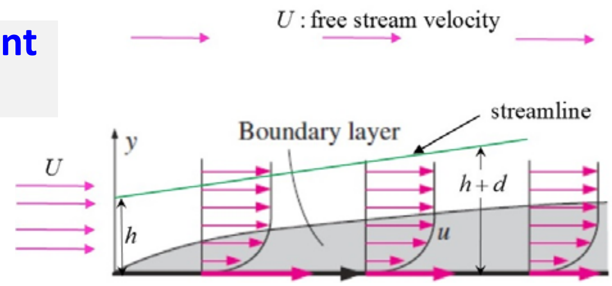

Boundary layer thickness $u_{y=\delta} = .99u_{\text{free stream}}$

Boundary layer shifts the streamlines upward

Displacement thickness $\delta^* = \int_0^\infty \left(1 - \frac{u}{U}\right) dy$

$U$ : free stream velocity

## Displacement thickness



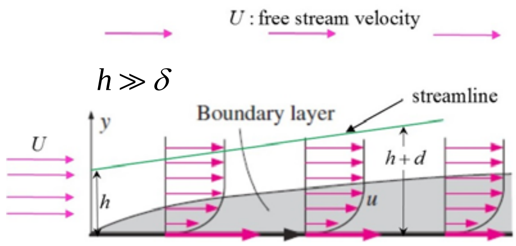

$h \gg \delta$

$$\left(\int_0^h \rho u dy\right)_{x=0} = \int_0^{h+d} \rho u dy \qquad \int_0^h U dy = \int_0^h u dy + \int_0^d u dy$$

$$\int_0^h U dy = \int_0^h u dy + \int_0^d U dy \qquad \int_0^h (U - u)\, dy = Ud$$

$$d = \int_0^h \left(1 - \frac{u}{U}\right) dy \qquad \delta^* = \int_0^\infty \left(1 - \frac{u}{U}\right) dy \approx \int_0^\delta \left(1 - \frac{u}{U}\right) dy$$

## BL flow causes momentum deficit, unlike potential flow

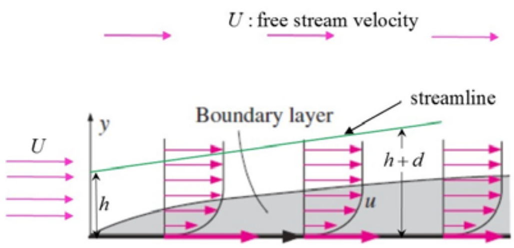



$$D = \left( \int_0^h \rho u^2 dy \right)_{x=0} - \int_0^{h+d} \rho u^2 dy \qquad \text{where } d = \int_0^h \left(1 - \frac{u}{U}\right) dy$$

$$= \rho U^2 h - \int_0^h \rho u^2 dy - \rho U^2 d \quad = \rho U^2 h - \int_0^h \rho u^2 dy - \rho U^2 \int_0^h \left(1 - \frac{u}{U}\right) dy$$

$$\frac{D}{\rho U^2} = h - \int_0^h \frac{u^2}{U^2} dy - \int_0^h \left(1 - \frac{u}{U}\right) dy \qquad \frac{D}{\rho U^2} = \int_0^h \frac{u}{U}\left(1 - \frac{u}{U}\right) dy$$

## Momentum thickness




$$\delta^{**} = \int_0^\infty \frac{u}{U}\left(1 - \frac{u}{U}\right) dy$$

$$\approx \int_0^\delta \frac{u}{U}\left(1 - \frac{u}{U}\right) dy$$

**Momentum thickness measures momentum deficit in terms of length; also indicates the shear stress**

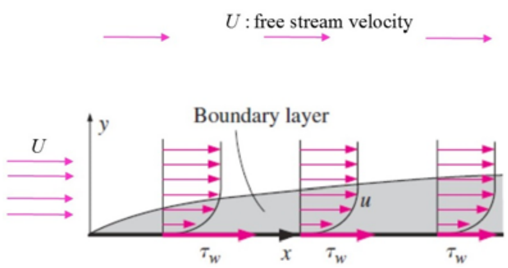
Boundary layer approximation ($\delta << x$) simplifies the N-S Eq leading to several exact solutions

