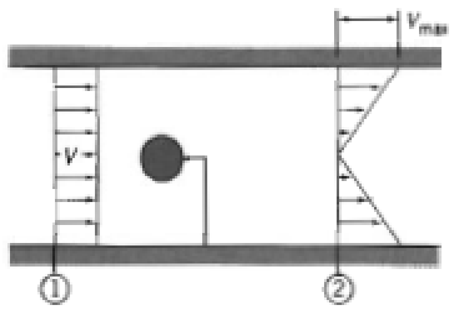
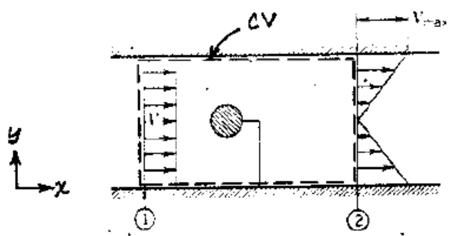
Solutions of practice problems on integral formulation (ESO-204A)

P1. A small round object is tested in a 1 m diameter wind tunnel. The pressure is uniform across section 1 and 2. The upstream pressure is 20mm H_2O (gage), the downstream pressure is 10mm H_2O (gage), and the mean air speed is 10m/s. the velocity profile at section 2 is linear, it varies from zero at the tunnel centerline to a maximum at the tunnel wall. Calculate (a) the mass flow rate in the wind tunnel, (b) the maximum velocity at section 2, and (c) the drag of the object and its supporting vane. Neglect viscous resistance at the tunnel wall.



P1



$p_1 = 20$ mm H_2O (gage) $\qquad p_2 = 10$ mm H_2O (gage)

$V_1 = 10$ m/s

Solution:

Basic equations:

$$0 = \underbrace{\frac{\partial}{\partial t}\int_{CV} \rho\, d\forall}_{=0(1)} + \int_{CS} \rho \vec{V}\cdot d\vec{A}$$

$$F_{S_x} + \underbrace{F_{B_x}}_{=0(4)} = \underbrace{\frac{\partial}{\partial t}\int_{CV} u\rho\, d\forall}_{=0(1)} + \int_{CS} u\rho \vec{V}\cdot d\vec{A}$$

$$p_1 = \rho g h_1 = 999\,\frac{kg}{m^3} \times 9.81\,\frac{m}{s^2} \times 0.02\,m = 196\ Pa\ (gage)$$

$$p_2 = 98.0\ Pa\ (gage)$$

Assumptions: (1) Steady flow
(2) Density uniform at each section
(3) Uniform flow at Section ①, so $\dot{m} = \rho V_1 A$
(4) Horizontal flow; $F_{B_x} = 0$

Then

$$\dot{m} = \rho_1 V_1 A = 1.23\,\frac{kg}{m^3} \times 10\,\frac{m}{s} \times \frac{\pi}{4}(1)^2 m^2 = 9.67\ kg/s$$ $\dot{m}$

From continuity,

$$\dot{m} = \int_{A_2} \rho_2 u_2\, dA_2 = \rho_2 \int_0^R V_{2,max}\frac{r}{R}\, 2\pi r\, dr = 2\pi \rho_2 V_{2,max} R^2 \int_0^1 \left(\frac{r}{R}\right)^2 d\left(\frac{r}{R}\right) = \frac{2\pi}{3}\rho_2 V_{2,max} R^2$$

$$V_{2,max} = \frac{3\dot{m}}{2\pi\rho_2 R^2} = \frac{3}{2\pi} \times 9.67\,\frac{kg}{s} \times \frac{m^3}{1.23\,kg} \times \frac{1}{(0.5)^2 m^2} = 15.0\ m/s$$ $V_{2,max}$

From the momentum equation,

$$R_x + p_1 A - p_2 A = u_1\{-\dot{m}\} + \int_{A_2} u \rho_2 V_2\, dA_2 = -V_1 \dot{m} + 2\pi \rho_2 V_{2,max}^2 R^2 \int_0^1 \left(\frac{r}{R}\right)^3 d\left(\frac{r}{R}\right)$$

$$u_1 = V_1 \qquad u_2 = V_{2,max}\frac{r}{R}$$

$$R_x = (p_2 - p_1)A - V_1 \dot{m} + 2\pi\rho_2 V_{2,max}^2 R^2 \left(\frac{1}{4}\right)$$

