ESO204A, Fluid Mechanics and rate Processes

Announcements

1. **Tutorial:** this Thursday, as usual; questions uploaded
2. **Quiz#1:** August 18, syllabus- as much as covered up to last class (Aug 08)

ESO204A, Fluid Mechanics and rate Processes

Conservation Laws: integral formulation

(Chapter 3 of F M White)

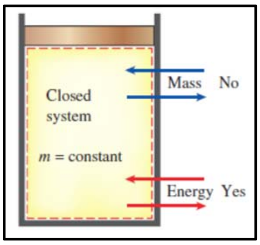
Reynolds Transport Theorem

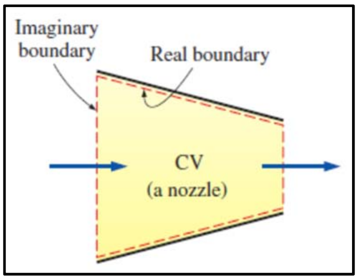
Connection between Eulerian and Lagrangian descriptions

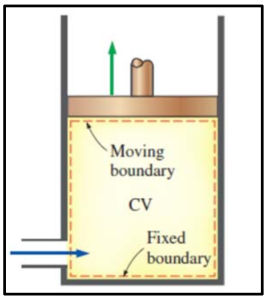
Mass conservation

System (control mass or closed system) **and control volume** (open system)

o System (sys): collection of matter with fixed identity, doesn't exchange mass with surroundings
o Control volume (CV): a geometric entity (fixed or moving, rigid or deformable) in space through which fluid may flow
o Control surface (CS): boundary of the CV

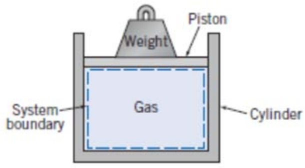




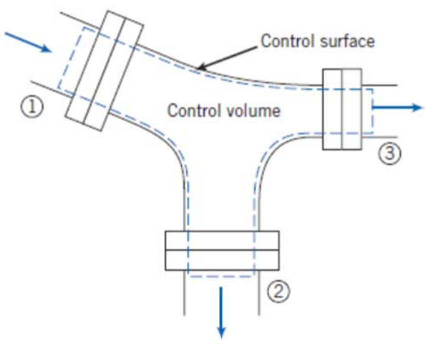


Apply conservation of mass to a system

$$\frac{dm_{sys}}{dt} = 0$$



How to use the above Eq. to find the Eq. of mass conservation in a CV?



$$\frac{\partial m_{CV}}{\partial t} = \text{mass inflow rate} - \text{mass outflow rate}$$

Extensive and intensive properties

extent ← $B = m\beta$ → intensity

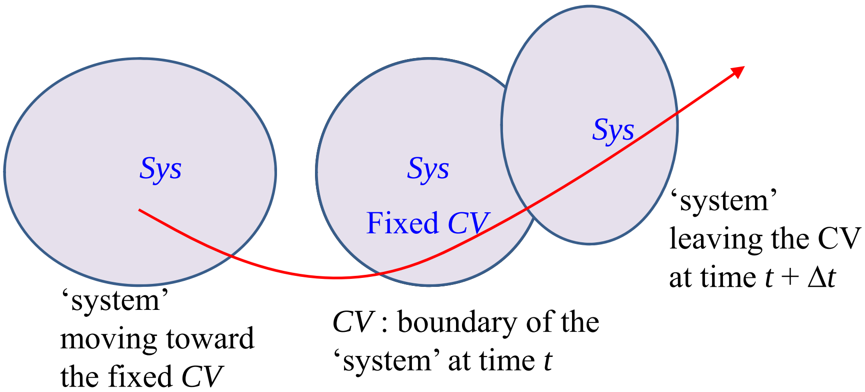
↓

mass

	Mass	Momentum	Kinetic Energy
B	m	$m\vec{u}$	$m\lvert\vec{u}\rvert^2/2$
β	1	$\vec{u}$	$\lvert\vec{u}\rvert^2/2$

In general, $B = \int_{V} \rho\beta d\mathcal{V}$

The moving ‘system’ may change its volume, shape



At time t system coincides with the fixed CV

We would like to make a connection between $\dfrac{dB_{sys}}{dt}$ and $\dfrac{\partial B_{CV}}{\partial t}$

Reynolds Transport Theorem (RTT)

For an extensive property $\boldsymbol{B}$ and corresponding intensive property $\boldsymbol{\beta}$, RTT states that

$$\frac{dB_{\text{sys}}}{dt} = \frac{\partial}{\partial t}\int_{\text{CV}} \rho\beta d\forall + \int_{\text{CS}} \rho\beta\left(\vec{u}.\vec{n}\right)dS$$

$$\text{where } B_{\text{sys}} = \int_{\text{mass}} \beta dm = \int_{\text{CV}(t)} \rho\beta d\forall$$

Material derivative, also written as $\dfrac{DB_{\text{sys}}}{Dt}$

$\vec{n}$: unit vector at the *CS* pointing outward from the *CV*

Connects Lagrangian (system-based) description to Eulerian (CV-based) description for an arbitrary volume of fluid

