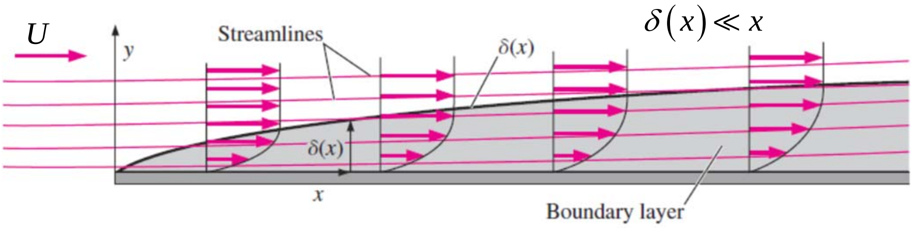
ESO204A, Fluid Mechanics and Rate Processes

Boundary layer and related topics

Chapter 7 of F M White
Chapter 9 of Fox McDonald

Boundary Layer over a flat plate

$\delta(x) \ll x$

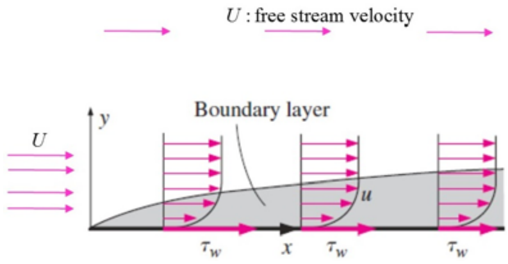


Boundary layer thickness $u_{y=\delta} = .99u_{\text{free stream}}$

Displacement thickness $\delta^* = \int_0^\infty \left(1 - \frac{u}{U}\right) dy$

Momentum thickness $\delta^{**} = \int_0^\infty \frac{u}{U}\left(1 - \frac{u}{U}\right) dy$

Important results for laminar boundary layer flow over a flat plate



$$\frac{\partial p}{\partial x}=0;\frac{\partial p}{\partial y}=0$$

$$\frac{u}{U}=f\left(\frac{y}{\delta}\right)$$

Self-similar profile

$$F_D=\rho bU^2\delta^{**}$$

Momentum Integral Eq.

$$\tau_w=\rho U^2\frac{d\delta^{**}}{dx}$$

$$C_{f,x}=\frac{2d\delta^{**}}{dx} \qquad C_D=\frac{2\delta^{**}}{L} \qquad C_D=\frac{1}{L}\int_0^L C_{f,x}dx$$

Approximate solution

$$\tau_w=\rho U^2\frac{d\delta^{**}}{dx} \qquad \frac{u}{U}=f\left(\frac{y}{\delta}\right) \Rightarrow u^*=f(\eta)$$

$$\delta^{**}=\int_0^\delta u^*\left(1-u^*\right)dy=\int_0^\delta f(1-f)dy=\delta\int_0^1 f(1-f)d\eta=c_1\delta$$

$$\tau_w=\mu\left(\frac{\partial u}{\partial y}\right)_{y=0}=\mu\left[\frac{\partial\left(Uu^*\right)}{\partial(\delta\eta)}\right]_{\eta=0}=\mu\frac{U}{\delta}f'(\eta=0)=c_2\mu\frac{U}{\delta}$$

$$\tau_w=\rho U^2\frac{d\delta^{**}}{dx} \Rightarrow c_2\mu\frac{U}{\delta}=\rho U^2c_1\frac{d\delta}{dx} \qquad \delta(x=0)=0$$

$$c_2\mu\frac{U}{\delta}=\rho U^2 c_1\frac{d\delta}{dx}\qquad \delta\left(x=0\right)=0$$

$$\Rightarrow\frac{c_2\mu U}{\rho U^2 c_1}dx=\delta d\delta\qquad \Rightarrow\delta^2=\frac{2c_2}{c_1}\frac{\mu x}{\rho U}+\overset{=0}{\cancel{2c_3}}$$

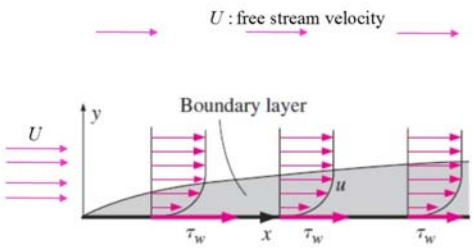
$$\delta^2=\frac{2c_2}{c_1}\frac{\mu x}{\rho U}=\frac{2c_2}{c_1}\frac{\mu}{\rho Ux}x^2=\frac{2c_2}{c_1}\frac{x^2}{\mathrm{Re}_x}\Rightarrow\frac{\delta^2}{x^2}=\frac{2c_2}{c_1}\frac{1}{\mathrm{Re}_x}$$

$$\boxed{\frac{\delta}{x}=\sqrt{\frac{2c_2}{c_1}}\mathrm{Re}_x^{-\frac{1}{2}}}$$

Recall $\delta^{**}=c_1\delta$

$$\boxed{\frac{\delta^{**}}{x}=\sqrt{2c_1c_2}\,\mathrm{Re}_x^{-\frac{1}{2}}}$$

Evaluate $\tau_w=\rho U^2\dfrac{d\delta^{**}}{dx}\qquad F_D=\rho bU^2\delta^{**}$



