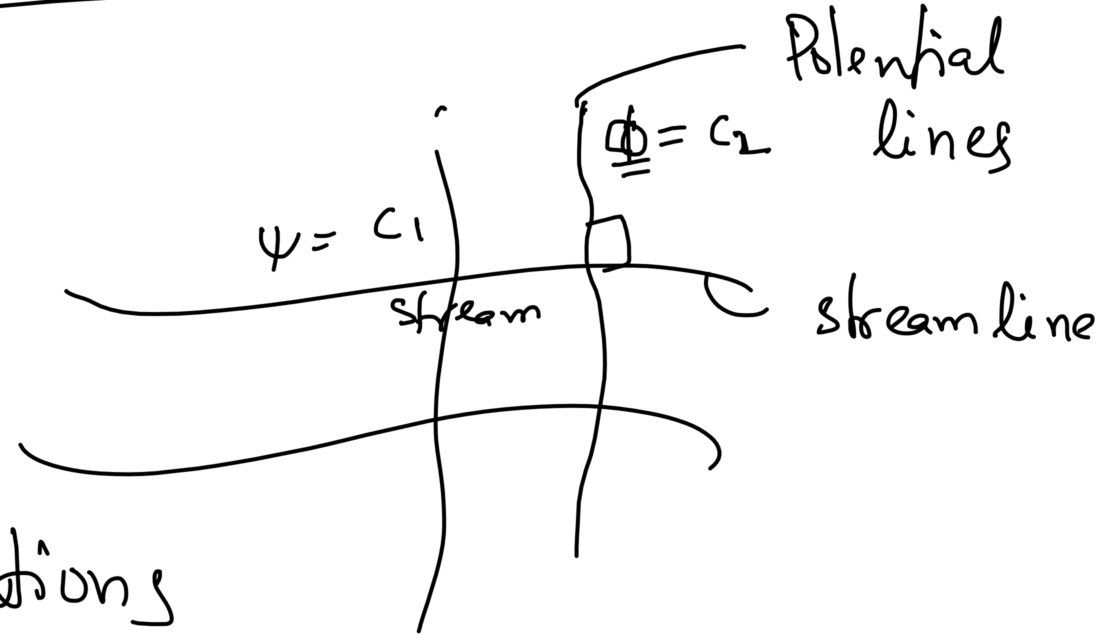
Velocity Potential (Φ)

- velocity potential concept is valid for irrotational flow

for irrotational flow $\Rightarrow \nabla \times V = 0$ $\Rightarrow$ so if we define a scalar function Φ such that $V = \nabla \Phi$

This eqn will be automatically satisfied

$\xi = \nabla \times V = 0$



$$u\hat{i} + v\hat{j} + w\hat{k} = \frac{\partial \phi}{\partial x}\hat{i} + \frac{\partial \phi}{\partial y}\hat{j} + \frac{\partial \phi}{\partial z}\hat{k}$$

$$u = \frac{\partial \phi}{\partial x} = \frac{\partial \psi}{\partial y}, \quad v = \frac{\partial \phi}{\partial y} = -\frac{\partial \psi}{\partial x}, \quad w = \frac{\partial \phi}{\partial z}$$

continuity $\nabla \cdot V = 0$, $V = \nabla \phi \Rightarrow \boxed{\nabla^2 \phi = 0}$ $\Leftarrow$ Boundary conditions

