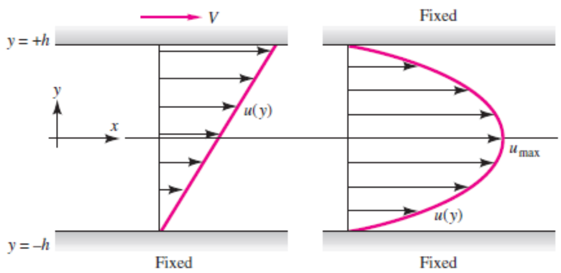
ESO204A, Fluid Mechanics and rate Processes

Laminar, incompressible, viscous flow: Exact Solutions



Chapter 4 of F M White
Chapter 5 of Fox McDonald

3-D, N-S eq., scalar form, Cartesian coordinate:

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}=g_x-\frac{1}{\rho}\frac{\partial p}{\partial x}+\nu\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right)$$

$$\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}=g_y-\frac{1}{\rho}\frac{\partial p}{\partial y}+\nu\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 v}{\partial z^2}\right)$$

$$\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z}=g_z-\frac{1}{\rho}\frac{\partial p}{\partial z}+\nu\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}+\frac{\partial^2 w}{\partial z^2}\right)$$

Incompressible continuity: $\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0$

Continuity: $\nabla.\vec{u} = 0$

Navier-Stokes: $\underbrace{\frac{\partial \vec{u}}{\partial t} + (\vec{u}.\nabla)\vec{u}}_{\text{inertia}} = \underbrace{\vec{g}}_{\text{body}} - \underbrace{\frac{1}{\rho}\nabla p}_{\text{pressure}} + \underbrace{\nu \nabla^2 \vec{u}}_{\text{viscous}}$

Solution of the above Equations provide velocity and pressure. Quantities of interest (force, mass flow rate etc.) can be calculated after we know the velocity, pressure

Properties such as density and viscosity need to be measured experimentally

$$\frac{\partial \vec{u}}{\partial t} + (\vec{u}.\nabla)\vec{u} = \vec{g} - \frac{1}{\rho}\nabla p + \nu \nabla^2 \vec{u}$$

$$0 = \vec{g} - \frac{1}{\rho}\nabla p_{hs}$$ Equation of fluid statics

subtracting

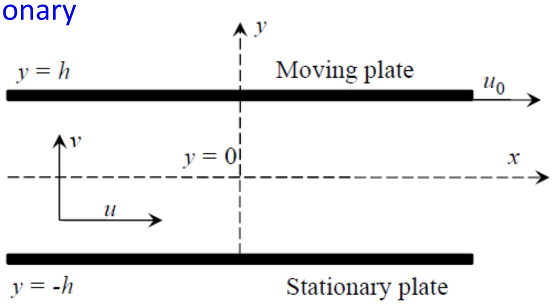
$$\frac{\partial \vec{u}}{\partial t} + (\vec{u}.\nabla)\vec{u} = -\frac{1}{\rho}\nabla p_m + \nu \nabla^2 \vec{u}; \; p_m = p - p_{hs}$$

N-S Eqn. with conservative body forces

Couette Flow

Laminar, **incompressible, steady** flow between two **infinitely** long parallel plates; top plate moving steadily and sustains the flow, bottom plate stationary

Let's say we wish to find the force at the top and bottom plates

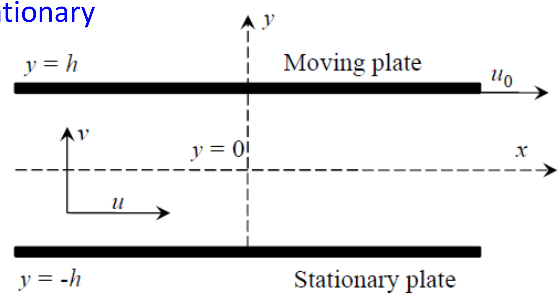


Can we solve this problem using integral analysis?

Couette Flow

Laminar, incompressible, steady flow between two **infinitely long** parallel plates; top plate moving steadily and **sustains the flow,** bottom plate stationary

Reasonable to assume 2-D, **fully developed** flow, why??



