Boundary Layer Theory – Summary

$$D(x) = \rho b \int_0^{\delta(x)} u\,(U_0 - u)\,dy$$

$$D(x) = b \int_0^{x} \tau_w\,dx \Rightarrow \frac{dD}{dx} = b\,\tau_w$$

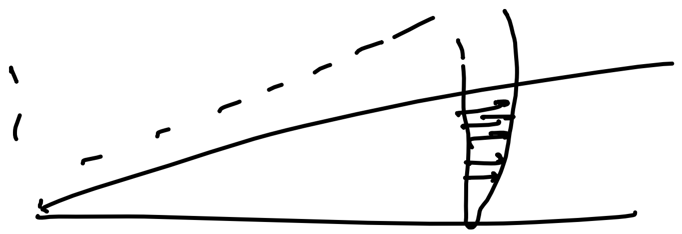
$$D(x) = \rho b U_0^2\, \theta(x)$$

valid for either laminar or turbulent flow

momentum integral relation

$$\tau_w = \rho U_0^2 \frac{d\theta}{dx}$$

Laminar sol^n



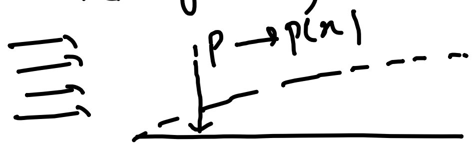
	Approximate sol^n	Exact sol^n
Velocity Profile	$u = U_0\left(\frac{2y}{\delta} - \frac{y^2}{\delta^2}\right)$ Karman	Blasius sol^n (PhD Prandtl)
BL thickness	$\frac{\delta}{x} = \frac{5.5}{Re_x^{1/2}}$	$\frac{\delta}{x} = \frac{5.0}{Re_x^{1/2}}$
Skin friction	$c_f = \frac{0.73}{Re_x^{1/2}}$	$c_f = \frac{0.664}{Re_x^{1/2}}$
Displacement thickness	$\frac{\delta^*}{x} = \frac{1.83}{Re_x^{1/2}}$	$\frac{\delta^*}{x} = \frac{1.721}{Re_x^{1/2}}$

Laminar flow sol^n (Exact) for BL (by Blasius)

$\nu = \frac{\mu}{\rho}$

Assumption: steady, incompressible, laminar flow, constant fluid properties,

$$\frac{dp^*}{dx} = 0$$



Continuity $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$

momentum $u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial x} = \nu \frac{\partial^2 u}{\partial y^2}$ → PDE → ODE

Sol^n Similarity Sol^n

$u = \frac{\partial \psi}{\partial y}$, $v = -\frac{\partial \psi}{\partial x}$ $u_0 \equiv u_\infty$

$$f(\eta) = \frac{\psi}{u_\infty\sqrt{\nu x/u_\infty}} \qquad \eta = y\sqrt{u_\infty/\nu x}$$

similarity variable → PDE → ODE

$$u = \frac{\partial \psi}{\partial y} = \frac{\partial \psi}{\partial \eta}\frac{d\eta}{dy} = u_\infty\cancel{\sqrt{\nu x/u_\infty}}\,\frac{df}{d\eta}\left(\cancel{\sqrt{u_\infty/\nu x}}\right) = u_\infty\frac{df}{d\eta}$$

$$v = -\frac{\partial \psi}{\partial x} = -\left(u_\infty\sqrt{\frac{\nu x}{u_\infty}}\,\frac{\partial f}{\partial x} + \frac{u_\infty}{2}\sqrt{\frac{\nu}{u_\infty x}}\, f\right)$$

$$\frac{\partial f}{\partial x} = \frac{df}{d\eta}\frac{\partial \eta}{\partial x} = \frac{df}{d\eta}\left(-\frac{1}{2}\right) y\sqrt{\frac{u_\infty}{\nu x^3}}$$

