2
Navier-Stokes Equations

The three basic coordinate systems encountered in applications, namely Cartesian, cylindrical and spherical are shown in Figure 2.1. The velocity components in the three orthogonal directions are uniformly represented as (u, v, w). The continuity equations in these three coordinate system are listed in Equations 2.1-2.3.

$$\text{Cartesian:} \quad \frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) + \frac{\partial}{\partial z}(\rho w) = 0 \tag{2.1}$$

$$\text{Cylindrical:} \quad \frac{\partial \rho}{\partial t} + \frac{1}{r}\frac{\partial}{\partial r}(r\rho u) + \frac{1}{r}\frac{\partial}{\partial \theta}(\rho v) + \frac{\partial}{\partial z}(\rho w) = 0 \tag{2.2}$$

$$\text{Spherical:} \quad \frac{\partial \rho}{\partial t} + \frac{1}{r^2}\frac{\partial}{\partial r}(r^2\rho u) + \frac{1}{r\sin\phi}\frac{\partial}{\partial \phi}(\rho v \sin\phi) + \frac{1}{r\sin\phi}\frac{\partial}{\partial \theta}(\rho w) = 0 \tag{2.3}$$

The incompressible form of the mass balance equation in the three coordi-

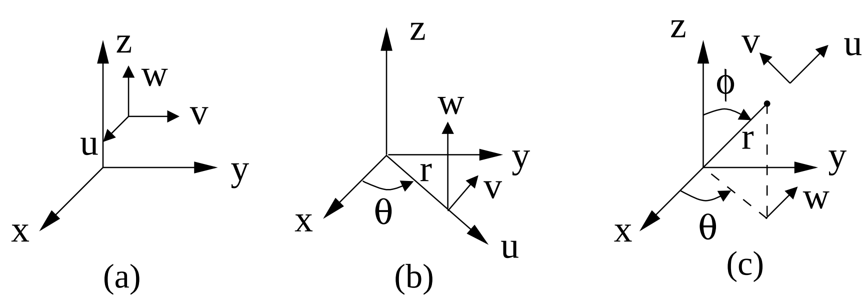

Figure 2.1: (a) Cartesian, (b) Cylindrical and (c) Spherical Coordinate Systems.

nate systems is stated below.

$$\text{Cartesian:} \quad \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

$$\text{Cylindrical:} \quad \frac{1}{r}\frac{\partial}{\partial r}(ru) + \frac{1}{r}\frac{\partial v}{\partial \theta} + \frac{\partial w}{\partial z} = 0$$

$$\text{Spherical:} \quad \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 u) + \frac{1}{r\sin\phi}\frac{\partial}{\partial \phi}(v\sin\phi) + \frac{1}{r\sin\phi}\frac{\partial w}{\partial \theta} = 0$$

2.1 Expanded Form of Stress-Strain Rate Equations in Cartesian Coordinates

The constitutive equations are written in component form as

$$\begin{aligned}
\sigma_{xx} &= -p_d + \lambda \nabla \cdot \mathbf{u} + 2 \mu u_x \\
\sigma_{yy} &= -p_d + \lambda \nabla \cdot \mathbf{u} + 2 \mu v_y \\
\sigma_{zz} &= -p_d + \lambda \nabla \cdot \mathbf{u} + 2 \mu w_z \\
\sigma_{xy} &= \mu(v_x + u_y) \\
\sigma_{yz} &= \mu(w_y + v_z) \\
\sigma_{xz} &= \mu(u_z + w_x)
\end{aligned} \tag{2.4}$$

2.2 Navier-Stokes Equations

Letting the body force vector $\chi = (X, Y, Z)$, the Cartesian form of the Navier-Stokes equations can be derived as:

x **component**

$$\rho \frac{du}{dt} = -p_x + \mu \nabla^2 u + X$$

y **component**

$$\rho \frac{dv}{dt} = -p_y + \mu \nabla^2 v + Y$$

z **component**

$$\rho \frac{dw}{dt} = -p_z + \mu \nabla^2 w + Z \tag{2.5}$$

where

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

$$\frac{d}{dt} \quad = \quad \frac{\partial}{\partial t} \; + \; u \, \frac{\partial}{\partial x} \; + \; v \, \frac{\partial}{\partial y} \; + \; w \, \frac{\partial}{\partial z}$$

Cylindrical coordinates

The coordinate-free form of Navier-Stokes equations can be used to obtain the component form in any other orthogonal coordinate system. Let (r, θ, z) be the cylindrical coordinate system and (u, v, w) be the velocities in the respective directions. The system of equations governing flow is obtained as follows.

