ESO204A, Fluid Mechanics and rate Processes

Laminar, incompressible, viscous flow: Exact Solutions

Couette flow, Poiseuille flow

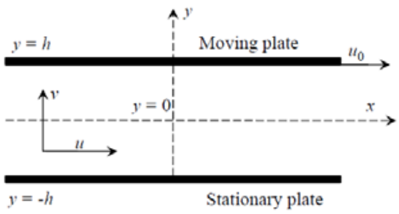
Chapter 4 of F M White
Chapter 5 of Fox McDonald

Couette Flow: summary

Laminar, **incompressible, steady** flow between two **infinitely** long parallel plates; top plate moving steadily and sustains the flow, bottom plate stationary

$$u = u(x, y), v = (x, y), w = 0$$

$$\frac{\partial \vec{u}}{\partial x} = 0 \qquad \frac{\partial p}{\partial x} = 0$$



no-slip: $u(x, y = h) = u_0; u(x, y = -h) = 0$

impermeability: $v(x, y = h) = 0$

$$u = \frac{u_0}{2}\left(1 + \frac{y}{h}\right); v = w = 0; p = \text{hydrostatic}$$

Stress components $\sigma_{yx} = \mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)$ $\sigma_{xx} = -p + 2\mu\frac{\partial u}{\partial x}$

$\sigma_{xy} = \mu\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)$ $\sigma_{yy} = -p + 2\mu\frac{\partial v}{\partial y}$

In general $\sigma_{ij} = \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$ for $i \neq j$

and $\sigma_{ij} = -p + \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$ for $i = j$

Overall: $\sigma_{ij} = -p\delta_{ij} + \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$

where $\delta_{ij} = 1$ for $i = j$; $\delta_{ij} = 0$ for $i \neq j$

Find the stress components in Couette flow

$\sigma_{ij} = -p\delta_{ij} + \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$ $u = \frac{u_0}{2}\left(1 + \frac{y}{h}\right), v = w = 0$

$\sigma_{yx} = \mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) = \frac{\mu u_0}{2h}$ $\sigma_{xy} = \mu\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right) = \frac{\mu u_0}{2h}$

$\sigma_{xx} = -p + 2\mu\frac{\partial u}{\partial x} = -p$ $\sigma_{yy} = -p + 2\mu\frac{\partial v}{\partial y} = -p$

Recall definition of normal and shear stresses

$\sigma_{xx} = \frac{F_x}{A_x} = \frac{F_{-x}}{A_{-x}}$ $\sigma_{yy} = \frac{F_y}{A_y} = \frac{F_{-y}}{A_{-y}}$ $\sigma_{yx} = \frac{F_x}{A_y} = \frac{F_{-x}}{A_{-y}}$ $\sigma_{xy} = \frac{F_y}{A_x} = \frac{F_{-y}}{A_{-x}}$

$$\sigma_{xx} = \frac{F_x}{A_x} = -p \quad \sigma_{yx} = \frac{F_x}{A_y} = \frac{\mu u_0}{2h} \quad \sigma_{yy} = \frac{F_y}{A_y} = -p \quad \sigma_{xy} = \frac{F_y}{A_x} = \frac{\mu u_0}{2h}$$

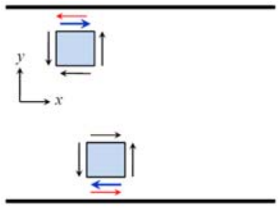
Assume a surface (unit area) normal: $\vec{n} = n_x \vec{i} + n_y \vec{j}; |\vec{n}| = 1$

$A_x = n_x, A_y = n_y$

Forces on this surface

$$F_x = A_x \sigma_{xx} + A_y \sigma_{yx} = -n_x p + n_y \frac{\mu u_0}{2h}$$

$$F_y = A_x \sigma_{xy} + A_y \sigma_{yy} = n_x \frac{\mu u_0}{2h} - n_y p$$

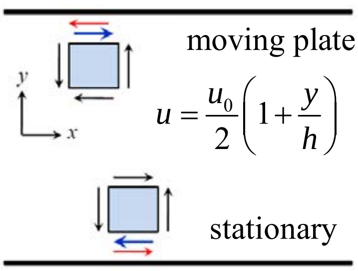


assume a particular case where:
$n_x = 0, n_y = 1$

$$F_x = \frac{\mu u_0}{2h}, F_y = -p$$

similarly for: $n_x = 0, n_y = -1$

$$F_x = -\frac{\mu u_0}{2h}, F_y = p$$



$$u = \frac{u_0}{2}\left(1 + \frac{y}{h}\right)$$

We now find the **forces** exerted on the top/bottom plates (per unit area) by the fluid

