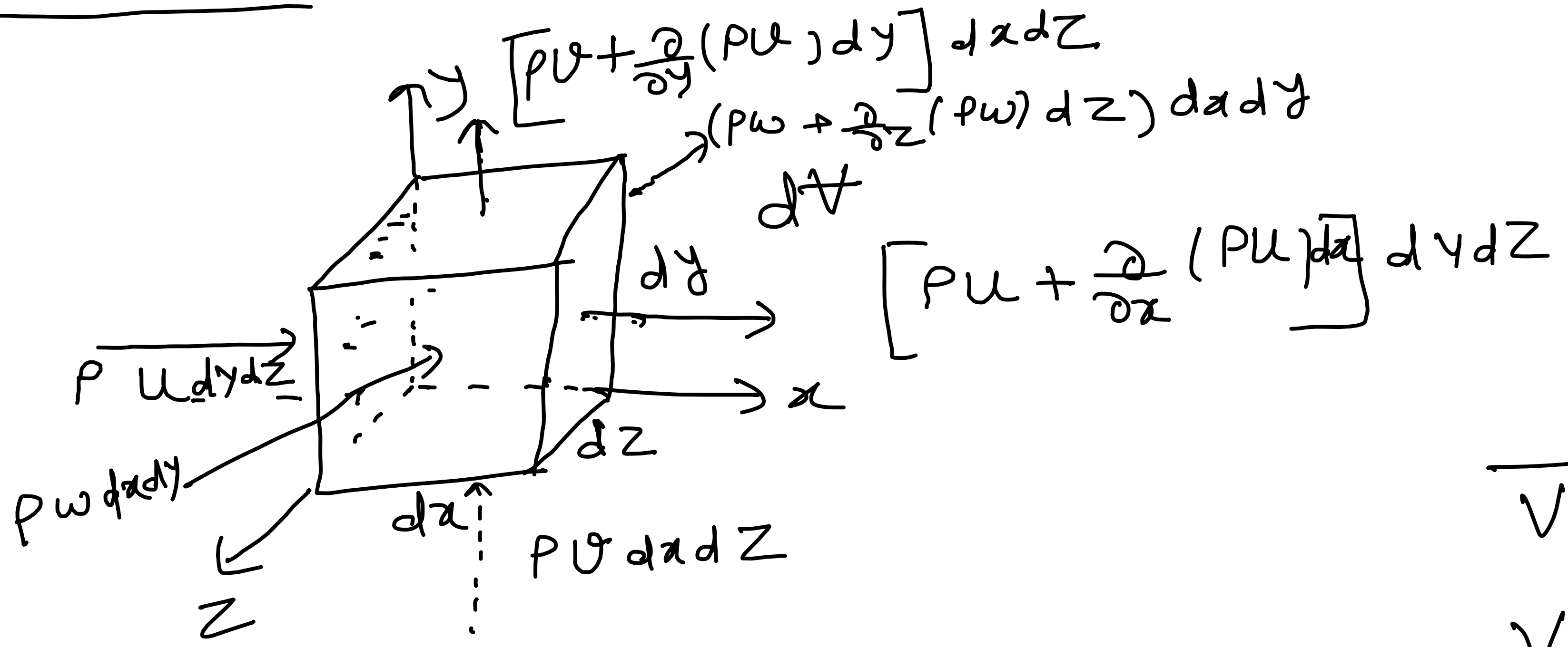
Differential balances : conservation at a point in space $\Leftarrow$ control volume $\to 0$

$d\forall \to 0$

Mass balance



$\vec{V}$

$V = u\hat{i} + v\hat{j} + w\hat{k}$

$0 = \frac{\partial}{\partial t}\int \rho\, \underline{d\forall} + \Sigma$ outlet rates $- \Sigma$ inlet rate

surface normal	inlet rate	outlet rate
x	$\rho u\, dy\,dz$	$\left[\rho u + \frac{\partial}{\partial x}(\rho u)\,dx\right] dy\,dz$
y	$\rho v\, dx\,dz$	$\left[\rho v + \frac{\partial}{\partial y}(\rho v)\,dy\right] dx\,dz$
z	$\rho w\, dx\,dy$	$\left[\rho w + \frac{\partial}{\partial z}(\rho w)\,dz\right] dx\,dy$

$$0 = \frac{\partial}{\partial t}\rho\, d\forall + \left[\frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) + \frac{\partial}{\partial z}(\rho w)\right] \frac{dx\,dy\,dz}{d\forall}$$