We can find important quantities if we know the velocity profile

Simplest example: $\dfrac{u}{U}=f\left(\dfrac{y}{\delta}\right)=f\left(\eta\right)=a+b\eta\;=\eta$

$$u\left(y=0\right)=0;u\left(y=\delta\right)=U\qquad a=0,b=1$$

$$\boxed{\begin{aligned}\frac{u}{U}&=\frac{y}{\delta}\text{ for }y\le\delta\\&=1\text{ for }y>\delta\end{aligned}}$$

Let us use this velocity profile

$$\frac{u}{U} = \frac{y}{\delta} \text{ for } y \le \delta$$
$$= 1 \text{ for } y > \delta$$

$$c_1 = \int_0^1 f\left(1 - f\right) d\eta \qquad c_2 = f'\left(\eta = 0\right)$$

$$\frac{\delta}{x} = \sqrt{\frac{2c_2}{c_1}} \mathrm{Re}_x^{-\frac{1}{2}} \qquad \frac{\delta^{**}}{x} = \sqrt{2c_1c_2}\, \mathrm{Re}_x^{-\frac{1}{2}}$$

$$c_1 = \int_0^1 \eta\left(1-\eta\right) d\eta = \frac{1}{6} \qquad c_2 = 1$$

$$\sqrt{\frac{2c_2}{c_1}} = 3.46 \qquad \sqrt{2c_1c_2} = .58$$

4.91
exact solution

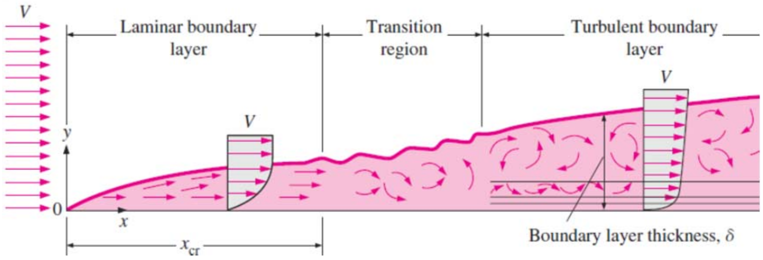
.664
exact solution

Exact vs. Approximate solution

Profile Character	$\delta \mathrm{Re}_x^{1/2}/x$	$c_f \mathrm{Re}_x^{1/2}$	$C_{Df} \mathrm{Re}_\ell^{1/2}$
a. Blasius solution	5.00	0.664	1.328
b. Linear $u/U = y/\delta$	3.46	0.578	1.156
c. Parabolic $u/U = 2y/\delta - (y/\delta)^2$	5.48	0.730	1.460
d. Cubic $u/U = 3(y/\delta)/2 - (y/\delta)^3/2$	4.64	0.646	1.292
e. Sine wave $u/U = \sin[\pi(y/\delta)/2]$	4.79	0.655	1.310

Blasius (exact) solution: exact solution of N-S Eq. after applying boundary layer approximation

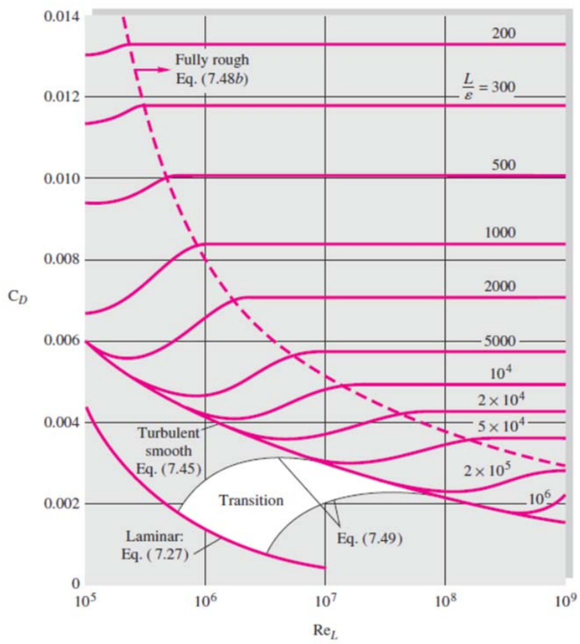
Laminar to Turbulent transition



Property	Laminar	Turbulent[†]
Boundary layer thickness	$\frac{\delta}{x} = \frac{4.91}{\sqrt{Re_x}}$	$\frac{\delta}{x} \cong \frac{0.16}{(Re_x)^{1/7}}$
Displacement thickness	$\frac{\delta^*}{x} = \frac{1.72}{\sqrt{Re_x}}$	$\frac{\delta^*}{x} \cong \frac{0.020}{(Re_x)^{1/7}}$
Momentum thickness	$\frac{\theta}{x} = \frac{0.664}{\sqrt{Re_x}}$	$\frac{\theta}{x} \cong \frac{0.016}{(Re_x)^{1/7}}$
Local skin friction coefficient	$C_{f,x} = \frac{0.664}{\sqrt{Re_x}}$	$C_{f,x} \cong \frac{0.027}{(Re_x)^{1/7}}$

Empirical Turbulent flow profile

$$\frac{u}{U} = \left(\frac{y}{\delta}\right)^{\frac{1}{7}}$$



Drag coefficient for flow over flat plate

Flow over cylinder and sphere

Boundary layer theory can be extended to all geometries including cylinder and sphere

Boundary layer theory accurately predicts the onset of flow separation, transition, drag/lift forces; potential flow theory fails here

Boundary layer approximation fails when flow separates, approximate solutions (such as flow over flat plate) are not possible in such cases

Flow Separation

Boundary layer separates from the solid surface