ESO204A, Fluid Mechanics and rate Processes

1-D Transient Conduction

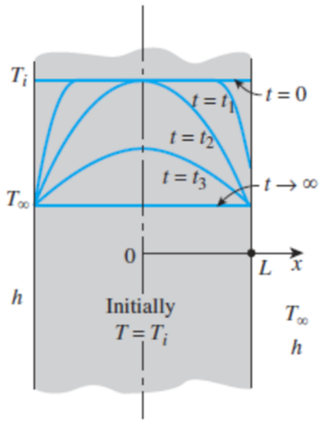
Chapter 4 of Cengel

1-D transient conduction, Cartesian

$$\frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial x^2}$$

$$T(t=0) = T_i \qquad x = 0 : \frac{\partial T}{\partial x} = 0$$

$$x = L : -k\frac{\partial T}{\partial x} = h(T - T_\infty)$$



Nondimensionalization

$$\theta = \frac{T - T_\infty}{T_i - T_\infty} \qquad X = \frac{x}{L} \qquad \tau = \frac{\alpha t}{L^2}$$

$$\text{length scale: } \frac{V}{A} = \frac{2L.H.1}{2H.1} = L$$

$$\theta = \frac{T - T_\infty}{T_i - T_\infty};\ X = \frac{x}{L};\ \tau = \frac{\alpha t}{L^2}$$

Dimensional

$$\frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial x^2} \qquad T(t=0) = T_i$$

$$x = 0: \frac{\partial T}{\partial x} = 0$$

$$x = L: -k\frac{\partial T}{\partial x} = h(T - T_\infty)$$

$$T = F(x, L, t, \alpha, k, h, T_i, T_\infty)$$

Nondimensional

$$\frac{\partial \theta}{\partial \tau} = \frac{\partial^2 \theta}{\partial X^2} \qquad \theta(\tau = 0) = 1$$

$$X = 0: \frac{\partial \theta}{\partial X} = 0$$

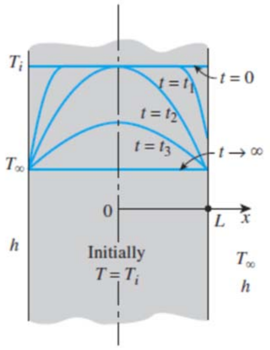
$$X = 1: \frac{\partial \theta}{\partial X} = -\mathrm{Bi}\,\theta$$

$$\theta = F(X, \tau, \mathrm{Bi})$$

1-D transient conduction, Cartesian

$$\frac{\partial \theta}{\partial \tau} = \frac{\partial^2 \theta}{\partial X^2} \qquad \theta(\tau = 0) = 1$$

$$X = 0: \frac{\partial \theta}{\partial X} = 0 \qquad X = 1: \frac{\partial \theta}{\partial X} = -\mathrm{Bi}\,\theta$$



Solution requires separation of variables or integral transform

$$\theta = \sum_{n=1}^{\infty} A_n \exp(-\lambda_n^2 \tau)\cos(\lambda_n X)$$

$$\text{where } A_n = \frac{4\sin\lambda_n}{2\lambda_n + \sin(2\lambda_n)}$$

and λ_n are the roots of the Eq. $\lambda \tan\lambda = \mathrm{Bi}$

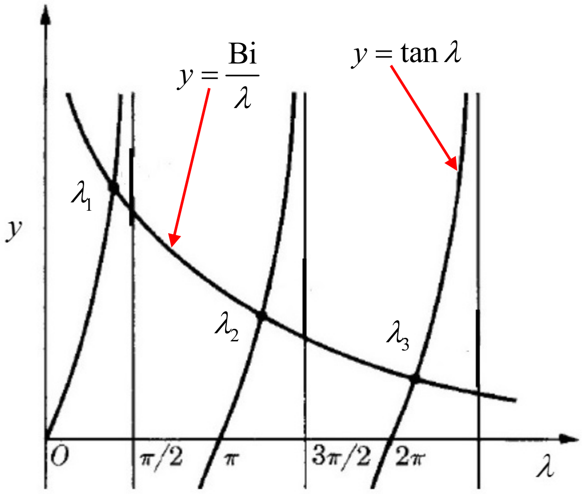
$$\theta = \frac{T - T_\infty}{T_i - T_\infty}$$

$$X = x/L$$

$$\tau = \alpha t / L^2$$

$$\theta = \sum_{n=1}^{\infty} A_n \exp\left(-\lambda_n^2 \tau\right)\cos\left(\lambda_n X\right) \quad A_n = \frac{4\sin\lambda_n}{2\lambda_n + \sin\left(2\lambda_n\right)} \quad \lambda_n \tan\lambda_n = \mathrm{Bi}$$