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) + \frac{\partial}{\partial z}(\rho w) = 0 \Rightarrow \boxed{\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho V) = 0}$$

Cartesian

Continuity eq^n $\to$ Continuum hypothesis

Cylindrical polar coordinates

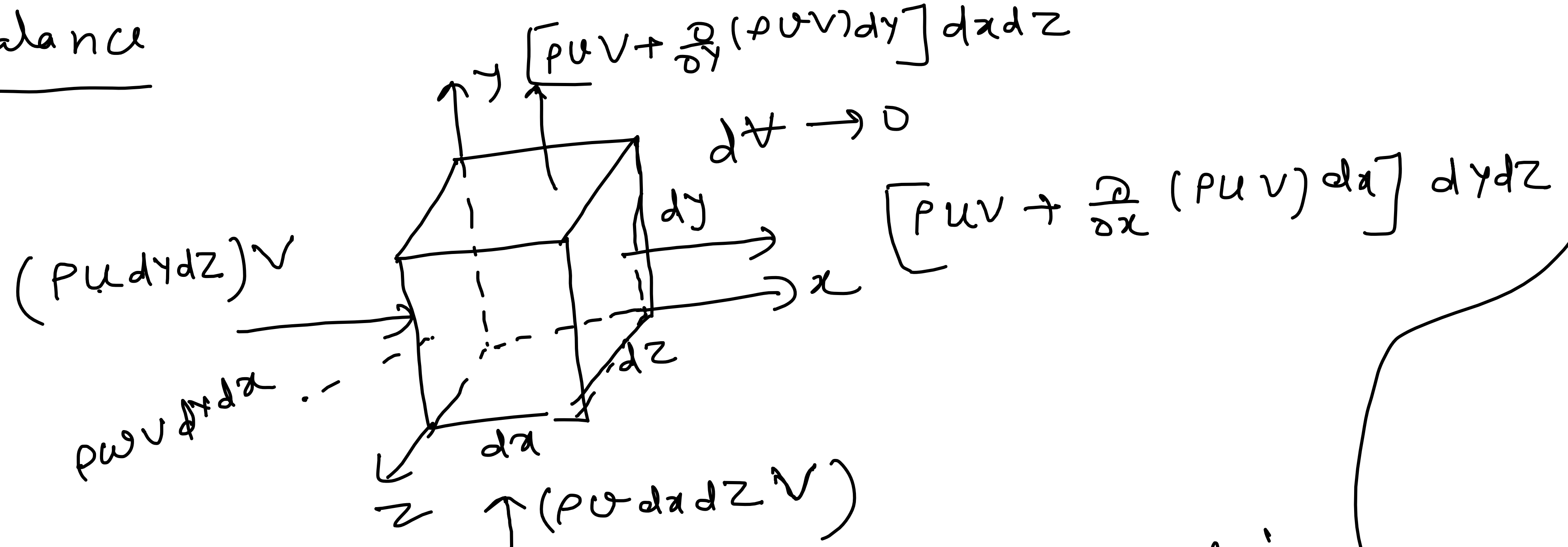
$$\frac{\partial \rho}{\partial t} + \frac{1}{r}\frac{\partial}{\partial r}(r\rho v_r) + \frac{1}{r}\frac{\partial}{\partial \theta}(\rho v_\theta) + \frac{\partial}{\partial z}(\rho v_z) = 0$$

$$V = v_r\,\hat{r} + v_\theta\,\hat{\theta} + v_z\,\hat{z}$$

for incompressible fluid $\rho = $ const

$$\to \nabla\cdot(\rho V) = 0$$

Momentum balance



$dF = \frac{\partial}{\partial t}\int \rho V\, d\forall + \sum$ momentum rate out $- \sum$ momentum rate in

$$dF = \frac{\partial}{\partial t}(\rho V)\, d\forall + \left[\frac{\partial}{\partial x}(\rho u V) + \frac{\partial}{\partial y}(\rho v V) + \frac{\partial}{\partial z}(\rho w V)\right] d\forall$$

