Chapter 4

BASIC EQUATIONS IN INTEGRAL FORM FOR A CONTROL VOLUME

We begin our study of fluids in motion by developing the basic equations in integral form for application to control volumes. Why the control volume formulation (i.e., fixed region) rather than the system (i.e., fixed mass) formulation? There are two basic reasons. First, it is extremely difficult to identify and follow the same mass of fluid at all times, as must be done to apply the system formulation. Second, we are often not interested in the motion of a given mass of fluid, but rather in the effect of the fluid motion on some device or structure (such as a wing section or a pipe elbow). Thus it is more convenient to apply the basic laws to a defined volume in space, using a control volume analysis.

The basic laws for a system are familiar to you from your earlier studies in physics, mechanics, and thermodynamics. We need to obtain mathematical expressions for these laws valid for a control volume, even though the laws actually apply to matter (i.e., to a system). This will involve deriving the mathematics that converts a system expression to an equivalent one for a control volume. Instead of deriving this conversion for each of the laws, we will do it once in general form, and then apply it to each law in turn.

4-1 BASIC LAWS FOR A SYSTEM

The basic laws for a system are summarized briefly; it turns out that we will need each of the basic equations for a system to be written as a rate equation.

Conservation of Mass

Since a system is, by definition, an arbitrary collection of matter of fixed identity, a system is composed of the same quantity of matter at all times. Conservation of mass requires that the mass, M, of the system be constant. On a rate basis, we have

$$\left.\frac{dM}{dt}\right)_{\text{system}} = 0 \tag{4.1a}$$

where

$$M_{\text{system}} = \int_{M(\text{system})} dm = \int_{\forall(\text{system})} \rho \, d\forall \tag{4.1b}$$

Newton's Second Law

For a system moving relative to an inertial reference frame, Newton's second law states that the sum of all external forces acting on the system is equal to the time rate of change of linear momentum of the system,

$$\vec{F} = \left.\frac{d\vec{P}}{dt}\right)_{\text{system}} \tag{4.2a}$$

where the linear momentum of the system is given by

$$\vec{P}_{\text{system}} = \int_{M(\text{system})} \vec{V}\, dm = \int_{V\!\!\!-(\text{system})} \vec{V}\, \rho\, dV\!\!\!- \tag{4.2b}$$

The Angular-Momentum Principle

The angular-momentum principle for a system states that the rate of change of angular momentum is equal to the sum of all torques acting on the system,

$$\vec{T} = \left.\frac{d\vec{H}}{dt}\right)_{\text{system}} \tag{4.3a}$$

where the angular momentum of the system is given by

$$\vec{H}_{\text{system}} = \int_{M(\text{system})} \vec{r} \times \vec{V}\, dm = \int_{V\!\!\!-(\text{system})} \vec{r} \times \vec{V}\, \rho\, dV\!\!\!- \tag{4.3b}$$

Torque can be produced by surface and body forces, and also by shafts that cross the system boundary,

$$\vec{T} = \vec{r} \times \vec{F}_s + \int_{M(\text{system})} \vec{r} \times \vec{g}\, dm + \vec{T}_{\text{shaft}} \tag{4.3c}$$

The First Law of Thermodynamics

The first law of thermodynamics is a statement of conservation of energy for a system,

$$\delta Q - \delta W = dE$$

The equation can be written in rate form as

$$\dot{Q} - \dot{W} = \left.\frac{dE}{dt}\right)_{\text{system}} \tag{4.4a}$$

where the total energy of the system is given by

$$E_{\text{system}} = \int_{M(\text{system})} e\, dm = \int_{V\!\!\!-(\text{system})} e\, \rho\, dV\!\!\!- \tag{4.4b}$$

and

$$e = u + \frac{V^2}{2} + gz \tag{4.4c}$$

In Eq. 4.4a, $\dot{Q}$ (the rate of heat transfer) is positive when heat is added to the system from the surroundings; $\dot{W}$ (the rate of work) is positive when work is done by the system on its surroundings. In Eq. 4.4c, u is the specific internal energy, V the speed, and z the height (relative to a convenient datum) of a particle of substance having mass dm.

The Second Law of Thermodynamics

If an amount of heat, δQ, is transferred to a system at temperature T, the second law of thermodynamics states that the change in entropy, dS, of the system satisfies

$$dS \geq \frac{\delta Q}{T}$$

On a rate basis we can write

$$\left.\frac{dS}{dt}\right)_{\text{system}} \geq \frac{1}{T}\dot{Q} \tag{4.5a}$$

where the total entropy of the system is given by

$$S_{\text{system}} = \int_{M(\text{system})} s\,dm = \int_{\forall(\text{system})} s\,\rho\,d\forall \tag{4.5b}$$

4-2 RELATION OF SYSTEM DERIVATIVES TO THE CONTROL VOLUME FORMULATION

In the previous section we summarized the basic equations for a system. We found that, when written on a rate basis, each equation involved the time derivative of an extensive property of the system—the mass (Eq. 4.1a), linear momentum (Eq. 4.2a), angular momentum (Eq. 4.3a), energy (Eq. 4.4a), or entropy (Eq. 4.5a) of the system. These are the equations we wish to convert to equivalent control volume equations. Let us use the symbol N to represent any one of these system extensive properties: more informally, we can think of N as the amount of "stuff" (mass, linear momentum, angular momentum, energy, or entropy) of the system. The corresponding intensive property (extensive property per unit mass) will be designated by η. Thus

$$N_{\text{system}} = \int_{M(\text{system})} \eta\,dm = \int_{\forall(\text{system})} \eta\,\rho\,d\forall \tag{4.6}$$

Comparing Eq. 4.6 with Eqs. 4.1b, 4.2b, 4.3b, 4.4b, and 4.5b, we see that if:

$$\begin{aligned}
N &= M, &&\text{then } \eta = 1\\
N &= \vec{P}, &&\text{then } \eta = \vec{V}\\
N &= \vec{H}, &&\text{then } \eta = \vec{r} \times \vec{V}\\
N &= E, &&\text{then } \eta = e\\
N &= S, &&\text{then } \eta = s
\end{aligned}$$

How can we derive a control volume description from a system description of a fluid flow? Before specifically answering this question, we can describe the derivation in general terms. We imagine selecting an arbitrary piece of the flowing fluid at some time t_0, as shown in Fig. 4.1*a*—we could imagine dyeing this piece of fluid, say, blue. This initial shape of the fluid system is chosen as our control volume, which is fixed in space relative to coordinates xyz. After an infinitesimal time Δt the system will have moved (probably changing shape as it does so) to a new location, as shown in Fig. 4.1*b*. The laws we discussed above apply to this piece of fluid—for example, its mass will be constant (Eq. 4.1a). By examining the geometry of the system/control volume pair at $t = t_0$ and at $t = t_0 + \Delta t$, we will be able to obtain control volume formulations of the basic laws.

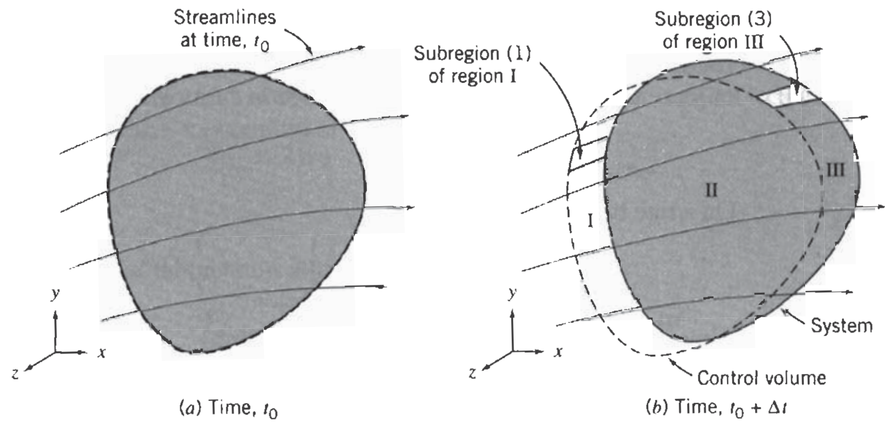


Fig. 4.1 System and control volume configuration.

Derivation

From Fig. 4.1 we see that the system, which was entirely within the control volume at time t_0, is partially out of the control volume at time $t_0 + \Delta t$. In fact, three regions can be identified. These are: regions I and II, which together make up the control volume, and region III, which, with region II, is the location of the system at time $t_0 + \Delta t$.

Recall that our objective is to relate the rate of change of any arbitrary extensive property, N, of the system to quantities associated with the control volume. From the definition of a derivative, the rate of change of N_{system} is given by

$$\left.\frac{dN}{dt}\right)_{\text{system}} \equiv \lim_{\Delta t \to 0} \frac{N_s)_{t_0+\Delta t} - N_s)_{t_0}}{\Delta t} \tag{4.7}$$

For convenience, subscript s has been used to denote the system in the definition of a derivative in Eq. 4.7.

From the geometry of Fig. 4.1,

$$N_s)_{t_0+\Delta t} = (N_{\text{II}} + N_{\text{III}})_{t_0+\Delta t} = (N_{\text{CV}} - N_{\text{I}} + N_{\text{III}})_{t_0+\Delta t}$$

and

$$N_s)_{t_0} = (N_{\text{CV}})_{t_0}$$

Substituting into the definition of the system derivative, Eq. 4.7, we obtain

$$\left.\frac{dN}{dt}\right)_s = \lim_{\Delta t \to 0} \frac{(N_{\text{CV}} - N_{\text{I}} + N_{\text{III}})_{t_0+\Delta t} - N_{\text{CV}})_{t_0}}{\Delta t}$$

Since the limit of a sum is equal to the sum of the limits, we can write

$$\left.\frac{dN}{dt}\right)_s = \underbrace{\lim_{\Delta t \to 0} \frac{N_{\text{CV}})_{t_0+\Delta t} - N_{\text{CV}})_{t_0}}{\Delta t}}_{①} + \underbrace{\lim_{\Delta t \to 0} \frac{N_{\text{III}})_{t_0+\Delta t}}{\Delta t}}_{②} - \underbrace{\lim_{\Delta t \to 0} \frac{N_{\text{I}})_{t_0+\Delta t}}{\Delta t}}_{③} \tag{4.8}$$

Our task now is to evaluate each of the three terms in Eq. 4.8.

Term ① in Eq. 4.8 simplifies to

$$\lim_{\Delta t \to 0} \frac{N_{\text{CV}})_{t_0+\Delta t} - N_{\text{CV}})_{t_0}}{\Delta t} = \frac{\partial N_{\text{CV}}}{\partial t} = \frac{\partial}{\partial t}\int_{\text{CV}} \eta \rho \, d\maltese \tag{4.9a}$$

To evaluate term ② we first develop an expression for $N_{\text{III}})_{t_0+\Delta t}$ by looking at the enlarged view of a typical subregion (subregion (3)) of region III shown in Fig. 4.2. The

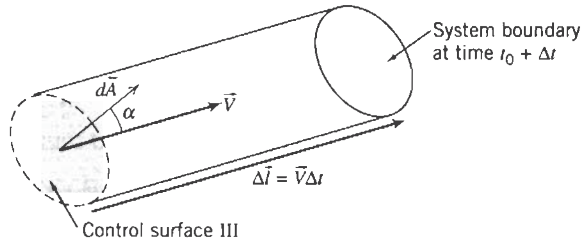


Fig. 4.2 Enlarged view of subregion (3) from Fig. 4.1.

vector area element $d\vec{A}$ of the control surface has magnitude dA, and its direction is the *outward* normal of the area element. In general, the velocity vector $\vec{V}$ will be at some angle α with respect to $d\vec{A}$.

For this subregion we have

$$dN_{\text{III}})_{t_0+\Delta t} = (\eta\rho\, d\mathcal{V})_{t_0+\Delta t}$$

We need to obtain an expression for the volume $d\mathcal{V}$ of this cylindrical element. The vector length of the cylinder is given by $\Delta\vec{l} = \vec{V}\Delta t$. The volume of a prismatic cylinder, whose area $d\vec{A}$ is at an angle α to its length $\Delta\vec{l}$, is given by $d\mathcal{V} = \Delta l\, dA\cos\alpha = \Delta\vec{l}\cdot d\vec{A} = \vec{V}\cdot d\vec{A}\Delta t$. Hence, for subregion (3) we can write

$$dN_{\text{III}})_{t_0+\Delta t} = \eta\rho\vec{V}\cdot d\vec{A}\Delta t$$

Then, for the entire region III we can integrate and for term ② in Eq. 4.8 obtain

$$\lim_{\Delta t\to 0}\frac{N_{\text{III}})_{t_0+\Delta t}}{\Delta t} = \lim_{\Delta t\to 0}\frac{\int_{\text{CS}_{\text{III}}} dN_{\text{III}})_{t_0+\Delta t}}{\Delta t} = \lim_{\Delta t\to 0}\frac{\int_{\text{CS}_{\text{III}}}\eta\rho\vec{V}\cdot d\vec{A}\Delta t}{\Delta t} = \int_{\text{CS}_{\text{III}}}\eta\rho\vec{V}\cdot d\vec{A} \tag{4.9b}$$

We can perform a similar analysis for subregion (1) of region I, and obtain for term ③ in Eq. 4.8

$$\lim_{\Delta t\to 0}\frac{N_{\text{I}})_{t_0+\Delta t}}{\Delta t} = -\int_{\text{CS}_{\text{I}}}\eta\rho\vec{V}\cdot d\vec{A} \tag{4.9c}$$

Why the minus sign in Eq. 4.9c? Term ③ in Eq. 4.8 is a measure of the amount of extensive property N (the amount of "stuff") that was in region I, and must be a positive number (e.g., we cannot have "negative matter"). However, for subregion (1), the velocity vector acts *into* the control volume, but the area normal *always* (by convention) points *outwards* (angle $\alpha > \pi/2$). Hence, the scalar product in Eq. 4.9c will be negative, requiring the additional negative sign to produce a positive result.

This concept of the sign of the scalar product is illustrated in Fig. 4.3 for (a) the general case of an inlet or exit, (b) an exit velocity parallel to the surface normal, and (c) an inlet velocity parallel to the surface normal. Cases (b) and (c) are obviously convenient special cases of (a); the value of the cosine in case (a) automatically generates the correct sign of either an inlet or an exit.

We can finally use Eqs. 4.9a, 4.9b, and 4.9c in Eq. 4.8 to obtain

$$\left.\frac{dN}{dt}\right)_{\text{system}} = \frac{\partial}{\partial t}\int_{\text{CV}}\eta\rho\, d\mathcal{V} + \int_{\text{CS}_{\text{I}}}\eta\rho\vec{V}\cdot d\vec{A} + \int_{\text{CS}_{\text{III}}}\eta\rho\vec{V}\cdot d\vec{A}$$

and the two last integrals can be combined because CS_{I} and CS_{III} constitute the entire control surface,

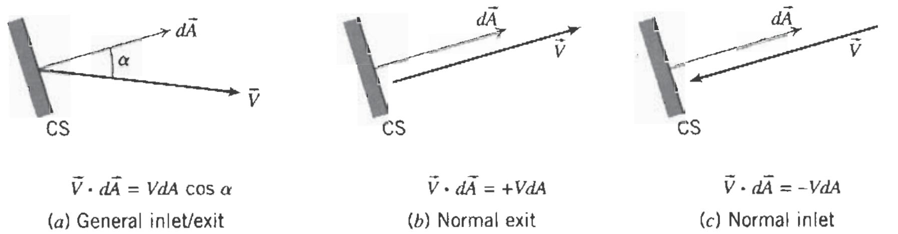


Fig. 4.3 Evaluating the scalar product.

$$\left.\frac{dN}{dt}\right)_{\text{system}} = \frac{\partial}{\partial t}\int_{\text{CV}} \eta\,\rho\,d\forall + \int_{\text{CS}} \eta\,\rho\vec{V}\cdot d\vec{A} \tag{4.10}$$

Equation 4.10 is the relation we set out to obtain. It is the fundamental relation between the rate of change of any arbitrary extensive property, N, of a system and the variations of this property associated with a control volume. Some authors refer to Eq. 4.10 as the *Reynolds Transport Theorem.*

Physical Interpretation

We have taken several pages to derive Eq. 4.10. Recall that our objective was to obtain a general relation between the rate of change of any arbitrary extensive property, N, of a system and variations of this property associated with the control volume. The main reason for deriving it was to reduce the algebra required to obtain the control volume formulations of each of the basic equations. Because we consider the equation itself to be "basic" we repeat it to emphasize its importance:

$$\left.\frac{dN}{dt}\right)_{\text{system}} = \frac{\partial}{\partial t}\int_{\text{CV}} \eta\,\rho\,d\forall + \int_{\text{CS}} \eta\,\rho\vec{V}\cdot d\vec{A} \tag{4.10}$$

It is important to recall that in deriving Eq. 4.10, the limiting process (taking the limit as $\Delta t \rightarrow 0$) ensured that the relation is valid at the instant when the system and the control volume coincide. In using Eq. 4.10 to go from the system formulations of the basic laws to the control volume formulations, we recognize that Eq. 4.10 relates the rate of change of any extensive property, N, of a system to variations of this property associated with a control volume at the instant when the system and the control volume coincide; this is true since, in the limit as $\Delta t \rightarrow 0$, the system and the control volume occupy the same volume and have the same boundaries.

Before using Eq. 4.10 to develop control volume formulations of the basic laws, let us make sure we understand each of the terms and symbols in the equation:

$\left.\dfrac{dN}{dt}\right)_{\text{system}}$ is the rate of change of any arbitrary extensive property (of the amount of "stuff", e.g., mass, energy) of the system.

$\dfrac{\partial}{\partial t}\displaystyle\int_{\text{CV}} \eta\,\rho\,d\forall$ is the time rate of change of arbitrary extensive property N within the control volume.

: η is the intensive property corresponding to N; $\eta = N$ per unit mass.

: $\rho\,d\forall$ is an element of mass contained in the control volume.

: $\int_{\text{CV}} \eta\,\rho\,d\forall$ is the total amount of extensive property N contained within the control volume.

