Fluid Mechanics and Rate Processes: Tutorial 7

P1. Consider fully-developed, incompressible flow (density ρ, viscosity μ, average velocity V) through a long, horizontal pipe, of circular cross-section (diameter D), as shown in Fig P1. Because of frictional forces between the fluid and the pipe wall, there exists a shear stress τ_w on the inside pipe wall as sketched. We assume some constant average roughness height ε along the inside wall of the pipe. Develop a non-dimensional relationship between shear stress τ_w and the other relevant parameters in the problem.

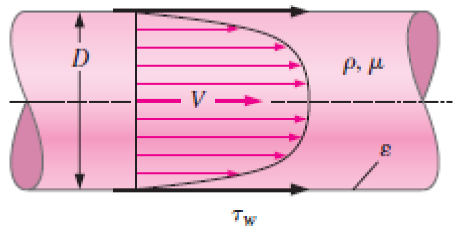
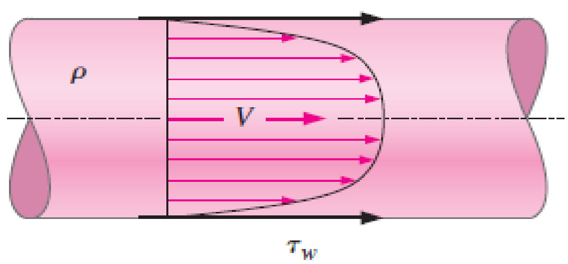


Fig. P1

Solution: We are to generate a non-dimensional relationship between shear stress and other parameters.



Darcy friction factor: $\boxed{f = \dfrac{8\tau_w}{\rho V^2}}$

Fanning friction factor: $\boxed{C_f = \dfrac{2\tau_w}{\rho V^2}}$

Assumptions **1** The flow is fully developed. **2** The fluid is incompressible. **3** No other parameters are significant in the problem.

Analysis The step-by-step method of repeating variables is employed to obtain the nondimensional parameters.

Step 1 There are six variables and constants in this problem; $n = 6$. They are listed in functional form, with the dependent variable listed as a function of the independent variables and constants:

List of relevant parameters: $\tau_w = f(V, \varepsilon, \rho, \mu, D) \qquad n = 6$

Step 2 The primary dimensions of each parameter are listed. Note that shear stress is a force per unit area, and thus has the same dimensions as pressure.

τ_w	V	ε	ρ	μ	D
$\{m^1L^{-1}t^{-2}\}$	$\{L^1t^{-1}\}$	$\{L^1\}$	$\{m^1L^{-3}\}$	$\{m^1L^{-1}t^{-1}\}$	$\{L^1\}$

Step 3 As a first guess, j is set equal to 3, the number of primary dimensions represented in the problem (m, L, and t).

Reduction: $j = 3$

If this value of j is correct, the expected number of Π's is $k = n - j = 6 - 3 = 3$.

Step 4 We choose three repeating parameters since $j = 3$. Following the guidelines of Table 7–3, we cannot pick the dependent variable τ_w. We cannot choose both ε and D since their dimensions are identical, and it would not be desirable to have μ or ε appear in all the Π's. The best choice of repeating parameters is thus V, D, and ρ.

Repeating parameters: V, D, and ρ

Step 5 The dependent Π is generated:

$$\Pi_1 = \tau_w V^{a_1} D^{b_1} \rho^{c_1} \quad \rightarrow \quad \{\Pi_1\} = \{(m^1L^{-1}t^{-2})(L^1t^{-1})^{a_1}(L^1)^{b_1}(m^1L^{-3})^{c_1}\}$$

from which $a_1 = -2$, $b_1 = 0$, and $c_1 = -1$, and thus the dependent Π is

$$\Pi_1 = \frac{\tau_w}{\rho V^2}$$

From Table 7–5, the established nondimensional parameter most similar to this Π_1 is the **Darcy friction factor,** defined with a factor of 8 in the numerator (Fig. 7–35). Thus, we may manipulate this Π according to the guidelines listed in Table 7–4 as follows:

Modified Π_1: $$\Pi_{1,\text{ modified}} = \frac{8\tau_w}{\rho V^2} = \text{Darcy friction factor} = f$$

Similarly, the two independent Π's are generated, the details of which are left for the reader:

$$\Pi_2 = \mu V^{a_2} D^{b_2} \rho^{c_2} \quad \rightarrow \quad \Pi_2 = \frac{\rho V D}{\mu} = \text{Reynolds number} = \text{Re}$$

$$\Pi_3 = \varepsilon V^{a_3} D^{b_3} \rho^{c_3} \quad \rightarrow \quad \Pi_3 = \frac{\varepsilon}{D} = \text{Roughness ratio}$$