Surface normal	momentum rate in	momentum rate out
x	$\rho u V\, dy\, dz$	$[\rho u V + \frac{\partial}{\partial x}(\rho u V)dx]\, dy\, dz$
y	$\rho v V\, dx\, dz$	$[\rho v V + \frac{\partial}{\partial y}(\rho v V)dy]\, dx\, dz$
z	$\rho w V\, dx\, dy$	$[\rho w V + \frac{\partial}{\partial z}(\rho w V)dz]\, dx\, dy$

$$\frac{dF}{d\forall} = \frac{\partial}{\partial t}(\rho V) + \frac{\partial}{\partial x}(\rho u V) + \frac{\partial}{\partial y}(\rho v V) + \frac{\partial}{\partial z}(\rho w V)$$

$$\frac{dF}{d\forall} = \underbrace{V\left[\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) + \frac{\partial}{\partial z}(\rho w)\right]}_{\to 0}$$

$$+ \rho\left[\frac{\partial V}{\partial t} + u\frac{\partial V}{\partial x} + v\frac{\partial V}{\partial y} + w\frac{\partial V}{\partial z}\right]$$

$$= \rho\left[\frac{\partial V}{\partial t} + (V\cdot\nabla)V\right]$$

$$\frac{dF}{d\forall} = \rho\left[\frac{dV}{dt}\right]$$

$$dF = \underbrace{\rho\, d\forall}_{dm}\, \frac{dV}{dt}$$

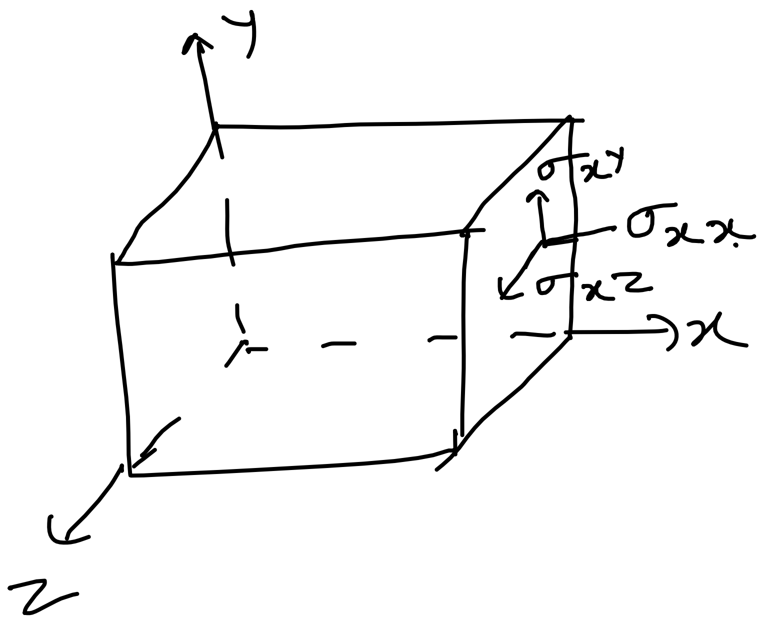
$$dF = (dF)_{body} + (dF)_{surface}$$

$$(dF)_{body} = \rho\vec{g}\, dx\, dy\, dz = \rho\vec{g}\, d\forall$$

$$\frac{(dF)_b}{d\forall} = \rho\vec{g}$$

$(dF)_{surface} \equiv$ stresses

$(dF)_S \equiv$ Pressure + Viscous stress tensor

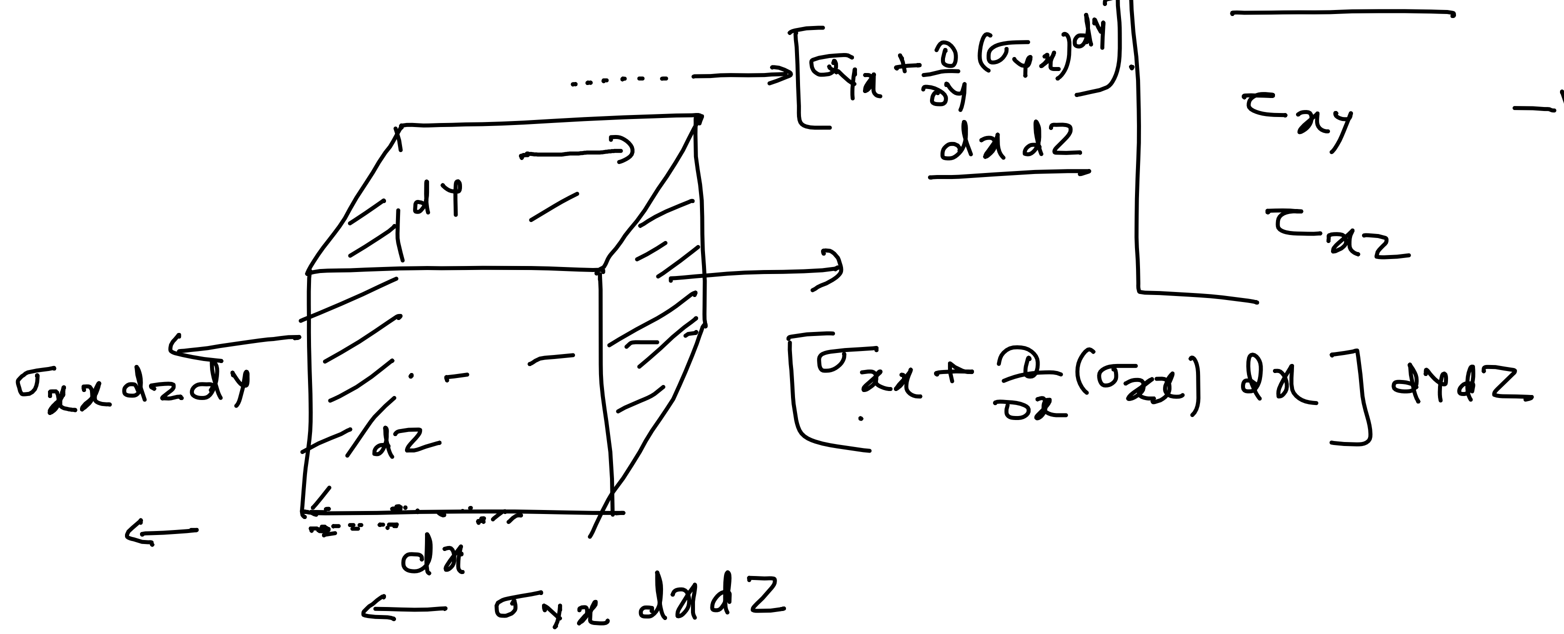


$$\longrightarrow \underset{\text{tenser}}{\sigma_{ij}} = \begin{bmatrix} \sigma_{xx} & \sigma_{yx} & \sigma_{zx} \\ \sigma_{xy} & \sigma_{yy} & \sigma_{zy} \\ \sigma_{xz} & \sigma_{yz} & \sigma_{zz} \end{bmatrix} \begin{array}{l} \to x\text{-direction } i \\ \to y- \\ \to z- \end{array}$$