$\int_{CS} \eta \rho \vec{V} \cdot d\vec{A}$ is the net rate of flux of extensive property N out through the control surface.

: $\rho \vec{V} \cdot d\vec{A}$ is the rate of mass flux through area element $d\vec{A}$ per unit time.

: $\eta \rho \vec{V} \cdot d\vec{A}$ is the rate of flux of extensive property N through area $d\vec{A}$.

An additional point should be made about Eq. 4.10. Velocity $\vec{V}$ is measured relative to the surface of the control volume. In developing Eq. 4.10, we considered a control volume fixed relative to coordinate system *xyz*. Since the velocity field was specified relative to the same reference coordinates, it follows that velocity $\vec{V}$ is measured relative to the control volume.

We shall further emphasize this point in deriving the control volume formulation of each of the basic laws. In each case, we begin with the familiar system formulation and use Eq. 4.10 to relate system derivatives to the time rates of change associated with a fixed control volume at the instant when the system and the control volume coincide.[1]

4-3 CONSERVATION OF MASS

The first physical principle to which we apply this conversion from a system to a control volume description is the mass conservation principle: The mass of the system remains constant,

$$\left.\frac{dM}{dt}\right)_{\text{system}} = 0 \tag{4.1a}$$

where

$$M_{\text{system}} = \int_{M(\text{system})} dm = \int_{\mathcal{V}(\text{system})} \rho \, d\mathcal{V} \tag{4.1b}$$

The system and control volume formulations are related by Eq. 4.10,

$$\left.\frac{dN}{dt}\right)_{\text{system}} = \frac{\partial}{\partial t}\int_{CV} \eta \rho \, d\mathcal{V} + \int_{CS} \eta \rho \vec{V} \cdot d\vec{A} \tag{4.10}$$

where

$$N_{\text{system}} = \int_{M(\text{system})} \eta \, dm = \int_{\mathcal{V}(\text{system})} \eta \rho \, d\mathcal{V} \tag{4.6}$$

To derive the control volume formulation of conservation of mass, we set

$$N = M \qquad \text{and} \qquad \eta = 1$$

With this substitution, we obtain

$$\left.\frac{dM}{dt}\right)_{\text{system}} = \frac{\partial}{\partial t}\int_{CV} \rho \, d\mathcal{V} + \int_{CS} \rho \vec{V} \cdot d\vec{A} \tag{4.11}$$

[1] Equation 4.10 has been derived for a control volume fixed in space relative to coordinates *xyz*. For the case of a *deformable* control volume, whose shape varies with time, Eq. 4.10 may be applied provided that the velocity, $\vec{V}$, in the flux integral is measured *relative* to the local control surface through which the flux occurs.

Comparing Eqs. 4.1a and 4.11, we arrive (after rearranging) at the control volume formulation of the conservation of mass:

$$\frac{\partial}{\partial t}\int_{\mathrm{CV}} \rho \, d\forall + \int_{\mathrm{CS}} \rho \vec{V} \cdot d\vec{A} = 0 \tag{4.12}$$

In Eq. 4.12 the first term represents the rate of change of mass within the control volume; the second term represents the net rate of mass flux out through the control surface. Equation 4.12 indicates that the rate of change of mass in the control volume plus the net outflow is zero. The mass conservation equation is also called the *continuity* equation. In common-sense terms, the rate of increase of mass in the control volume is due to the net inflow of mass:

$$\begin{array}{ccc} \text{Rate of increase} & = & \text{Net influx of} \\ \text{of mass in CV} & & \text{mass} \\ \dfrac{\partial}{\partial t}\displaystyle\int_{\mathrm{CV}} \rho \, d\forall & = & -\displaystyle\int_{\mathrm{CS}} \rho \vec{V} \cdot d\vec{A} \end{array}$$

In using Eq. 4.12, care should be taken in evaluating the scalar product $\vec{V} \cdot d\vec{A} = V dA \cos\alpha$: It could be positive (outflow, $\alpha < \pi/2$), negative (inflow, $\alpha > \pi/2$), or even zero ($\alpha = \pi/2$). Recall that Fig. 4.3 illustrates the general case as well as the convenient cases $\alpha = 0$ and $\alpha = \pi$.

Special Cases

In special cases it is possible to simplify Eq. 4.12. Consider first the case of an incompressible fluid, in which density remains constant. When ρ is constant, it is not a function of space or time. Consequently, for *incompressible fluids*, Eq. 4.12 may be written as

$$\rho \frac{\partial}{\partial t}\int_{\mathrm{CV}} d\forall + \rho \int_{\mathrm{CS}} \vec{V} \cdot d\vec{A} = 0$$

The integral of $d\forall$ over the control volume is simply the volume of the control volume. Thus, on dividing through by ρ, we write

$$\frac{\partial \forall}{\partial t} + \int_{\mathrm{CS}} \vec{V} \cdot d\vec{A} = 0$$

For a nondeformable control volume of fixed size and shape, $\forall = \text{constant}$. The conservation of mass for incompressible flow through a fixed control volume becomes

$$\int_{\mathrm{CS}} \vec{V} \cdot d\vec{A} = 0 \tag{4.13}$$

Note that we have not assumed the flow to be steady in reducing Eq. 4.12 to the form 4.13. We have only imposed the restriction of incompressible fluid. Thus Eq. 4.13 is a statement of conservation of mass for flow of an incompressible fluid that may be steady or unsteady.

The dimensions of the integrand in Eq. 4.13 are L^3/t. The integral of $\vec{V} \cdot d\vec{A}$ over a section of the control surface is commonly called the *volume flow rate* or *volume* *rate of flow.* Thus, for incompressible flow, the volume flow rate into a fixed control volume must be equal to the volume flow rate out of the control volume. The volume flow rate Q, through a section of a control surface of area A, is given by

$$Q = \int_A \vec{V} \cdot d\vec{A} \tag{4.14a}$$

The average velocity magnitude, $\bar{V}$, at a section is defined as

$$\bar{V} = \frac{Q}{A} = \frac{1}{A}\int_A \vec{V} \cdot d\vec{A} \tag{4.14b}$$

Consider now the general case of *steady, compressible flow* through a fixed control volume. Since the flow is steady, this means that at most $\rho = \rho(x, y, z)$. By definition, no fluid property varies with time in a steady flow. Consequently, the first term of Eq. 4.12 must be zero and, hence, for steady flow, the statement of conservation of mass reduces to

$$\int_{\text{CS}} \rho\vec{V} \cdot d\vec{A} = 0 \tag{4.15}$$

Thus, for steady flow, the mass flow rate into a control volume must be equal to the mass flow rate out of the control volume.

As we noted in our previous discussion of velocity fields in Section 2-2, the idealization of uniform flow at a section frequently provides an adequate flow model. Uniform flow at a section implies the velocity is constant across the entire area at a section. When the density also is constant at a section, the flux integral in Eq. 4.12 may be replaced by a product. Thus, when we have uniform flow through some area $\vec{A}$ of the control volume

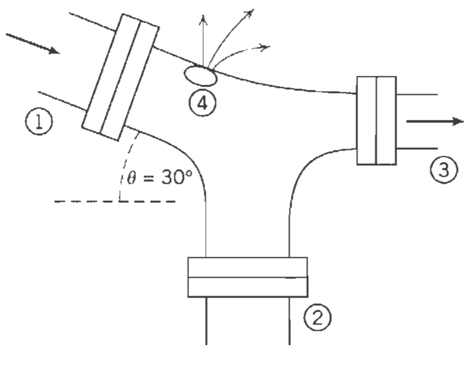
$$\int_A \rho\vec{V} \cdot d\vec{A} = \rho\vec{V} \cdot \vec{A}$$

where once again we remember that the sign of the scalar product will be positive for outflow, negative for inflow.

We will now look at three Example Problems to illustrate some features of the various forms of the conservation of mass equation for a control volume. Example Problem 4.1 involves a problem in which we have uniform flow at each section, Example Problem 4.2 involves a problem in which we do not have uniform flow at a location, and Example Problem 4.3 involves a problem in which we have unsteady flow.

EXAMPLE 4.1 Mass Flow at a Pipe Junction

Consider the steady flow in a water pipe joint shown in the diagram. The areas are: $A_1 = 0.2$ m^2, $A_2 = 0.2$ m^2, and $A_3 = 0.15$ m^2. In addition, fluid is lost out of a hole at ④, estimated at a rate of 0.1 m^3/s. The average speeds at sections ① and ③ are $V_1 = 5$ m/s and $V_3 = 12$ m/s, respectively. Find the velocity at section ②.



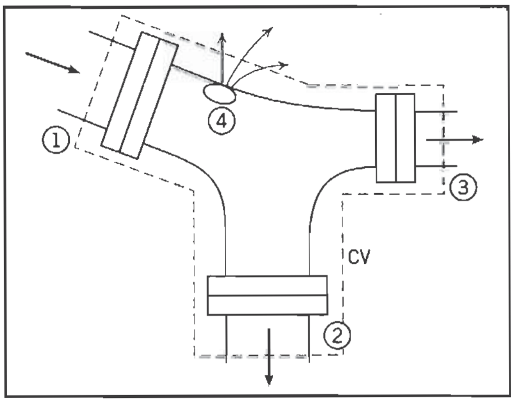
EXAMPLE PROBLEM 4.1

GIVEN: Steady flow of water through the device.

$A_1 = 0.2\ \text{m}^2$ $\quad A_2 = 0.2\ \text{m}^2$ $\quad A_3 = 0.15\ \text{m}^2$

$V_1 = 5\ \text{m/s}$ $\quad V_3 = 12\ \text{m/s}$ $\quad \rho = 999\ \text{kg/m}^3$

Volume flow rate at ④ $= 0.1\ \text{m}^3/\text{s}$



FIND: Velocity at section ②.

SOLUTION:

Choose a fixed control volume as shown. Make an assumption that the flow at section ② is outwards, and label the diagram accordingly (if this assumption is incorrect our final result will tell us).

Governing equation:

The general control volume equation is Eq. 4.12, but we can go immediately to Eq. 4.13 because of assumption (2),

$$\int_{\text{CS}} \vec{V} \cdot d\vec{A} = 0$$

Assumptions: (1) Steady flow (given).
(2) Incompressible flow.
(3) Uniform properties at each section.

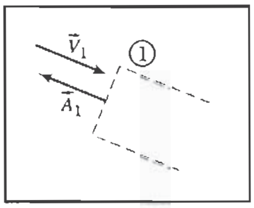
Assumption (3) (and use of Eq. 4.14a for the leak) leads to

$$\vec{V}_1 \cdot \vec{A}_1 + \vec{V}_2 \cdot \vec{A}_2 + \vec{V}_3 \cdot \vec{A}_3 + Q_4 = 0 \qquad (1)$$

where Q_4 is the flow rate out of the leak.

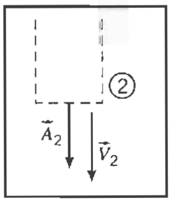
Let us examine the first three terms in Eq. 1 in light of the discussion of Fig. 4.3, and the directions of the velocity vectors:

$\vec{V}_1 \cdot \vec{A}_1 = -V_1 A_1$



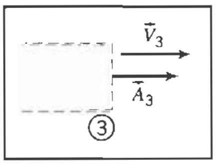
$\left\{\begin{array}{l}\text{Sign of } \vec{V}_1 \cdot \vec{A}_1 \text{ is} \\ \text{negative at surface ①}\end{array}\right\}$

$\vec{V}_2 \cdot \vec{A}_2 = +V_2 A_2$



$\left\{\begin{array}{l}\text{Sign of } \vec{V}_2 \cdot \vec{A}_2 \text{ is} \\ \text{positive at surface ②}\end{array}\right\}$

$\vec{V}_3 \cdot \vec{A}_3 = +V_3 A_3$



$\left\{\begin{array}{l}\text{Sign of } \vec{V}_3 \cdot \vec{A}_3 \text{ is} \\ \text{positive at surface ③}\end{array}\right\}$

Using these results in Eq. 1,

$$-V_1A_1 + V_2A_2 + V_3A_3 + Q_4 = 0$$

or

$$V_2 = \frac{V_1A_1 - V_3A_3 - Q_4}{A_2} = \frac{\frac{5\text{ m}}{\text{s}} \times 0.2\text{ m}^2 - \frac{12\text{ m}}{\text{s}} \times 0.15\text{ m}^2 - 0.1\frac{\text{m}^3}{\text{s}}}{0.2\text{ m}^2} = -4.5\text{ m/s} \longleftarrow V_2$$

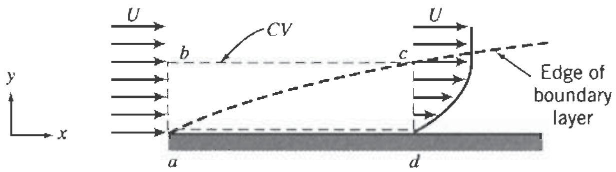
Recall that V_2 represents the magnitude of the velocity, which we assumed was outwards from the control volume. The fact that V_2 is negative means that in fact we have an *inflow* at location ②—our initial assumption was invalid.

> This problem demonstrates use of the sign convention for evaluating $\int_A \vec{V} \cdot d\vec{A}$. In particular, the area normal is *always* drawn *outwards* from the control surface.

EXAMPLE 4.2 Mass Flow Rate in Boundary Layer

The fluid in direct contact with a stationary solid boundary has zero velocity; there is no slip at the boundary. Thus the flow over a flat plate adheres to the plate surface and forms a boundary layer, as depicted below. The flow ahead of the plate is uniform with velocity, $\vec{V} = U\hat{i}$; $U = 30$ m/s. The velocity distribution within the boundary layer ($0 \le y \le \delta$) along *cd* is approximated as $u/U = 2(y/\delta) - (y/\delta)^2$.

The boundary-layer thickness at location *d* is $\delta = 5$ mm. The fluid is air with density $\rho = 1.24$ kg/m^3. Assuming the plate width perpendicular to the paper to be $w = 0.6$ m, calculate the mass flow rate across surface *bc* of control volume *abcd*.



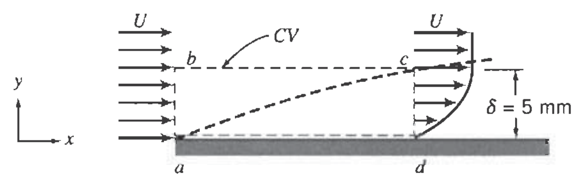
EXAMPLE PROBLEM 4.2

GIVEN: Steady, incompressible flow over a flat plate, $\rho = 1.24$ kg/m^3. Width of plate, $w = 0.6$ m. Velocity ahead of plate is uniform: $\vec{V} = U\hat{i}$, $U = 30$ m/s.

At $x = x_d$:

$$\delta = 5\text{ mm}$$

$$\frac{u}{U} = 2\left(\frac{y}{\delta}\right) - \left(\frac{y}{\delta}\right)^2$$



FIND: Mass flow rate across surface *bc*.

SOLUTION:

The fixed control volume is shown by the dashed lines.

Governing equation:

The general control volume equation is Eq. 4.12, but we can go immediately to Eq. 4.15 because of assumption (1),

$$\int_{\mathrm{CS}} \rho \vec{V} \cdot d\vec{A} = 0$$

Assumptions: (1) Steady flow (given).
(2) Incompressible flow (given).
(3) Two-dimensional flow, given properties are independent of z.

Assuming that there is no flow in the z direction, then

$$\int_{A_{ab}} \rho \vec{V} \cdot d\vec{A} + \int_{A_{bc}} \rho \vec{V} \cdot d\vec{A} + \int_{A_{cd}} \rho \vec{V} \cdot d\vec{A} + \overbrace{\int_{A_{da}} \cancel{\rho \vec{V}} \cdot d\vec{A}}^{\left(\text{no flow across } da\right)} = 0$$

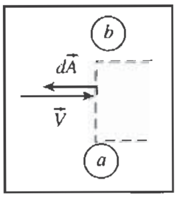
$$\therefore \dot{m}_{bc} = \int_{A_{bc}} \rho \vec{V} \cdot d\vec{A} = -\int_{A_{ab}} \rho \vec{V} \cdot d\vec{A} - \int_{A_{cd}} \rho \vec{V} \cdot d\vec{A} \qquad (1)$$

We need to evaluate the integrals on the right side of the equation.
For depth w in the z direction, we obtain

$$\int_{A_{ab}} \rho \vec{V} \cdot d\vec{A} = -\int_{A_{ab}} \rho u \, dA = -\int_{y_a}^{y_b} \rho u w \, dy$$

$$= -\int_0^{\delta} \rho u w \, dy = -\int_0^{\delta} \rho U w \, dy$$

$$\int_{A_{ab}} \rho \vec{V} \cdot d\vec{A} = -\left[\rho U w y\right]_0^{\delta} = -\rho U w \delta$$



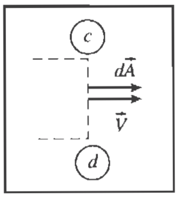
$\left\{\begin{array}{c}\vec{V} \cdot d\vec{A} \text{ is negative} \\ dA = w\,dy\end{array}\right\}$

$\{u = U \text{ over area } ab\}$

$$\int_{A_{cd}} \rho \vec{V} \cdot d\vec{A} = \int_{A_{cd}} \rho u \, dA = \int_{y_d}^{y_c} \rho u w \, dy$$

$$= \int_0^{\delta} \rho u w \, dy = \int_0^{\delta} \rho w U \left[2\left(\frac{y}{\delta}\right) - \left(\frac{y}{\delta}\right)^2\right] dy$$

$$\int_{A_{cd}} \rho \vec{V} \cdot d\vec{A} = \rho w U \left[\frac{y^2}{\delta} - \frac{y^3}{3\delta^2}\right]_0^{\delta} = \rho w U \delta \left[1 - \frac{1}{3}\right] = \frac{2\rho U w \delta}{3}$$



$\left\{\begin{array}{c}\vec{V} \cdot d\vec{A} \text{ is positive} \\ dA = w\,dy\end{array}\right\}$

Substituting into Eq. 1, we obtain

$$\therefore \dot{m}_{bc} = \rho U w \delta - \frac{2\rho U w \delta}{3} = \frac{\rho U w \delta}{3}$$

$$= \frac{1}{3} \times 1.24 \frac{\mathrm{kg}}{\mathrm{m}^3} \times 30 \frac{\mathrm{m}}{\mathrm{s}} \times 0.6 \text{ m} \times 5 \text{ mm} \times \frac{\mathrm{m}}{1000 \text{ mm}}$$

$$\dot{m}_{bc} = 0.0372 \text{ kg/s} \longleftarrow \left\{\begin{array}{l}\text{Positive sign indicates flow} \\ \text{out across surface } bc.\end{array}\right\} \qquad \dot{m}_b$$

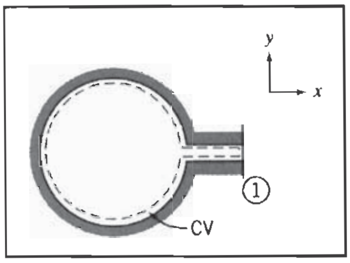
This problem demonstrates use of the conservation of mass equation when we have nonuniform flow at a section.

