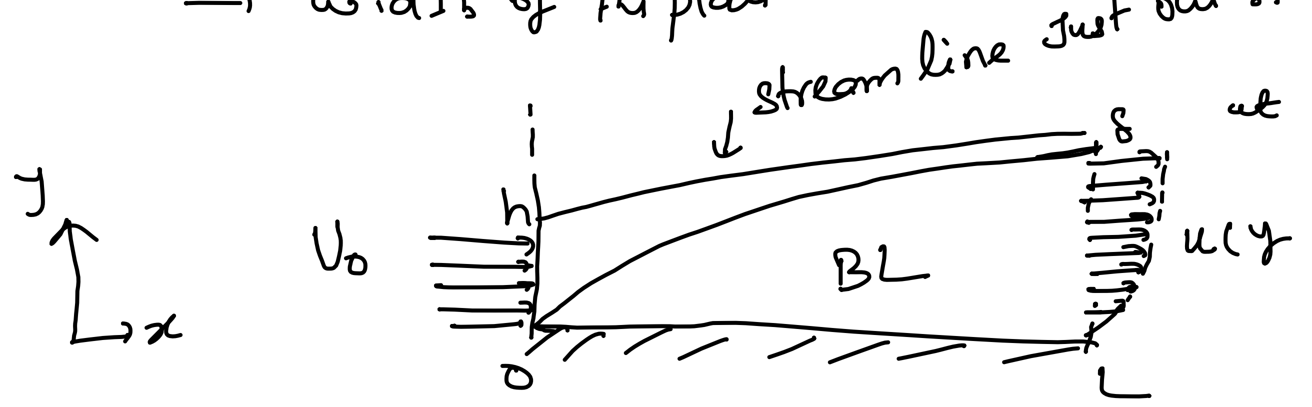
Boundary layer theory.... continued

Drag on a flat plate

- Newton's law of motion is valid for any kind of flow (laminar or turbulent)

→ width of the plate $= b$



at BL $u = U_0$ at $y = \infty$
$u = 0.99U_0$ at $y = \delta$

control volume $ohδLo$ → Apply newton's law of motion

Drag force on the surface $= D\hat{i}$

x-force on OL surface $= -D\hat{i}$

The four sides of CV

(i) oh side ⇒ inlet $\vec{V}\cdot\vec{n} = -U_0$

(ii) hδ side ⇒ inlet/outlet $\vec{V}\cdot\vec{n} = 0$ $\vec{V} \parallel \vec{dS}$

(iii) δL side ⇒ outlet $\vec{V}\cdot\vec{n} = u(y)$

(iv) LO side ⇒ surface $\vec{V}\cdot\vec{n} = 0$

x-momentum balance at steady state

$$-D = \cancel{\frac{d}{dt}(\ )} + \int_{(i)} u\rho(V.n)\,dA + \int_{(iii)} u\rho(V.n)\,dA$$

$$-D = U_0\rho(-U_0)\,b\,h + \int u(y)\,\rho\,u(y)\,b\,dy\Big|_{x=L} \quad \text{— (i)}$$

→ we want to replace h in terms of δ.

→ mass balance on CV

$$0 = \cancel{\frac{d}{dt}(\ )} + \int_0^\delta \rho b\,u(y)\,dy\Big|_{x=L} - \rho b U_0 h \quad \text{— (ii)}$$

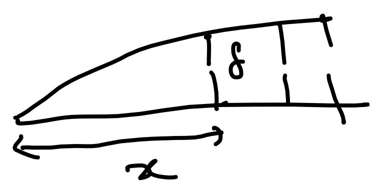
$$U_0 h = \int_0^\delta u\,dy\Big|_{x=L}$$

$$D = \rho b U_0 \int_0^\delta u\,dy\Big|_{x=L} - \rho b \int_0^\delta u^2\,dy\Big|_{x=L}$$

