4.12

Tutorial 5

a) $u = x,\ v = -y$

$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$ at steady state

$1 - 1 = 0$ satisfied

b) $u = x^2 y + y^3,\ v = x^3 - xy^2$

$\frac{\partial u}{\partial x} = 2xy \quad \frac{\partial v}{\partial y} = -2xy \quad \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$ satisfied

(c) $u = \sin^2(xy),\ v = \cos^2(xy)$

$\frac{\partial u}{\partial x} = y\, 2\sin(xy)\cos(xy)\ ,\quad \frac{\partial v}{\partial y} = x\, 2\cos(xy)\,(-)\sin(xy)$

$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \neq 0$ Not satisfied

d) $u = \frac{x}{x^2+y^2}\ ,\ v = \frac{y}{x^2+y^2}$

$\frac{\partial u}{\partial x} = \frac{(x^2+y^2) - x(2x)}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}\ ,\quad \frac{\partial v}{\partial y} = \frac{(x^2+y^2) - y(2y)}{(x^2+y^2)^2} = \frac{x^2-y^2}{(x^2+y^2)^2}$

$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$ satisfied

(e) $u_r = r\sin\theta\cos\theta\ ,\ u_\theta = -r\sin^2\theta$

$\frac{1}{r}\frac{\partial}{\partial r}(r u_r) + \frac{1}{r}\frac{\partial}{\partial \theta}(u_\theta) + \frac{\partial}{\partial z}(u_z) = 0$

$\Rightarrow \cancel{\frac{1}{r}}\, 2\cancel{r}\sin\theta\cos\theta - \cancel{\frac{1}{r}}\cancel{r}\, 2\sin\theta\cos\theta = 0$ Satisfied

4.16

$u = 2V\left(\frac{x}{L} - \frac{y}{L}\right) \qquad v = -2V\frac{y}{L}$

$\cancel{\frac{\partial \psi}{\partial y} = 2V\left(\frac{x}{L} - \frac{y}{L}\right)}$

$\cancel{\psi = 2V\left(\frac{xy}{L} - \frac{y^2}{2L}\right) + f_1(x)}\ ,\quad -\frac{\partial \psi}{\partial x} = -2V\frac{y}{L}$

$\psi = \frac{Vy^2}{L} + f(y)$

$\psi = 2V\frac{xy}{L} - \frac{Vy^2}{L} + \frac{Vy^2}{L} + C = 2V\frac{xy}{L} + C$

$$\frac{\partial \phi}{\partial x} = 2V\left(\frac{x}{L} - \frac{y}{L}\right) \Rightarrow \phi = 2V\left(\frac{x^2}{2L} - \frac{xy}{L}\right) + f_1(y)$$

$$\frac{\partial \phi}{\partial y} = -2V\frac{y}{L} \Rightarrow \phi = -\frac{\cancel{2}Vy^2}{L}$$

$$\phi = \frac{Vx^2}{L} - \frac{2Vxy}{L} - \frac{Vy^2}{L} + C$$

$$= V\frac{(x^2-y^2)}{L} - \frac{2Vxy}{L} + C$$

<u>4.66</u> $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = \frac{2V}{L} - \frac{2V}{L} \equiv 0$ ψ exists

$$\nabla \times V = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j}\right) \times \left(2V\left(\frac{x}{L} - \frac{y}{L}\right)\hat{i} - 2V\frac{y}{L}\hat{j}\right)$$

$$= 0 + \frac{2V}{L} \neq 0 \quad \phi \text{ does not exist}$$

$$\frac{\partial \psi}{\partial y} = u = 2V\left(\frac{x}{L} - \frac{y}{L}\right)$$

$$\psi = 2V\left(\frac{xy}{L} - \frac{y^2}{2L}\right) + f(x)$$

$$\frac{\partial \psi}{\partial x} = 2V\left(\frac{y}{L}\right) + f'(x) = -v$$

$$v = -2V\frac{y}{L}$$

$$\psi = 2V\left(\frac{xy}{L} - \frac{y^2}{2L}\right) + C$$

<u>4.67</u> $\nabla^2\phi = 0$

(a) $\nabla^2\phi = 0$

(b) $\nabla^2\phi = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)(x^2 - y^2) = 2 - 2 \equiv 0$

(c) $\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)(\ln(x+y)) = \frac{\partial}{\partial x}\left(\frac{1}{x+y}\right) + \frac{\partial}{\partial y}\left(\frac{1}{x+y}\right) = \frac{-1}{(x+y)^2} - \frac{1}{(x+y)^2} \neq 0$

(d) $\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)(\sin x + \cos y) = \frac{\partial}{\partial x}(\cos x) + \frac{\partial}{\partial y}(-\sin y) = -\sin x - \cos y \neq 0$

