ESO204A, Fluid Mechanics and rate Processes

Conservation laws: differential formulation

Useful for calculation of 'field detail', as opposed to integral formulation that deals with 'average' and 'net' quantities

Chapter 4 of F M White
Chapter 5 of Fox McDonald

Conservation laws: differential formulation

Mass conservation

Momentum conservation

Exact solution

This Sec. requires some vector calculus recap, use your Maths books, if necessary

Reynolds Transport Theorem

$$\frac{dB_{sys}}{dt} = \frac{\partial}{\partial t}\int_{CV} \rho\beta d\forall + \int_{CS} \rho\beta(\vec{u}.\vec{n})dA; \quad B_{sys} = \int_{mass} \beta dm$$

Assuming CV does not change over time

$$\frac{dB_{sys}}{dt} = \int_{CV} \frac{\partial(\rho\beta)}{\partial t} d\forall + \int_{CS} \rho\beta(\vec{u}.\vec{n})dA$$

Now changing surface integral to volume integral

$$\frac{dB_{sys}}{dt} = \int_{CV}\left[\frac{\partial(\rho\beta)}{\partial t} + \nabla.(\rho\beta\vec{u})\right]d\forall$$

Review Gauss divergence theorem from your Maths book

For now, this form of RTT will be more useful

Mass conservation: $\frac{dm}{dt} = 0$

RTT: $\frac{dB_{sys}}{dt} = \int_{CV}\left[\frac{\partial(\rho\beta)}{\partial t} + \nabla.(\rho\beta\vec{u})\right]d\forall$

using $B_{sys} = m$ we have $\beta = 1$ and $\frac{dB_{sys}}{dt} = 0$

$$\int_{CV}\left[\frac{\partial\rho}{\partial t} + \nabla.(\rho\vec{u})\right]d\forall = 0$$

This relation is true for arbitrary CV

$$\frac{\partial\rho}{\partial t} + \nabla.(\rho\vec{u}) = 0$$

Mass conservation Equation
also known as
Continuity Equation

Continuity: $\frac{\partial \rho}{\partial t}+\nabla.\left(\rho\vec{u}\right)=0$

Splitting the second term $\nabla.\left(\rho\vec{u}\right)=\rho\nabla.\vec{u}+\left(\vec{u}.\nabla\right)\rho$

$$\frac{\partial \rho}{\partial t}+\nabla.\left(\rho\vec{u}\right)=0 \quad \Rightarrow \frac{\partial \rho}{\partial t}+\left(\vec{u}.\nabla\right)\rho+\rho\nabla.\vec{u}=0$$

Rearranging $\frac{d\rho}{dt}+\rho\nabla.\vec{u}=0$ $\quad \frac{1}{\rho}\frac{d\rho}{dt}+\nabla.\vec{u}=0$

Continuity for incompressible flow: $\nabla.\vec{u}=0$ Steady or unsteady

Continuity for incompressible flow: $\nabla.\vec{u}=0$

Condition for incompressible flow: $\frac{1}{\rho}\frac{d\rho}{dt}=0$

Happens when

- density is constant everywhere (although this is not necessary)
- density of a fluid particle remains constant but density may vary from point to point (known as variable density incompressible flow)

In majority of incompressible flow cases, density is considered to be constant everywhere

From thermodynamics: $\rho = \rho(p, T)$

$$d\rho = \left(\frac{\partial \rho}{\partial p}\right)_T dp + \left(\frac{\partial \rho}{\partial T}\right)_p dT \qquad \text{speed of sound } c = \sqrt{\left(\frac{\partial p}{\partial \rho}\right)_T}$$

$$\text{Mach number Ma} = \frac{u}{c}$$

Incompressibility requires high values of c (small Ma)

Ma < 0.3 is usually considered incompressibe

Speed of sound (sonic speed) c is usually high in liquids; liquid flows are, therefore, almost always incompressible

Speed of sound (sonic speed) c is also quite high in gases; many engineering applications, involving air flows, are in incompressible regime

- Usain Bolt ~44.7km/h
- F1 cars ~300km/h
- Category 5 hurricane ~300km/h

Examples of compressible flows

- Cruise speed of passenger aircraft ~270m/s
- AK47 bullet, MIG-29 ~700m/s (supersonic)
- Reentry vehicle ~2000-7000m/s (supersonic)

