Tutorial-2 Solutions
(ESO-204A)

P1. A two- dimensional flow is described in the Lagrangian system as

$$x = x_0 e^{-kt} + y_0(1-e^{-2kt})$$
$$y = y_0 e^{kt}$$

Find **(a)** the equation of a fluid particle in the flow field and **(b)** the velocity components in the Eulerian system.

Soln. (a) Trajectory of fluid particle in the flow field is found by eliminating *t* from the equations describing its motion as follows:

$$e^{kt} = y / y_0$$

Hence, $$x = x_0(y_0 / y) + y_0(1 - y_0^2 / y^2)$$

which finally gives after some arrangement,

$$(x - y_0)y^2 - x_0 y_0 y + y_0^3 = 0$$

This is the required equation.

(b) u (the x component of velocity)

$$\begin{aligned}
&= \frac{dx}{dt} \\
&= \frac{d}{dt}\left[x_0 e^{-kt} + y_0(1-e^{-2kt})\right] \\
&= -kx_0 e^{-kt} + 2ky_0 e^{-2kt} \\
&= -k[x - y_0(1-e^{-2kt})] + 2ky_0 e^{-2kt} \\
&= -kx + ky_0(1+e^{-2kt}) \\
&= -kx + ky(e^{-kt} + e^{-3kt})
\end{aligned}$$

Ans.

v (the y component of velocity)

$$\begin{aligned}
&= \frac{dy}{dt} = \frac{d}{dt}(y_0 e^{kt}) \\
&= ky_0 e^{kt} = ky
\end{aligned}$$

Ans.

P2. Find the acceleration components at point (1, 1, 1) for the following flow field:

$$u = 2x^2 + 3y\,,\;\; v = -2xy + 3y^2 + 3zy\,,\;\; w = -\frac{3}{2}z^2 + 2xz - 9y^2 z$$

Soln. x component of acceleration,

$$a_x = \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}$$

$$= 0 + (2x^2 + 3y)4x + (-2xy + 3y^2 + 3zy)3 + 0$$

Therefore, $$(a_x)_{at(1,1,1)} = 0 + 5\times 4 + 4\times 3 + 0 = 32 units$$

y component of acceleration,

$$a_y = \frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z}$$

$$= 0 + (2x^2 + 3y)(-2y) + (-2xy + 3y^2 + 3zy)(-2x + 6y + 3z) + (-\frac{3}{2}z^2 + 2xz - 9y^2 z)3y$$

Therefore, $$(a_y)_{at(1,1,1)} = 5\times(-2) + 4\times 7 + (-8.5)\times 3 = -7.5 units$$

z component of acceleration,

$$a_z = \frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z}$$

$$= 0 + (2x^2 + 3y)2z + (-2xy + 3y^2 + 3zy)(-18yz) + (-\frac{3}{2}z^2 + 2xz - 9y^2 z)(-3z + 2x - 9y^2)$$

Therefore, $$(a_z)_{at(1,1,1)} = 0 + (2+3)\times 2 - (-2+3+3)18 + (-\frac{3}{2} + 2 - 9)(-3 + 2 - 9) = 23 units$$

P3. The velocity and density field in a diffuser are given by

$$u = u_0 e^{-2x/L} \qquad \text{and} \qquad \rho = \rho_0 e^{-x/L}$$

Find the rate of change of density at x=L.

Soln. The rate of change of density in this case can be written as

$$\frac{D\rho}{Dt} = \frac{\partial \rho}{\partial t} + u\frac{\partial \rho}{\partial x}$$

$$= 0 + u_0 e^{-2x/L}\frac{\partial}{\partial x}(\rho_0 e^{-x/L})$$

$$= u_0 e^{-2x/L}\left(-\frac{\rho_0}{L}\right)e^{-x/L}$$

$$= \left(-\frac{\rho_0 u_0}{L}\right)e^{-3x/L}$$

At x=L, $$\frac{D\rho}{Dt} = \left(-\frac{\rho_0 u_0}{L}\right)e^{-3}$$

P4. The velocity field is defined by $\vec{V} = ay\hat{i} + b\hat{j}$, where $a = 1s^{-1}$, and $b = 2m/s$; coordinates are measured in meters.

Find: **(a)** Equation of streamline through (x, y) = (b, b)

(b) At t=1s, coordinates of particle that passed through point (x_0, y_0) = (1, 4) at t=0

(c) At t=3s, coordinates of particle that passed through point (x_0, y_0) = (-3, 0) at t_0=1s

Soln. (a) The velocity vector is tangent to the streamlines,

$$\left.\frac{dy}{dx}\right)_{streamline} = \frac{v}{u} = \frac{b}{ay}$$

Or $$\int_b^y ay\,dy = \int_b^x b\,dx$$

Then, $$\left.\frac{1}{2}ay^2\right]_b^y = bx\Big]_b^x, \qquad 2b(x-b) = a(y^2 - b^2)$$

And $$4(x-b) = y^2 - b^2 \qquad \text{or} \qquad x = \frac{y^2}{4} + 1$$

This is the equation for streamline.

(b) Follow particle that passed through (1, 4) at t=0,

$$u = \frac{dx}{dt} = ay, \qquad \int_{x_0}^x dx = \int_0^t ay\,dt \qquad \text{\{need y=y(t) \}}$$

$$v = \frac{dy}{dt} = b, \qquad \int_{y_0}^y dy = \int_0^t b\,dt \quad \text{and} \quad y = y_0 + bt \qquad 1(a)$$

Then $$\int_{x_0}^x dx = x - x_0 = \int_0^t a(y_0 + bt)dt = ay_0 t + \frac{1}{2}bt^2$$

$$x = x_0 + ay_0 t + \frac{1}{2}bt^2 \qquad 1(b)$$

Following particle through (1, 4) at t=0, then at t=1s,

$$x_P = 1 + (1\times 4\times 1) + (\frac{1}{2}\times 2\times 1^2) = 6 \quad \text{and} \quad y_P = 4 + 2(1) = 6$$

Position of particle (x_P, y_P) = (6, 6)

(c) Streak line at t=3s, locate position of particle that passed through point (x_0, y_0) = (-3,0) at earlier time t_0=1s.

For a particle,

$$v = \frac{dy}{dt} = b \ , \qquad \int_{y_0}^{y} dy = \int_{t_0}^{t} b dt \qquad \text{and} \quad y = y_0 + b(t - t_0) \qquad \text{2(a)}$$

$$u = \frac{dx}{dt} = ay \ , \qquad \int_{x_0}^{x} dx = \int_{t_0}^{t} ay dt = \int_{t_0}^{t} a[y_0 + b(t - t_0)]dt$$

And $$x = x_0 + ay_0(t - t_0) + \frac{ab}{2}(t^2 - t_0^2) - abt_0(t - t_0) \qquad \text{2(b)}$$

Then from equation 2a and 2b, for t=3s and t_0=1s

$$x = -3 + 0 + \frac{1.(2)}{2}[3^2 - 1^2] - (1)(2)(1)(3 - 1) = 1$$

$$y = 0 + 2(3 - 1) = 4$$

(x, y) = (1, 4)

Since points (b, b), (1, 4) and (-3, 0) are all on the same streamline ($x = \frac{y^2}{4} + 1$), then path line, streak lines and streamlines coincide.