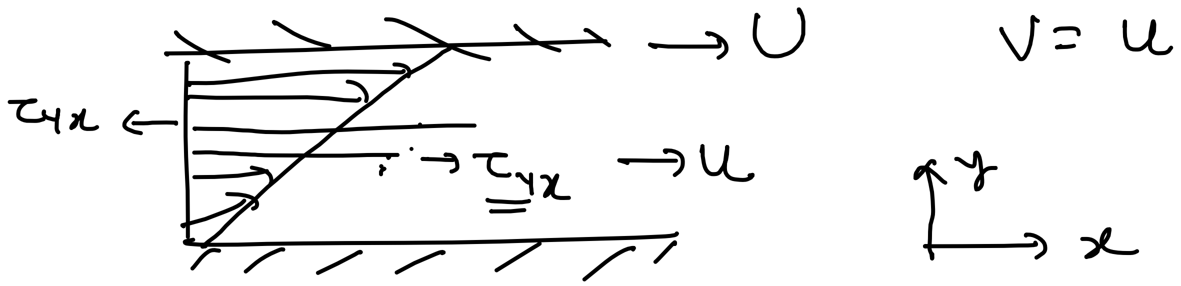
Newton's Law of viscosity (Revisit)



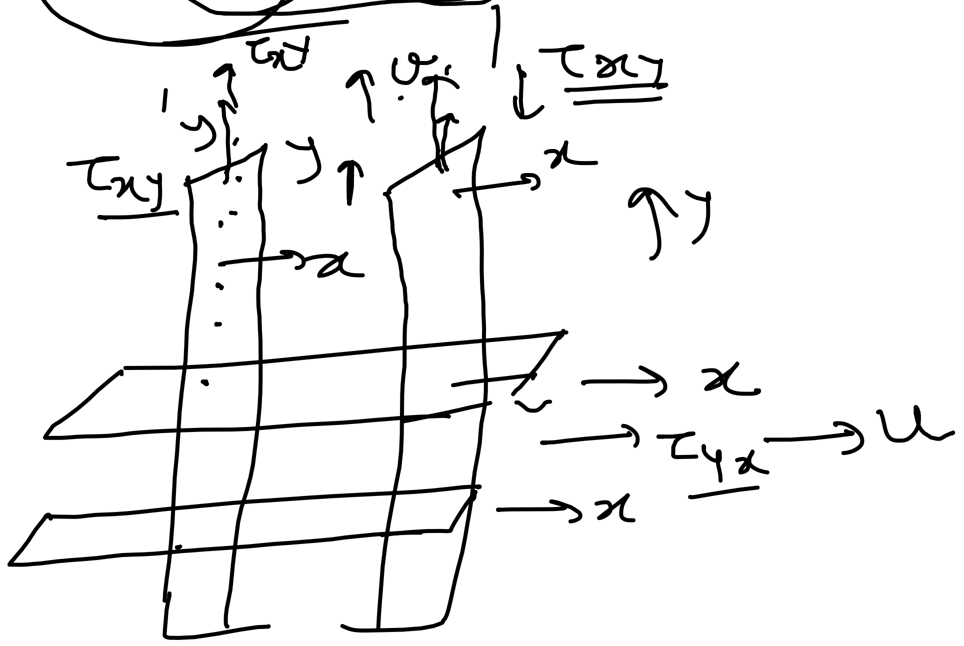
$V = u\hat{i}$

$\tau_{yx} = \mu\left(\frac{\partial u}{\partial y}\right) \leftarrow$ first Lecture

$V = u\hat{i} + v\hat{j} + w\hat{k}$

Yesterday's Lecture $\to$ generalization of Newton's Law of viscosity

✓ $\tau_{yx} = \tau_{xy} = \mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)$; $\tau_{zx} = \tau_{xz} = \mu\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right)$; $\tau_{yz} = \tau_{zy} = \mu\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right)$



$$\rho \frac{dV}{dt} = -\nabla p + \rho g + \nabla \cdot \tau_{ij}$$

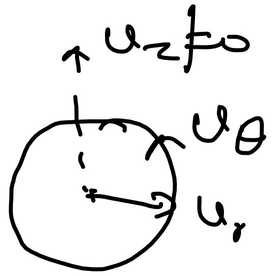
Inviscid $\Rightarrow$ No viscous stress $\tau_{ij} = 0$

✓ $\rho \frac{dV}{dt} = -\nabla p + \rho g$ Euler's eqn

Stream function → 2D u, v,

Incompressible plane flow in polar coordinates (Cylindrical)