Due to exponential decay, only **first few terms,** of the infinite series, will be important



$$\theta = \sum_{n=1}^{\infty} A_n \exp\left(-\lambda_n^2 \tau\right)\cos\left(\lambda_n X\right)$$

$$\lambda_n \tan\lambda_n = \mathrm{Bi} \quad A_n = \frac{4\sin\lambda_n}{2\lambda_n + \sin\left(2\lambda_n\right)}$$

Only the first term is important, unless the value of τ is very small

Example: $\mathrm{Bi} = 5,\ \mathrm{X} = 1,\ \tau = 0.2$

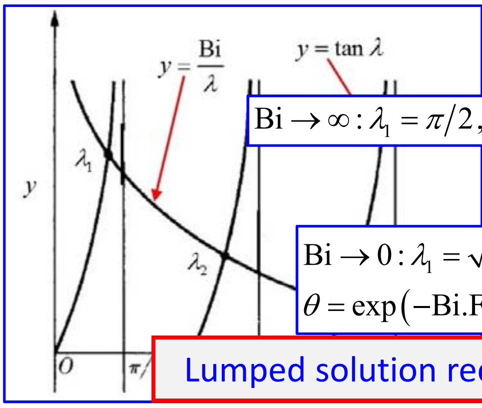
n	λ_n	A_n	θ_n
1	1.3138	1.2402	0.22321
2	4.0336	–0.3442	0.00835
3	6.9096	0.1588	0.00001
4	9.8928	–0.876	0.00000

First term approximation is applicable for $\tau > 0.2$

For $\tau > 0.2,\ \theta \approx A_1 \exp\left(-\lambda_1^2 \tau\right) \cos\left(\lambda_1 X\right)$

$$A_1 = \frac{4 \sin \lambda_1}{2\lambda_1 + \sin\left(2\lambda_1\right)}$$

λ_1 is the smallest root of $\lambda \tan \lambda = \mathrm{Bi}$



$\mathrm{Bi} \to \infty : \lambda_1 = \pi/2,\, A_1 = 4/\pi$

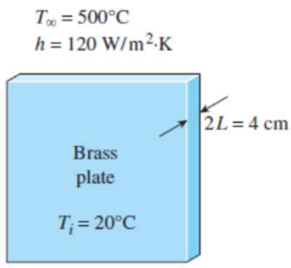
$\mathrm{Bi} \to 0 : \lambda_1 = \sqrt{\mathrm{Bi}},\, A_1 = 1$

$\theta = \exp\left(-\mathrm{Bi.Fo}\right)$

Lumped solution recovered

Bi	λ_1
0.01	0.0998
0.02	0.1410
0.04	0.1987
0.06	0.2425
0.08	0.2791
0.1	0.3111
0.2	0.4328
0.3	0.5218
0.4	0.5932
0.5	0.6533
0.6	0.7051
0.7	0.7506
0.8	0.7910
0.9	0.8274
1.0	0.8603
2.0	1.0769
3.0	1.1925
4.0	1.2646
5.0	1.3138
6.0	1.3496
7.0	1.3766
8.0	1.3978
9.0	1.4149
10.0	1.4289
20.0	1.4961
30.0	1.5202
40.0	1.5325
50.0	1.5400
100.0	1.5552
∞	1.5708

Find the surface temperature after $t = 7\mathrm{min}$



$k = 110\,\mathrm{W/m\text{-}K},\, \rho = 8530\,\mathrm{kg/m^3}$

$c = 380\,\mathrm{J/kg\text{-}K},\, \alpha = 33.9 \times 10^{-6}\,\mathrm{m^2/s}$