Let the gradient operator be $\nabla = (\partial/\partial r,\ 1/r\ \partial/\partial\theta,\ \partial/\partial z)$. Let $\mathbf{e}_r, \mathbf{e}_\theta$ and $\mathbf{e}_z$ be unit vectors in r, θ and z directions respectively. We choose $\mathbf{e}_z$ to be a fixed vector normal to the plane of the paper, though $\mathbf{e}_r$ and $\mathbf{e}_\theta$ change directions with θ. Specifically

$$\begin{aligned}
\frac{\partial}{\partial r} \, (\mathbf{e}_r, \mathbf{e}_\theta) &= 0 \\
\frac{\partial}{\partial \theta} \, (\mathbf{e}_r, \mathbf{e}_\theta) &= (\mathbf{e}_\theta, -\mathbf{e}_r)
\end{aligned}$$

Using these relations we evaluate the dot products in cylindrical coordinates as follows.

$$\begin{aligned}
\nabla \cdot \mathbf{u} &= \left(\mathbf{e}_r \, \frac{\partial}{\partial r} \; + \; \frac{\mathbf{e}_\theta}{r} \, \frac{\partial}{\partial \theta} \; + \; \mathbf{e}_z \, \frac{\partial}{\partial z} \right) \cdot \\
& \quad (u \; \mathbf{e}_r \; + \; v \; \mathbf{e}_\theta \; + \; w \; \mathbf{e}_z) \\
&= \frac{\partial u}{\partial r} \; + \; \frac{u}{r} \, \mathbf{e}_\theta . \, \frac{\partial}{\partial \theta} \, \mathbf{e}_r \; + \; \frac{1}{r} \, \frac{\partial v}{\partial \theta} \; + \; \frac{\partial w}{\partial z} \\
&= u_r \; + \; \frac{u}{r} + \frac{1}{r} \, v_\theta \; + \; w_z \; = \; 0 \;\; \text{(incompressibility)} \qquad (2.6) \\
\nabla^2 &= \nabla \cdot \nabla \; = \; \left(\mathbf{e}_r \, \frac{\partial}{\partial r} \; + \; \frac{\mathbf{e}_\theta}{r} \, \frac{\partial}{\partial \theta} \; + \; \mathbf{e}_z \, \frac{\partial}{\partial z} \right) \cdot \\
& \quad \left(\mathbf{e}_r \, \frac{\partial}{\partial r} \; + \; \frac{\mathbf{e}_\theta}{r} \, \frac{\partial}{\partial \theta} \; + \; \mathbf{e}_z \, \frac{\partial}{\partial z} \right) \\
&= \frac{\partial^2}{\partial r^2} \; + \; \frac{1}{r} \, \frac{\partial}{\partial r} \, \left[\mathbf{e}_\theta \cdot \frac{\partial}{\partial \theta} \, \mathbf{e}_r \right] \; + \; \frac{1}{r^2} \, \frac{\partial^2}{\partial \theta^2} \; + \; \frac{\partial^2}{\partial z^2} \\
&= \frac{\partial^2}{\partial r^2} \; + \; \frac{1}{r} \, \frac{\partial}{\partial r} \; + \; \frac{1}{r^2} \, \frac{\partial^2}{\partial \theta^2} \; + \; \frac{\partial^2}{\partial z^2} \\
\nabla^2 \mathbf{u} &= \left(\frac{\partial^2}{\partial r^2} \; + \; \frac{1}{r} \, \frac{\partial}{\partial r} \; + \; \frac{1}{r^2} \, \frac{\partial^2}{\partial \theta^2} \; + \; \frac{\partial^2}{\partial z^2} \right) \\
& \quad (u \; \mathbf{e}_r \; + \; v \; \mathbf{e}_\theta \; + w \; \mathbf{e}_z) \\
&= \left(\nabla^2 u, \; \nabla^2 v, \; \nabla^2 w \right) \; + \; \frac{1}{r^2} \, \frac{\partial^2}{\partial \theta^2} \; \mathbf{u}
\end{aligned}$$

The second term is

$$\begin{aligned}
& \frac{\partial}{\partial\theta}\ (u_\theta\ \mathbf{e}_r\ +\ u\ \mathbf{e}_\theta\ \ + v_\theta\ \mathbf{e}_\theta\ -\ v\ \mathbf{e}_r) \\
= \ & u_{\theta\theta}\ \mathbf{e}_r\ +\ u_\theta\ \mathbf{e}_\theta\ +\ u_\theta\ \mathbf{e}_\theta\ -\ u\ \mathbf{e}_r\ +\ v_{\theta\theta}\ \mathbf{e}_\theta \\
& - v_\theta\ \mathbf{e}_r\ -\ v_\theta\ \mathbf{e}_r\ -\ v\ \mathbf{e}_\theta \\
= \ & (u_{\theta\theta}\ -\ u\ -\ 2v_\theta)\ \mathbf{e}_r\ +\ (v_{\theta\theta}\ +\ 2u_\theta\ -\ v)\ \mathbf{e}_\theta
\end{aligned}$$

