ESO204A, Fluid Mechanics and rate Processes

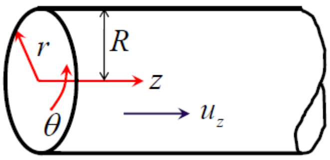
Laminar, incompressible, viscous flow: Exact Solutions

Hagen-Poiseuille flow

Chapter 4 of F M White
Chapter 5 of Fox McDonald

Hagen-Poiseuille Flow

Steady, fully-developed, axisymmetric flow, with no-swirl, in a circular pipe
Starting from conservation Equations in cylindrical coordinate



$$\text{fully-dev: } \frac{\partial \vec{u}}{\partial z} = 0; \text{ axisymmetric } \frac{\partial (\;)}{\partial \theta} = 0; \text{no swirl}: \; u_\theta = 0$$

$$\text{Continuity: } \frac{1}{r}\frac{\partial (r u_r)}{\partial r} + \frac{1}{r}\frac{\partial (u_\theta)}{\partial \theta} + \frac{\partial (u_z)}{\partial z} = 0 \quad \Rightarrow \frac{\partial (r u_r)}{\partial r} = 0$$

$$\Rightarrow r u_r = f(\theta, z) \quad \text{BC: } u_r (r = R) = 0 \quad \Rightarrow u_r = 0 \text{ everywhere}$$

$$\vec{u}.\nabla \equiv u_r \frac{\partial}{\partial r} + \frac{u_\theta}{r}\frac{\partial}{\partial \theta} + u_z \frac{\partial}{\partial z} = 0$$

$$\nabla^2 \equiv \frac{1}{r}\frac{\partial}{\partial r}\left(r \frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2} = \frac{1}{r}\frac{\partial}{\partial r}\left(r \frac{\partial}{\partial r}\right)$$

$$\frac{\partial \vec{u}}{\partial z}=0, \frac{\partial(\ )}{\partial \theta}=0, u_\theta=u_r=0 \qquad \vec{u}.\nabla(\ )=0 \quad \nabla^2=\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right)$$

$$r\text{-mom: } \frac{\partial u_r}{\partial t}+(\vec{u}.\nabla)u_r-\frac{u_\theta{}^2}{r}=-\frac{1}{\rho}\frac{\partial p}{\partial r}+\nu\left(\nabla^2 u_r-\frac{u_r}{r^2}-\frac{2}{r^2}\frac{\partial u_\theta}{\partial \theta}\right) \Rightarrow \frac{\partial p}{\partial r}=0$$

$$\theta\text{-mom: } \frac{\partial u_\theta}{\partial t}+(\vec{u}.\nabla)u_\theta-\frac{u_r u_\theta}{r}=-\frac{1}{\rho r}\frac{\partial p}{\partial \theta}+\nu\left(\nabla^2 u_\theta-\frac{u_\theta}{r^2}+\frac{2}{r^2}\frac{\partial u_r}{\partial \theta}\right) \Rightarrow \frac{\partial p}{\partial \theta}=0$$

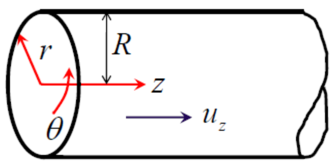
$$z\text{-mom: } \frac{\partial u_z}{\partial t}+(\vec{u}.\nabla)u_z=-\frac{1}{\rho}\frac{\partial p}{\partial z}+\nu\nabla^2 u_z \Rightarrow \frac{1}{r}\frac{d}{dr}\left(r\frac{du_z}{dr}\right)=\frac{1}{\mu}\frac{dp}{dz}=\text{constant}$$

$$\Rightarrow r\frac{du_z}{dr}=\frac{1}{\mu}\frac{dp}{dz}\frac{r^2}{2}+c_1 \qquad \Rightarrow \frac{du_z}{dr}=\frac{1}{2\mu}\frac{dp}{dz}r+\frac{c_1}{r}$$

$$u_z=\frac{r^2}{4\mu}\frac{dp}{dz}+c_1\ln r+c_2$$

$$\frac{\partial \vec{u}}{\partial z}=0, \frac{\partial(\ )}{\partial \theta}=0, u_\theta=u_r=0$$

