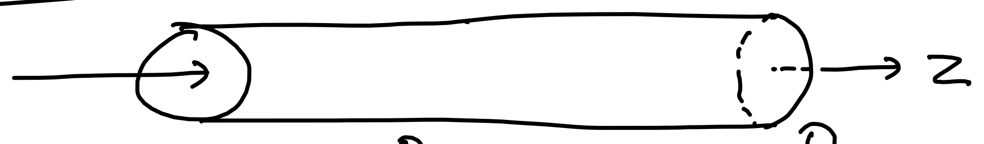
Pipe flow: Find the Velocity Profile and derive Hagen-Poiseuille eq^n



incompressible $\frac{\partial \rho}{\partial t}$

$V = \underline{\underline{v_r}}\,\hat{r} + v_\theta\,\hat{\theta} + \underline{\underline{v_z}}\,\hat{z}$

r, θ, z

Appendix D

continuity $\quad \frac{1}{r}\frac{\partial}{\partial r}(r\rho v_r) + \frac{1}{r}\cancel{\frac{\partial}{\partial \theta}}(\rho v_\theta) + \cancel{\frac{\partial}{\partial z}}(\rho v_z) = 0$ FD

$V \rightarrow$ stresses $\rightarrow$ Force $\quad$ symmetric $\rightarrow$

$\frac{\partial}{\partial \theta} = 0$

$\frac{\partial}{\partial r}(r v_r) = 0$

$v_r = \frac{f(z,\cancel{\theta})}{r} = \frac{f(z)}{r}$

$v_r = 0$ at $r = R$ for all value z

$\rightarrow f(z) = 0 \Rightarrow v_r = 0$

$\cancel{v_\theta}$, $\cancel{v_r}$, $v_z = v_z(r, \cancel{\theta}, \cancel{z})$ (sym, FD)

No circulation

$\underline{v_z = v_z(r)}$

$z = 0$

$r = 0 \quad \boxed{\frac{\partial v_z}{\partial r} = 0}$

$\tau_{rz} = 0$

$\frac{\partial v_z}{\partial r} = 0$

z-momentum

$\cancel{\frac{\partial v_z}{\partial t}}^{SS} + (V\cdot\nabla)v_z = -\frac{1}{\rho}\frac{\partial p}{\partial z} + g_z + \nu\nabla^2 v_z \qquad \nu = \frac{\mu}{\rho}$

$(V\cdot\nabla)v_z = \cancel{v_r}\frac{\partial v_z}{\partial r} + \frac{1}{r}\cancel{v_\theta}\cancel{\frac{\partial v_z}{\partial \theta}} + v_z\cancel{\frac{\partial v_z}{\partial z}}^{FD} = 0$

$g_z = 0$

$\nabla^2 v_z = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right) + \frac{1}{r^2}\cancel{\frac{\partial^2 v_z}{\partial \theta^2}} + \cancel{\frac{\partial^2 v_z}{\partial z^2}}$ FD

Symmetry

$0 = -\frac{1}{\rho}\frac{\partial p}{\partial z} + \nu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right)\right] \Rightarrow$

B.C.1 $\quad r = 0, \quad \frac{\partial v_z}{\partial r} = 0 \quad \Big| \quad r = R, \quad v_z = 0$

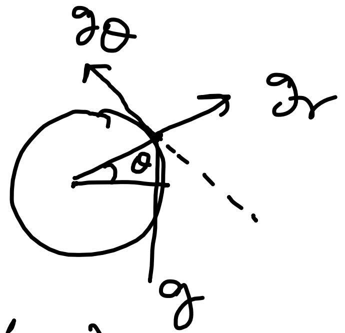
r-momentum

$\cancel{\frac{\partial v_r}{\partial t}} + \cancel{(V\cdot\nabla)v_r} - \frac{1}{r}\cancel{v_\theta^2} = -\frac{1}{\rho}\frac{\partial p}{\partial r} + g_r + \nu\left(\cancel{\nabla^2 v_r} - \cancel{\frac{v_r}{r^2}} - \frac{2}{r^2}\cancel{\frac{\partial v_\theta}{\partial \theta}}\right)$