Heating of a large, thin brass plate in a furnace

$$\mathrm{Bi} = \frac{hL}{k} = .02$$

$$\theta = A_1 \exp\left(-\lambda_1^2 \tau\right) \cos\left(\lambda_1 X\right)$$

$$\lambda \tan \lambda = \mathrm{Bi} \Rightarrow \lambda_1 = 0.14$$

$$\theta = \frac{T - T_\infty}{T_i - T_\infty} \qquad X = x/L$$

$$\tau = \frac{\alpha t}{L^2} = 35.6$$

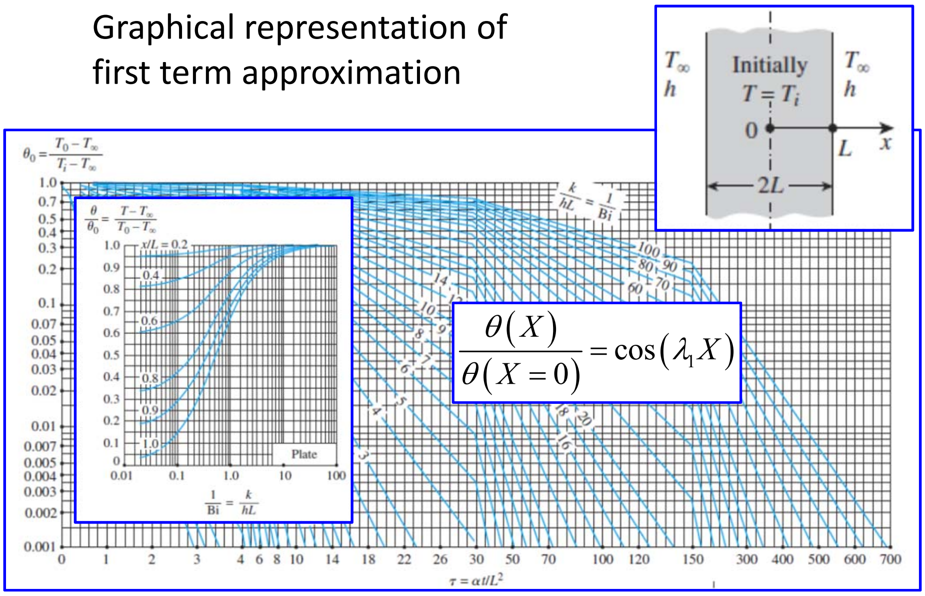
$$\theta\left(X = 1\right) = A_1 \exp\left(-\lambda_1^2 \tau\right) \cos\left(\lambda_1\right) \qquad T\left(x = L\right) = 282^\circ C$$

This problem may as well be solved by lumped approximation

Heisler Chart

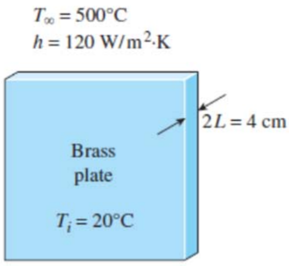
$$\theta(X=0) = A_1 \exp\left(-\lambda_1^2 \tau\right)$$

Graphical representation of first term approximation



$$\frac{\theta(X)}{\theta(X=0)} = \cos(\lambda_1 X)$$

Find the surface temperature after $t = 7\text{min}$



Heating of a large, thin brass plate in a furnace

$$k = 110\,\text{W/m-K}, \rho = 8530\,\text{kg/m}^3$$

$$c = 380\,\text{J/kg-K}, \alpha = 33.9\times10^{-6}\,\text{m}^2/\text{s}$$

$$\frac{1}{\text{Bi}} = \frac{k}{hL} = 45.8 \qquad \tau = \frac{\alpha t}{L^2} = 35.6$$

$$\frac{T_{X=0} - T_\infty}{T_i - T_\infty} = 0.46 \qquad \frac{T_{X=1} - T_\infty}{T_{X=0} - T_\infty} = 0.99$$

Combining $\quad \dfrac{T_{X=1} - T_\infty}{T_i - T_\infty} = 0.46 \times 0.99 \qquad T_{X=1} = 282^\circ\text{C}$