(e) $\nabla^2\phi = \left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2} + \frac{\partial^2}{\partial z^2}\right]\left(r + \frac{1}{r}\right)\cos\theta$

$$\frac{1}{r}\frac{\partial}{\partial r} r\left(1 - \frac{1}{r^2}\right)\cos\theta - \frac{1}{r^2}\left(r + \frac{1}{r}\right)\cos\theta$$

$$+ \left(1 + \frac{1}{r^2}\right)\cos\theta - \frac{1}{r^2}\left(r + \frac{1}{r}\right)\cos\theta = \left(\frac{1}{r} + \frac{1}{r^3}\right)\cos\theta - \left(\frac{1}{r} + \frac{1}{r^3}\right)\cos\theta \equiv 0$$

4.91

(a) $w = w(x)$

z-momentum

$$0 = -\frac{\partial p}{\partial z} + \mu\left(\frac{d^2w}{dx^2}\right) - \rho g$$

y-momentum

$$0 = -\frac{\partial p}{\partial y} + 0 \qquad p = p(x, z)$$

x-momentum

$$0 = -\frac{\partial p}{\partial x} + 0 \qquad p = p(z)$$

$$\left(\frac{\partial p}{\partial z} + \rho g\right) = \mu \frac{d^2w}{dx^2}$$

at $x = \delta$ $\quad \frac{dp}{dz} = \frac{d}{dz} p_a = 0$

$\frac{\partial p}{\partial z}$ = constant

$$\frac{dw}{dx} = \frac{1}{\mu}\left(\frac{dp}{dz} + \rho g\right)x + c_1$$

$$w = \frac{1}{2\mu}\left(\frac{dp}{dz} + \rho g\right)x^2 + c_1 x + c_2$$

B.C. 1 $\quad x = 0 \quad w = 0$

B.C. 2 $\quad x = \delta \quad \frac{dw}{dx} = 0$

$$0 = c_2$$

$$0 = \frac{1}{\mu}\left(\frac{dp}{dz} + \rho g\right)\delta + c_1$$

$$c_1 = -\frac{1}{\mu}\left(\frac{dp}{dz} + \rho g\right)\delta$$

$$w = \frac{1}{2\mu}\left(\frac{dp}{dz} + \rho g\right)x^2 - \frac{1}{\mu}\left(\frac{dp}{dz} + \rho g\right)\delta x$$

$$= \frac{1}{2\mu}\rho g x^2 - \frac{1}{\mu}\rho g \delta x$$

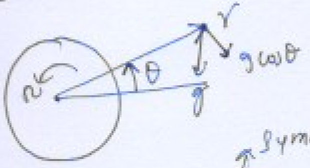
$$= \frac{\rho g x}{2\mu}\left(x - 2\delta\right)$$

(b) $\left.\frac{\partial w}{\partial x}\right|_{x=0} = -\frac{1}{\mu}\rho g \delta$

$$\mu = -\frac{\rho g \delta}{\left.\frac{dw}{dx}\right|_{wall}}$$

4.93

$u_\theta(r,\theta,t)$



symmetry

continuity: $\cancel{\frac{\partial \rho}{\partial t}} + \frac{1}{r}\frac{\partial}{\partial r}(\rho r u_r) + \frac{1}{r}\cancel{\frac{\partial}{\partial \theta}(u_\theta)} + \cancel{\frac{\partial}{\partial z}(u_z)} = 0$

SS

$\frac{1}{r}\frac{\partial}{\partial r}(r u_r) = 0$

$r u_r = f(z,\cancel{\theta})$

$u_r = \frac{f(z)}{r}$

at $r=R$ $u_r = 0 = \frac{f(z)}{R} \Rightarrow f(z) = 0$

$u_r = 0$

$u_\theta = u_\theta(r,\cancel{\theta},\cancel{z})$

$u_\theta = u_\theta(r)$

θ momentum

$$\frac{\partial u_\theta}{\partial t} + (v\cdot\nabla)u_\theta + \frac{1}{r}\cancel{u_r}u_\theta = -\frac{1}{\rho r}\cancel{\frac{\partial p}{\partial\theta}} + g_\theta + \nu\left(\nabla^2 u_\theta - \frac{u_\theta}{r^2} + \frac{2}{r^2}\cancel{\frac{\partial u_r}{\partial\theta}}\right)$$

$(v\cdot\nabla)u_\theta = \cancel{u_r}\frac{\partial u_\theta}{\partial r} + \frac{1}{r}u_\theta\cancel{\frac{\partial u_\theta}{\partial\theta}} + u_z\cancel{\frac{\partial u_\theta}{\partial z}} = 0$

$\nabla^2 u_\theta = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_\theta}{\partial r}\right) + \frac{1}{r^2}\cancel{\frac{\partial^2 u_\theta}{\partial\theta^2}} + \cancel{\frac{\partial^2 u_\theta}{\partial z^2}}$; $g_\theta = -g\cos\theta$

$\Rightarrow \frac{\partial u_\theta}{\partial t} = -g\cos\theta + \nu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_\theta}{\partial r}\right) - \frac{u_\theta}{r^2}\right]$

B.C. $u_\theta(R,t) = \Omega R$

$u_\theta(\infty,t) = 0$