EXAMPLE 4.3 Density Change in Venting Tank

A tank of 0.05 m^3 volume contains air at 800 kPa (absolute) and 15°C. At $t = 0$, air begins escaping from the tank through a valve with a flow area of 65 mm^2. The air passing through the valve has a speed of 300 m/s and a density of 6 kg/m^3. Determine the instantaneous rate of change of density in the tank at $t = 0$.

EXAMPLE PROBLEM 4.3

GIVEN: Tank of volume $\mathcal{V} = 0.05\ \text{m}^3$ contains air at $p = 800$ kPa (absolute), $T = 15°\text{C}$. At $t = 0$, air escapes through a valve. Air leaves with speed $V = 300$ m/s and density $\rho = 6\ \text{kg/m}^3$ through area $A = 65\ \text{mm}^2$.



FIND: Rate of change of air density in the tank at $t = 0$.

SOLUTION:

Choose a fixed control volume as shown by the dashed line.

Governing equation: $$\frac{\partial}{\partial t}\int_{\text{CV}} \rho\, d\mathcal{V} + \int_{\text{CS}} \rho \vec{V} \cdot d\vec{A} = 0$$

Assumptions: (1) Properties in the tank are uniform, but time-dependent.
(2) Uniform flow at section ①.

Since properties are assumed uniform in the tank at any instant, we can take ρ out from within the integral of the first term,

$$\frac{\partial}{\partial t}\left[\rho_{\text{CV}}\int_{\text{CV}} d\mathcal{V}\right] + \int_{\text{CS}} \rho \vec{V} \cdot d\vec{A} = 0$$

Now, $\int_{\text{CV}} d\mathcal{V} = \mathcal{V}$, and hence

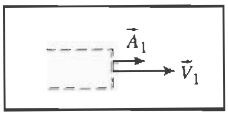
$$\frac{\partial}{\partial t}(\rho\mathcal{V})_{\text{CV}} + \int_{\text{CS}} \rho \vec{V} \cdot d\vec{A} = 0$$

The only place where mass crosses the boundary of the control volume is at surface ①. Hence

$$\int_{\text{CS}} \rho \vec{V} \cdot d\vec{A} = \int_{A_1} \rho \vec{V} \cdot d\vec{A} \qquad \text{and} \qquad \frac{\partial}{\partial t}(\rho\mathcal{V}) + \int_{A_1} \rho \vec{V} \cdot d\vec{A} = 0$$

At surface ① the sign of $\rho \vec{V} \cdot d\vec{A}$ is positive, so

$$\frac{\partial}{\partial t}(\rho\mathcal{V}) + \int_{A_1} \rho V\, dA = 0$$



Since flow is assumed uniform over surface ①, then

$$\frac{\partial}{\partial t}(\rho\mathcal{V}) + \rho_1 V_1 A_1 = 0 \qquad \text{or} \qquad \frac{\partial}{\partial t}(\rho\mathcal{V}) = -\rho_1 V_1 A_1$$

Since the volume, $\mathcal{V}$, of the tank is not a function of time,

$$\mathcal{V}\frac{\partial \rho}{\partial t} = -\rho_1 V_1 A_1$$

and

$$\frac{\partial \rho}{\partial t} = -\frac{\rho_1 V_1 A_1}{\mathcal{V}}$$

At $t = 0$,

$$\frac{\partial \rho}{\partial t} = \frac{-6\ \text{kg}}{\text{m}^3} \times \frac{300\ \text{m}}{\text{s}} \times 65\ \text{mm}^2 \times \frac{1}{0.05\ \text{m}^3} \times \frac{\text{m}^2}{10^6\ \text{mm}^2}$$

$$\frac{\partial \rho}{\partial t} = -2.34\ (\text{kg/m}^3)/\text{s} \qquad \{\text{The density is decreasing.}\} \qquad \longleftarrow \frac{\partial \rho}{\partial t}$$

This problem demonstrates use of the conservation of mass equation for unsteady flow problems.

4-4 MOMENTUM EQUATION FOR INERTIAL CONTROL VOLUME

We wish to develop a mathematical formulation of Newton's second law suitable for application to a control volume. In this section our derivation will be restricted to an inertial control volume fixed in space relative to coordinate system xyz that is not accelerating relative to stationary reference frame XYZ.

In deriving the control volume form of Newton's second law, the procedure is analogous to the procedure followed in deriving the mathematical form of the conservation of mass for a control volume. We begin with the mathematical formulation for a system and then use Eq. 4.10 to go from the system to the control volume formulation.

Recall that Newton's second law for a system moving relative to an inertial coordinate system was given by Eq. 4.2a as

$$\vec{F} = \left.\frac{d\vec{P}}{dt}\right)_{\text{system}} \tag{4.2a}$$

where the linear momentum of the system is given by

$$\vec{P}_{\text{system}} = \int_{M(\text{system})} \vec{V}\, dm = \int_{\unicode{x2C65}(\text{system})} \vec{V}\rho\, d\unicode{x2C65} \tag{4.2b}$$

and the resultant force, $\vec{F}$, includes all surface and body forces acting on the system,

$$\vec{F} = \vec{F}_S + \vec{F}_B$$

The system and control volume formulations are related using Eq. 4.10,

$$\left.\frac{dN}{dt}\right)_{\text{system}} = \frac{\partial}{\partial t}\int_{\text{CV}} \eta\rho\, d\unicode{x2C65} + \int_{\text{CS}} \eta\rho\vec{V}\cdot d\vec{A} \tag{4.10}$$

To derive the control volume formulation of Newton's second law, we set

$$N = \vec{P} \quad \text{and} \quad \eta = \vec{V}$$

From Eq. 4.10, with this substitution, we obtain

$$\left.\frac{d\vec{P}}{dt}\right)_{\text{system}} = \frac{\partial}{\partial t}\int_{\text{CV}} \vec{V}\rho\, d\unicode{x2C65} + \int_{\text{CS}} \vec{V}\rho\vec{V}\cdot d\vec{A} \tag{4.16}$$

From Eq. 4.2a

$$\left.\frac{d\vec{P}}{dt}\right)_{\text{system}} = \vec{F})_{\text{on system}} \tag{4.2a}$$

Since, in deriving Eq. 4.10, the system and the control volume coincided at t_0, then

$$\vec{F})_{\text{on system}} = \vec{F})_{\text{on control volume}}$$

In light of this, Eqs. 4.2a and 4.16 may be combined to yield the control volume formulation of Newton's second law for a nonaccelerating control volume

$$\vec{F} = \vec{F}_S + \vec{F}_B = \frac{\partial}{\partial t}\int_{\text{CV}} \vec{V}\,\rho\, dV\!\!\!\!- + \int_{\text{CS}} \vec{V}\,\rho\vec{V}\cdot d\vec{A} \tag{4.17}$$

This equation states that the sum of all forces (surface and body forces) acting on a nonaccelerating control volume is equal to the sum of the rate of change of momentum inside the control volume and the net rate of flux of momentum out through the control surface.

The derivation of the momentum equation for a control volume was straightforward. Application of this basic equation to the solution of problems will not be difficult if we exercise care in using the equation.

In using any basic equation for a control volume analysis, the first step must be to draw the boundaries of the control volume and label appropriate coordinate directions. In Eq. 4.17, the force, $\vec{F}$, represents all forces acting on the control volume. It includes both surface forces and body forces. As in the case of the free-body diagram of basic mechanics, all forces (and moments) acting on the control volume should be shown so that they can be systematically accounted for in the application of the basic equations. If we denote the body force per unit mass as $\vec{B}$, then

$$\vec{F}_B = \int \vec{B}\, dm = \int_{\text{CV}} \vec{B}\,\rho\, dV\!\!\!\!-$$

When the force of gravity is the only body force, then the body force per unit mass is $\vec{g}$. The surface force due to pressure is given by

$$\vec{F}_S = \int_A -p\, d\vec{A}$$

Note that these surface forces always act *onto* the control surface ($d\vec{A}$ points outwards, and the negative sign reverses this direction). The nature of the forces acting on the control volume undoubtedly will influence the choice of control volume boundaries.

All velocities, $\vec{V}$, in Eq. 4.17 are measured relative to the control volume. The momentum flux, $\vec{V}\rho\vec{V}\cdot d\vec{A}$, through an element of the control surface area, $d\vec{A}$, is a vector. As we previously discussed (refer to Fig. 4.3), the sign of the scalar product, $\rho\vec{V}\cdot d\vec{A}$, depends on the direction of the velocity vector, $\vec{V}$, relative to the area vector, $d\vec{A}$. The signs of the components of the velocity, $\vec{V}$, depend on the coordinate system chosen.

The momentum equation is a vector equation. As with all vector equations, it may be written as three scalar component equations. The scalar components of Eq. 4.17, relative to an *xyz* coordinate system, are

$$F_x = F_{S_x} + F_{B_x} = \frac{\partial}{\partial t}\int_{\text{CV}} u\,\rho\, dV\!\!\!\!- + \int_{\text{CS}} u\,\rho\vec{V}\cdot d\vec{A} \tag{4.18a}$$

$$F_y = F_{S_y} + F_{B_y} = \frac{\partial}{\partial t}\int_{\text{CV}} v\,\rho\, dV\!\!\!\!- + \int_{\text{CS}} v\,\rho\vec{V}\cdot d\vec{A} \tag{4.18b}$$

$$F_z = F_{S_z} + F_{B_z} = \frac{\partial}{\partial t}\int_{\text{CV}} w\,\rho\, dV\!\!\!\!- + \int_{\text{CS}} w\,\rho\vec{V}\cdot d\vec{A} \tag{4.18c}$$

Note that, as we found for the mass conservation equation (Eq. 4.12), the control surface integrals in Eq. 4.17 and Eqs. 4.18 can be replaced with simple algebraic expressions when we have uniform flow at a each inlet or exit, and that for steady flow the first term on the right side is zero.

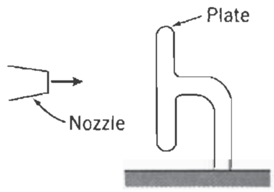
In Eq. 4.17 and Eqs. 4.18 we must be careful in evaluating the signs of the control surface integrands:

1. The sign of $\rho \vec{V} \cdot d\vec{A}$ is determined as per our discussion of Fig. 4.3—outflows are positive, inflows are negative.
2. The sign of the velocity components u, v, and w must be carefully evaluated based on the sketch of the control volume and choice of coordinate system—unknown velocity directions are selected arbitrarily (the mathematics will indicate the validity of the assumption).

We will now look at five Example Problems to illustrate some features of the various forms of the momentum equation for a control volume. Example Problem 4.4 demonstrates how intelligent choice of the control volume can simplify analysis of a problem, Example Problem 4.5 involves a problem in which we have significant body forces, Example Problem 4.6 explains how to simplify surface force evaluations by working in gage pressures, Example Problem 4.7 involves nonuniform surface forces, and Example Problem 4.8 involves a problem in which we have unsteady flow.

EXAMPLE 4.4 Choice of Control Volume for Momentum Analysis

Water from a stationary nozzle strikes a flat plate as shown. The water leaves the nozzle at 15 m/s; the nozzle area is 0.01 m^2. Assuming the water is directed normal to the plate, and flows along the plate, determine the horizontal force on the support.

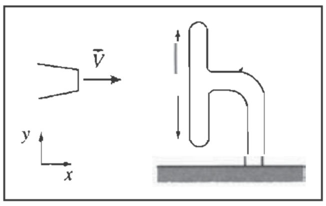


EXAMPLE PROBLEM 4.4

GIVEN: Water from a stationary nozzle is directed normal to the plate; subsequent flow is parallel to plate.

Jet velocity, $\vec{V} = 15\hat{i}$ m/s

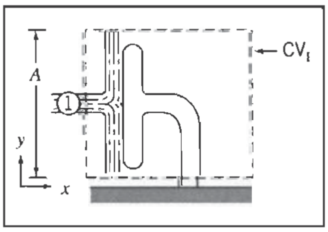
Nozzle area, $A_n = 0.01$ m^2

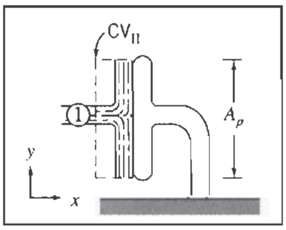


FIND: Horizontal force on the support.

SOLUTION:

We chose a coordinate system in defining the problem above. We must now choose a suitable control volume. Two possible choices are shown by the dashed lines below.





In both cases, water from the nozzle crosses the control surface through area A_1 (assumed equal to the nozzle area) and is assumed to leave the control volume tangent to the plate surface in the $+y$ or $-y$ direction. Before trying to decide which is the "best" control volume to use, let us write the governing equations.

$$\vec{F} = \vec{F}_S + \vec{F}_B = \frac{\partial}{\partial t}\int_{\mathrm{CV}} \vec{V}\,\rho\, d\forall + \int_{\mathrm{CS}} \vec{V}\,\rho\vec{V}\cdot d\vec{A} \quad \text{and} \quad \frac{\partial}{\partial t}\int_{\mathrm{CV}} \rho\, d\forall + \int_{\mathrm{CS}} \rho\vec{V}\cdot d\vec{A} = 0$$

Assumptions: (1) Steady flow.
(2) Incompressible flow.
(3) Uniform flow at each section where fluid crosses the CV boundaries.

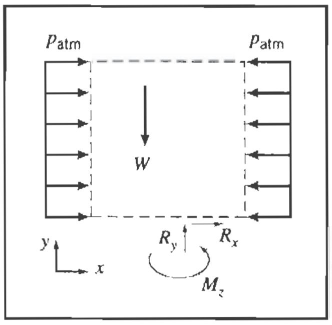
Regardless of our choice of control volume, the flow is steady and the basic equations become

$$\vec{F} = \vec{F}_S + \vec{F}_B = \int_{\mathrm{CS}} \vec{V}\,\rho\vec{V}\cdot d\vec{A} \quad \text{and} \quad \int_{\mathrm{CS}} \rho\vec{V}\cdot d\vec{A} = 0$$

Evaluating the momentum flux term will lead to the same result for both control volumes. We should choose the control volume that allows the most straightforward evaluation of the forces.

Remember in applying the momentum equation that the force, $\vec{F}$, represents all forces acting *on* the control volume.

Let us solve the problem using each of the control volumes.



$\mathrm{CV_I}$

The control volume has been selected so that the area of the left surface is equal to the area of the right surface. Denote this area by A.

The control volume cuts through the support. We denote the components of the reaction force of the support on the control volume as R_x and R_y and assume both to be positive. (The force of the control volume on the support is equal and opposite to R_x and R_y.) M_z is the reaction moment (about the z axis) from the support on the control volume.

Atmospheric pressure acts on all surfaces of the control volume. Note that *the pressure in a free jet is ambient*, i.e., in this case atmospheric. (The distributed force due to atmospheric pressure has been shown on the vertical faces only.)

The body force on the control volume is denoted as W.

Since we are looking for the horizontal force, we write the x component of the steady flow momentum equation

$$F_{S_x} + F_{B_x} = \int_{\mathrm{CS}} u\,\rho\vec{V}\cdot d\vec{A}$$

There are no body forces in the x direction, so $F_{B_x} = 0$, and

$$F_{S_x} = \int_{\mathrm{CS}} u\,\rho\vec{V}\cdot d\vec{A}$$

To evaluate F_{S_x}, we must include all surface forces acting on the control volume

$F_{S_x} =$	$p_{atm}A$	$-$	$p_{atm}A$	$+$	R_x
	force due to atmospheric pressure acts to right (positive direction) on left surface		force due to atmospheric pressure acts to left (negative direction) on right surface		force of support on control volume (assumed positive)

Consequently, $F_{S_x} = R_x$, and

$$R_x = \int_{\mathrm{CS}} u\,\rho\vec{V}\cdot d\vec{A} = \int_{A_1} u\,\rho\vec{V}\cdot d\vec{A} \qquad \left\{\begin{matrix}\text{For mass crossing top and bottom}\\ \text{surfaces, } u = 0.\end{matrix}\right\}$$

$$= \int_{A_1} u(-\rho V_1\, dA_1) \qquad \left\{\begin{matrix}\text{At (1), } \rho\vec{V}\cdot d\vec{A} = -\rho V_1\, dA_1\text{, since direction}\\ \text{of } \vec{V}_1 \text{ and } d\vec{A}_1 \text{ are } 180^\circ \text{ apart.}\end{matrix}\right\}$$

$$R_x = -u_1\rho V_1 A_1 \qquad \{\text{properties uniform over } A_1\}$$

$$R_x = -\frac{15\text{ m}}{\text{s}} \times \frac{999\text{ kg}}{\text{m}^3} \times \frac{15\text{ m}}{\text{s}} \times 0.01\text{ m}^2 \times \frac{\text{N}\cdot\text{s}^2}{\text{kg}\cdot\text{m}} \qquad \{u_1 = 15\text{ m/s}\}$$

$$R_x = -2.25\text{ kN} \qquad \{R_x \text{ acts opposite to positive direction assumed.}\}$$

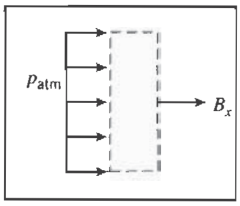
The horizontal force on the support is

$$K_x = -R_x = 2.25\text{ kN} \qquad \{\text{force on support acts to the right}\} \qquad K_x$$

CV_{II} with Horizontal Forces Shown

The control volume has been selected so the areas of the left surface and of the right surface are equal to the area of the plate. Denote this area by A_p.