$$\vec{F}_T = -\frac{\mu u_0}{2h}\vec{i} + p\vec{j} \qquad \vec{F}_B = \frac{\mu u_0}{2h}\vec{i} - p\vec{j}$$

Note that, the wall shear stress relation follows the Newton's law

$$\tau_w = \mu \left(\frac{\partial u}{\partial y}\right)_{y=-h} = \frac{\mu u_0}{2h} \Rightarrow \frac{\tau_w}{\frac{1}{2}\rho u_0^2} = \frac{\mu}{\rho u_0 h}$$

Skin friction coefficient $C_f = \frac{1}{\mathrm{Re_h}}$

Re : Reynolds number $= \frac{\rho u_0 h}{\mu}$; Poiseuille number $\mathrm{Po} = C_f \,\mathrm{Re} = 1$

Couette Flow: Summary

$$u = \frac{u_0}{2}\left(1 + \frac{y}{h}\right)$$
$$v = w = 0$$
$$p = \text{constant}$$

Skin friction coefficient $C_f = \dfrac{1}{\text{Re}_h}$

Re : Reynolds number

Poiseuille number $\text{Po} = C_f \text{ Re} = 1$

$$\text{Wall shear force per unit area} = \frac{\mu u_0}{2h}$$

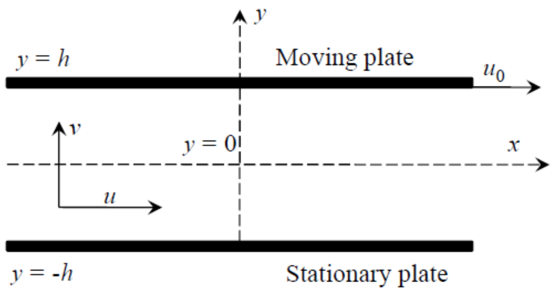
Applications:

- Lubrication
- Geological systems
- Painting, cleaning etc. (thin-film applications)

Couette-Poiseuille Flow

Laminar, incompressible, steady flow between two infinitely long parallel plates; top plate moving steadily, bottom plate stationary

We continue to assume 2-D, fully developed flow

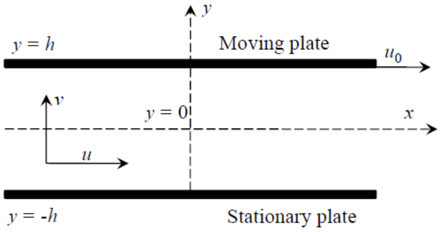


$$u = u(x, y), v = (x, y), w = 0 \qquad \frac{\partial \vec{u}}{\partial x} = 0 \qquad \frac{\partial p}{\partial x} \text{ may be non-zero}$$

How the velocity field should look like?

$$u = u(x, y), v = (x, y), w = 0, \frac{\partial \vec{u}}{\partial x} = 0$$

Now, our goal is to find three unknowns (u, v, p) from continuity and momentum Equations



Applying continuity Eq. $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \quad \Rightarrow \frac{\partial v}{\partial y} = 0$

$$v = f(x) \quad \text{BC: } v(y = h) = 0$$

$$v = 0$$

$$u = u(x, y), v = w = 0; \frac{\partial \vec{u}}{\partial x} = 0$$

$$z\text{-mom: } \frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial z} + \nu\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2}\right)$$

$$\frac{\partial p}{\partial z} = 0$$

$$y\text{-mom: } \frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial y} + \nu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right) \Rightarrow \frac{\partial p}{\partial y} = 0$$

$$\Rightarrow p = p(x)$$

$$x\text{-mom: } \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) \qquad \nu = \frac{\mu}{\rho}$$

$$\Rightarrow \frac{d^2 u}{dy^2} = \frac{1}{\mu}\frac{dp}{dx} = \text{constant}$$