$$u_z=\frac{r^2}{4\mu}\frac{dp}{dz}+c_1\ln r+c_2$$



$$r=0:\ u_z=\text{finite} \Rightarrow c_1=0$$

$$r=R:\ u_z=0\ \text{(no-slip)} \Rightarrow c_2=-\frac{R^2}{4\mu}\frac{dp}{dx}$$

$$u_z=-\frac{R^2}{4\mu}\frac{dp}{dz}\left(1-\frac{r^2}{R^2}\right) \qquad u_{\text{av}}=\frac{1}{\pi R^2}\int_0^R u.2\pi r dr=-\frac{R^2}{8\mu}\frac{dp}{dz}$$

$$u_{\max}=-\frac{R^2}{4\mu}\frac{dp}{dz}=2u_{\text{av}} \qquad u_z=2u_{\text{av}}\left(1-\frac{r^2}{R^2}\right)=u_{\max}\left(1-\frac{r^2}{R^2}\right)$$

$$u_z = -\frac{R^2}{4\mu}\frac{dp}{dz}\left(1-\frac{r^2}{R^2}\right) = 2u_{\text{av}}\left(1-\frac{r^2}{R^2}\right)$$

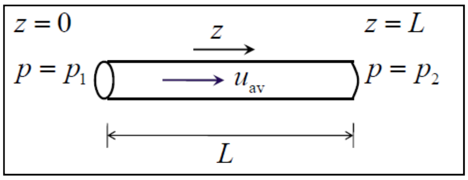
Shear stress on the pipe wall:

$$\tau_w = -\mu\left[\frac{du}{dr}\right]_{r=R} = \frac{4\mu u_{\text{av}}}{R} = \frac{8\mu u_{\text{av}}}{d} \qquad \frac{\tau_w}{\frac{1}{2}\rho u_{\text{av}}{}^2} = \frac{8\mu u_{\text{av}}}{\frac{1}{2}\rho u_{\text{av}}{}^2 d}$$

$$C_f = \frac{16}{\mathrm{Re}_{\mathrm{d}}} \qquad \mathrm{Po} = C_f\,\mathrm{Re} = 16$$

Pressure field in H-P flow

$$\frac{\partial p}{\partial r} = 0;\ \frac{\partial p}{\partial \theta} = 0;\ \frac{\partial p}{\partial z} = \text{constant}$$

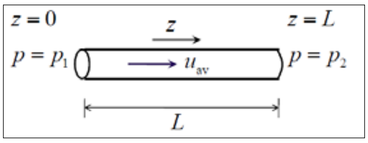


$$\frac{dp}{dz} = c_1,\ \text{say} \Rightarrow p = c_1 z + c_2$$

$$z=0:\ p=p_1;\ z=L:\ p=p_2 \qquad \Rightarrow c_2 = p_1;\ c_1 = \frac{p_2-p_1}{L}$$

$$p = p_1 - (p_1 - p_2)\frac{z}{L} \qquad \frac{dp}{dz} = \frac{p_2-p_1}{L}$$

$$\text{we know } u_{\text{av}} = -\frac{R^2}{8\mu}\frac{dp}{dz} \qquad \Rightarrow \frac{dp}{dz} = -\frac{8\mu u_{\text{av}}}{R^2} = -\frac{32\mu u_{\text{av}}}{d^2}$$



$$\frac{dp}{dz}=\frac{p_2-p_1}{L} \qquad \frac{dp}{dz}=-\frac{32\mu u_{av}}{d^2}$$

$$p_1-p_2=-L\frac{dp}{dz}=\frac{32\mu L u_{av}}{d^2} \quad \Rightarrow \frac{p_1-p_2}{\rho g}=\frac{32\mu L u_{av}}{\rho g d^2}$$

$$\frac{p_1-p_2}{\rho g}=h_f,\ \text{head loss}$$

Indicates energy spent to overcome friction

$$\Rightarrow h_f=\frac{32\mu L u_{av}}{\rho g d^2}=64\frac{\mu}{\rho u_{av} d}\frac{L}{d}\frac{u_{av}^2}{2g}=\frac{64}{\mathrm{Re}}\frac{L}{d}\frac{u_{av}^2}{2g} \Rightarrow h_f=f\frac{L}{d}\frac{u_{av}^2}{2g}$$