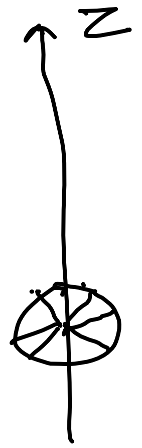
$u_r,\ v_\theta,\ v_z$



$$\frac{1}{r}\frac{\partial}{\partial r}(r u_r) + \frac{1}{r}\frac{\partial}{\partial \theta}(v_\theta) = 0$$

$$v_r = \frac{1}{r}\frac{\partial \psi}{\partial \theta}\ ,\quad v_\theta = -\frac{\partial \psi}{\partial r}$$

Incompressible axissymmetric flow



$$\frac{\partial}{\partial \theta} = 0$$

$$2D \rightarrow v_\theta = 0$$

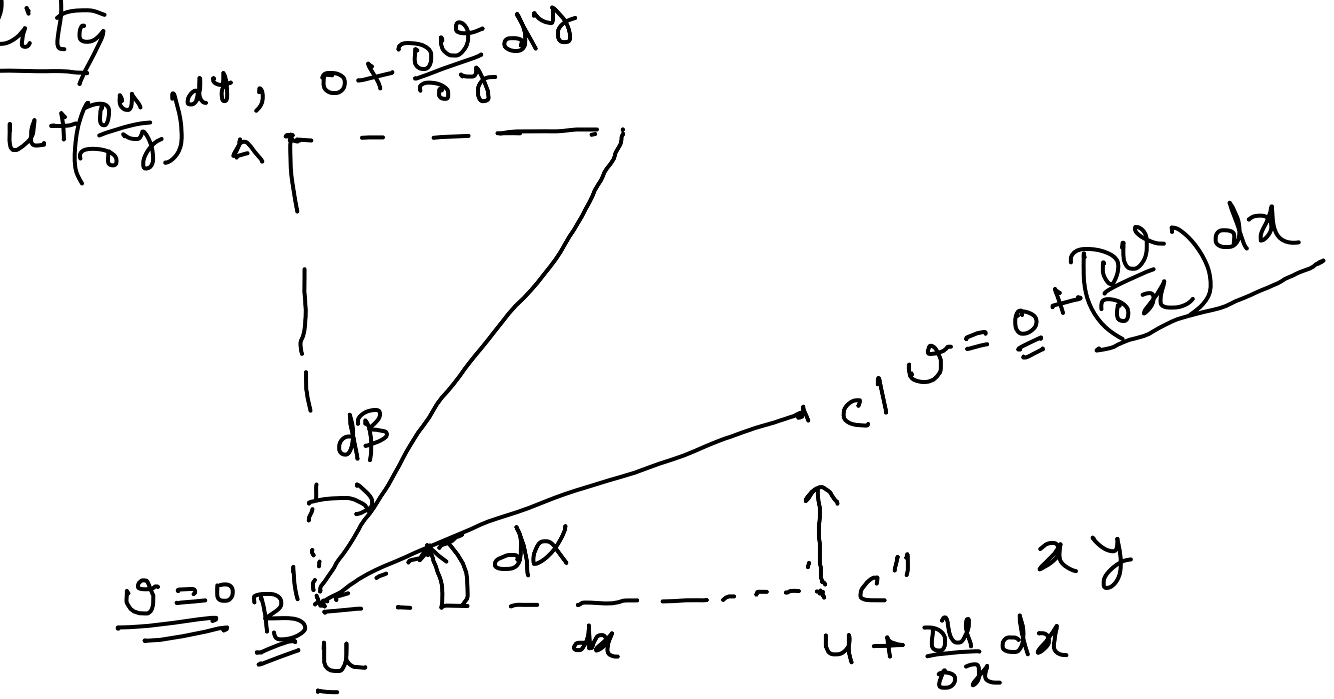
$$v_z,\ v_r$$

$$\frac{1}{r}\frac{\partial}{\partial r}(r v_r) + \frac{\partial}{\partial z}(v_z) = 0$$

$$v_r = -\frac{1}{r}\frac{\partial \psi}{\partial z}\ ,\quad v_z = \frac{1}{r}\frac{\partial \psi}{\partial r}$$