$$= \begin{bmatrix} -p+\tau_{xx} & \tau_{yx} & \tau_{zx} \\ \tau_{xy} & -p+\tau_{yy} & \tau_{zy} \\ \tau_{xz} & \tau_{yz} & -p+\tau_{zz} \end{bmatrix}$$

x- comp of surface forces

$$dF_{sx} = \left[\frac{\partial}{\partial x}(\sigma_{xx}) + \frac{\partial}{\partial y}(\sigma_{yx}) + \frac{\partial}{\partial z}(\sigma_{zx})\right] dx\,dy\,dz$$



$$\Rightarrow \frac{df_{sx}}{d\forall} = \frac{\partial}{\partial x}(\sigma_{xx}) + \frac{\partial}{\partial y}(\sigma_{yx}) + \frac{\partial}{\partial z}(\sigma_{zx})$$

$$\frac{df_{sx}}{d\forall} = -\frac{\partial p}{\partial x} + \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z}$$

similarly

$$\frac{df_{sy}}{d\forall} = -\frac{\partial p}{\partial y} + \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \tau_{yy}}{\partial y} + \frac{\partial \tau_{zy}}{\partial z}$$

$$\frac{df_{sz}}{d\forall} = -\frac{\partial p}{\partial z} + \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \tau_{zz}}{\partial z}$$

— multiply by corresponding unit

$$\frac{df_s}{d\forall} = -\left(\frac{\partial p}{\partial x}\hat{i} + \frac{\partial p}{\partial y}\hat{j} + \frac{\partial p}{\partial z}\hat{k}\right) + \left(\frac{\partial}{\partial x}\tau_{xx} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z}\right)\hat{i}$$

$$+ \left[\frac{\partial}{\partial x}\tau_{xy} + \frac{\partial}{\partial y}\tau_{yy} + \frac{\partial}{\partial z}\tau_{zy}\right]\hat{j}$$

$$+ \left[\frac{\partial}{\partial x}\tau_{xz} + \frac{\partial}{\partial y}\tau_{yz} + \frac{\partial}{\partial z}\tau_{zz}\right]\hat{k}$$

$$\frac{df_s}{d\forall} = -\underbrace{\nabla p}_{\text{Pressure}} + \underbrace{\nabla\cdot(\tau_{ij})}_{\text{Viscous}}$$