The nonlinear acceleration terms are

$$\begin{aligned}
\mathbf{u}\cdot\nabla\mathbf{u} &= \left[u\frac{\partial}{\partial r}\ +\ \frac{v}{r}\ \frac{\partial}{\partial\theta}\ +\ w\ \frac{\partial}{\partial z}\right]\ \{u\ \mathbf{e}_r\ +\ v\ \mathbf{e}_\theta\ +\ w\ \mathbf{e}_z\} \\
&= u\ u_r\ \mathbf{e}_r\ +\ u\ v_r\ \mathbf{e}_\theta\ +\ \frac{v\ u_\theta}{r}\ \mathbf{e}_r\ +\ \frac{uv}{r}\ \mathbf{e}_\theta \\
&+ \frac{v}{r}\ v_\theta\ \mathbf{e}_\theta\ -\ \frac{v^2}{r}\ \mathbf{e}_r\ +\mathbf{e}_r\ w\ \frac{\partial u}{\partial z}\ +\ \mathbf{e}_\theta\ w\ \frac{\partial v}{\partial z} \\
&+ \mathbf{e}_z\ w\ \frac{\partial w}{\partial z}\ +\ \mathbf{e}_z\ u\ \frac{\partial w}{\partial r}\ +\ \mathbf{e}_z\ \frac{v}{r}\ \frac{\partial w}{\partial\theta} \\
&= \mathbf{e}_r\ \left[u\ u_r\ +\ \frac{v}{r}\ u_\theta\ -\ \frac{v^2}{r}\ +\ w\ u_z\right] \\
&+ \mathbf{e}_\theta\ \left[u\ v_r\ +\ \frac{uv}{r}\ +\ \frac{1}{r}\ v\ v_\theta\ +\ w\ v_z\right] \\
&+ \mathbf{e}_z\ \left[u\ w_r\ +\ \frac{v}{r}\ w_\theta\ +\ w\ w_z\right]
\end{aligned}$$

Combining the above calculations, the momentum equations in component form are:

r component

$$\rho\ \left(\frac{du}{dt}\ -\ \frac{v^2}{r}\right)\ = -p_r\ +\ \mu\ \left(\nabla^2\ u - \frac{2}{r^2}\ v_\theta\ -\ \frac{u}{r^2}\right)\ +\ X_r \qquad (2.7)$$

θ component

$$\rho\ \left(\frac{dv}{dt}\ +\ \frac{uv}{r}\right)\ = -\frac{1}{r}\ p_\theta\ \ +\mu\ \left(\nabla^2\ v + \frac{2}{r^2}\ u_\theta\ -\ \frac{v}{r^2}\right)\ +\ Y_\theta \qquad (2.8)$$

z component

$$\rho\ \left(\frac{dw}{dt}\right) = -\ p_z\ +\ \mu\ \nabla^2\ w\ +\ Z \qquad (2.9)$$

where

$$\begin{aligned}
\frac{du}{dt} &= \frac{\partial u}{\partial t}\ +\ u\ u_r\ +\ \frac{v}{r}\ u_\theta\ +\ w\ u_z \\
\nabla^2 u &= u_{rr}\ +\ \frac{1}{r}\ u_r\ +\ \frac{1}{r^2}\ u_{\theta\theta}\ +\ u_{zz}
\end{aligned}$$

and subscripts refer to partial differentiation. For example, $u_r = \partial u / \partial r$, $u_\theta = \partial u / \partial \theta$ and so on. X_r, Y_θ and Z are body force components in r, θ and z directions respectively.