$0 = -\frac{1}{\rho}\frac{\partial p}{\partial r} + g_r$

$g_\theta = -g\cos\theta$

$g_r = -g\sin\theta$



$\underline{p = -\rho g r\sin\theta + f_1(z)}$

$\frac{\partial p}{\partial z} = f_1'(z)$

$\boxed{v_z = \left(-\frac{\partial p}{\partial z}\right)\frac{1}{4\mu}\left(R^2 - r^2\right)}$

$\tau \rightarrow F$

Hagen-Poiseuille eqn $\rightarrow$ $\underline{\Delta p = f(Q)}$ ✓

p_1 ———— p_2 $\qquad p_1 > p_2$

$\Delta p = \underline{(p_1 - p_2)}$

$v_z(max) \Rightarrow r = 0 \qquad v_z(max) = \left(-\frac{\partial p}{\partial z}\right)\frac{R^2}{4\mu}$

$$v_{z\,avg} = \frac{\int_0^R \underline{v_z}\, 2\pi r\, dr}{\pi R^2} = \left(-\frac{\partial p}{\partial z}\right)\frac{R^2}{8\mu}$$

$\Rightarrow v_{z,avg} = \frac{1}{2} v_{z,max}$

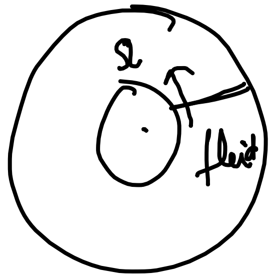
$$-\frac{\partial p}{\partial z} = \frac{p_1 - p_2}{L} = \frac{\Delta p}{L}$$

$$\frac{Q}{\pi R^2} = \left(\frac{\Delta p}{L}\right)\frac{R^2}{8\mu}$$

$$\Rightarrow \boxed{\frac{\Delta p}{L} = \frac{128\, Q\, \mu}{\pi D^4}}$$

Hagen-Poiseuille eqn

Example



Ω

Symmetry

continuity $\quad \frac{1}{r}\frac{\partial}{\partial r}(r v_r) + \cancel{\frac{1}{r}\frac{\partial}{\partial \theta}(v_\theta)} + \cancel{\frac{\partial}{\partial z} v_z} = 0$

$v_\theta,\ v_r,\ v_z \qquad \Rightarrow r v_r = f(\cancel{\theta}, \cancel{z}) \qquad v_r = 0$

$\underline{v_\theta = v_\theta(r, \cancel{\theta}, \cancel{z})}$ driving force $\qquad g_\theta = -g\cos\theta$

No gravity

θ-momentum

$$\cancel{\rho\frac{\partial v_\theta}{\partial t}} + \rho\cancel{(v\cdot\nabla)} v_\theta + \cancel{\frac{\rho v_r v_\theta}{r}} = -\cancel{\frac{1}{r}\frac{\partial p}{\partial \theta}} + \cancel{\rho g_\theta} + \mu\left(\nabla^2 v_\theta - \frac{v_\theta}{r^2}\right) + \frac{2}{r^2}\frac{\partial v_r}{\partial \theta}$$

$$(v\cdot\nabla) v_\theta = \cancel{v_r}\frac{\partial v_\theta}{\partial r} + \frac{1}{r} v_\theta \cancel{\frac{\partial v_\theta}{\partial \theta}} + \cancel{v_z}\frac{\partial v_\theta}{\partial z} = 0$$

$$\nabla^2 v_\theta = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_\theta}{\partial r}\right) + \frac{1}{r^2}\cancel{\frac{\partial^2 v_\theta}{\partial \theta^2}} + \cancel{\frac{\partial^2 v_\theta}{\partial z^2}}$$

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r \frac{\partial v_\theta}{\partial r}\right) - \frac{v_\theta}{r^2} = 0$$

No gravity

$$\frac{d^2 v_\theta}{dr^2} + \frac{1}{r}\frac{d v_\theta}{dr} - \frac{v_\theta}{r^2} = 0$$

