Fluid Mechanics and Rate Processes: Tutorial 6

P1. Consider a steady, two-dimensional, incompressible flow of a Newtonian fluid with the velocity field $u = -2xy$, $v = y^2 - x^2$, and $w = 0$. **(a)** Does this flow satisfy conservation of mass? **(b)** Find the pressure field $p(x, y)$ if the pressure at point ($x = 0$, $y = 0$) is equal to P_a.

Solution: Evaluate and check the incompressible continuity equation:

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0=-2y+2y+0\equiv 0 \quad \textbf{Yes!} \quad \textit{Ans.}\ \text{(a)}$$

(b) Find the pressure gradients from the Navier-Stokes *x*- and *y*-relations:

$$\rho\left(u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}\right)=-\frac{\partial p}{\partial x}+\mu\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right), \quad or:$$

$$\rho[-2xy(-2y)+(y^2-x^2)(-2x)]=-\frac{\partial p}{\partial x}+\mu(0+0+0), \quad or: \quad \frac{\partial p}{\partial x}=-2\rho(xy^2+x^3)$$

and, similarly for the y-momentum relation,

$$\rho\left(u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}\right)=-\frac{\partial p}{\partial y}+\mu\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 v}{\partial z^2}\right), \quad or:$$

$$\rho[-2xy(-2x)+(y^2-x^2)(2y)]=-\frac{\partial p}{\partial y}+\mu(-2+2+0), \quad or: \quad \frac{\partial p}{\partial y}=-2\rho(x^2y+y^3)$$

The two gradients $\partial p/\partial x$ and $\partial p/\partial y$ may be integrated to find $p(x, y)$:

$$p=\int\frac{\partial p}{\partial x}dx\Big|_{y=Const}=-2\rho\left(\frac{x^2y^2}{2}+\frac{x^4}{4}\right)+f(y), \quad then\ differentiate:$$

$$\frac{\partial p}{\partial y}=-2\rho(x^2y)+\frac{df}{dy}=-2\rho(x^2y+y^3), \quad whence \quad \frac{df}{dy}=-2\rho y^3, \quad or: \quad f(y)=-\frac{\rho}{2}y^4+C$$

$$Thus: \quad p=-\frac{\rho}{2}(2x^2y^2+x^4+y^4)+C=p_a \quad at\ (x,y)=(0,0), \quad or: \quad \mathbf{C=p_a}$$

Finally, the pressure field for this flow is given by

$$\mathbf{p=p_a-\frac{1}{2}\rho(2x^2y^2+x^4+y^4)} \quad \textit{Ans.}\ \text{(b)}$$

P2. A constant-thickness film of viscous liquid flows in laminar motion down a plate inclined at angle θ, as in Fig. P2. The velocity profile is

$$u = C\,y\,(2h - y) \quad \text{and} \qquad v = w = 0$$

Find the constant *C* in terms of the specific weight and viscosity and the angle θ. Find the volume flux *Q* per unit width in terms of these parameters.

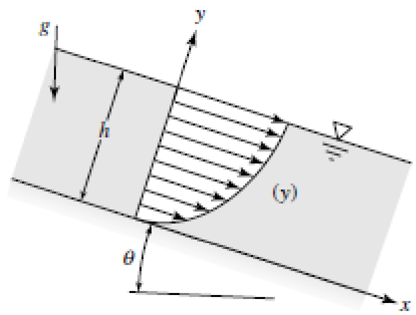

Fig. P2

Solution: There is atmospheric pressure all along the surface at y = h, hence $\partial p/\partial x = 0$. The x-momentum equation can easily be evaluated from the known velocity profile:

$$\rho\left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right) = -\frac{\partial p}{\partial x} + \rho g_x + \mu\nabla^2 u, \quad \text{or:} \quad 0 = 0 + \rho g \sin\theta + \mu(-2C)$$

$$\text{Solve for} \quad \mathbf{C = \frac{\rho g \sin\theta}{2\mu}} \quad \textit{Ans.}\text{ (a)}$$

The flow rate per unit width is found by integrating the velocity profile and using C:

$$Q = \int_0^h u\,dy = \int_0^h Cy(2h - y)\,dy = \frac{2}{3}Ch^3 = \mathbf{\frac{\rho g h^3 \sin\theta}{3\mu}} \text{ per unit width} \quad \textit{Ans.}\text{ (b)}$$

P3. For the fully developed laminar-pipe-flow solution (as already done in class), find the axisymmetric stream function $\psi(r, z)$. Use this result to determine the average velocity $V = Q/A$ in the pipe as a ratio of u_{max}.

Solution: The given velocity distribution, $v_z = u_{max}(1 - r^2/R^2)$, $v_r = 0$, satisfies continuity, so a stream function <u>does</u> exist and is found as follows:

$$v_z = u_{max}(1 - r^2/R^2) = \frac{1}{r}\frac{\partial \psi}{\partial r}, \quad \text{solve for } \psi = u_{max}\left(\frac{r^2}{2} - \frac{r^4}{4R^2}\right) + f(z), \text{ now use in}$$

$$v_r = 0 = -\frac{1}{r}\frac{\partial \psi}{\partial z} = 0 + \frac{df}{dz}, \quad \text{thus } f(z) = \text{const}, \quad \boldsymbol{\psi = u_{max}\left(\frac{r^2}{2} - \frac{r^4}{4R^2}\right)} \quad \textit{Ans.}$$

We can find the flow rate and average velocity from the text for polar coordinates:

$$Q_{1\text{-}2} = 2\pi(\psi_2 - \psi_1), \quad \text{or:} \quad Q_{0\text{-}R} = 2\pi\left[u_{max}\left(\frac{R^2}{2} - \frac{R^4}{4R^2}\right) - u_{max}(0-0)\right] = \frac{\pi}{2}R^2 u_{max}$$

$$\text{Then} \quad V_{avg} = Q/A_{pipe} = [(\pi/2)R^2 u_{max}/(\pi R^2)] = \boldsymbol{\frac{1}{2}u_{max}} \quad \textit{Ans.}$$