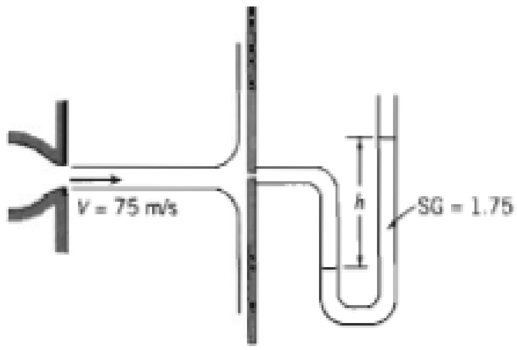
$$= (98.0 - 196)\,\frac{N}{m^2} \times \frac{\pi}{4}(1)^2 m^2 + \left[-10\,\frac{m}{s} \times 9.67\,\frac{kg}{s} + \frac{\pi}{2} \times 1.23\,\frac{kg}{m^3} \times (15)^2 \frac{m^2}{s^2} \times (0.5)^2 m^2\right] \frac{N\cdot s^2}{kg\cdot m}$$

$$R_x = -65.0\ N$$

R_x is force to hold CV in place. CV cuts strut, so R_x is force needed to hold object. Drag of object and strut is

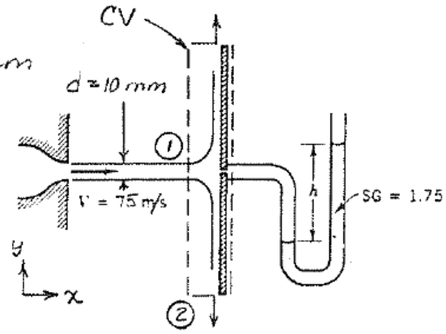
$$F_D = |R_x| = 65.0\ N$$ F_D

P2. A horizontal axisymmetric jet of air with 10mm diameter strikes a stationary vertical disk of 200mm diameter. The jet speed is 75m/s at the nozzle exit. A manometer is connected to the center of disk. Calculate (a) the deflection, h, if the manometer liquid has SG=1.75 and (b) the force exerted by the jet on the disk.



P2

Solution: Apply Bernoulli and momentum equations. Use CV shown.



Basic equations: $\frac{p}{\rho} + \frac{V^2}{2} + g\cancel{z}^{(5)} = \text{constant}$

$=0(5) \quad =0(1)$

$$F_{S_x} + \cancel{F_{B_x}} = \frac{\partial}{\partial t}\int_{CV} u\rho\, d\forall + \int_{CS} u\rho \vec{V}\cdot d\vec{A}$$

Assumptions: (1) Steady flow
(2) Incompressible flow
(3) Flow along a streamline
(4) No friction
(5) $F_{Bx} = 0$; horizontal flow
(6) Uniform flow in jet

Apply Bernoulli between jet exit and stagnation point

$$\frac{p}{\rho} + \frac{V^2}{2} = \frac{p_0}{\rho} + 0 \;;\; p_0 - p = \tfrac{1}{2}\rho V^2$$

From hydrostatics, $p_0 - p = SG\,\rho_{H_2O}\, g\,\Delta h$

Thus $\Delta h = \dfrac{\frac{1}{2}\rho V^2}{SG\,\rho_{H_2O}\, g} = \dfrac{\rho V^2}{2SG\,\rho_{H_2O}\, g}$

$$\Delta h = 1.23\,\frac{kg}{m^3} \times (75)^2\frac{m^2}{s^2} \times \frac{1}{2(1.75)} \times \frac{m^3}{999\,kg} \times \frac{s^2}{9.81\,m} = 0.202\ m \text{ or } 202\ mm$$ Δh

From momentum,

$$R_x = u_1\{-\rho VA\} + u_2\{\rho VA\} = -\rho V^2 A$$

$$u_1 = V \qquad u_2 = 0$$

$$R_x = -1.23\,\frac{kg}{m^3} \times (75)^2\frac{m^2}{s^2} \times \frac{\pi}{4}(0.01)^2 m^2 \times \frac{N\cdot s^2}{kg\cdot m} = -0.543\ N \text{ (to left)}$$ R_x

This is the force needed to hold the plate.