Step 6 We write the final functional relationship as

$$f = \frac{8\tau_w}{\rho V^2} = f\left(\text{Re}, \frac{\varepsilon}{D}\right) \tag{1}$$

Discussion The result applies to both laminar and turbulent fully developed pipe flow; it turns out, however, that the second independent Π (roughness ratio ε/D) is not nearly as important in laminar pipe flow as in turbulent pipe flow. This problem presents an interesting connection between geometric similarity and dimensional analysis. Namely, it is necessary to match ε/D since it is an independent Π in the problem. From a different perspective, thinking of roughness as a geometric property, it is necessary to match ε/D to ensure *geometric similarity* between two pipes.

P2. The power P generated by a certain windmill design depends upon its diameter D, the air density ρ, the wind velocity V, the rotation rate Ω, and the number of blades n. (a) Write this relationship in dimensionless form. A model windmill, of diameter 50 cm, develops 2.7 kW at sea level when V = 40 m/s and when rotating at 4800 rev/min. (b) What power will be developed by a *similar* prototype, of diameter 5 m, in winds of 12 m/s at 2000 m standard altitude? (c) What is the appropriate rotation rate of the prototype?

Solution: (a) For the function $P = \text{fcn}(D, \rho, V, \Omega, n)$ the appropriate dimensions are $\{P\} = \{ML^2T^{-3}\}$, $\{D\} = \{L\}$, $\{\rho\} = \{ML^{-3}\}$, $\{V\} = \{L/T\}$, $\{\Omega\} = \{T^{-1}\}$, and $\{n\} = \{1\}$. Using (D, ρ, V) as repeating variables, we obtain the desired dimensionless function:

$$\frac{P}{\rho D^2 V^3} = fcn\left(\frac{\Omega D}{V}, n\right) \quad \textit{Ans. (a)}$$

(c) "Geometrically similar" means that n is the same for both windmills. For "dynamic similarity," the advance ratio ($\Omega D/V$) must be the same:

$$\left(\frac{\Omega D}{V}\right)_{model} = \frac{(4800\ r/\text{min})(0.5\ m)}{(40\ m/s)} = 1.0 = \left(\frac{\Omega D}{V}\right)_{proto} = \frac{\Omega_{proto}(5\ m)}{12\ m/s},$$

$$\text{or:} \quad \Omega_{proto} = \mathbf{144\ \frac{rev}{min}} \quad \textit{Ans. (c)}$$

(b) At 2000 m altitude, $\rho = 1.0067$ kg/m^3. At sea level, $\rho = 1.2255$ kg/m^3. Since $\Omega D/V$ and n are the same, it follows that the power coefficients equal for model and prototype:

$$\frac{P}{\rho D^2 V^3} = \frac{2700W}{(1.2255)(0.5)^2(40)^3} = \frac{P_{proto}}{(1.0067)(5)^2(12)^3},$$

$$\textit{solve} \quad \mathbf{P_{proto}} = 5990\ W \approx \mathbf{6\ kW} \quad \textit{Ans. (b)}$$

P3. A one-twelfth-scale model of an airplane is to be tested at 20°C in a pressurized wind tunnel. The prototype is to fly at 240 m/s at 10-km standard altitude. Assuming air to be an ideal gas, find the tunnel pressure (in atm) to scale both the Mach number and the Reynolds number accurately?

Solution: For air at 10000-m standard altitude (Table A-6), take $\rho = 0.4125$ kg/m^3, $\mu = 1.47\text{E}{-5}$ kg/m·s, and sound speed $a = 299$ m/s. At sea level, unless the pressure rise is vast (it isn't), T = 288°K and $\rho = 1.225$ kg/m^3, $\mu = 1.80\text{E}{-5}$ kg/m·s. Equate Ma and Re:

$$\text{Ma}_p = V_p/a_p = \frac{240}{299} = 0.803 = \text{Ma}_m = V_m/340, \quad \text{solve for } V_{model} \approx 273 \text{ m/s}$$

$$\text{Re}_p = \frac{\rho_p V_p L_p}{\mu_p} = \frac{0.4125(240)L_p}{1.47\text{E}{-5}} = \text{Re}_m = \frac{\rho_m (273)(L_p/12)}{1.80\text{E}{-5}}, \quad \text{solve for } \rho_m = 5.33\,\frac{\text{kg}}{\text{m}^3}$$

Then, for an ideal gas, $p_{model} = \rho_m R T_m = (5.33)(287)(293) \approx \mathbf{448{,}000\ Pa = 4.42\ atm}$ *Ans.*