The control volume is in contact with the plate over the entire plate surface. We denote the horizontal reaction force from the plate on the control volume as B_x (and assume it to be positive).



Atmospheric pressure acts on the left surface of the control volume (and on the two horizontal surfaces). The body force on this control volume has no component in the x direction.

Then the x component of the momentum equation,

$$F_{S_x} = \int_{\text{CS}} u\,\rho\vec{V}\cdot d\vec{A}$$

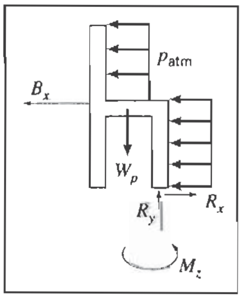
yields

$$F_{S_x} = p_{\text{atm}}A_p + B_x = \int_{A_1} u\,\rho\vec{V}\cdot d\vec{A} = \int_{A_1} u\left(-\rho V_1\,dA\right) = -2.25\text{ kN}$$

Then

$$B_x = -p_{\text{atm}}A_p - 2.25\text{ kN}$$

To determine the net force on the plate, we need a free-body diagram of the plate:



$$\Sigma F_x = 0 = -B_x - p_{\text{atm}}A_p + R_x$$
$$R_x = p_{\text{atm}}A_p + B_x$$
$$R_x = p_{\text{atm}}A_p + (-p_{\text{atm}}A_p - 2.25\text{ kN}) = -2.25\text{ kN}$$

Then the horizontal force on the support is $K_x = -R_x = 2.25$ kN.

Note that the choice of CV_{II} resulted in the need for an additional free-body diagram. In general it is advantageous to select the control volume so that the force sought acts explicitly on the control volume.

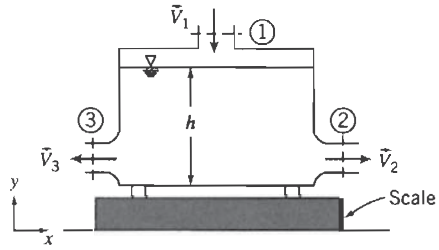
Notes:

- ✓ This problem demonstrates how thoughtful choice of the control volume can simplify use of the momentum equation.
- ✓ The analysis would have been greatly simplified if we had worked in gage pressures (see Example Problem 4.6).
- ✓ For this problem the force generated was entirely due to the plate absorbing the jet's horizontal momentum.

EXAMPLE 4.5 Tank on Scale: Body Force

A metal container 2 ft high, with an inside cross-sectional area of 1 ft^2, weighs 5 lbf when empty. The container is placed on a scale and water flows in through an opening in the top and out through the two equal-area openings in the sides, as shown in the diagram. Under steady flow conditions, the height of the water in the tank is $h = 1.9$ ft. Your boss claims that the scale will read the weight of the volume of water in the tank plus the tank weight, i.e., that we can treat this as a simple statics problem. You disagree, claiming that a fluid flow analysis is required. Who is right, and what does the scale indicate?

$$A_1 = 0.1\,\text{ft}^2$$
$$\vec{V}_1 = -10\hat{j}\ \text{ft/s}$$
$$A_2 = A_3 = 0.1\,\text{ft}^2$$

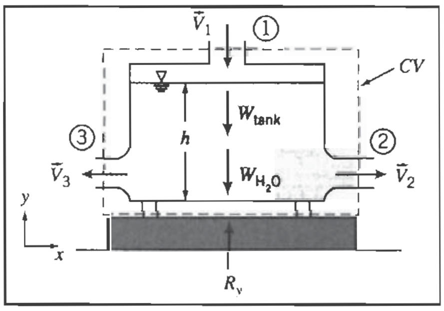


EXAMPLE PROBLEM 4.5

GIVEN:

Metal container, of height 2 ft and cross-sectional area $A = 1$ ft^2, weighs 5 lbf when empty. Container rests on scale. Under steady flow conditions water depth is $h = 1.9$ ft. Water enters vertically at section ① and leaves horizontally through sections ② and ③.

$$A_1 = 0.1\,\text{ft}^2$$
$$\vec{V}_1 = -10\hat{j}\ \text{ft/s}$$
$$A_2 = A_3 = 0.1\,\text{ft}^2$$



FIND: Scale reading.

SOLUTION:

Choose a control volume as shown; R_y is the force of the scale on the control volume (exerted on the control volume through the supports) and is assumed positive.

The weight of the tank is designated W_{tank}; the weight of the water in the tank is W_{H_2O}.

Atmospheric pressure acts uniformly on the entire control surface, and therefore has no net effect on the control volume. Because of this null effect we have not shown the pressure distribution in the diagram.

Governing equations:

The general control volume momentum and mass conservation equations are Eqs. 4.17 and 4.12, respectively,

$$\vec{F}_S + \vec{F}_B = \underbrace{\cancel{\frac{\partial}{\partial t}}\int_{CV} \vec{V}\rho\, d\Psi}_{=0(1)} + \int_{CS} \vec{V}\rho\vec{V}\cdot d\vec{A}$$

$$\underbrace{\cancel{\frac{\partial}{\partial t}}\int_{CV} \rho\, d\Psi}_{=0(1)} + \int_{CS} \rho\vec{V}\cdot d\vec{A} = 0$$

Note that for brevity we usually start with the simplest forms (based on the problem assumptions, e.g., steady flow) of the mass conservation and momentum equations. However, in this problem, for illustration purposes, we start with the most general forms of the equations.

Assumptions: (1) Steady flow (given).
(2) Incompressible flow.
(3) Uniform flow at each section where fluid crosses the CV boundaries.

We are only interested in the y component of the momentum equation

$$F_{S_y} + F_{B_y} = \int_{CS} v\,\rho\vec{V}\cdot d\vec{A} \qquad (1)$$

$$F_{S_y} = R_y \qquad \{\text{There is no net force due to atmospheric pressure.}\}$$

$$F_{B_y} = -W_{\text{tank}} - W_{H_2O} \qquad \{\text{Both body forces act in negative } y \text{ direction.}\}$$

$$W_{H_2O} = \rho g \Psi = \gamma A h$$

$$\int_{CS} v\,\rho\vec{V}\cdot d\vec{A} = \int_{A_1} v\,\rho\vec{V}\cdot d\vec{A} = \int_{A_1} v(-\rho V_1 dA_1) \qquad \left\{\begin{array}{l}\vec{V}\cdot d\vec{A} \text{ is negative at } ①\\ v = 0 \text{ at sections } ② \text{ and } ③\end{array}\right\}$$

$$= v_1(-\rho V_1 A_1) \qquad \left\{\begin{array}{l}\text{We are assuming uniform}\\ \text{properties at } ①\end{array}\right\}$$

Using these results in Eq. 1 gives

$$R_y - W_{\text{tank}} - \gamma A h = v_1(-\rho V_1 A_1)$$

Note that v_1 is the y component of the velocity, so that $v_1 = -V_1$, where we recall that $V_1 = 10$ m/s is the magnitude of velocity $\vec{V}_1$. Hence, solving for R_y,

$$R_y = W_{\text{tank}} + \gamma A h + \rho V_1^2 A_1$$

$$= 5\,\text{lbf} + 62.4\,\frac{\text{lbf}}{\text{ft}^3} \times 1\,\text{ft}^2 \times 1.9\,\text{ft} + 1.94\,\frac{\text{slug}}{\text{ft}^3} \times 100\,\frac{\text{ft}^2}{\text{s}^2} \times 0.1\,\text{ft}^2 \times \frac{\text{lbf}\cdot\text{s}^2}{\text{slug}\cdot\text{ft}}$$

$$= 5\,\text{lbf} + 118.6\,\text{lbf} + 19.4\,\text{lbf}$$

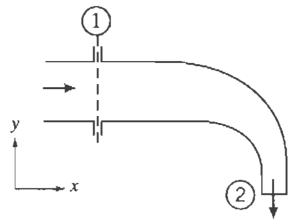
$$R_y = 143\,\text{lbf} \qquad R_y$$

Note that this is the force of the scale on the control volume; it is also the reading, on the scale. We can see that the scale reading is due to: the tank weight (5 lbf), the weight of water instantaneously in the tank (118.6 lbf), and the force involved in absorbing the downward momentum of the fluid at section ① (19.4 lbf). Hence your boss is wrong—neglecting the momentum results in an error of almost 15%.

> This problem illustrates use of the momentum equation including significant body forces.

EXAMPLE 4.6 Flow through Elbow: Use of Gage Pressures

Water flows steadily through the 90° reducing elbow shown in the diagram. At the inlet to the elbow, the absolute pressure is 220 kPa and the cross-sectional area is 0.01 m². At the outlet, the cross-sectional area is 0.0025 m² and the velocity is 16 m/s. The elbow discharges to the atmosphere. Determine the force required to hold the elbow in place.



EXAMPLE PROBLEM 4.6

GIVEN: Steady flow of water through 90° reducing elbow.

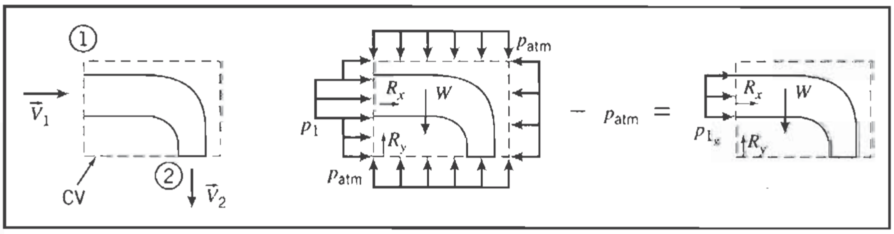
$$p_1 = 220 \text{ kPa (abs)} \qquad A_1 = 0.01 \text{ m}^2 \qquad \vec{V}_2 = -16\hat{j} \text{ m/s} \qquad A_2 = 0.0025 \text{ m}^2$$

FIND: Force required to hold elbow in place.

SOLUTION:

Choose a fixed control volume as shown. Note that we have several surface force computations: p_1 on area A_1 and p_{atm} everywhere else. The exit at section ② is to a free jet, and so at ambient (i.e., atmospheric) pressure. We can use a simplification here: If we subtract p_{atm} from the entire surface (a null effect as far as forces are concerned) we can work in gage pressures, as shown.

Note that since the elbow is anchored to the supply line, in addition to the reaction forces R_x and R_y (shown), there would also be a reaction moment (not shown).



Governing equations:

$$\vec{F} = \vec{F}_S + \vec{F}_B = \underbrace{\cancel{\frac{\partial}{\partial t}}\int_{CV} \vec{V}\rho\, dV\!\!\!\!-}_{=0(4)} + \int_{CS} \vec{V}\rho\vec{V}\cdot d\vec{A}$$

$$\underbrace{\cancel{\frac{\partial}{\partial t}}\int_{CV} \rho\, dV\!\!\!\!-}_{=0(4)} + \int_{CS} \rho\vec{V}\cdot d\vec{A} = 0$$

Assumptions:
(1) Uniform flow at each section.
(2) Atmospheric pressure, $p_{atm} = 101$ kPa (abs).
(3) Incompressible flow.
(4) Steady flow (given).
(5) Neglect weight of elbow and water in elbow.

Once again we started with the most general form of the governing equations. Writing the x component of the momentum equation results in

$$F_{S_x} = \int_{CS} u\rho\vec{V}\cdot d\vec{A} = \int_{A_1} u\rho\vec{V}\cdot d\vec{A} \qquad \{F_{B_x} = 0 \text{ and } u_2 = 0\}$$

$$p_{1_g}A_1 + R_x = \int_{A_1} u\rho\vec{V}\cdot d\vec{A}$$

so

$$R_x = -p_{1_g}A_1 + \int_{A_1} u\rho\vec{V}\cdot d\vec{A}$$
$$= -p_{1_g}A_1 + u_1(-\rho V_1 A_1)$$
$$R_x = -p_{1_g}A_1 - \rho V_1^2 A_1$$

Note that u_1 is the x component of the velocity, so that $u_1 = V_1$. To find V_1, use the mass conservation equation:

$$\int_{CS}\rho\vec{V}\cdot d\vec{A} = \int_{A_1}\rho\vec{V}\cdot d\vec{A} + \int_{A_2}\rho\vec{V}\cdot d\vec{A} = 0$$
$$\therefore\ (-\rho V_1 A_1) + (\rho V_2 A_2) = 0$$

and

$$V_1 = V_2\frac{A_2}{A_1} = \frac{16\ \text{m}}{\text{s}}\times\frac{0.0025}{0.01} = 4\ \text{m/s}$$

We can now compute R_x

$$R_x = -p_{1_g}A_1 - \rho V_1^2 A_1$$
$$= -1.19\times10^5\,\frac{\text{N}}{\text{m}^2}\times 0.01\ \text{m}^2 - 999\,\frac{\text{kg}}{\text{m}^3}\times 16\,\frac{\text{m}^2}{\text{s}^2}\times 0.01\ \text{m}^2\times\frac{\text{N}\cdot\text{s}^2}{\text{kg}\cdot\text{m}}$$
$$R_x = -1.35\ \text{kN} \qquad R_x$$

Writing the y component of the momentum equation gives

$$F_{S_y} + F_{B_y} = R_y + F_{B_y} = \int_{CS} v\rho\vec{V}\cdot d\vec{A} = \int_{A_2} v\rho\vec{V}\cdot d\vec{A} \qquad \{v_1 = 0\}$$

or

$$R_y = -F_{B_y} + \int_{A_2} v\rho\vec{V}\cdot d\vec{A}$$
$$= -F_{B_y} + v_2(\rho V_2 A_2)$$
$$R_y = -F_{B_y} - \rho V_2^2 A_2$$

Note that v_2 is the y component of the velocity, so that $v_2 = -V_2$, where V_2 is the magnitude of the exit velocity.

Substituting known values

$$R_y = -F_{B_y} - \rho V_2^2 A_2$$
$$= -F_{B_y} - 999\,\frac{\text{kg}}{\text{m}^3}\times(16)^2\,\frac{\text{m}^2}{\text{s}^2}\times 0.0025\ \text{m}^2\times\frac{\text{N}\cdot\text{s}^2}{\text{kg}\cdot\text{m}}$$
$$= -F_{B_y} - 639\ \text{N} \qquad R_y$$

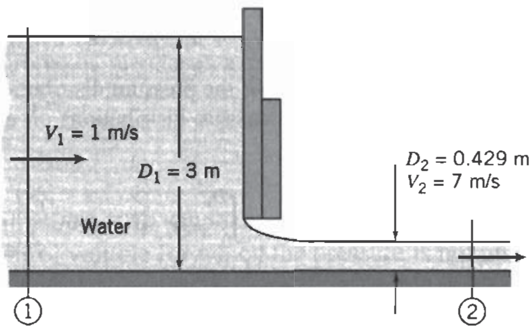
Neglecting F_{B_y} gives

$$R_y = -639\ \text{N} \qquad R_y$$

This problem illustrates how using gage pressures simplifies evaluation of the surface forces in the momentum equation.

EXAMPLE 4.7 Flow under a Sluice Gate: Hydrostatic Pressure Force

Water in an open channel is held in by a sluice gate. Compare the horizontal force of the water on the gate (a) when the gate is closed and (b) when it is open (assuming steady flow, as shown). Assume the flow at sections ① and ② is incompressible and uniform, and that (because the streamlines are straight there) the pressure distributions are hydrostatic.



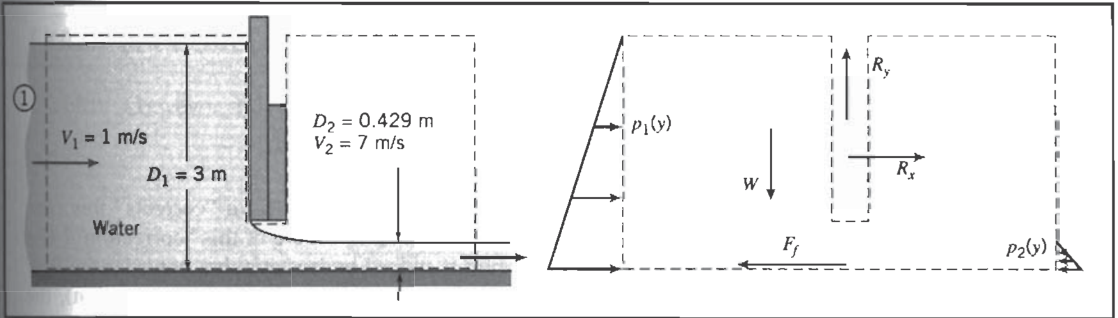
EXAMPLE PROBLEM 4.7

GIVEN: Flow under sluice gate. Width $= w$.

FIND: Horizontal force (per unit width) on the closed and open gate.

SOLUTION:

Choose a control volume as shown for the open gate. Note that it is much simpler to work in gage pressures, as we learned in Example Problem 4.6.



The forces acting on the control volume include:

- Force of gravity W.
- Friction force F_f.
- Components R_x and R_y of reaction force from gate.
- Hydrostatic pressure distribution on vertical surfaces, assumption (6).
- Pressure distribution $p_b(x)$ along bottom surface (not shown).

Apply the x component of the momentum equation.

Governing equation:

$$F_{S_x} + \underbrace{\cancel{F_{B_x}}}_{=0(2)} = \underbrace{\cancel{\frac{\partial}{\partial t}}\int_{\mathrm{CV}} u\, \rho d\forall}_{=0(3)} + \int_{\mathrm{CS}} u\, \rho \vec{V} \cdot d\vec{A}$$

Assumptions: (1) F_f negligible (neglect friction on channel bottom).
(2) $F_{B_x} = 0$.
(3) Steady flow.
(4) Incompressible flow (given).
(5) Uniform flow at each section (given).
(6) Hydrostatic pressure distributions at ① and ② (given).