Present course primarily involves incompressible flows only

Incompressible Continuity: $\nabla . \vec{u} = 0$

Vector form or coordinate free form

In Cartesian coordinate system

1-D: $\frac{\partial u}{\partial x} = 0$

2-D: $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$

3-D: $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$

For incompressible flow, steady/unsteady continuity Eqns. are just the same

For a 2-D incompressible flow field: $u = bx, \; b = \text{constant}$

Find v, when $v(y=0)=0$

$= 0$ since $v(y=0)=0$

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0 \quad \Rightarrow \frac{\partial v}{\partial y}=-b \quad \Rightarrow v=-by+\cancel{f}(t,x)$$

How the streamlines should look like?

$$\frac{dx}{u}=\frac{dy}{v} \quad \Rightarrow \frac{dx}{bx}=\frac{dy}{-by} \quad \Rightarrow xy=c \;\; (\text{constant})$$

Different values of c will provide different streamlines

For **2-D incompressible flow,** there exists a function, called **streamfunction**

$$\psi(x,y,t) \text{ such that } u=\frac{\partial \psi}{\partial y}, v=-\frac{\partial \psi}{\partial x}$$

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=\frac{\partial^2 \psi}{\partial x \partial y}-\frac{\partial^2 \psi}{\partial x \partial y}=0$$ Continuity satisfied!!

Along a streamline $\dfrac{dx}{u}=\dfrac{dy}{v} \quad \Rightarrow udy - vdx = 0$

$$\Rightarrow \frac{\partial \psi}{\partial y}dy+\frac{\partial \psi}{\partial x}dx=0 \quad \Rightarrow d\psi = 0 \;\; \Rightarrow \psi = \text{constant}$$

Find the streamfunction and hence the Eq. of streamline for the following 2-D incompressible flow field

$$u = bx, v = -by;\ b = \text{constant}$$

$$u = \frac{\partial \psi}{\partial y}, v = -\frac{\partial \psi}{\partial x} \qquad \Rightarrow bx = \frac{\partial \psi}{\partial y}, by = \frac{\partial \psi}{\partial x}$$

$$\Rightarrow \psi = bxy + f_1(x), \psi = bxy + f_2(y) \qquad f_1(x) = f_2(y) = c_1 \ (\text{constant})$$

Streamfunction: $\psi = bxy + c_1$ $\Rightarrow xy = \frac{\psi - c_1}{b} = c$

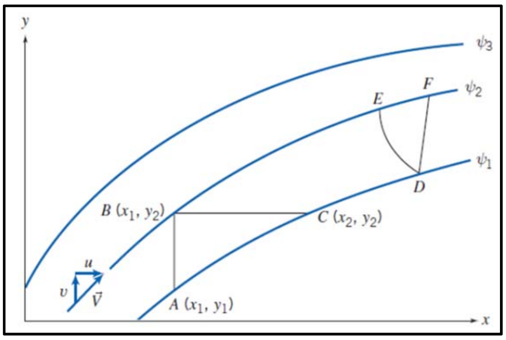
Streamline: $xy = c$ Different values of c will provide different streamlines

Along a streamline

$$d\psi = 0 \quad \Rightarrow \psi = \text{constant}$$

$$Q_{\text{AB}} = \int_{y=y_1}^{y=y_2} u dy = \int_{y=y_1}^{y=y_2} \frac{\partial \psi}{\partial y} dy$$

$$= \psi_2 - \psi_1$$



$$Q_{\text{CD}} = \int_{x=x_1}^{x=x_2} v dx = \int_{x=x_1}^{x=x_2} -\frac{\partial \psi}{\partial x} dx = \psi_2 - \psi_1$$

$\Delta\psi$: Volume flow rate between two streamlines

RTT: $$\frac{dB_{sys}}{dt} = \int_{CV} \left[\frac{\partial(\rho\beta)}{\partial t} + \nabla.(\rho\beta\vec{u}) \right] d\forall$$

$$\frac{\partial(\rho\beta)}{\partial t} + \nabla.(\rho\beta\vec{u}) = \left(\beta\frac{\partial\rho}{\partial t} + \rho\frac{\partial\beta}{\partial t} \right) + \left[\beta\nabla.(\rho\vec{u}) + \rho(\vec{u}.\nabla)\beta \right]$$

$$= \beta\left[\frac{\partial\rho}{\partial t} + \nabla.(\rho\vec{u}) \right] + \rho\left[\frac{\partial\beta}{\partial t} + (\vec{u}.\nabla)\beta \right]$$

Mass conservation: $$\frac{\partial\rho}{\partial t} + \nabla.(\rho\vec{u}) = 0$$

RTT: $$\frac{dB_{sys}}{dt} = \int_{CV} \rho\left[\frac{\partial\beta}{\partial t} + (\vec{u}.\nabla)\beta \right] d\forall$$