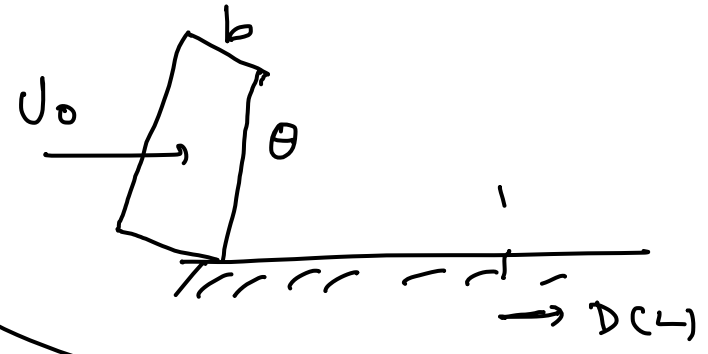
$$\boxed{\underline{D} = \rho b \int_0^{\delta} \underline{u}\,(U_0 - \underline{u})\, dy \Big|_{x=L}}$$

Alternative way of calculating Drag on a flat plate

$$D(x) = \rho b \int_0^{\delta(x)} u\,(U_0 - u)\, dy$$



Momentum thickness θ



momentum in $= \rho U_0 b \theta \, U_0 = \rho U_0^2 b \theta$

$$D(x) = \rho U_0^2 b \theta$$

θ = is measure of drag

$$\Rightarrow \quad \boxed{\theta = \int_0^{\delta} \frac{u}{U_0}\left(1 - \frac{u}{U_0}\right) dy}$$

$$D(x) = b \int_0^{x} \tau_w \, dx \Rightarrow \frac{dD(x)}{dx} = b\,\tau_w$$

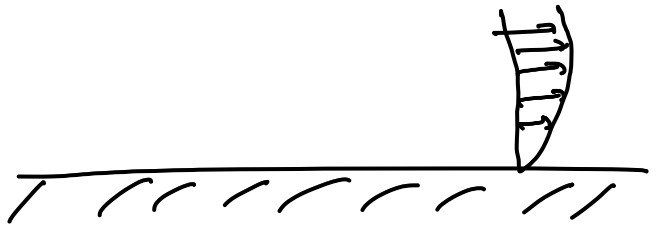
$$\frac{dD(x)}{dx} = b\,\tau_w = \rho U_0^2 b \frac{d\theta}{dx} \Rightarrow \boxed{\tau_w = \rho U_0^2 \frac{d\theta}{dx}}$$

- momentum-integral eq^n for flat plate
- valid laminar / turbulent

→ goal → find $\delta(x)$

→ Drag force → friction coefficient C_f → (similar to friction factor for pressure drop)

Laminar flow



Karman → impirical
$\underline{u(y)}$
$Re \to$ High

Assumption: $u(y)$ = Parabolic f^n of y
→ Approximal sol^n
→ Exact sol^n by solving Navier-stoke's for BL

Approximate sol^n

$$u(x,y) = U_0\left(\frac{2y}{\delta} - \frac{y^2}{\delta^2}\right) \qquad 0 \le y \le \delta(x)$$

$$\theta = \int_0^{\delta} \frac{u}{U_0}\left(1 - \frac{u}{U_0}\right) dy$$

$$= \frac{2}{15}\delta$$

$$\tau_w = \mu \left.\frac{\partial u}{\partial y}\right)_{y=0} = \frac{2U_0\mu}{\delta}$$

$$\tau_w = \rho U_0^2 \frac{d\theta}{dx} = \frac{2U_0\mu}{\delta}$$

$$\rho U_0^2 \left(\frac{2}{15}\right)\frac{d\delta}{dx} = \frac{2U_0\mu}{\delta}$$

$$\frac{1}{2}\delta^2 = \frac{15\nu}{U_0} x$$

$$\boxed{\frac{\delta}{x} \simeq 5.5\left(\frac{\nu}{U_0 x}\right)^{1/2} = \frac{5.5}{Re_x^{1/2}}}$$ Approximate

$$\boxed{\frac{\delta}{x} = \frac{5.0}{Re_x^{1/2}}}$$ Exact

Drag force

Skin-friction Coefficient C_f

$$C_f = \frac{\tau_w}{\frac{1}{2}\rho U_0^2} = \frac{2U_0\mu}{\delta\left(\frac{1}{2}\rho U_0^2\right)} = \frac{4\mu}{\delta \rho U_0}$$

$$\delta =$$

$$C_f = \frac{4}{(15\times 2)^{1/2}}\left(\frac{\nu}{x U_0}\right)^{1/2} = \frac{0.73}{Re_x^{1/2}} \text{ (Approximate)}$$

$$C_f = \frac{0.664}{Re_x^{1/2}} \text{ (Exact)}$$

$Re \to$ High $\Rightarrow$ $\delta << L$ (thin) $\Rightarrow$ Boundary Layer theory