Then

$$F_{S_x} = F_{R_1} + F_{R_2} + R_x = u_1(-\rho V_1 w D_1) + u_2(\rho V_2 w D_2)$$

The surface forces acting on the CV are due to the pressure distributions and the unknown force R_x. From assumption (6), we can integrate the gage pressure distributions on each side to compute the hydrostatic forces F_{R_1} *and* F_{R_2},

$$F_{R_1} = \int_0^{D_1} p_1\, dA = w\int_0^{D_1} \rho g y\, dy = \rho g w \left.\frac{y^2}{2}\right|_0^{D_1} = \frac{1}{2}\rho g w D_1^2$$

where y is measured downward from the free surface of location ①, and

$$F_{R_2} = \int_0^{D_2} p_2\, dA = w\int_0^{D_2} \rho g y\, dy = \rho g w \left.\frac{y^2}{2}\right|_0^{D_2} = \frac{1}{2}\rho g w D_2^2$$

where y is measured downward from the free surface of location ②. (Note that we could have used the hydrostatic force equation, Eq. 3.10b, directly to obtain these forces.)

Evaluating F_{S_x} gives

$$F_{S_x} = R_x + \frac{\rho g w}{2}(D_1^2 - D_2^2)$$

Substituting into the momentum equation, with $u_1 = V_1$ and $u_2 = V_2$, gives

$$R_x + \frac{\rho g w}{2}(D_1^2 - D_2^2) = -\rho V_1^2 w D_1 + \rho V_2^2 w D_2$$

or

$$R_x = \rho w(V_2^2 D_2 - V_1^2 D_1) - \frac{\rho g w}{2}(D_1^2 - D_2^2)$$

The second term on the right is the net hydrostatic force on the gate; the first term "corrects" this (and leads to a smaller net force) for the case when the gate is open. What is the nature of this "correction"? The pressure in the fluid far away from the gate in either direction is indeed hydrostatic, but consider the flow close to the gate: Because we have significant velocity variations here (in magnitude and direction), the pressure distributions deviate significantly from hydrostatic—for example, as the fluid accelerates under the gate there will be a significant pressure drop on the lower left side of the gate. Deriving this pressure field would be a difficult task, but by careful choice of our CV we have avoided having to do so!

We can now compute the horizontal force per unit width,

$$\frac{R_x}{w} = \rho\left(V_2^2 D_2 - V_1^2 D_1\right) - \frac{\rho g}{2}\left(D_1^2 - D_2^2\right)$$

$$= 999\,\frac{\text{kg}}{\text{m}^3} \times \left[(7)^2(0.429) - (1)^2(3)\right]\frac{\text{m}^2}{\text{s}^2}\,\text{m} \times \frac{\text{N}\cdot\text{s}^2}{\text{kg}\cdot\text{m}}$$

$$-\frac{1}{2} \times 999\,\frac{\text{kg}}{\text{m}^3} \times 9.81\,\frac{\text{m}}{\text{s}^2} \times \left[(3)^2 - (0.429)^2\right]\text{m}^2 \times \frac{\text{N}\cdot\text{s}^2}{\text{kg}\cdot\text{m}}$$

$$\frac{R_x}{w} = 18.0\ \text{kN/m} - 43.2\ \text{kN/m}$$

$$\frac{R_x}{w} = -25.2 \text{ kN/m}$$

R_x is the external force acting on the control volume, applied to the CV by the gate. Therefore, the force of the water on the gate is K_x, where $K_x = -R_x$. Thus,

$$\frac{K_x}{w} = -\frac{R_x}{w} = 25.2 \text{ kN/m} \longleftarrow \frac{K_x}{w}$$

This force can be compared to the force on the closed gate of 43.2 kN (obtained from the second term on the right in the equation above, evaluated with D_2 set to zero because for the closed gate there is no fluid on the right of the gate)—the force on the open gate is significantly less as the water accelerates out under the gate.

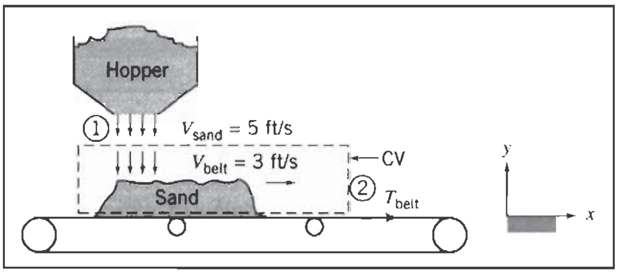
> This problem illustrates the application of the momentum equation to a control volume for which the pressure is not uniform on the control surface.

EXAMPLE 4.8 Conveyor Belt Filling: Rate of Change of Momentum in Control Volume

A horizontal conveyor belt moving at 3 ft/s receives sand from a hopper. The sand falls vertically from the hopper to the belt at a speed of 5 ft/s and a flow rate of 500 lbm/s (the density of sand is approximately 2700 lbm/cubic yard). The conveyor belt is initially empty but begins to fill with sand. If friction in the drive system and rollers is negligible, find the tension required to pull the belt while the conveyor is filling.

EXAMPLE PROBLEM 4.8

GIVEN: Conveyor and hopper shown in sketch.



FIND: T_{belt} at the instant shown.

SOLUTION:

Use the control volume and coordinates shown. Apply the x component of the momentum equation.

Governing equations:

$$F_{S_x} + \underbrace{\cancel{F_{B_x}}}_{=0(2)} = \frac{\partial}{\partial t}\int_{\text{CV}} u\,\rho\, dV + \int_{\text{CS}} u\,\rho \vec{V}\cdot d\vec{A} \qquad \frac{\partial}{\partial t}\int_{\text{CV}} \rho\, dV + \int_{\text{CS}} \rho \vec{V}\cdot d\vec{A} = 0$$

Assumptions: (1) $F_{S_x} = T_{\text{belt}} = T$.
(2) $F_{B_x} = 0$.
(3) Uniform flow at section ①.
(4) All sand on belt moves with $V_{\text{belt}} = V_b$.

Then

$$T = \frac{\partial}{\partial t}\int_{\text{CV}} u\,\rho\, d\mathcal{V} + u_1(-\rho V_1 A_1) + u_2(\rho V_2 A_2)$$

Since $u_1 = 0$, and there is no flow at section ②,

$$T = \frac{\partial}{\partial t}\int_{\text{CV}} u\,\rho\, d\mathcal{V}$$

From assumption (4), inside the CV, $u = V_b =$ constant, and hence

$$T = V_b \frac{\partial}{\partial t}\int_{\text{CV}} \rho\, d\mathcal{V} = V_b \frac{\partial M_s}{\partial t}$$

where M_s is the mass of sand on the belt (inside the control volume). This result is perhaps not surprising—the tension in the belt is the force required to increase the momentum inside the CV (which is increasing because even though the velocity of the mass in the CV is constant, the mass is not). From the continuity equation,

$$\frac{\partial}{\partial t}\int_{\text{CV}} \rho\, d\mathcal{V} = \frac{\partial}{\partial t} M_s = -\int_{\text{CS}} \rho \vec{V} \cdot d\vec{A} = \dot{m}_s = 500 \text{ lbm/s}$$

Then

$$T = V_b \dot{m}_s = \frac{3 \text{ ft}}{\text{s}} \times \frac{500 \text{ lbm}}{\text{s}} \times \frac{\text{slug}}{32.2 \text{ lbm}} \times \frac{\text{lbf} \cdot \text{s}^2}{\text{slug} \cdot \text{ft}}$$

$$T = 46.6 \text{ lbf} \qquad T$$

This problem illustrates application of the momentum equation to a control volume in which the momentum is changing.

*Differential Control Volume Analysis

We have considered a number of examples in which conservation of mass and the momentum equation have been applied to finite control volumes. However, the control volume chosen for analysis need not be finite in size.

Application of the basic equations to a differential control volume leads to differential equations describing the relationships among properties in the flow field. In some cases, the differential equations can be solved to give detailed information about property variations in the flow field. For the case of steady, incompressible, frictionless flow along a streamline, integration of one such differential equation leads to a useful (and famous) relationship among speed, pressure, and elevation in a flow field. This case is presented to illustrate the use of differential control volumes.

*This section may be omitted without loss of continuity in the text material.

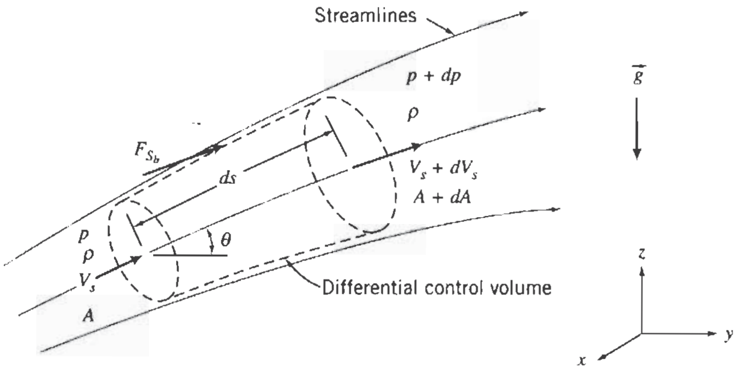


Fig. 4.4 Differential control volume for momentum analysis of flow through a stream tube.

Let us apply the continuity and momentum equations to a steady incompressible flow without friction, as shown in Fig. 4.4. The control volume chosen is fixed in space and bounded by flow streamlines, and is thus an element of a stream tube. The length of the control volume is ds.

Because the control volume is bounded by streamlines, flow across the bounding surfaces occurs only at the end sections. These are located at coordinates s and $s + ds$, measured along the central streamline.

Properties at the inlet section are assigned arbitrary symbolic values. Properties at the outlet section are assumed to increase by differential amounts. Thus at $s + ds$, the flow speed is assumed to be $V_s + dV_s$, and so on. The differential changes, dp, dV_s, and dA, all are assumed to be positive in setting up the problem. (As in a free-body analysis in statics or dynamics, the actual algebraic sign of each differential change will be determined from the results of the analysis.)

Now let us apply the continuity equation and the s component of the momentum equation to the control volume of Fig. 4.4.

a. Continuity Equation

Basic equation:

$$\underbrace{\cancel{\frac{\partial}{\partial t}}\int_{\mathrm{CV}} \rho \, d\forall}_{=0(1)} + \int_{\mathrm{CS}} \rho \vec{V} \cdot d\vec{A} = 0 \tag{4.12}$$

Assumptions:
(1) Steady flow.
(2) No flow across bounding streamlines.
(3) Incompressible flow, ρ = constant.

Then

$$(-\rho V_s A) + \{\rho(V_s + dV_s)(A + dA)\} = 0$$

so

$$\rho(V_s + dV_s)(A + dA) = \rho V_s A \tag{4.19a}$$

On expanding the left side and simplifying, we obtain

$$V_s \, dA + A \, dV_s + dA \, dV_s = 0$$

But $dA\, dV_s$ is a product of differentials, which may be neglected compared with $V_s\, dA$ or $A\, dV_s$. Thus

$$V_s\, dA + A\, dV_s = 0 \tag{4.19b}$$

b. Streamwise Component of the Momentum Equation

Basic equation:
$$F_{S_s} + F_{B_s} = \overbrace{\cancel{\frac{\partial}{\partial t}\int_{\text{CV}} u_s\, \rho\, d\forall}}^{=0(1)} + \int_{\text{CS}} u_s\, \rho \vec{V} \cdot d\vec{A} \tag{4.20}$$

Assumption: (4) No friction, so F_{S_b} is due to pressure forces only.

The surface force (due only to pressure) will have three terms:

$$F_{S_s} = pA - (p + dp)(A + dA) + \left(p + \frac{dp}{2}\right) dA \tag{4.21a}$$

The first and second terms in Eq. 4.21a are the pressure forces on the end faces of the control surface. The third term is F_{S_b}, the pressure force acting in the s direction on the bounding stream surface of the control volume. Its magnitude is the product of the average pressure acting on the stream surface, $p + \frac{1}{2} dp$, times the area component of the stream surface in the s direction, dA. Equation 4.21a simplifies to

$$F_{S_s} = -A\, dp - \frac{1}{2}\, dp\, dA \tag{4.21b}$$

The body force component in the s direction is

$$F_{B_s} = \rho g_s\, d\forall = \rho(-g \sin\theta)\left(A + \frac{dA}{2}\right) ds$$

But $\sin\theta\, ds = dz$, so that

$$F_{B_s} = -\rho g\left(A + \frac{dA}{2}\right) dz \tag{4.21c}$$

The momentum flux will be

$$\int_{\text{CS}} u_s\, \rho \vec{V} \cdot d\vec{A} = V_s(-\rho V_s A) + (V_s + dV_s)\{\rho(V_s + dV_s)(A + dA)\}$$

since there is no mass flux across the bounding stream surfaces. The mass flux factors in parentheses and braces are equal from continuity, Eq. 4.19a, so

$$\int_{\text{CS}} u_s\, \rho \vec{V} \cdot d\vec{A} = V_s(-\rho V_s A) + (V_s + dV_s)(\rho V_s A) = \rho V_s A\, dV_s \tag{4.22}$$

Substituting Eqs. 4.21b, 4.21c, and 4.22 into Eq. 4.20 (the momentum equation) gives

$$-A\, dp - \tfrac{1}{2}\, dp\, dA - \rho g A\, dz - \tfrac{1}{2}\, \rho g\, dA\, dz = \rho V_s A\, dV_s$$

Dividing by ρA and noting that products of differentials are negligible compared with the remaining terms, we obtain

$$-\frac{dp}{\rho} - g\,dz = V_s\,dV_s = d\left(\frac{V_s^2}{2}\right)$$

or

$$\frac{dp}{\rho} + d\left(\frac{V_s^2}{2}\right) + g\,dz = 0 \tag{4.23}$$

For incompressible flow, this equation may be integrated to obtain

$$\frac{p}{\rho} + \frac{V_s^2}{2} + gz = \text{constant}$$

or, dropping subscript s,

$$\frac{p}{\rho} + \frac{V^2}{2} + gz = \text{constant} \tag{4.24}$$

This equation is subject to the restrictions:

1. Steady flow.
2. No friction.
3. Flow along a streamline.
4. Incompressible flow.

By applying the momentum equation to an infinitesimal stream tube control volume, for steady incompressible flow without friction, we have derived a relation among pressure, speed, and elevation. This relationship is very powerful and useful. It also makes a lot of sense. Imagine, for example, a horizontal frictionless flow. The only horizontal force a fluid particle in this flow can experience is that due to a net pressure force. Hence, the only way such a particle could accelerate (i.e., increase its velocity) is by moving from a higher to a lower pressure region; an increase in velocity correlates with a decrease in pressure (and vice versa). This trend is what is indicated in Eq. 4.24: For z = constant, if V increases p must decrease (and vice versa) in order that the left side of the equation remains constant.

Equation 4.24 has many practical applications. For example, it could have been used to evaluate the pressure at the inlet of the reducing elbow analyzed in Example Problem 4.6 or to determine the velocity of water leaving the sluice gate of Example Problem 4.7. In both of these flow situations the restrictions required to derive Eq. 4.24 are reasonable idealizations of the actual flow behavior. The restrictions must be emphasized heavily because they do not always form a realistic model for flow behavior; consequently, they must be justified carefully each time Eq. 4.24 is applied.

Equation 4.24 is a form of the *Bernoulli equation.* It will be derived again in detail in Chapter 6 because it is such a useful tool for flow analysis and because an alternative derivation will give added insight into the need for care in applying the equation.

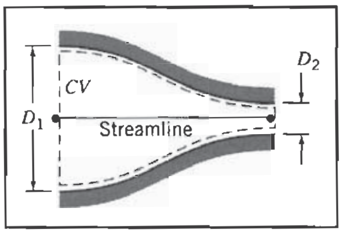
EXAMPLE 4.9 Nozzle Flow: Application of Bernoulli Equation

Water flows steadily through a horizontal nozzle, discharging to the atmosphere. At the nozzle inlet the diameter is D_1; at the nozzle outlet the diameter is D_2. Derive an expression for the minimum gage pressure required at the nozzle inlet to produce a given volume flow rate, Q. Evaluate the inlet gage pressure if $D_1 = 3.0$ in., $D_2 = 1.0$ in., and the desired flow rate is 0.7 ft³/s.

EXAMPLE PROBLEM 4.9

GIVEN: Steady flow of water through a horizontal nozzle, discharging to the atmosphere.

$D_1 = 3.0$ in. $\quad D_2 = 1.0$ in. $\quad p_2 = p_{atm}$



FIND: (a) p_{1g} as a function of volume flow rate, Q.
(b) p_{1g} for $Q = 0.7$ ft³/s.

SOLUTION:

Governing equations:

$$\frac{p_1}{\rho} + \frac{V_1^2}{2} + gz_1 = \frac{p_2}{\rho} + \frac{V_2^2}{2} + gz_2$$

$$= 0(1)$$
$$\cancel{\frac{\partial}{\partial t}} \int_{CV} \rho \, d\forall + \int_{CS} \vec{V} \cdot d\vec{A} = 0$$

Assumptions:
(1) Steady flow (given).
(2) Incompressible flow.
(3) Frictionless flow.
(4) Flow along a streamline.
(5) $z_1 = z_2$.
(6) Uniform flow at sections ① and ②.