The above Equation is known as Darcy-Weisbach Equation

$$f=\frac{64}{\mathrm{Re}}$$

f : Darcy friction factor, $f=4C_f$

C_f : Fanning friction factor (skin friction coefficient)

$$A=\frac{\pi}{4}d^2;P=\pi d \ \text{for circular cross section}$$

$$d=\frac{4A}{P}$$

For non-circular cross-section, **hydraulic diameter** is considered as an appropriate length-scale, defined as:

$$d_h=\frac{4A}{P}$$

Both velocity and pressure drop depends on average velocity, which, in real applications, may be calculated from mass flow rate

Nondimensionalization of Governing Equation

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=-\frac{1}{\rho}\frac{\partial p}{\partial x}+\nu\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)$$

Reference velocity and length (scales): u_0, L

Define dimensionless quantities:

$$u^*=u/u_0, v^*=v/u_0, x^*=x/L, y^*=y/L$$

Now use: $u=u_0u^*, v=u_0v^*, x=Lx^*, =Ly^*$

$$\partial u=u_0\partial u^*, \partial v=u_0\partial v^*, \partial x=L\partial x^*, \partial y=L\partial y^*$$

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=-\frac{1}{\rho}\frac{\partial p}{\partial x}+\nu\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)$$

$u=u_0u^*, v=u_0v^*$ $\quad \partial u=u_0\partial u^*, \partial v=u_0\partial v^*, \partial x=L\partial x^*, \partial y=L\partial y^*$

$$\frac{u_0\partial u^*}{\partial t}+u_0u^*\frac{u_0\partial u^*}{L\partial x^*}+u_0v^*\frac{u_0\partial u^*}{L\partial y^*}=-\frac{1}{\rho}\frac{\partial p}{L\partial x^*}+\nu\left(\frac{u_0\partial^2 u^*}{L^2\partial x^{*2}}+\frac{u_0\partial^2 u^*}{L^2\partial y^{*2}}\right)$$

Multiply both sides by: L/u_0^2

$$\frac{L\partial u^*}{u_0\partial t}+u^*\frac{\partial u^*}{\partial x^*}+v^*\frac{\partial u^*}{\partial y^*}=-\frac{1}{\rho u_0^2}\frac{\partial p}{\partial x^*}+\frac{\nu}{u_0L}\left(\frac{\partial^2 u^*}{\partial x^{*2}}+\frac{\partial^2 u^*}{\partial y^{*2}}\right)$$

Further define dimensionless time, pressure: $\quad t^*=\dfrac{t}{L/u_0}, p^*=\dfrac{p}{\rho u_0^2}$

$$\frac{\partial u^*}{\partial t^*}+u^*\frac{\partial u^*}{\partial x^*}+v^*\frac{\partial u^*}{\partial y^*}=-\frac{\partial p^*}{\partial x^*}+\frac{1}{\mathrm{Re}}\left(\frac{\partial^2 u^*}{\partial x^{*2}}+\frac{\partial^2 u^*}{\partial y^{*2}}\right)$$

Where Reynolds number $\mathrm{Re}=\frac{u_0 L}{\nu}=\frac{\rho u_0 L}{\mu}$

$$\mathrm{Re}=\frac{\rho u_0 L}{\mu}=\frac{\rho u_0^2 L^2}{\mu\frac{u_0}{L}L^2}\sim\frac{\text{inertia force}}{\text{viscous force}}$$

$$p^*=\frac{p}{\rho u_0^2}=\frac{pL^2}{\rho u_0^2 L^2}\sim\frac{\text{pressure force}}{\text{inertia force}}$$

Nondimensional numbers indicate relative dominance of different forces predicting a variety of flow regimes

N-S Equation as force balance

$$\text{N-S:}\ \frac{\partial \mathbf{u}}{\partial t}+(\mathbf{u}.\nabla)\mathbf{u}=-\frac{1}{\rho}\nabla p+\nu\nabla^2\mathbf{u}$$

inertia pressure viscous

- When one force is small enough, balance of two others are sufficient
- Dimensionless numbers are useful in comparing the forces

creeping flow: small inertia, pressure ~ viscous

inviscid flow: small viscous, pressure ~ inertia

some cases of boundary layer flow: small pressure, viscous ~ inertia