$\nabla^2 \psi = 0$ $\leftarrow$ irrotational flow

$$= d\phi = \frac{\partial \phi}{\partial x}dx + \frac{\partial \phi}{\partial y}dy = 0 \text{ for a potential line}$$

$$= u\,dx + v\,dy = 0$$

$$\left(\frac{dy}{dx}\right)_{\phi = \text{Const}} = -\frac{u}{v}$$

$$d\psi = \frac{\partial \psi}{\partial x}dx + \frac{\partial \psi}{\partial y}dy = 0$$

$$= -v\,dx + u\,dy = 0$$

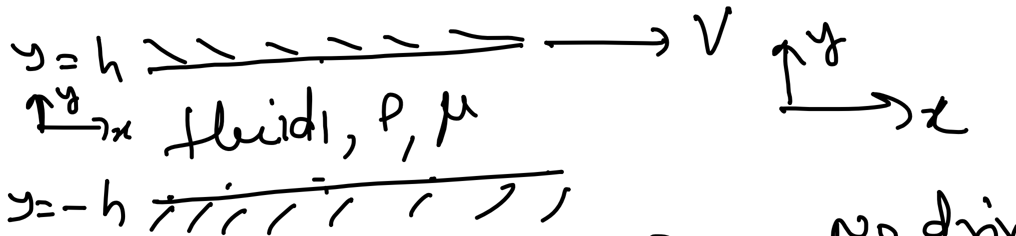
$$\left(\frac{dy}{dx}\right)_{\psi = \text{Const}} = \frac{v}{u} \Rightarrow \boxed{\left(\frac{dy}{dx}\right)_{\phi = \text{const}} = -\frac{1}{\left(\frac{dy}{dx}\right)_{\psi = \text{const}}}}$$

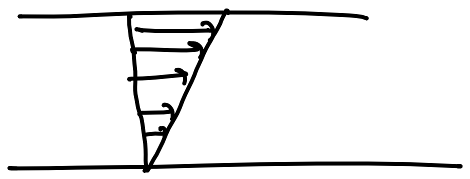
$\phi = c \perp \psi = c_1$

Examples on Solving Navier-stokes' for simplified problems

1. Couette flow between a fixed and a moving plate

$(u, v, w) = (x, y, z)$

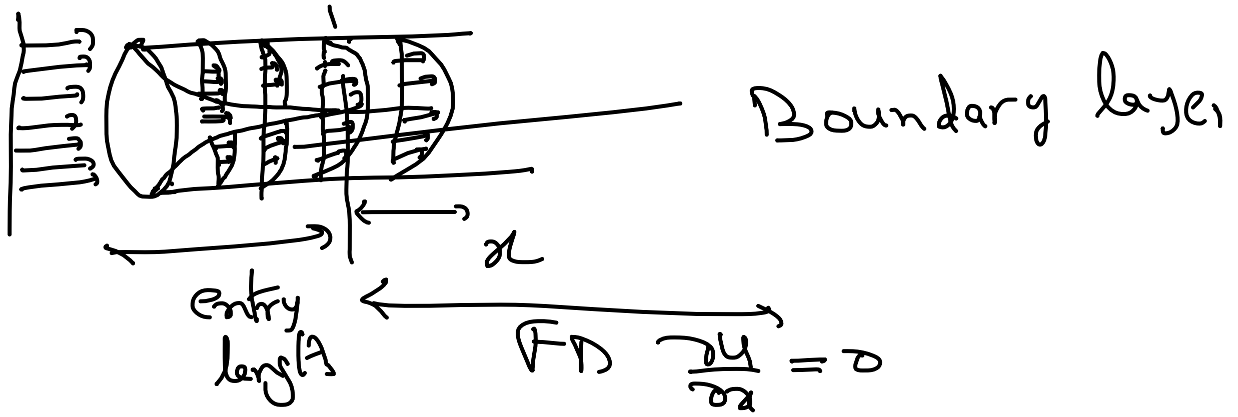




No driving force in z-dir

Continuity $\quad \cancel{\frac{\partial u}{\partial x}}^{FD} + \frac{\partial v}{\partial y} + \cancel{\frac{\partial w}{\partial z}}^{0} = 0 \qquad z \to \infty$

fully developed flow (FD) $\quad \frac{\partial u}{\partial x} = 0$



$\frac{\partial v}{\partial y} = 0$

$v = f(x, z)$

v at any (x, z) for $y = \pm h$, $v = 0$

all $x, y, z \Rightarrow v = 0$

$w = 0$ (No driving force)