Important results for laminar Boundary layer flow over a flat plate, known as Blasius solution

$$\frac{\delta}{x} = 4.91\,\mathrm{Re}_x^{-\frac{1}{2}} \qquad \frac{\delta^*}{x} = 1.72\,\mathrm{Re}_x^{-\frac{1}{2}} \qquad \frac{\delta^{**}}{x} = 0.664\,\mathrm{Re}_x^{-\frac{1}{2}}$$

BL approximation is applicable for High Re flow only; additional restrictions will follow

Important results for laminar boundary layer flow over a flat plate

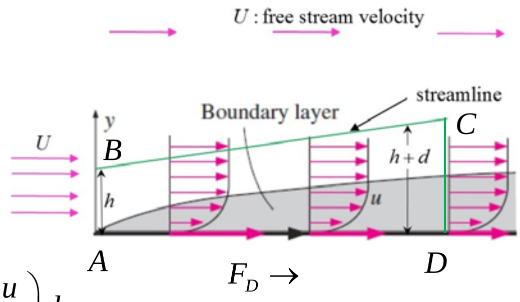



$$\frac{\delta}{x} = 4.91\,\mathrm{Re}_x^{-\frac{1}{2}} \qquad \frac{\delta^*}{x} = 1.72\,\mathrm{Re}_x^{-\frac{1}{2}} \qquad \frac{\delta^{**}}{x} = 0.664\,\mathrm{Re}_x^{-\frac{1}{2}}$$

$$\frac{\partial p}{\partial x} = 0; \frac{\partial p}{\partial y} = 0 \qquad \frac{u}{U} = f\left(\frac{y}{\delta}\right)$$

Self-similar velocity profile

$$D = \left( \int_0^h \rho u^2 dy \right)_{x=0} - \int_0^{h+d} \rho u^2 dy$$




$$\delta^{**} = \frac{D}{\rho U^2} = \int_0^{\delta} \frac{u}{U}\left(1 - \frac{u}{U}\right) dy$$

**Drag force**

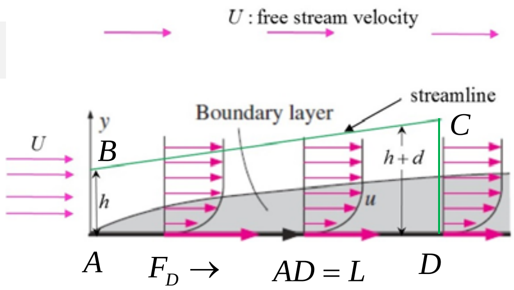
$$F_D = \left( \int_0^h \rho u^2 b dy \right)_{x=0} - \int_0^{h+d} \rho u^2 b dy \quad = bD$$

$$F_D = \rho b U^2 \delta^{**} \quad \Rightarrow \frac{dF_D}{dx} = \rho b U^2 \frac{d\delta^{**}}{dx} \qquad dF_D = \tau_w b dx$$

$$\tau_w = \rho U^2 \frac{d\delta^{**}}{dx}$$

**Karman's Momentum Integral Eq.**

**Wall shear stress**




$$\tau_w = \rho U^2 \frac{d\delta^{**}}{dx}$$

**Skin-friction coefficient**

$$C_{f,x} = \frac{\tau_w}{\frac{1}{2}\rho U^2} \qquad C_{f,x} = \frac{2d\delta^{**}}{dx}$$

**Drag force** $F_D = \rho b U^2 \delta^{**}$

**Drag coefficient**

$$C_D = \frac{F_D}{\frac{1}{2}\rho U^2 (bL)} \qquad C_D = \frac{2\delta^{**}}{L}$$

**Combining**

$$C_D = \frac{1}{L}\int_0^L C_{f,x} dx$$