Vorticity $= \nabla \times \vec{V}$ →

Vorticity and irrotationality



Avg angular velocity $\vec{\omega}_z = \frac{1}{2}\left(\frac{d\alpha}{dt} - \frac{d\beta}{dt}\right)$

$$d\alpha = \lim_{dt\to 0} \tan^{-1} \frac{\left(\frac{\partial v}{\partial x}\right) dx\, dt}{dx + \left(\frac{\partial u}{\partial x} dx\right) dt} = \lim_{dt\to 0} \tan^{-1} \frac{\left(\frac{\partial v}{\partial x}\right) \cancel{dx}\, dt}{\cancel{dx}} = \frac{\partial v}{\partial x} dt$$

$$d\beta = \lim_{dt\to 0} \tan^{-1} \frac{\frac{\partial u}{\partial y} dy\, dt}{dy} = \frac{\partial u}{\partial y} dt$$

$$\omega_z = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)$$

Similarly $\omega_x = \frac{1}{2}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right)$, $\omega_y = \frac{1}{2}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right)$

$$\omega = \omega_x \hat{i} + \omega_y \hat{j} + \omega_z \hat{k}$$

$$\omega = \frac{1}{2} \text{curl } V$$

$$= \frac{1}{2}\begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ u & v & w \end{vmatrix}$$

$$=$$

$$\text{Vorticity} = 2\omega = \nabla \times V$$

Bernoulli eq^n ← momentum balance → energy balance along streamline

→ Frictionless flow ⇒ Viscous terms are not present

→ Euler's eq^n $\quad\quad$ ξ = vorticity $= \nabla \times v$

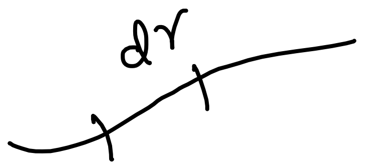
$$\rho \frac{dv}{dt} = \rho g - \nabla p$$

$$\frac{dv}{dt} = \frac{\partial v}{\partial t} + (v \cdot \nabla) v = \frac{\partial v}{\partial t} + \nabla \tfrac{1}{2} v^2 + \xi \times v$$

→ vector identity

$$\rho \frac{dv}{dt} = \rho g - \nabla p = \rho \frac{\partial v}{\partial t} + \rho \nabla \tfrac{1}{2} v^2 + \rho(\xi \times v)$$

$$\left(\rho \frac{\partial v}{\partial t} + \nabla \tfrac{1}{2} \rho v^2 + \rho(\xi \times v) + \nabla p - \rho g\right) = 0$$



$$\left[\rho \frac{\partial v}{\partial t} + \nabla \tfrac{1}{2} \rho v^2 + \rho(\xi \times v) + \nabla p - \rho g\right] \cdot dr = 0$$

$(\nabla \times v) \times v$

$$(\xi \times v) \cdot dr = 0$$

~~i)~~ $v = 0$ ⇒ hydrostatic $\quad \nabla p = -\rho \vec{g}$

✓ ii) $\xi = 0$ ⇒ vorticity $= 0$ = irrotational flow ←

✗ iii) $(\xi \times v) \perp dr$ → specialized condition ← rarely valid

iv) $(\xi \times v) \perp dr$

⇒ $\vec{v} \parallel \vec{dr}$

streamline

$\xi \times v = 0$

condition (iv)

$$\frac{\partial v}{\partial t}\cdot dr + \left(\frac{\partial}{\partial x}\tfrac{1}{2}v^2\,\hat{i} + \frac{\partial}{\partial y}\tfrac{1}{2}v^2\,\hat{j} + \frac{\partial}{\partial z}\tfrac{1}{2}v^2\,\hat{k}\right)\cdot(dx\,\hat{i} + dy\,\hat{j} + dz\,\hat{k}) + \frac{1}{\rho}\left(\frac{\partial p}{\partial x}\hat{i} + \frac{\partial p}{\partial y}\hat{j} + \frac{\partial p}{\partial z}\hat{k}\right)\cdot(dx\,\hat{i} + dy\,\hat{j} + dz\,\hat{k})$$

$$- \vec{g}\cdot(dx\,\hat{i} + dy\,\hat{j} + dz\,\hat{k}) = 0$$

$$\frac{\partial V}{\partial t}ds + d\left(\tfrac{1}{2}v^2\right) + \frac{1}{\rho}dP - (-g\hat{k})\cdot(dx\,\hat{i} + dy\,\hat{j} + dz\,\hat{k}) = 0$$

$$\frac{\partial v}{\partial t}ds + d\left(\tfrac{1}{2}v^2\right) + \frac{dp}{\rho} + g\,dz = 0$$

Steady state

$$d\left(\tfrac{1}{2}v^2\right) + \frac{dp}{\rho} + g\,dz = 0 \rightarrow \text{Bernoulli eq}$$

incompressible flow

$$\tfrac{1}{2}(V_2^2 - V_1^2) + \left(\frac{P_2 - P_1}{\rho}\right) + g(Z_2 - Z_1) = 0$$

$\tfrac{1}{2}v^2 + \frac{p}{\rho} + gz =$ constant along a stream line → frictionless flow incompressible

(ii) $\xi = 0$ $(\xi \times V)\cdot dr = 0$ → irrotational

every where