$u = u(\cancel{x}, y, \cancel{z})$ No driving force

$u = u(y)$

momentum eq^n

x- momentum

S.S

$$\rho\left(\cancel{\frac{\partial u}{\partial t}}^{0} + \underbrace{u\cancel{\frac{\partial u}{\partial x}}}_{FD} + \cancel{v}^{0}\frac{\partial u}{\partial y} + \cancel{w}^{0}\frac{\partial u}{\partial z}\right) = -\frac{\partial p}{\partial x} + \mu\left(\cancel{\frac{\partial^2 u}{\partial x^2}}^{FD} + \frac{\partial^2 u}{\partial y^2} + \cancel{\frac{\partial^2 u}{\partial z^2}}^{\text{No driving force}}\right) + \cancel{\rho g_x}$$

$$0 = -\frac{\partial p}{\partial x} + \mu \frac{\partial^2 u}{\partial y^2}$$

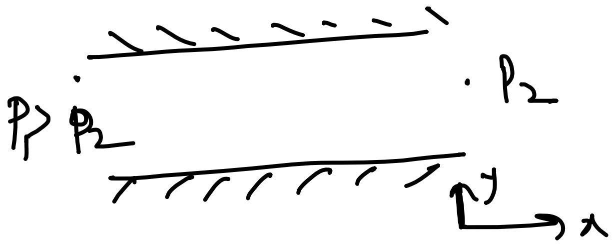
Assuming no pressure gradient $\frac{\partial p}{\partial x} = 0$

$$\boxed{\frac{d^2 u}{dy^2} = 0 \qquad \text{B.C. } 1 \quad y = -h \quad u = 0 \qquad 2 \quad y = +h \quad u = V}$$ (no slip condition

$$\boxed{u = \frac{V}{2h} y + \frac{V}{2}} \qquad -h \le y \le h$$

$$\tau_{yx} = \mu \frac{\partial u}{\partial y}$$

Example 2: Flow due to pressure gradient between two fixed plates



$$\frac{\partial p}{\partial x} < 0$$

continuity $\quad \cancel{\frac{\partial u}{\partial x}} + \frac{\partial v}{\partial y} + \cancel{\frac{\partial w}{\partial z}} = 0$

$\Rightarrow v = 0$

$w = 0$ (no driving force)

$u = u(\cancel{x}, y, \cancel{z}) \Rightarrow u = u(y)$

x- momentum

$$0 = -\frac{\partial p}{\partial x} + \mu \left(\frac{\partial^2 u}{\partial y^2} \right) \quad \text{— (i)}$$

y- momentum

$$\rho \left(\cancel{\frac{\partial v}{\partial t}} + u \cancel{\frac{\partial v}{\partial x}} + v \cancel{\frac{\partial v}{\partial y}} + w \cancel{\frac{\partial v}{\partial z}} \right) = -\frac{\partial p}{\partial y} + \mu \left(\cancel{\frac{\partial^2 v}{\partial x^2}} + \cancel{\frac{\partial^2 v}{\partial y^2}} + \cancel{\frac{\partial^2 v}{\partial z^2}} \right) + \rho g_y$$

$$\frac{\partial p}{\partial y} = \rho(-g) \qquad \downarrow g$$

$$p = -\rho g y + f(x, \cancel{z})$$

$$p = -\rho g y + f(x)$$

$$\frac{\partial p}{\partial x} = f'(x) \Leftarrow \text{constant}$$

$$\frac{d^2 u}{d y^2} = \frac{1}{\mu} \left(\frac{\partial p}{\partial x} \right)$$

B.C. $y = -h \quad u = 0$
$y = +h \quad u = 0$] No slip condition

$$\boxed{u = \left(-\frac{\partial p}{\partial x} \right) \frac{h^2}{2\mu} \left(1 - \frac{y^2}{h^2} \right)} \Leftarrow \text{Parabolic}$$