$= \rho g \cos\theta$

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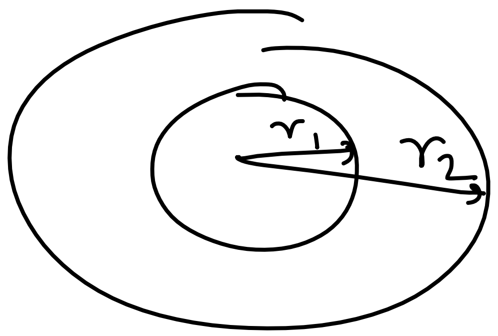
gravity

$$v_\theta = c\, r^n$$

$n = \pm 1$

$$\boxed{v_\theta = c_1 r + \frac{c_2}{r}}$$



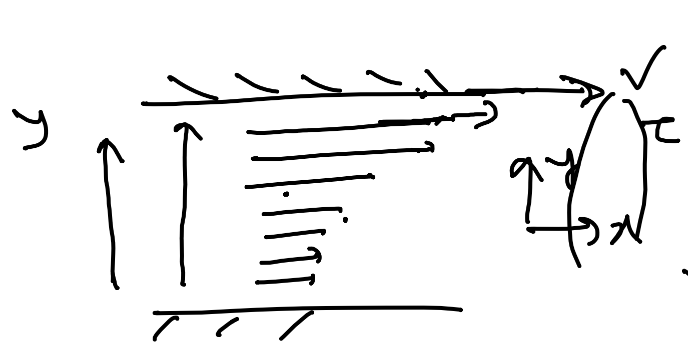
B.C. 1 $\quad v_\theta = \Omega r_1 \quad r = r_1$

$v_\theta = 0 \quad r = R_2$

$$\boxed{v_\theta = \Omega_i r_1 \frac{\left(\frac{r_2}{r} - \frac{r}{r_2}\right)}{\frac{r_2}{r_1} - \frac{r_1}{r_2}}}$$