Apply the Bernoulli equation along a streamline between points ① and ② to evaluate p_1. Then

$$p_{1g} = p_1 - p_{atm} = p_1 - p_2 = \frac{\rho}{2}(V_2^2 - V_1^2) = \frac{\rho}{2}V_1^2\left[\left(\frac{V_2}{V_1}\right)^2 - 1\right]$$

Apply the continuity equation

$$(-\rho V_1 A_1) + (\rho V_2 A_2) = 0 \qquad \text{or} \qquad V_1 A_1 = V_2 A_2 = Q$$

so that

$$\frac{V_2}{V_1} = \frac{A_1}{A_2} \qquad \text{and} \qquad V_1 = \frac{Q}{A_1}$$

Then

$$p_{1g} = \frac{\rho Q^2}{2A_1^2}\left[\left(\frac{A_1}{A_2}\right)^2 - 1\right]$$

Since $A = \pi D^2/4$, then

$$p_{1g} = \frac{8\rho Q^2}{\pi^2 D_1^4}\left[\left(\frac{D_1}{D_2}\right)^4 - 1\right] \longleftarrow p_{1g}$$

(Note that for a given nozzle the pressure required is proportional to the square of the flow rate—not surpising since we have used Eq. 4.24, which shows that $p \sim V^2 \sim Q^2$.) With $D_1 = 3.0$ in., $D_2 = 1.0$ in., and $\rho = 1.94$ slug/ft³,

$$p_{1g} = \frac{8}{\pi^2} \times \frac{1.94\ \text{slug}}{\text{ft}^3} \times \frac{1}{(3)^4\ \text{in.}^4} \times Q^2\left[(3.0)^4 - 1\right]\frac{\text{lbf}\cdot\text{s}^2}{\text{slug}\cdot\text{ft}} \times \frac{144\ \text{in.}^2}{\text{ft}^2}$$

$$p_{1g} = 224Q^2\ \frac{\text{lbf}\cdot\text{s}^2}{\text{in.}^2\cdot\text{ft}^6}$$

With $Q = 0.7\ \text{ft}^3/\text{s}$, then $\quad p_{1g} = 110\ \text{lbf/in.}^2 \longleftarrow$ p_{1g}

This problem illustrates application of the Bernoulli equation to a flow where the restrictions of steady, incompressible, frictionless flow along a streamline are reasonable.

Control Volume Moving with Constant Velocity

In the preceding problems, which illustrate applications of the momentum equation to inertial control volumes, we have considered only stationary control volumes. Suppose we have a control volume moving at constant speed. We can set up two coordinate systems: *XYZ*, our original stationary (and therefore inertial) coordinates, and *xyz*, coordinates attached to the control volume (also inertial because the control volume is not accelerating with respect to *XYZ*).

Equation 4.10, which expresses system derivatives in terms of control volume variables, is valid for any motion of coordinate system *xyz* (fixed to the control volume), provided that all velocities are measured *relative* to the control volume. To emphasize this point, we rewrite Eq. 4.10 as

$$\left.\frac{dN}{dt}\right)_{\text{system}} = \frac{\partial}{\partial t}\int_{\text{CV}} \eta\,\rho\, d\forall + \int_{\text{CS}} \eta\,\rho\,\vec{V}_{xyz}\cdot d\vec{A} \tag{4.25}$$

Since all velocities must be measured relative to the control volume, in using this equation to obtain the momentum equation for an inertial control volume from the system formulation, we must set

$$N = \vec{P}_{xyz} \qquad \text{and} \qquad \eta = \vec{V}_{xyz}$$

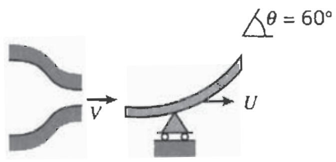
The control volume equation is then written as

$$\vec{F} = \vec{F}_S + \vec{F}_B = \frac{\partial}{\partial t}\int_{\text{CV}} \vec{V}_{xyz}\,\rho\, d\forall + \int_{\text{CS}} \vec{V}_{xyz}\,\rho\vec{V}_{xyz}\cdot d\vec{A} \tag{4.26}$$

Equation 4.26 is the formulation of Newton's second law applied to any inertial control volume (stationary or moving with a constant velocity). It is identical to Eq. 4.17 except that we have included subscript *xyz* to emphasize that velocities must be measured relative to the control volume. (It is helpful to imagine that the velocities are those that would be seen by an observer moving with the control volume.) Example Problem 4.10 illustrates the use of Eq. 4.26 for a control volume moving at constant velocity.

EXAMPLE 4.10 Vane Moving with Constant Velocity

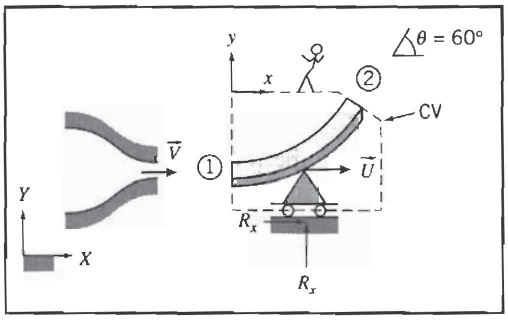
The sketch shows a vane with a turning angle of 60°. The vane moves at constant speed, $U = 10$ m/s, and receives a jet of water that leaves a stationary nozzle with speed $V = 30$ m/s. The nozzle has an exit area of 0.003 m^2. Determine the force components that act on the vane.



EXAMPLE PROBLEM 4.10

GIVEN: Vane, with turning angle $\theta = 60°$, moves with constant velocity, $\vec{U} = 10\hat{i}$ m/s. Water from a constant area nozzle, $A = 0.003$ m^2, with velocity $\vec{V} = 30\hat{i}$ m/s, flows over the vane as shown.

FIND: Force components acting on the vane.



SOLUTION:

Select a control volume moving with the vane at constant velocity, $\vec{U}$, as shown by the dashed lines. R_x and R_y are the components of force required to maintain the velocity of the control volume at $10\hat{i}$ m/s.

The control volume is inertial, since it is not accelerating (U = constant). Remember that all velocities must be measured relative to the control volume in applying the basic equations.

Governing equations:

$$\vec{F}_S + \vec{F}_B = \frac{\partial}{\partial t}\int_{CV} \vec{V}_{xyz}\,\rho\, d\mathcal{V} + \int_{CS} \vec{V}_{xyz}\,\rho \vec{V}_{xyz}\cdot d\vec{A}$$

$$\frac{\partial}{\partial t}\int_{CV} \rho\, d\mathcal{V} + \int_{CS} \rho \vec{V}_{xyz}\cdot d\vec{A} = 0$$

Assumptions:
(1) Flow is steady relative to the vane.
(2) Magnitude of relative velocity along the vane is constant: $|\vec{V}_1| = |\vec{V}_2| = V - U$.
(3) Properties are uniform at sections ① and ②.
(4) $F_{B_x} = 0$.
(5) Incompressible flow.

The x component of the momentum equation is

$$F_{S_x} + \underbrace{\cancel{F_{B_x}}}_{=0(4)} = \underbrace{\cancel{\frac{\partial}{\partial t}}\int_{CV} u_{xyz}\,\rho\, d\mathcal{V}}_{=0(1)} + \int_{CS} u_{xyz}\,\rho \vec{V}_{xyz}\cdot d\vec{A}$$

There is no net pressure force, since p_{atm} acts on all sides of the CV. Thus

$$R_x = \int_{A_1} u(-\rho V dA) + \int_{A_2} u(\rho V dA) = +u_1(-\rho V_1 A_1) + u_2(\rho V_2 A_2)$$

(All velocities are measured relative to xyz.) From the continuity equation

$$\int_{A_1} (-\rho V dA) + \int_{A_2} (\rho V dA) = (-\rho V_1 A_1) + (\rho V_2 A_2) = 0$$

or

$$\rho V_1 A_1 = \rho V_2 A_2$$

Therefore,

$$R_x = (u_2 - u_1)(\rho V_1 A_1)$$

All velocities must be measured relative to the CV, so we note that

$$V_1 = V - U \qquad V_2 = V - U$$
$$u_1 = V - U \qquad u_2 = (V - U)\cos\theta$$

Substituting yields

$$R_x = [(V-U)\cos\theta - (V-U)](\rho(V-U)A_1) = (V-U)(\cos\theta - 1)\{\rho(V-U)A_1\}$$
$$= (30-10)\,\frac{\text{m}}{\text{s}} \times (0.50-1) \times \left(999\,\frac{\text{kg}}{\text{m}^3}\,(30-10)\,\frac{\text{m}}{\text{s}} \times 0.003\,\text{m}^2\right) \times \frac{\text{N}\cdot\text{s}^2}{\text{kg}\cdot\text{m}}$$

$$R_x = -599\ \text{N}\ \{\text{to the left}\}$$

Writing the y component of the momentum equation, we obtain

$$F_{S_y} + F_{B_y} = \underbrace{\cancel{\frac{\partial}{\partial t}}\int_{\text{CV}} v_{xyz}\,\rho\,d\forall}_{=0(1)} + \int_{\text{CS}} v_{xyz}\,\rho\vec{V}_{xyz}\cdot d\vec{A}$$

Denoting the mass of the CV as M gives

$$R_y - Mg = \int_{\text{CS}} v\,\rho\vec{V}\cdot d\vec{A} = \int_{A_2} v\,\rho\vec{V}\cdot d\vec{A} \qquad \{v_1 = 0\} \qquad \left\{\begin{array}{l}\text{All velocities are}\\ \text{measured relative to}\\ xyz.\end{array}\right\}$$

$$= \int_{A_2} v\,(\rho V dA) = v_2(\rho V_2 A_2) = v_2(\rho V_1 A_1) \qquad \{\text{Recall } \rho V_2 A_2 = \rho V_1 A_1.\}$$

$$= (V-U)\sin\theta\,\{\rho(V-U)A_1\}$$

$$= (30-10)\,\frac{\text{m}}{\text{s}} \times (0.866) \times \left((999)\,\frac{\text{kg}}{\text{m}^3}\,(30-10)\,\frac{\text{m}}{\text{s}} \times 0.003\,\text{m}^2\right) \times \frac{\text{N}\cdot\text{s}^2}{\text{kg}\cdot\text{m}}$$

$$R_y - Mg = 1.04\ \text{kN} \qquad \{\text{upward}\}$$

Thus the vertical force is

$$R_y = 1.04\ \text{kN} + Mg \qquad \{\text{upward}\}$$

Then the net force on the vane (neglecting the weight of the vane and water within the CV) is

$$\vec{R} = -0.599\hat{i} + 1.04\hat{j}\ \text{kN} \qquad \vec{R}$$

This problem illustrates how to evaluate the momentum equation for a control volume in constant velocity motion by evaluating all velocities relative to the control volume.

4-5 MOMENTUM EQUATION FOR CONTROL VOLUME WITH RECTILINEAR ACCELERATION

For an inertial control volume (having no acceleration relative to a stationary frame of reference), the appropriate formulation of Newton's second law is given by Eq. 4.26,

$$\vec{F} = \vec{F}_S + \vec{F}_B = \frac{\partial}{\partial t}\int_{\text{CV}} \vec{V}_{xyz}\,\rho\,d\forall + \int_{\text{CS}} \vec{V}_{xyz}\,\rho\vec{V}_{xyz}\cdot d\vec{A} \tag{4.26}$$

Not all control volumes are inertial; for example, a rocket must accelerate if it is to get off the ground. Since we are interested in analyzing control volumes that may accelerate relative to inertial coordinates, it is logical to ask whether Eq. 4.26 can be used for an accelerating control volume. To answer this question, let us briefly review the two major elements used in developing Eq. 4.26.

First, in relating the system derivatives to the control volume formulation (Eq. 4.25 or 4.10), the flow field, $\vec{V}(x, y, z, t)$, was specified relative to the control volume's coordinates x, y, and z. No restriction was placed on the motion of the xyz reference frame. Consequently, Eq. 4.25 (or Eq. 4.10) is valid at any instant for any arbitrary motion of the coordinates x, y, and z provided that all velocities in the equation are measured relative to the control volume.

Second, the system equation

$$\vec{F} = \left.\frac{d\vec{P}}{dt}\right)_{\text{system}} \tag{4.2a}$$

where the linear momentum of the system is given by

$$\vec{P}_{\text{system}} = \int_{M(\text{system})} \vec{V}\, dm = \int_{V\!\!\!\!-(\text{system})} \vec{V}\,\rho\, dV\!\!\!\!- \tag{4.2b}$$

is valid only for velocities measured relative to an inertial reference frame. Thus, if we denote the inertial reference frame by XYZ, then Newton's second law states that

$$\vec{F} = \left.\frac{d\vec{P}_{XYZ}}{dt}\right)_{\text{system}} \tag{4.27}$$

Since the time derivatives of $\vec{P}_{XYZ}$ and $\vec{P}_{xyz}$ are not equal when the control volume reference frame xyz is accelerating relative to the inertial reference frame, Eq. 4.26 is not valid for an accelerating control volume.

To develop the momentum equation for a linearly accelerating control volume, it is necessary to relate $\vec{P}_{XYZ}$ of the system to $\vec{P}_{xyz}$ of the system. The system derivative $d\vec{P}_{xyz}/dt$ can then be related to control volume variables through Eq. 4.25. We begin by writing Newton's second law for a system, remembering that the acceleration must be measured relative to an inertial reference frame that we have designated XYZ. We write

$$\vec{F} = \left.\frac{d\vec{P}_{XYZ}}{dt}\right)_{\text{system}} = \frac{d}{dt}\int_{M(\text{system})} \vec{V}_{XYZ}\, dm = \int_{M(\text{system})} \frac{d\vec{V}_{XYZ}}{dt}\, dm \tag{4.28}$$

The velocities with respect to the inertial (XYZ) and the control volume coordinates (xyz) are related by the relative-motion equation

$$\vec{V}_{XYZ} = \vec{V}_{xyz} + \vec{V}_{rf} \tag{4.29}$$

where $\vec{V}_{rf}$ is the velocity of the control volume reference frame.

Since we are assuming the motion of xyz is pure translation, without rotation, relative to inertial reference frame XYZ, then

$$\frac{d\vec{V}_{XYZ}}{dt} = \vec{a}_{XYZ} = \frac{d\vec{V}_{xyz}}{dt} + \frac{d\vec{V}_{rf}}{dt} = \vec{a}_{xyz} + \vec{a}_{rf} \tag{4.30}$$

where

$\vec{a}_{XYZ}$ is the rectilinear acceleration of the system relative to inertial reference frame XYZ,

$\vec{a}_{xyz}$ is the rectilinear acceleration of the system relative to noninertial reference frame xyz (i.e., relative to the control volume), and

$\vec{a}_{rf}$ is the rectilinear acceleration of noninertial reference frame xyz (i.e., of the control volume) relative to inertial frame XYZ.

Substituting from Eq. 4.30 into Eq. 4.28 gives

$$\vec{F} = \int_{M(\text{system})} \vec{a}_{rf}\, dm + \int_{M(\text{system})} \frac{d\vec{V}_{xyz}}{dt}\, dm$$

or

$$\vec{F} - \int_{M(\text{system})} \vec{a}_{rf}\, dm = \left.\frac{d\vec{P}_{xyz}}{dt}\right)_{\text{system}} \tag{4.31a}$$

where the linear momentum of the system is given by

$$\left.\vec{P}_{xyz}\right)_{\text{system}} = \int_{M(\text{system})} \vec{V}_{xyz}\, dm = \int_{\forall(\text{system})} \vec{V}_{xyz}\, \rho\, d\forall \tag{4.31b}$$

and the force, $\vec{F}$, includes all surface and body forces acting on the system.

To derive the control volume formulation of Newton's second law, we set

$$N = \vec{P}_{xyz} \quad \text{and} \quad \eta = \vec{V}_{xyz}$$

From Eq. 4.25, with this substitution, we obtain

$$\left.\frac{d\vec{P}_{xyz}}{dt}\right)_{\text{system}} = \frac{\partial}{\partial t}\int_{\text{CV}} \vec{V}_{xyz}\, \rho\, d\forall + \int_{\text{CS}} \vec{V}_{xyz}\, \rho \vec{V}_{xyz} \cdot d\vec{A} \tag{4.32}$$

Combining Eq. 4.31a (the linear momentum equation for the system) and Eq. 4.32 (the system–control volume conversion), and recognizing that at time t_0 the system and control volume coincide, Newton's second law for a control volume accelerating, without rotation, relative to an inertial reference frame is

$$\vec{F} - \int_{\text{CV}} \vec{a}_{rf}\, \rho\, d\forall = \frac{\partial}{\partial t}\int_{\text{CV}} \vec{V}_{xyz}\, \rho\, d\forall + \int_{\text{CS}} \vec{V}_{xyz}\, \rho \vec{V}_{xyz} \cdot d\vec{A}$$

Since $\vec{F} = \vec{F}_S + \vec{F}_B$, this equation becomes

$$\vec{F}_S + \vec{F}_B - \int_{\text{CV}} \vec{a}_{rf}\, \rho\, d\forall = \frac{\partial}{\partial t}\int_{\text{CV}} \vec{V}_{xyz}\, \rho\, d\forall + \int_{\text{CS}} \vec{V}_{xyz}\, \rho \vec{V}_{xyz} \cdot d\vec{A} \tag{4.33}$$

Comparing this momentum equation for a control volume with rectilinear acceleration to that for a nonaccelerating control volume, Eq. 4.26, we see that the only difference is the presence of one additional term in Eq. 4.33. When the control volume is not accelerating relative to inertial reference frame XYZ, then $\vec{a}_{rf} = 0$, and Eq. 4.33 reduces to Eq. 4.26.

The precautions concerning the use of Eq. 4.26 also apply to the use of Eq. 4.33. Before attempting to apply either equation, one must draw the boundaries of the control volume and label appropriate coordinate directions. For an accelerating control volume, one must label two coordinate systems: one (xyz) on the control volume and the other (XYZ) an inertial reference frame.

In Eq. 4.33, $\vec{F}_S$ represents all surface forces acting on the control volume. Since the mass within the control volume may vary with time, both the remaining terms on the left side of the equation may be functions of time. Furthermore, the acceleration,

$\vec{a}_{rf}$, of the reference frame xyz relative to an inertial frame will in general be a function of time.

All velocities in Eq. 4.33 are measured relative to the control volume. The momentum flux, $\vec{V}_{xyz}\,\rho\vec{V}_{xyz}\cdot d\vec{A}$, through an element of the control surface area, $d\vec{A}$, is a vector. As we saw for the nonaccelerating control volume, the sign of the scalar product, $\rho\vec{V}_{xyz}\cdot d\vec{A}$, depends on the direction of the velocity vector, $\vec{V}_{xyz}$, relative to the area vector, $d\vec{A}$.