The "force" of the jet on the plate is

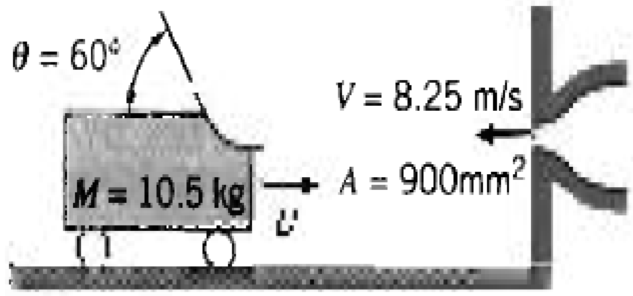
$$K_x = -R_x = 0.543\ N \text{ (to right)}$$

t

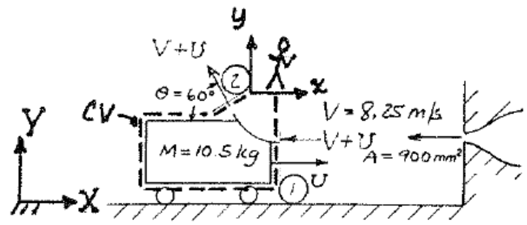
$$-U\frac{dU}{d\theta}M = \rho(V+U)^2 A(1-\cos\theta)$$

Separating variables $\quad \dfrac{U\,dU}{(V+U)^2} = -\dfrac{\rho A(1-\cos\theta)}{M}d\theta \qquad (3)$

P3. A small cart that carries a single turning vane rolls on a level track. The cart mass is M=10.5kg and its initial speed is U_o=12.5m/s. At t=0, the vane is struck by an opposing jet of water, as shown. Neglect any external forces due to air or rolling resistance. Determine the time and distance needed for the liquid jet to bring the cart to rest. Plot the cart speed (nondimensionalized on U_o) and the distance traveled as function of time.



P3



Solution: Apply x component of momentum using CS and CV shown.

Basic equation: $\overset{=0(1)}{F_{S_x}} + \overset{=0(2)}{F_{B_x}} - \int_{CV} a_{rf_x} \rho \, dV\!\!\!\!- = \overset{\approx 0(3)}{\frac{\partial}{\partial t}} \int_{CV} u_{xyz} \rho \, dV\!\!\!\!- + \int_{CS} u_{xyz} \rho \vec{V} \cdot d\vec{A}$

Assumptions: (1) No resistance; $F_{S_x} = 0$
(2) Horizontal; $F_{B_x} = 0$
(3) Neglect mass of water on vane; $\partial/\partial t \approx 0$
(4) No change in speed w.r.to vane
(5) Uniform flow at each cross-section

Then

$$-a_{rf_x} M_{CV} = u_1 \{ -|\rho (V+U) A| \} + u_2 \{ +|\rho (V+U) A| \}$$

$$a_{rf_x} = \frac{dU}{dt} \qquad u_1 = -(V+U) \qquad u_2 = -(V+U)\cos\theta \quad (\text{w.r.to CV})$$

So $$-\frac{dU}{dt} M = \rho (V+U)^2 A - \rho (V+U)^2 A \cos\theta = \rho (V+U)^2 A (1-\cos\theta) \qquad (1)$$

Note V = constant, so $dU = d(V+U)$. Substituting

$$-\frac{d(V+U)}{(V+U)^2} = \frac{\rho A (1-\cos\theta)}{M} dt \qquad (2)$$

Integrate from U_0 at $t=0$ to stop, when $U=0$

$$\left. \frac{1}{V+U} \right]_{U=U_0}^{U=0} = \frac{1}{V} - \frac{1}{V+U_0} = \frac{V+U_0-V}{V(V+U_0)} = \frac{U_0}{V(V+U_0)} = \frac{\rho A (1-\cos\theta) t}{M}$$

Thus $$t = \frac{U_0 M}{\rho (V+U_0) V A (1-\cos\theta)}$$

$$= 12.5 \frac{m}{sec} \times 10.5\,kg \times \frac{m^3}{999\,kg} \times \frac{sec}{(12.5+8.25)\,m} \times \frac{sec}{8.25\,m} \times \frac{1}{900 \times 10^{-6}\,m^2} \times \frac{1}{(1-\cos 60°)}$$

t = 1.71 sec (to stop) $\qquad t$

To find distance note $\frac{dU}{dt} = \frac{dU}{ds}\frac{ds}{dt} = \frac{dU}{ds} U = U\frac{dU}{ds}$; so from Eq. 1

$$-U \frac{dU}{ds} M = \rho (V+U)^2 A (1-\cos\theta)$$

Separating variables $$\frac{U\,dU}{(V+U)^2} = -\frac{\rho A (1-\cos\theta)}{M} ds \qquad (3)$$