$$v = -\left[u_\infty\sqrt{\frac{\nu x}{u_\infty}}\,\frac{df}{d\eta}\left(-\frac{1}{2}\right) y\sqrt{\frac{u_\infty}{\nu x}} + \frac{1}{2}\sqrt{\frac{\nu u_\infty}{x}}\, f\right]$$

$$= \frac{1}{2}\sqrt{\frac{\nu u_\infty}{x}}\left[y\sqrt{\frac{u_\infty}{x\nu}}\,\frac{df}{d\eta} - f\right] = \frac{1}{2}\sqrt{\frac{\nu u_\infty}{x}}\left[\eta\frac{df}{d\eta} - f\right]$$

$$-\frac{\partial u}{\partial x} = u_\infty \frac{d^2f}{d\eta^2}\frac{\partial \eta}{\partial x} = u_\infty\, y\sqrt{\frac{u_\infty}{\nu x}}\;\frac{1}{x}\left(-\frac{1}{2}\right)\frac{d^2f}{d\eta^2}$$

$$\frac{\partial u}{\partial x} = -\frac{u_\infty}{2x}\,\eta\,\frac{d^2f}{d\eta^2}$$

$$\frac{\partial u}{\partial y} = u_\infty \frac{d^2f}{d\eta^2}\frac{\partial \eta}{\partial y} = \underline{u_\infty\sqrt{u_\infty/\nu x}\;\frac{d^2f}{d\eta^2}}$$

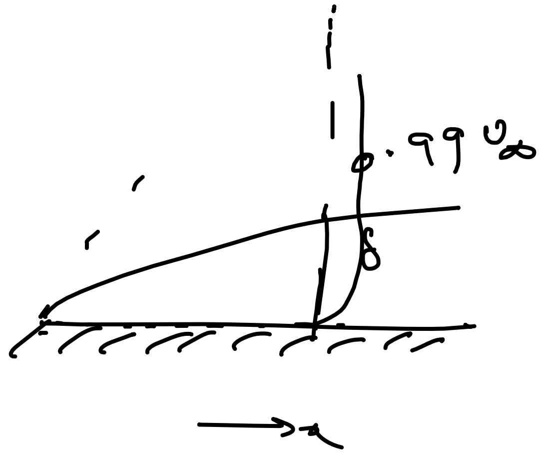
$$\frac{\partial^2 u}{\partial y^2} = u_\infty\sqrt{u_\infty/\nu x}\;\sqrt{u_\infty/\nu x}\;\frac{d^3f}{d\eta^3} = \frac{u_\infty^2}{\nu x}\frac{d^3f}{d\eta^3}$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \nu\frac{\partial^2 u}{\partial y^2}$$

$$\Rightarrow u_\infty\frac{df}{d\eta}\left(-\frac{u_\infty}{2x}\right)\eta\frac{d^2f}{d\eta^2} + \frac{1}{2}\sqrt{\frac{\nu u_\infty}{x}}\left(\eta\frac{df}{d\eta} - f\right)u_\infty\sqrt{\frac{u_\infty}{\nu x}}\frac{d^2f}{d\eta^2} = \nu\frac{u_\infty^2}{\nu x}\frac{d^3f}{d\eta^3}$$

$$\Rightarrow -\frac{\eta}{2}\cancel{\frac{df}{d\eta}}\frac{d^2f}{d\eta^2} + \frac{\eta}{2}\cancel{\frac{df}{d\eta}\frac{d^2f}{d\eta^2}} - \frac{f}{2}\frac{d^2f}{d\eta^2} = \frac{d^3f}{d\eta^3}$$

$$\boxed{2\frac{d^3f}{d\eta^3} + f\frac{d^2f}{d\eta^2} = 0} \quad \text{— (i)}$$



B.C. x, y $\quad y=0,\; y=\delta,\; y=\infty$

$y=0 \quad u=v=0 \Rightarrow \dfrac{df}{d\eta}=0$ at $\eta=0$, $f=0$

$\Rightarrow y=\infty \quad u=u_\infty$

~~$y=\delta \quad u=u_\infty$~~

$y=\infty$, $\eta\to\infty$, $\quad u_\infty\dfrac{df}{d\eta} = u_\infty \Rightarrow \dfrac{df}{d\eta}\to 1$

numerical sol^n of (i)