The momentum equation is a vector equation. As with all vector equations, it may be written as three scalar component equations. The scalar components of Eq. 4.33 are

$$F_{S_x} + F_{B_x} - \int_{\mathrm{CV}} a_{rf_x}\,\rho\,d\forall = \frac{\partial}{\partial t}\int_{\mathrm{CV}} u_{xyz}\,\rho\,d\forall + \int_{\mathrm{CS}} u_{xyz}\,\rho\vec{V}_{xyz}\cdot d\vec{A} \tag{4.34a}$$

$$F_{S_y} + F_{B_y} - \int_{\mathrm{CV}} a_{rf_y}\,\rho\,d\forall = \frac{\partial}{\partial t}\int_{\mathrm{CV}} v_{xyz}\,\rho\,d\forall + \int_{\mathrm{CS}} v_{xyz}\,\rho\vec{V}_{xyz}\cdot d\vec{A} \tag{4.34b}$$

$$F_{S_z} + F_{B_z} - \int_{\mathrm{CV}} a_{rf_z}\,\rho\,d\forall = \frac{\partial}{\partial t}\int_{\mathrm{CV}} w_{xyz}\,\rho\,d\forall + \int_{\mathrm{CS}} w_{xyz}\,\rho\vec{V}_{xyz}\cdot d\vec{A} \tag{4.34c}$$

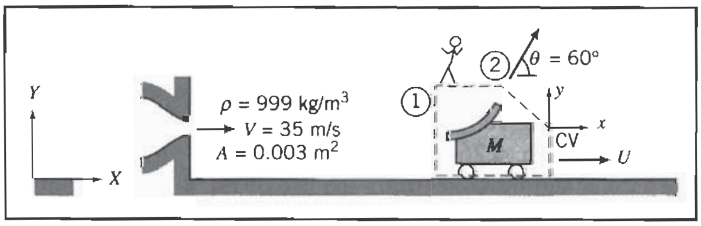
We will consider two applications of the linearly accelerating control volume: Example Problem 4.11 will analyze an accelerating control volume in which the mass contained in the control volume is constant; Example Problem 4.12 will analyze an accelerating control volume in which the mass contained varies with time.

EXAMPLE 4.11 Vane Moving with Rectilinear Acceleration

A vane, with turning angle $\theta = 60°$, is attached to a cart. The cart and vane, of mass $M = 75$ kg, roll on a level track. Friction and air resistance may be neglected. The vane receives a jet of water, which leaves a stationary nozzle horizontally at $V = 35$ m/s. The nozzle exit area is $A = 0.003$ m^2. Determine the velocity of the cart as a function of time and plot the results.

EXAMPLE PROBLEM 4.11

GIVEN: Vane and cart as sketched, with $M = 75$ kg.



FIND: $U(t)$ and plot results.

SOLUTION:
Choose the control volume and coordinate systems shown for the analysis. Note that XY is a fixed frame, while frame xy moves with the cart. Apply the x component of the momentum equation.

Governing equation:

$$\underbrace{\cancel{F_{S_x}}}_{=0(1)} + \underbrace{\cancel{F_{B_x}}}_{=0(2)} - \int_{CV} a_{rf_x} \rho \, d\forall = \underbrace{\cancel{\frac{\partial}{\partial t}}\int_{CV} u_{xyz} \rho \, d\forall}_{\simeq 0(4)} + \int_{CS} u_{xyz} \rho \vec{V}_{xyz} \cdot d\vec{A}$$

Assumptions: (1) $F_{S_x} = 0$, since no resistance is present.
(2) $F_{B_x} = 0$.
(3) Neglect the mass of water in contact with the vane compared to the cart mass.
(4) Neglect rate of change of momentum of liquid inside the CV.

$$\frac{\partial}{\partial t}\int_{CV} u_{xyz} \rho \, d\forall \simeq 0$$

(5) Uniform flow at sections ① and ②.
(6) Speed of water stream is not slowed by friction on the vane, so $|\vec{V}_{xyz_1}| = |\vec{V}_{xyz_2}|$.
(7) $A_2 = A_1 = A$.

Then, dropping subscripts rf and xyz for clarity (but remembering that all velocities are measured relative to the moving coordinates of the control volume),

$$\begin{aligned} -\int_{CV} a_x \rho \, d\forall &= u_1(-\rho V_1 A_1) + u_2(\rho V_2 A_2) \\ &= (V-U)\{-\rho(V-U)A\} + (V-U)\cos\theta\{\rho(V-U)A\} \\ &= -\rho(V-U)^2 A + \rho(V-U)^2 A\cos\theta \end{aligned}$$

For the left side of this equation we have

$$-\int_{CV} a_x \rho \, d\forall = -a_x M_{CV} = -a_x M = -\frac{dU}{dt} M$$

so that

$$-M\frac{dU}{dt} = -\rho(V-U)^2 A + \rho(V-U)^2 A\cos\theta$$

or

$$M\frac{dU}{dt} = (1-\cos\theta)\rho(V-U)^2 A$$

Separating variables, we obtain

$$\frac{dU}{(V-U)^2} = \frac{(1-\cos\theta)\rho A}{M} dt = b\,dt \qquad \text{where } b = \frac{(1-\cos\theta)\rho A}{M}$$

Note that since $V =$ constant, $dU = -d(V-U)$. Integrating between limits $U = 0$ at $t = 0$, and $U = U$ at $t = t$,

$$\int_0^U \frac{dU}{(V-U)^2} = \int_0^U \frac{-d(V-U)}{(V-U)^2} = \left.\frac{1}{(V-U)}\right]_0^U = \int_0^t b\,dt = bt$$

or

$$\frac{1}{(V-U)} - \frac{1}{V} = \frac{U}{V(V-U)} = bt$$

Solving for U, we obtain

$$\frac{U}{V} = \frac{Vbt}{1+Vbt}$$

Evaluating Vb gives

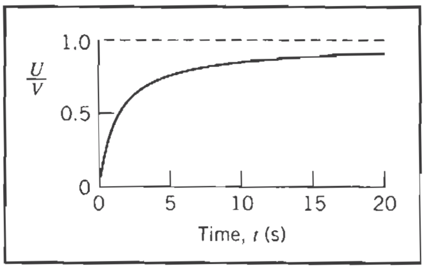
$$Vb = V\frac{(1-\cos\theta)\rho A}{M}$$

$$Vb = \frac{35\text{ m}}{\text{s}} \times \frac{(1-0.5)}{75\text{ kg}} \times \frac{999\text{ kg}}{\text{m}^3} \times 0.003\text{ m}^2 = 0.699\text{ s}^{-1}$$

Thus

$$\frac{U}{V} = \frac{0.699t}{1+0.699t} \qquad (t \text{ in seconds}) \qquad U(t)$$

Plot:



The graph was generated from an *Excel* workbook. This workbook is interactive: It allows one to see the effect of different values of ρ, A, M, and θ on U/V against time t, and also to determine the time taken for the cart to reach, for example, 95% of jet speed.

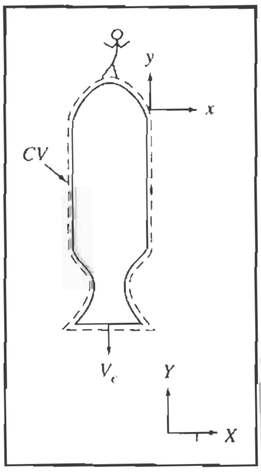
EXAMPLE 4.12 Rocket Directed Vertically

A small rocket, with an initial mass of 400 kg, is to be launched vertically. Upon ignition the rocket consumes fuel at the rate of 5 kg/s and ejects gas at atmospheric pressure with a speed of 3500 m/s relative to the rocket. Determine the initial acceleration of the rocket and the rocket speed after 10 s, if air resistance is neglected.

EXAMPLE PROBLEM 4.12

GIVEN: Small rocket accelerates vertically from rest.
Initial mass, $M_0 = 400$ kg.
Air resistance may be neglected.
Rate of fuel consumption, $\dot{m}_e = 5$ kg/s.
Exhaust velocity, $V_e = 3500$ m/s, relative to rocket, leaving at atmospheric pressure.

FIND: (a) Initial acceleration of the rocket.
(b) Rocket velocity after 10 s.



SOLUTION:

Choose a control volume as shown by dashed lines. Because the control volume is accelerating, define inertial coordinate system XY and coordinate system xy attached to the CV. Apply the y component of the momentum equation.

Governing equation: $F_{S_y} + F_{B_y} - \int_{CV} a_{rf_y}\, \rho\, d\forall = \frac{\partial}{\partial t}\int_{CV} v_{xyz}\, \rho\, d\forall + \int_{CS} v_{xyz}\, \rho \vec{V}_{xyz} \cdot d\vec{A}$

Assumptions: (1) Atmospheric pressure acts on all surfaces of the CV; since air resistance is neglected, $F_{S_y} = 0$.
(2) Gravity is the only body force; g is constant.
(3) Flow leaving the rocket is uniform, and V_e is constant.

Under these assumptions the momentum equation reduces to

$$\underset{Ⓐ}{F_{B_y}} - \underset{Ⓑ}{\int_{CV} a_{rf_y}\rho\, d\forall} = \underset{Ⓒ}{\frac{\partial}{\partial t}\int_{CV} v_{xyz}\, \rho\, d\forall} + \underset{Ⓓ}{\int_{CS} v_{xyz}\, \rho \vec{V}_{xyz} \cdot d\vec{A}} \tag{1}$$

Let us look at the equation term by term:

Ⓐ $\quad F_{B_y} = -\int_{CV} g\, \rho\, d\forall = -g\int_{CV} \rho\, d\forall = -gM_{CV} \qquad$ {since g is constant}

The mass of the CV will be a function of time because mass is leaving the CV at rate $\dot{m}_e$. To determine M_{CV} as a function of time, we use the conservation of mass equation

$$\frac{\partial}{\partial t}\int_{CV} \rho\, d\forall + \int_{CS} \rho \vec{V} \cdot d\vec{A} = 0$$

Then

$$\frac{\partial}{\partial t}\int_{CV} \rho\, d\forall = -\int_{CS} \rho \vec{V} \cdot d\vec{A} = -\int_{CS} \left(\rho V_{xyz} dA\right) = -\dot{m}_e$$

The minus sign indicates that the mass of the CV is decreasing with time. Since the mass of the CV is only a function of time, we can write

$$\frac{dM_{CV}}{dt} = -\dot{m}_e$$

To find the mass of the CV at any time, t, we integrate

$$\int_{M_0}^{M} dM_{CV} = -\int_0^t \dot{m}_e\, dt \qquad \text{where at } t = 0,\ M_{CV} = M_0, \text{ and at } t = t,\ M_{CV} = M$$

Then, $M - M_0 = -\dot{m}_e t$, or $M = M_0 - \dot{m}_e t$.

Substituting the expression for M into term Ⓐ, we obtain

$$F_{B_y} = -\int_{CV} g\, \rho\, d\forall = -gM_{CV} = -g(M_0 - \dot{m}_e t)$$

Ⓑ $\quad -\int_{CV} a_{rf_y}\, \rho\, d\forall$

The acceleration, a_{rf_y}, of the CV is that seen by an observer in the XY coordinate system. Thus a_{rf_y} is not a function of the coordinates xyz, and

$$-\int_{CV} a_{rf_y}\, \rho\, d\forall = -a_{rf_y}\int_{CV} \rho\, d\forall = -a_{rf_y} M_{CV} = -a_{rf_y}(M_0 - \dot{m}_e t)$$

Ⓒ $\quad \frac{\partial}{\partial t}\int_{CV} v_{xyz}\, \rho\, d\forall$

This is the time rate of change of the y momentum of the fluid in the control volume measured relative to the control volume.

Even though the y momentum of the fluid inside the CV, measured relative to the CV, is a large number, it does not change appreciably with time. To see this, we must recognize that:

(1) The unburned fuel and the rocket structure have zero momentum relative to the rocket.
(2) The velocity of the gas at the nozzle exit remains constant with time as does the velocity at various points in the nozzle.

Consequently, it is reasonable to assume that

$$\frac{\partial}{\partial t}\int_{\mathrm{CV}} v_{xyz}\,\rho\,d\mathcal{V} \approx 0$$

Ⓓ $$\int_{\mathrm{CS}} v_{xyz}\,\rho\vec{V}_{xyz}\cdot d\vec{A} = \int_{\mathrm{CS}} v_{xyz}\left(\rho V_{xyz}\,dA\right) = -V_e\int_{\mathrm{CS}}\left(\rho V_{xyz}\,dA\right)$$

The velocity v_{xyz} (relative to the control volume) is $-V_e$ (it is in the negative y direction), and is a constant, so was taken outside the integral. The remaining integral is simply the mass flow rate at the exit (positive because flow is out of the control volume),

$$\int_{\mathrm{CS}}\left(\rho V_{xyz}\,dA\right) = \dot{m}_e$$

and so

$$\int_{\mathrm{CS}} v_{xyz}\,\rho\vec{V}_{xyz}\cdot d\vec{A} = -V_e\dot{m}_e$$

Substituting terms Ⓐ through Ⓓ into Eq. 1, we obtain

$$-g(M_0 - \dot{m}_e t) - a_{rf_y}(M_0 - \dot{m}_e t) = -V_e\dot{m}_e$$

or

$$a_{rf_y} = \frac{V_e\dot{m}_e}{M_0 - \dot{m}_e t} - g \qquad (2)$$

At time $t = 0$,

$$a_{rf_y}\Big)_{t=0} = \frac{V_e\dot{m}_e}{M_0} - g = \frac{3500}{}\frac{\mathrm{m}}{\mathrm{s}} \times \frac{5\ \mathrm{kg}}{\mathrm{s}} \times \frac{1}{400\ \mathrm{kg}} - \frac{9.81\ \mathrm{m}}{\mathrm{s}^2}$$

$$a_{rf_y}\Big)_{t=0} = 33.9\ \mathrm{m/s^2} \longleftarrow \qquad a_{rf_y}\Big)_{t=0}$$

The acceleration of the CV is by definition

$$a_{rf_y} = \frac{dV_{\mathrm{CV}}}{dt}$$

Substituting from Eq. 2,

$$\frac{dV_{\mathrm{CV}}}{dt} = \frac{V_e\dot{m}_e}{M_0 - \dot{m}_e t} - g$$

Separating variables and integrating gives

$$V_{\mathrm{CV}} = \int_0^{V_{\mathrm{CV}}} dV_{\mathrm{CV}} = \int_0^t \frac{V_e\dot{m}_e\,dt}{M_0 - \dot{m}_e t} - \int_0^t g\,dt = -V_e\ln\left[\frac{M_0 - \dot{m}_e t}{M_0}\right] - gt$$

At $t = 10$ s,

$$V_{\mathrm{CV}} = -3500\frac{\mathrm{m}}{\mathrm{s}} \times \ln\left[\frac{350\ \mathrm{kg}}{400\ \mathrm{kg}}\right] - 9.81\frac{\mathrm{m}}{\mathrm{s}^2} \times 10\ \mathrm{s}$$

$$V_{\mathrm{CV}} = 369\ \mathrm{m/s} \longleftarrow \qquad V_{\mathrm{CV}}\Big)_{t} = 10\ \mathrm{s}$$

The position vector, $\vec{r}$, locates each mass or volume element of the system with respect to the coordinate system. The torque, $\vec{T}$, applied to a system may be written

$$\vec{T} = \vec{r} \times \vec{F}_s + \int_{M(\text{system})} \vec{r} \times \vec{g}\, dm + \vec{T}_{\text{shaft}} \tag{4.3c}$$

where $\vec{F}_s$ is the surface force exerted on the system.

The relation between the system and fixed control volume formulations is

$$\left.\frac{dN}{dt}\right)_{\text{system}} = \frac{\partial}{\partial t}\int_{\text{CV}} \eta \rho \, d\mathcal{V} + \int_{\text{CS}} \eta \rho \vec{V} \cdot d\vec{A} \tag{4.10}$$

where

$$N_{\text{system}} = \int_{M(\text{system})} \eta \, dm$$

If we set $N = \vec{H}$, then $\eta = \vec{r} \times \vec{V}$, and

$$\left.\frac{d\vec{H}}{dt}\right)_{\text{system}} = \frac{\partial}{\partial t}\int_{\text{CV}} \vec{r} \times \vec{V} \rho \, d\mathcal{V} + \int_{\text{CS}} \vec{r} \times \vec{V} \rho \vec{V} \cdot d\vec{A} \tag{4.45}$$

Combining Eqs. 4.3a, 4.3c, and 4.45, we obtain

$$\vec{r} \times \vec{F}_s + \int_{M(\text{system})} \vec{r} \times \vec{g}\, dm + \vec{T}_{\text{shaft}} = \frac{\partial}{\partial t}\int_{\text{CV}} \vec{r} \times \vec{V} \rho \, d\mathcal{V} + \int_{\text{CS}} \vec{r} \times \vec{V} \rho \vec{V} \cdot d\vec{A}$$

Since the system and control volume coincide at time t_0,

$$\vec{T} = \vec{T}_{\text{CV}}$$

and

$$\vec{r} \times \vec{F}_s + \int_{\text{CV}} \vec{r} \times \vec{g}\, \rho \, d\mathcal{V} + \vec{T}_{\text{shaft}} = \frac{\partial}{\partial t}\int_{\text{CV}} \vec{r} \times \vec{V} \rho \, d\mathcal{V} + \int_{\text{CS}} \vec{r} \times \vec{V} \rho \vec{V} \cdot d\vec{A} \tag{4.46}$$

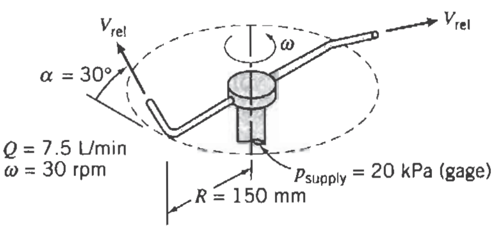
Equation 4.46 is a general formulation of the angular-momentum principle for an inertial control volume. The left side of the equation is an expression for all the torques that act on the control volume. Terms on the right express the rate of change of angular momentum within the control volume and the net rate of flux of angular momentum from the control volume. All velocities in Eq. 4.46 are measured relative to the fixed control volume.