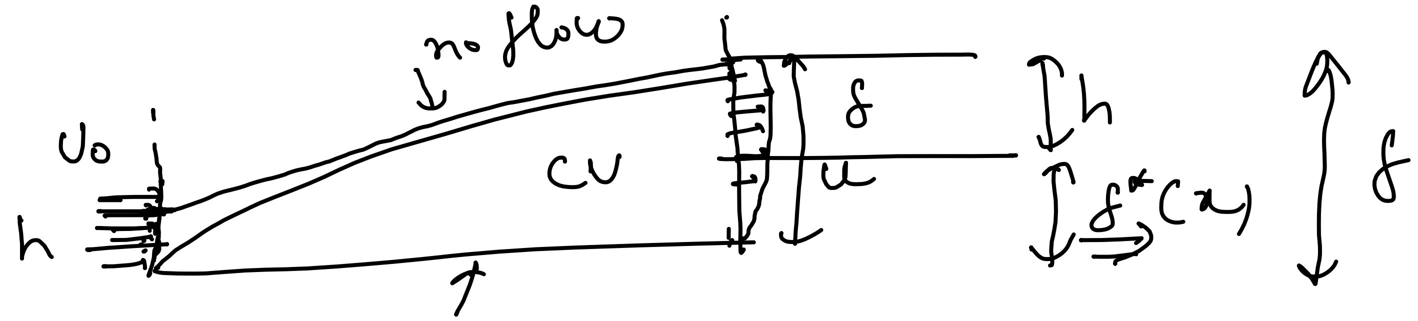
$$\frac{\delta}{x} \le 0.1 \Rightarrow 0.1 = \frac{5}{(Re_x)^{1/2}} \Rightarrow Re_x > 2500$$

Thin

— For thick BL, boundary layer theory fails because thick layer has significant effect on the outer - inviscid flow   Re → High but laminar

Displacement thickness ($\delta^*(x)$)

— stream lines outside of BL are deflected by a distance $\delta^*(x)$ to satisfy mass conservation between the inlet and outlet



$\delta = \delta^* + h$

mass balance on CV

$$0 = \cancel{\frac{d}{dt}}(0) + \int \cancel{\rho} u \cancel{b}\, dy - \cancel{\rho} U_0 \cancel{b} h$$

$$U_0 h = \int_0^\delta u\, dy = \int_0^\delta (U_0 + u - U_0)\, dy =$$

$$= U_0 \delta + \int (u - U_0)\, dy$$

$$U_0 h = U_0(\delta^* + h) + \int (u - U_0)\, dy$$

$$-U_0 \delta^* = \int (u - U_0)\, dy \Rightarrow \boxed{\delta^* = \int_0^\delta \left(1 - \frac{u}{U_0}\right) dy}$$

Approximate

$$u = U_0\left(\frac{2y}{\delta} - \frac{y^2}{\delta^2}\right)$$

$$\delta^* = \int_0^\delta \left(1 - \frac{2y}{\delta} + \frac{y^2}{\delta^2}\right) dy = \frac{1}{3}\delta$$

$$\boxed{\frac{\delta^*}{x} = \frac{1}{3}\frac{\delta}{x} = \frac{1}{3}\frac{5.5}{Re_x^{1/2}} = \frac{1.83}{Re_x^{1/2}}}$$

Exact

$$\delta^* = 0.344\,\delta$$

$$\frac{\delta}{x} = \frac{5.0}{Re_x^{1/2}}$$

$$\boxed{\frac{\delta^*}{x} = \frac{1.721}{Re_x^{1/2}}}$$