$\eta = y\sqrt{\frac{u_\infty}{\nu x}}$	f	$\frac{df}{d\eta}=\frac{u}{u_\infty}$	$\frac{d^2f}{d\eta^2}$
0	0	0	0.332
0.4	0.027	0.183	0.331
–	–	–	–
4.8	3.085	0.988	0.022
5.2	3.482	0.994	0.011
–	–	–	–
6.8	5.079	1.000	0.00

$\frac{u}{u_\infty} \simeq 1 \qquad \eta = 5.0$

$\hookrightarrow y = \delta$

$$\eta = y\sqrt{u_\infty/\nu x} = \delta\sqrt{\frac{u_\infty}{\nu x}} = 5.0$$

$$\delta = \frac{5.0}{\sqrt{\frac{u_\infty}{\nu x}}}$$

$$\boxed{\frac{\delta}{x} = \frac{5.0}{\sqrt{\frac{u_\infty x}{\nu}}} = \frac{5.0}{Re_x^{1/2}}}$$

$$\tau_w = \mu \frac{\partial u}{\partial y}\Big|_{y=0} = \mu u_\infty \sqrt{\frac{u_\infty}{\nu x}}\, \frac{d^2 f}{d\eta^2}\Big|_{\eta=0} = \mu u_\infty \sqrt{\frac{u_\infty \rho}{\mu x}}\,(0.332) = 0.332\, u_\infty \sqrt{\frac{\rho \mu u_\infty}{x}}$$

$$\boxed{C_{f,x} = \frac{\tau_{w,x}}{\rho u_\infty^2 / 2} = 0.664 \sqrt{\frac{\mu}{\rho u_\infty x}} = \frac{0.664}{Re_x^{1/2}}}$$

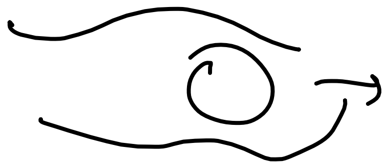
$C_D \to$ Drag coefficient $\quad C_D = \frac{F_D\,(L)}{\frac{1}{2}\rho u_\infty^2 A_p}$

$$D(x) = b \int_0^x \tau_w \, dx$$

$$C_f = \frac{\tau_w}{\frac{1}{2}\rho u_\infty^2} = \frac{0.664}{\left(\frac{u_\infty x \rho}{\mu}\right)^{1/2}} \Rightarrow \tau_w = 0.332\, \frac{\rho^{1/2} \mu^{1/2} u_\infty^{1.5}}{x^{1/2}}$$

$$D(x) = b \int_0^x 0.332\, \frac{\rho^{1/2} \mu^{1/2} u_\infty^{1.5}}{x^{1/2}}\, dx \qquad \text{(Drag force on one side)}$$

$$= 0.664\, b\, \rho^{1/2} \mu^{1/2} u_\infty^{1.5} x^{1/2}$$



$$C_D(L) = \frac{D(L)}{\frac{1}{2}\rho u_\infty^2 b L}$$

$$C_D(x) = \frac{D(x)}{\frac{1}{2}\rho u_\infty^2 b x}$$

$$D(x) = \rho b \int_0^\delta u(u_\infty - u)\, dy$$

$$\boxed{\frac{\theta}{x} = \frac{0.664}{Re_x^{1/2}}} \quad (?)$$

$$\boxed{C_D(x) = \frac{2}{x} \int \frac{u}{u_\infty}\left(1 - \frac{u}{u_\infty}\right) dy = \frac{2}{x}\,\theta(x) = 2 \times \frac{0.664}{Re_x^{1/2}} = \frac{1.328}{Re_x^{1/2}}}$$