The notation (u, v, w) can be used to represent velocity components in the **spherical** coordinates r, ϕ and θ. Here, θ is the polar angle measured in the x-y plane from the positive x-axis, and ϕ, the azimuthal angle measured with respect to the positive z-axis. With this notation, angle θ in cylindrical and spherical coordinates are identical. The continuity and Navier-Stokes equations in spherical coordinates are the following:

Incompressibility:

$$\nabla \cdot \mathbf{u} = \frac{1}{r} \frac{\partial}{\partial r} \left(r^2 \ u \right) \ + \ \frac{1}{\sin \phi} \frac{\partial}{\partial \phi} \left(v \ \sin \phi \right) \ + \ \frac{1}{\sin \phi} \frac{\partial w}{\partial \theta} = 0 \tag{2.10}$$

$$\begin{aligned}
r \ \textbf{component} \qquad \rho \left[\frac{du}{dt} - \frac{v^2 + w^2}{r} \right] \quad &= \quad - \frac{\partial p}{\partial r} \ + \ \mu \ \nabla^2 \ u \\
&- \quad \mu \left(\frac{2u}{r^2} + \frac{2}{r^2} \frac{\partial v}{\partial \phi} + \frac{2v \ \cot \ \phi}{r^2} \right) \\
&- \quad \mu \left(\frac{2}{r^2 \ \sin \phi} \frac{\partial w}{\partial \theta} \right) + X_r \qquad (2.11) \\
\phi \ \textbf{component} \quad \rho \left[\frac{dv}{dt} + \frac{uv}{r} - \frac{w^2 \ \cot \ \phi}{r} \right] \quad &= \quad - \frac{1}{r} \frac{\partial p}{\partial \phi} \\
&+ \quad \mu \left[\nabla^2 \ v \ + \ \frac{2}{r^2} \frac{\partial u}{\partial \phi} \right] \\
&- \quad \mu \left[\frac{v}{r^2 \ \sin^2 \ \phi} + \frac{2 \ \cot \ \phi}{r^2 \ \sin \phi} \frac{\partial w}{\partial \theta} \right] \\
&+ \quad Y_\phi \qquad (2.12) \\
\theta \ \textbf{component} \quad \rho \left[\frac{dw}{dt} + \frac{uw}{r} + \frac{vw \ \cot \ \phi}{r} \right] \quad &= \quad \frac{-1}{r \ \sin \phi} \frac{\partial p}{\partial \theta} \\
&+ \quad \mu \left[\nabla^2 \ w \ - \ \frac{w}{r^2 \ \sin^2 \ \phi} \right] \\
&+ \quad \mu \left[\frac{2}{r^2 \ \sin^2 \phi} \frac{\partial u}{\partial \theta} + \frac{2 \ \cot \phi}{r^2 \ \sin \phi} \frac{\partial v}{\partial \theta} \right] \\
&+ \quad Z_\theta \qquad (2.13)
\end{aligned}$$

where

$$\begin{aligned}
\nabla^2 &= \frac{1}{r^2}\,\frac{\partial}{\partial r}\left(r^2\,\frac{\partial}{\partial r}\right) + \frac{1}{r^2\,\sin\phi}\,\frac{\partial}{\partial \phi}\left(\sin\phi\,\frac{\partial}{\partial \phi}\right) \\
&+ \frac{1}{r^2\,\sin^2\phi}\,\frac{\partial^2}{\partial \theta^2} \\
\frac{d}{dt} &= \frac{\partial}{\partial t} + u\,\frac{\partial}{\partial r} + \frac{v}{r}\,\frac{\partial}{\partial \phi} + \frac{w}{r\,\sin\phi}\,\frac{\partial}{\partial \theta}
\end{aligned}$$

and X_r, Y_ϕ and Z_θ are the body force components in r, ϕ, and θ directions respectively.

Constitutive Relationship in Cylindrical Polar Coordinates

Starting from the result

$$\dot{\varepsilon} = \frac{1}{2}(D + D^T)$$

where $D = \nabla \mathbf{u}$, the rate of strain tensor in two dimensional polar coordinates has components

$$\dot{\varepsilon}_{rr} = u_r,\ \dot{\varepsilon}_{\theta\theta} = \frac{1}{r}\,v_\theta + \frac{u}{r} \text{ and } \dot{\varepsilon}_{r\theta} = \frac{1}{2}(v_r - \frac{v}{r} + \frac{1}{r}\,u_\theta)$$

where u and v are velocity components in the r and θ directions respectively. The explicit forms of the stress components are

$$\sigma_{rr} = -p_d + 2\mu u_r$$

$$\sigma_{r\theta} = \mu\left(v_r + \frac{1}{r}u_\theta - \frac{v}{r}\right)$$

$$\sigma_{\theta\theta} = -p_d + 2\mu\left(\frac{1}{r}v_\theta + \frac{u}{r}\right)$$

On the surface of a circular cylinder ($r = R$), these equations simplify to

$$\sigma_{rr} = -p_d$$

$$\sigma_{r\theta} = \mu v_r$$

$$\sigma_{\theta\theta} = -p_d$$