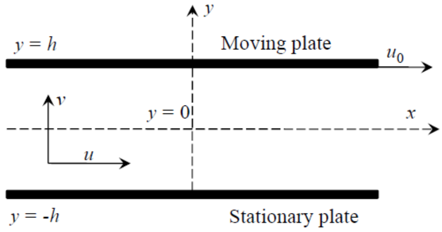
2-D: $u = u(x, y), v = (x, y), w = 0$

Fully developed: $\frac{\partial \vec{u}}{\partial x} = 0$ Further assume $\frac{\partial p}{\partial x} = 0$ why??

Can we, using integral analysis, find the force at the top plate?

$$u = u(x,y), v = (x,y), w = 0 \qquad \frac{\partial \vec{u}}{\partial x} = 0; \frac{\partial p}{\partial x} = 0$$

Now, our goal is to find three unknowns (u, v, p) from continuity and momentum Equations



How the velocity field should look like?

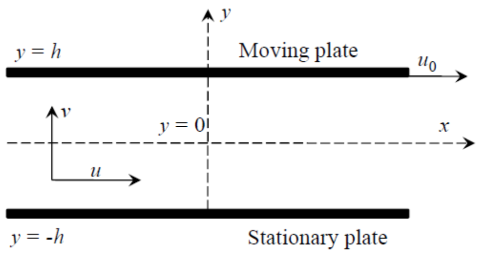
Applying z-momentum Eq.

$$\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial z} + \nu\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2}\right)$$

$$\frac{\partial p}{\partial z} = 0$$

$$u = u(x,y), v = (x,y), w = 0$$

$$\frac{\partial \vec{u}}{\partial x} = 0; \frac{\partial p}{\partial x} = \frac{\partial p}{\partial z} = 0$$



$$\text{Continuity:} \quad \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

$$\text{using fully developed condition } \frac{\partial u}{\partial x} = 0, \text{ we have } \frac{\partial v}{\partial y} = 0$$

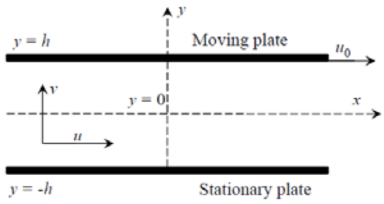
$$\Rightarrow v = f(x)$$

$$\text{Boundary condition (impermeablility): } v(x, y = h) = 0$$

$$\Rightarrow v = 0 \text{ everywhere}$$

$$u = u(x, y), v = w = 0$$

$$\frac{\partial \vec{u}}{\partial x} = 0; \frac{\partial p}{\partial x} = \frac{\partial p}{\partial z} = 0$$



$$y\text{-mom: } \frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial y} + \nu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right) \Rightarrow \frac{\partial p}{\partial y} = 0$$

$\Rightarrow p = \text{constant}$

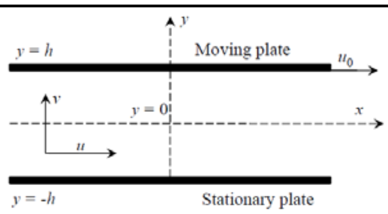
We mean pressure is hydrostatic in all directions

$$x\text{-mom: } \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$$

$$\Rightarrow \frac{d^2 u}{dy^2} = 0 \qquad \Rightarrow u = by + c$$

$$u = by + c$$

$$\text{BCs (no-slip): } u(x, y = h) = u_0$$
$$u(x, y = -h) = 0$$



$$u = \frac{u_0}{2}\left(1 + \frac{y}{h}\right)$$

Velocity field is indeed linear!!

Complete Solution

$$u = \frac{u_0}{2}\left(1 + \frac{y}{h}\right) \qquad v = w = 0 \qquad p = \text{constant}$$

The purpose of the differential formulation is to find the complete information of the field

Stress components $\sigma_{yx} = \mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)$ $\quad \sigma_{xx} = -p + 2\mu\frac{\partial u}{\partial x}$

$\sigma_{xy} = \mu\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)$ $\quad \sigma_{yy} = -p + 2\mu\frac{\partial v}{\partial y}$

In general $\sigma_{ij} = \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \text{ for } i \neq j$

and $\quad \sigma_{ij} = -p + \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \text{ for } i = j$

Overall: $\quad \sigma_{ij} = -p\delta_{ij} + \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$

where $\quad \delta_{ij} = 1 \text{ for } i = j;\ \delta_{ij} = 0 \text{ for } i \neq j$