Equation 3 may be integrated. Using tables, and integrating from U_0 at $t = 0$ to stop (when $U = 0$),

$$\int_{U_0}^{0} \frac{U\,dU}{(V+U)^2} = \left[\ln(V+U) + \frac{V}{V+U}\right]_{U_0}^{0} = \ln\left(\frac{V}{V+U_0}\right) + \frac{V}{V} - \frac{V}{V+U_0} = -\frac{\rho A(1-\cos\theta)}{M}s$$

Simplifying and solving for s,

$$s = -\frac{M}{\rho A(1-\cos\theta)}\left[\ln\left(\frac{V}{V+U_0}\right) + 1 - \frac{V}{V+U_0}\right]$$

$$= -10.5\,kg \times \frac{m^3}{999\,kg} \times \frac{1}{900 \times 10^{-6}\,m^2} \times \frac{1}{(1-\cos 60°)}\left[\ln\left(\frac{8.25}{8.25+12.5}\right) + 1 - \frac{8.25}{8.25+12.5}\right]$$

$s = 7.47$ m (to stop) $\qquad s$

From Eq. 2 the general solution is

$$\int_{U_0}^{U} -\frac{d(V+U)}{(V+U)^2} = \frac{1}{V+U}\bigg]_{U_0}^{U} = \frac{1}{V+U} - \frac{1}{V+U_0} = \frac{(V+U_0)-(V+U)}{(V+U)(V+U_0)} = \frac{\rho A(1-\cos\theta)t}{M} = at$$

Thus $U_0 - U = a(V+U)(V+U_0)t = aV(V+U_0)t + aU(V+U_0)t \qquad \{\text{Let } b = V+U_0\}$

Simplifying, $U = \dfrac{U_0 - aVbt}{1+abt} \qquad (4) \qquad U(t)$

Acceleration is found from Eq. 1

$$a_x = \frac{dU}{dt} = \frac{\rho A(1-\cos\theta)(V+U)^2}{M} = a(V+U)^2 \qquad a_x(U)$$

Integrate Eq. 4 to get $X(t)$:

$$U = \frac{dX}{dt} = \frac{U_0 - abVt}{1+abt}$$

$$dX = \frac{U_0}{1+abt}dt - \frac{abVt}{1+abt}dt$$

Integrating

$$X = \frac{U_0}{ab}\ln(1+abt)\Big]_0^t - \frac{V}{ab}\int_0^t \frac{x}{1+x}dx = \left[\frac{U_0}{ab}\ln(1+abt) - \frac{V}{ab}\left(1+abt - \ln(1+abt)\right)\right]_0^t$$

$$X = \frac{U_0}{ab}\ln(1+abt) - \frac{V}{ab}\left[abt - \ln(1+abt)\right] \qquad X(t)$$

Numerical values and plots are on the next page.

Acceleration, Velocity, and Position of Cart vs. Time:

Input Parameters:

A =	900	mm^2	9.00E-04	m^2
M =	10.5	kg		
U_0 =	12.5	m/s		
V =	8.25	m/s		
θ =	60	degrees	1.047	rad
ρ =	999	kg/m^3		

Calculated Parameters:

a =	0.0428	m^{-1}
b =	20.75	m/s

Calculated Results:

Time, t (s)	Velocity, U (m/s)	Accel., a_x (m/s)	Accel., a_x (gs)	Position, X (m)
0	12.5	-18.4	-1.88	0.00
0.1	10.8	-15.5	-1.58	1.16
0.2	9.37	-13.3	-1.35	2.17
0.3	8.13	-11.5	-1.17	3.04
0.4	7.06	-10.0	-1.02	3.80
0.5	6.12	-8.84	-0.901	4.46
0.6	5.29	-7.84	-0.800	5.03
0.7	4.54	-7.01	-0.714	5.52
0.8	3.88	-6.30	-0.642	5.94
0.9	3.28	-5.69	-0.580	6.30
1.0	2.74	-5.17	-0.527	6.60
1.1	2.24	-4.72	-0.481	6.85
1.2	1.79	-4.32	-0.440	7.05
1.3	1.38	-3.97	-0.405	7.21
1.4	0.998	-3.66	-0.373	7.33
1.5	0.646	-3.39	-0.345	7.41
1.6	0.319	-3.14	-0.320	7.46
1.7	0.0160	-2.93	-0.298	7.47
1.705	0.00000	-2.91	-0.297	7.47
1.8	-0.267	-2.73	-0.278	7.46
1.9	-0.530	-2.55	-0.260	7.42
2.0	-0.777	-2.39	-0.244	7.35