$\Rightarrow \tau$

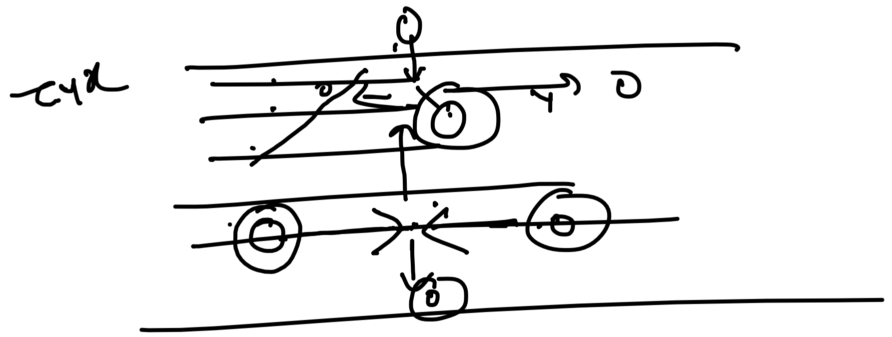
Laminar & Turbulent flow



$\tau_{yx} = \frac{1}{A}\frac{dP_x}{dt}$

y- momentum

Laminar →



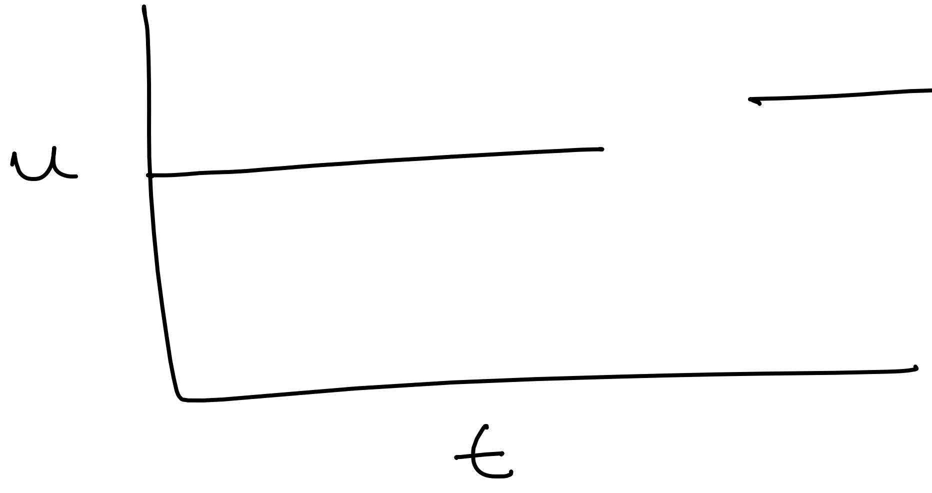
eddies

gaseous molecule ≡ eddies

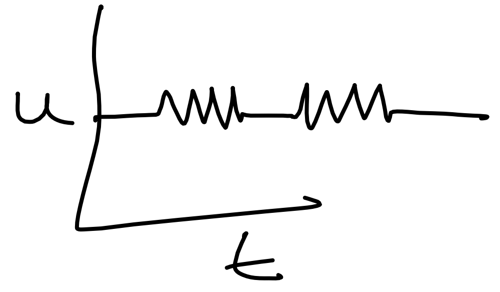
Turbulent = Laminar + Superimposed with eddies

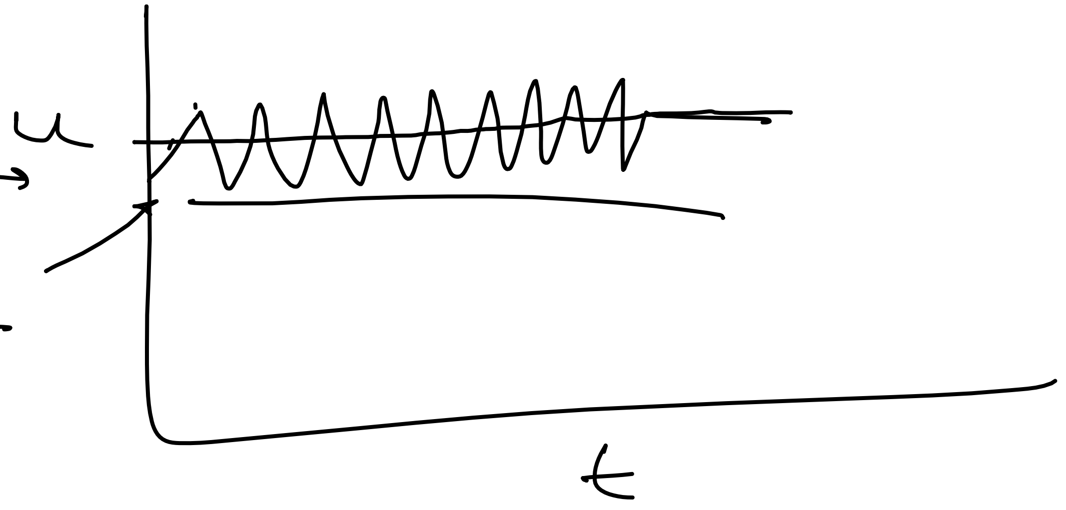
Steady

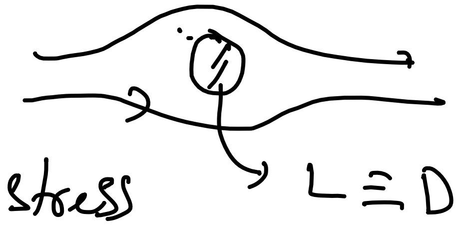
Steady flow = $\frac{V}{t}$ = constant



→ Transition →







Laminar

↑

Re = low

Reynolds number (Re) = $\frac{\rho V L}{\mu}$ = $\frac{\rho V^2}{\left(\frac{\mu V}{L}\right)}$ ← stress ← viscous stress

= $\frac{\text{inertial stress}}{\text{Viscous}}$ →

$\cancel{A}$ ρV = mass flux

($\rho V.V$) = momentum flux

inertial stress >> viscous stress

$Re_{cr} = 2300$ (Pipe

$Re < 2300$ ⇒ laminar

$Re > 4000$ turbulent

$Re >> 1$ ⇒ Turbulent