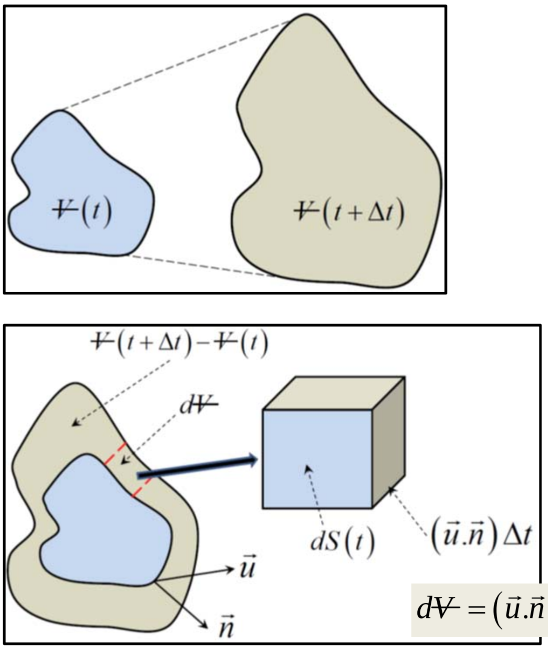
$$\frac{dB_{\text{sys}}}{dt} = \lim_{\Delta t\to 0}\left(\frac{1}{\Delta t}\left[\int_{\forall(t+\Delta t)} \rho\beta\left(t+\Delta t\right)d\forall - \int_{\forall(t)} \rho\beta\left(t\right)d\forall\right]\right)$$

$$= \lim_{\Delta t\to 0}\left(\frac{1}{\Delta t}\left[\int_{\forall(t)} \rho\beta\left(t+\Delta t\right)d\forall - \int_{\forall(t)} \rho\beta\left(t\right)d\forall\right]\right)$$

$$+ \lim_{\Delta t\to 0}\left(\frac{1}{\Delta t}\left[\int_{\forall(t+\Delta t)} \rho\beta\left(t+\Delta t\right)d\forall - \int_{\forall(t)} \rho\beta\left(t+\Delta t\right)d\forall\right]\right)$$

$$= \frac{\partial}{\partial t}\int_{\forall(t)} \rho\beta\left(t\right)d\forall + \lim_{\Delta t\to 0}\left(\frac{1}{\Delta t}\left[\int_{\forall(t+\Delta t)-\forall(t)} \rho\beta\left(t+\Delta t\right)d\forall\right]\right)$$

$$= \frac{\partial}{\partial t}\int_{\text{CV}} \rho\beta d\forall + \int_{\text{CS}} \rho\beta\left(\vec{u}.\vec{n}\right)dS$$



$$d\mathcal{V} = (\vec{u}.\vec{n})\Delta t dS$$

$$\frac{1}{\Delta t}\left[\int_{\mathcal{V}(t+\Delta t)-\mathcal{V}(t)} \beta d\mathcal{V}\right]$$

$$= \frac{1}{\Delta t}\int_{\text{CS}} \beta(\vec{u}.\vec{n})\Delta t dS$$

$$= \int_{\text{CS}} \beta(\vec{u}.\vec{n}) dS$$

Reynolds Transport Theorem (RTT)

Rate of change of B of the system

$$\frac{dB_{\text{sys}}}{dt} = \frac{\partial}{\partial t}\int_{\text{CV}} \rho\beta d\mathcal{V} + \int_{\text{CS}} \rho\beta(\vec{u}.\vec{n}) dS$$

Rate of change of B in the CV

Rate of flow of B through the CS

The theorem can also be proved for moving/deformable CVs

What's the difference between

$$\frac{dB_{sys}}{dt} \text{ and } \frac{\partial B_{CV}}{\partial t}; B \equiv \text{mass}$$

Consider fixed CV just enclosing the fire extinguisher (in service)

$$\frac{\partial B_{CV}}{\partial t} < 0 \qquad \frac{dB_{sys}}{dt} = 0$$

Conservation of mass: integral form

$$\frac{dB_{sys}}{dt} = \frac{\partial}{\partial t}\int_{CV} \rho\beta dV\!\!\!/ + \int_{CS} \rho\beta(\vec{u}.\vec{n})\,dS \qquad \text{here } B = m \Rightarrow \beta = 1$$

$$\frac{dm_{sys}}{dt} = \frac{\partial}{\partial t}\int_{CV} \rho dV\!\!\!/ + \int_{CS} \rho(\vec{u}.\vec{n})\,dS$$

$$\boxed{\frac{\partial}{\partial t}\int_{CV} \rho dV\!\!\!/ + \int_{CS} \rho(\vec{u}.\vec{n})\,dS = 0}$$

Equation of mass conservation
(in integral form)

Equation of mass conservation: integral form

$$\frac{\partial}{\partial t}\int_{\mathrm{CV}} \rho d\cancel{V} + \int_{\mathrm{CS}} \rho\left(\vec{u}.\vec{n}\right) dS = 0$$

Constant density (incompressible) flow:

$$\frac{\partial \cancel{V}_{\mathrm{CV}}}{\partial t} + \int_{\mathrm{CS}} \left(\vec{u}.\vec{n}\right) dS = 0 \Rightarrow \int_{\mathrm{CS}} \left(\vec{u}.\vec{n}\right) dS = 0$$

If CV doesn't change

Similarly for steady flow:

$$\rho\frac{\partial \cancel{V}_{\mathrm{CV}}}{\partial t} + \int_{\mathrm{CS}} \rho\left(\vec{u}.\vec{n}\right) dS = 0 \Rightarrow \int_{\mathrm{CS}} \rho\left(\vec{u}.\vec{n}\right) dS = 0$$

If CV doesn't change