For analysis of rotating machinery, Eq. 4.46 is often used in scalar form by considering only the component directed along the axis of rotation. This application is illustrated in Chapter 10.

The application of Eq. 4.46 to the analysis of a simple lawn sprinkler is illustrated in Example Problem 4.14. This same problem is considered in Example Problem 4.15 (on the CD) using the angular-momentum principle expressed in terms of a *rotating* control volume.

EXAMPLE 4.14 Lawn Sprinkler: Analysis Using Fixed Control Volume

A small lawn sprinkler is shown in the sketch below. At an inlet gage pressure of 20 kPa, the total volume flow rate of water through the sprinkler is 7.5 liters per minute and it rotates at 30 rpm. The diameter of each jet is 4 mm. Calculate the jet speed relative to each sprinkler nozzle. Evaluate the friction torque at the sprinkler pivot.



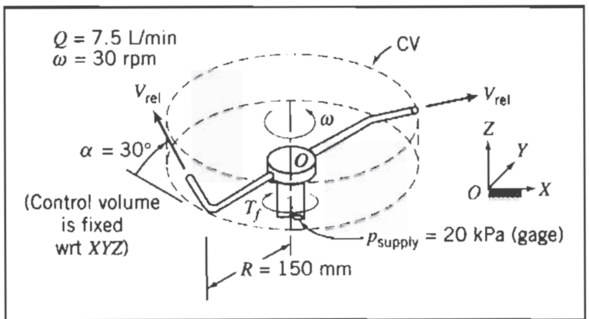
EXAMPLE PROBLEM 4.14

GIVEN: Small lawn sprinkler as shown.

FIND: (a) Jet speed relative to each nozzle.
(b) Friction torque at pivot.

SOLUTION:

Apply continuity and angular momentum equations using fixed control volume enclosing sprinkler arms.



Governing equations:

$$\overbrace{\cancel{\frac{\partial}{\partial t}}\int_{CV}\rho\, d\forall}^{=0(1)} + \int_{CS}\rho\vec{V}\cdot d\vec{A} = 0$$

$$\vec{r}\times\vec{F}_s + \int_{CV}\vec{r}\times\vec{g}\,\rho\, d\forall + \vec{T}_{shaft} = \frac{\partial}{\partial t}\int_{CV}\vec{r}\times\vec{V}\rho\, d\forall + \int_{CS}\vec{r}\times\vec{V}\rho\vec{V}\cdot d\vec{A} \tag{1}$$

where all velocities are measured relative to the inertial coordinates XYZ.

Assumptions: (1) Incompressible flow.
(2) Uniform flow at each section.
(3) $\vec{\omega}$ = constant.

From continuity, the jet speed relative to the nozzle is given by

$$V_{rel} = \frac{Q}{2A_{jet}} = \frac{Q}{2}\frac{4}{\pi D_{jet}^2}$$

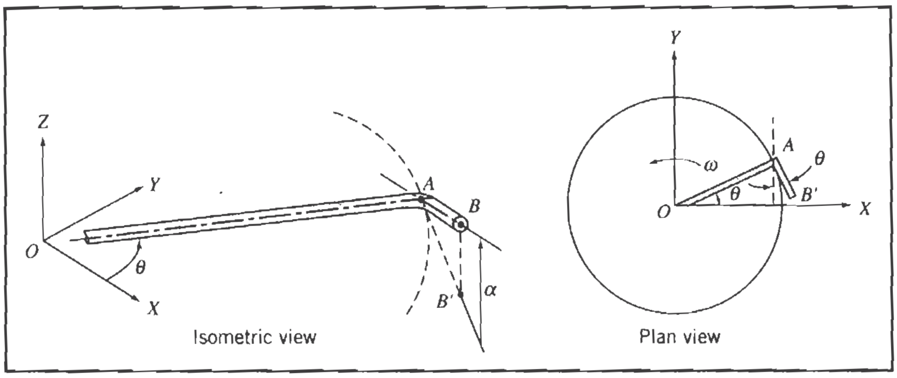
$$= \frac{1}{2}\times\frac{7.5\ \text{L}}{\text{min}}\times\frac{4}{\pi}\frac{1}{(4)^2\ \text{mm}^2}\times\frac{\text{m}^3}{1000\ \text{L}}\times\frac{10^6\ \text{mm}^2}{\text{m}^2}\times\frac{\text{min}}{60\ \text{s}}$$

$$V_{rel} = 4.97\ \text{m/s}$$ V_{rel}

Consider terms in the angular momentum equation separately. Since atmospheric pressure acts on the entire control surface, and the pressure force at the inlet causes no moment about O, $\vec{r}\times\vec{F}_s = 0$. The moments of the body (i.e., gravity) forces in the two arms are equal and opposite and hence the second term on the left side of the equation is zero. The only external torque acting on the CV is friction in the pivot. It opposes the motion, so

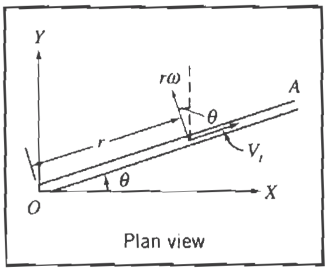
$$\vec{T}_{shaft} = -T_f\hat{K} \tag{2}$$

Our next task is to determine the two angular momentum terms on the right side of Eq. 1. Consider the unsteady term: This is the rate of change of angular momentum in the control volume. It is clear that although the position $\vec{r}$ and velocity $\vec{V}$ of fluid particles are functions of time in XYZ coordinates, because the sprinkler rotates at constant speed the control volume angular momentum *is* constant in *XYZ* coordinates, so this term is zero; however, as an exercise in manipulating vector quantities, let us derive this result. Before we can evaluate the control volume integral, we need to develop expressions for the instantaneous position vector, $\vec{r}$, and velocity vector, $\vec{V}$ (measured relative to the fixed coordinate system *XYZ*) of each element of fluid in the control volume.



OA lies in the *XY* plane; *AB* is inclined at angle α to the *XY* plane; point B' is the projection of point *B* on the *XY* plane.

We assume that the length, *L*, of the tip *AB* is small compared with the length, *R*, of the horizontal arm *OA*. Consequently we neglect the angular momentum of the fluid in the tips compared with the angular momentum in the horizontal arms.



Consider flow in the horizontal tube *OA* of length *R*. Denote the radial distance from *O* by *r*. At any point in the tube the fluid velocity relative to fixed coordinates *XYZ* is the sum of the velocity relative to the tube $\vec{V}_t$ and the tangential velocity $\vec{\omega} \times \vec{r}$. Thus

$$\vec{V} = \hat{i}(V_t \cos\theta - r\omega \sin\theta) + \hat{j}(V_t \sin\theta + r\omega\cos\theta)$$

(Note that θ is a function of time.) The position vector is

$$\vec{r} = \hat{i}r\cos\theta + \hat{j}r\sin\theta$$

and

$$\vec{r} \times \vec{V} = \hat{K}(r^2\omega\cos^2\theta + r^2\omega\sin^2\theta) = \hat{K}r^2\omega$$

Then

$$\int_{\mathcal{V}_{OA}} \vec{r} \times \vec{V} \rho \, d\mathcal{V} = \int_O^R \hat{K}r^2\omega\rho A \, dr = \hat{K}\frac{R^3\omega}{3}\rho A$$

and

$$\frac{\partial}{\partial t}\int_{\mathcal{V}_{OA}} \vec{r} \times \vec{V} \rho \, d\mathcal{V} = \frac{\partial}{\partial t}\left[\hat{K}\frac{R^3\omega}{3}\rho A\right] = 0 \qquad (3)$$

where *A* is the cross-sectional area of the horizontal tube. Identical results are obtained for the other horizontal tube in the control volume. We have confirmed our insight that the angular momentum within the control volume does not change with time.

Now we need to evaluate the second term on the right, the flux of momentum across the control surface. There are three surfaces through which we have mass and therefore momentum flux: the supply line (for which $\vec{r} \times \vec{V} = 0$ because $\vec{r} = 0$) and the two nozzles. Consider the nozzle at the end of branch *OAB*. For $L \ll R$, we have

$$\vec{r}_{\text{jet}} = \vec{r}_B \approx \vec{r}\Big|_{r=R} = (\hat{I}r\cos\theta + \hat{J}r\sin\theta)\Big|_{r=R} = \hat{I}R\cos\theta + \hat{J}R\sin\theta$$

and for the instantaneous jet velocity $\vec{V}_j$ we have

$$\vec{V}_j = \vec{V}_{\text{rel}} + \vec{V}_{\text{tip}} = \hat{I}V_{\text{rel}}\cos\alpha\sin\theta - \hat{J}V_{\text{rel}}\cos\alpha\cos\theta + \hat{K}V_{\text{rel}}\sin\alpha - \hat{I}\omega R\sin\theta + \hat{J}\omega R\cos\theta$$

$$\vec{V}_j = \hat{I}(V_{\text{rel}}\cos\alpha - \omega R)\sin\theta - \hat{J}(V_{\text{rel}}\cos\alpha - \omega R)\cos\theta + \hat{K}V_{\text{rel}}\sin\alpha$$

$$\vec{r}_B \times \vec{V}_j = \hat{I}RV_{\text{rel}}\sin\alpha\sin\theta - \hat{J}RV_{\text{rel}}\sin\alpha\cos\theta - \hat{K}R(V_{\text{rel}}\cos\alpha - \omega R)(\sin^2\theta + \cos^2\theta)$$

$$\vec{r}_B \times \vec{V}_j = \hat{I}RV_{\text{rel}}\sin\alpha\sin\theta - \hat{J}RV_{\text{rel}}\sin\alpha\cos\theta - \hat{K}R(V_{\text{rel}}\cos\alpha - \omega R)$$

The flux integral evaluated for flow crossing the control surface at location B is then

$$\int_{\text{CS}} \vec{r} \times \vec{V}_j\, \rho\vec{V}\cdot d\vec{A} = \left[\hat{I}RV_{\text{rel}}\sin\alpha\sin\theta - \hat{J}RV_{\text{rel}}\sin\alpha\cos\theta - \hat{K}R(V_{\text{rel}}\cos\alpha - \omega R)\right]\rho\frac{Q}{2}$$

The velocity and radius vectors for flow in the left arm must be described in terms of the same unit vectors used for the right arm. In the left arm the $\hat{I}$ and $\hat{J}$ components of the cross product are of opposite sign, since $\sin(\theta + \pi) = -\sin(\theta)$ and $\cos(\theta + \pi) = -\cos(\theta)$. Thus for the complete CV,

$$\int_{\text{CS}} \vec{r} \times \vec{V}_j\, \rho\vec{V}\cdot d\vec{A} = -\hat{K}R(V_{\text{rel}}\cos\alpha - \omega R)\rho Q \tag{4}$$

Substituting terms (2), (3), and (4) into Eq. 1, we obtain

$$-T_f\hat{K} = -\hat{K}R(V_{\text{rel}}\cos\alpha - \omega R)\rho Q$$

or

$$T_f = R(V_{\text{rel}}\cos\alpha - \omega R)\rho Q$$

This expression indicates that when the sprinkler runs at constant speed the friction torque at the sprinkler pivot just balances the torque generated by the angular momentum of the two jets.

From the data given,

$$\omega R = 30\,\frac{\text{rev}}{\text{min}} \times 150\text{ mm} \times 2\pi\,\frac{\text{rad}}{\text{rev}} \times \frac{\text{min}}{60\text{ s}} \times \frac{\text{m}}{1000\text{ mm}} = 0.471\text{ m/s}$$

Substituting gives

$$T_f = 150\text{ mm} \times \left(4.97\,\frac{\text{m}}{\text{s}} \times \cos 30^\circ - 0.471\,\frac{\text{m}}{\text{s}}\right) 999\,\frac{\text{kg}}{\text{m}^3} \times 7.5\,\frac{\text{L}}{\text{min}}$$

$$\times \frac{\text{m}^3}{1000\text{ L}} \times \frac{\text{min}}{60\text{ s}} \times \frac{\text{N}\cdot\text{s}^2}{\text{kg}\cdot\text{m}} \times \frac{\text{m}}{1000\text{ mm}}$$

$$T_f = 0.0718\text{ N}\cdot\text{m} \qquad T_f$$

This problem illustrates use of the angular momentum principle for an inertial control volume. Note that in this example the fluid particle position vector $\vec{r}$ and velocity vector $\vec{V}$ are time-dependent (through θ) in XYZ coordinates. This problem will be solved again using a noninertial (rotating) xyz coordinate system in Example Problem 4.15 (on the CD).

Equation for Rotating Control Volume (CD-ROM)

4-8 THE FIRST LAW OF THERMODYNAMICS

The first law of thermodynamics is a statement of conservation of energy. Recall that the system formulation of the first law was

$$\dot{Q} - \dot{W} = \left.\frac{dE}{dt}\right)_{\text{system}} \tag{4.4a}$$

where the total energy of the system is given by

$$E_{\text{system}} = \int_{M(\text{system})} e\, dm = \int_{\forall(\text{system})} e\, \rho\, d\forall \tag{4.4b}$$

and

$$e = u + \frac{V^2}{2} + gz$$

In Eq. 4.4a, the rate of heat transfer, $\dot{Q}$, is positive when heat is added *to* the system from the surroundings; the rate of work, $\dot{W}$, is positive when work is done *by* the system on its surroundings.

To derive the control volume formulation of the first law of thermodynamics, we set

$$N = E \qquad \text{and} \qquad \eta = e$$

in Eq. 4.10 and obtain

$$\left.\frac{dE}{dt}\right)_{\text{system}} = \frac{\partial}{\partial t}\int_{\text{CV}} e\, \rho\, d\forall + \int_{\text{CS}} e\, \rho \vec{V} \cdot d\vec{A} \tag{4.53}$$

Since the system and the control volume coincide at t_0,

$$\left[\dot{Q} - \dot{W}\right]_{\text{system}} = \left[\dot{Q} - \dot{W}\right]_{\text{control volume}}$$

In light of this, Eqs. 4.4a and 4.53 yield the control volume form of the first law of thermodynamics,

$$\dot{Q} - \dot{W} = \frac{\partial}{\partial t}\int_{\text{CV}} e\, \rho\, d\forall + \int_{\text{CS}} e\, \rho \vec{V} \cdot d\vec{A} \tag{4.54}$$

where

$$e = u + \frac{V^2}{2} + gz$$

Note that for steady flow the first term on the right side of Eq. 4.54 is zero.

Is Eq 4.54 the form of the first law used in thermodynamics? Even for steady flow, Eq. 4.54 is not quite the same form used in applying the first law to control volume problems. To obtain a formulation suitable and convenient for problem solutions, let us take a closer look at the work term, $\dot{W}$.

Rate of Work Done by a Control Volume

The term $\dot{W}$ in Eq. 4.54 has a positive numerical value when work is done by the control volume on the surroundings. The rate of work done *on* the control volume is of opposite sign to the work done *by* the control volume.

The rate of work done by the control volume is conveniently subdivided into four classifications,

$$\dot{W} = \dot{W}_s + \dot{W}_{\text{normal}} + \dot{W}_{\text{shear}} + \dot{W}_{\text{other}}$$

Let us consider these separately:

1. Shaft Work

We shall designate shaft work W_s and hence the rate of work transferred out through the control surface by shaft work is designated $\dot{W}_s$. Examples of shaft work are the work produced by the steam turbine (positive shaft work) of a power plant, and the work input required to run the compressor of a refrigerator (negative shaft work).

2. Work Done by Normal Stresses at the Control Surface

Recall that work requires a force to act through a distance. Thus, when a force, $\vec{F}$, acts through an infinitesimal displacement, $d\vec{s}$, the work done is given by

$$\delta W = \vec{F} \cdot d\vec{s}$$

To obtain the rate at which work is done by the force, divide by the time increment, Δt, and take the limit as $\Delta t \to 0$. Thus the rate of work done by the force, $\vec{F}$, is

$$\dot{W} = \lim_{\Delta t \to 0} \frac{\delta W}{\Delta t} = \lim_{\Delta t \to 0} \frac{\vec{F} \cdot d\vec{s}}{\Delta t} \qquad \text{or} \qquad \dot{W} = \vec{F} \cdot \vec{V}$$

We can use this to compute the rate of work done by the normal and shear stresses. Consider the segment of control surface shown in Fig. 4.6. For an elementary area $d\vec{A}$ we can write an expression for the normal stress force $d\vec{F}_{\text{normal}}$: It will be given by the normal stress σ_{nn} multiplied by the vector area element $d\vec{A}$ (normal to the control surface).

Hence the rate of work done on the area element is

$$d\vec{F}_{\text{normal}} \cdot \vec{V} = \sigma_{nn}\, d\vec{A} \cdot \vec{V}$$

Since the work out across the boundaries of the control volume is the negative of the work done on the control volume, the total rate of work out of the control volume due to normal stresses is

$$\dot{W}_{\text{normal}} = -\int_{\text{CS}} \sigma_{nn}\, d\vec{A} \cdot \vec{V} = -\int_{\text{CS}} \sigma_{nn} \vec{V} \cdot d\vec{A}$$

3. Work Done by Shear Stresses at the Control Surface

Just as work is done by the normal stresses at the boundaries of the control volume, so may work be done by the shear stresses.

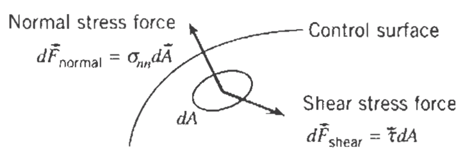


Fig. 4.6 Normal and shear stress forces.

As shown in Fig. 4.6, the shear force acting on an element of area of the control surface is given by

$$d\vec{F}_{\text{shear}} = \vec{\tau}\, dA$$

where the shear stress vector, $\vec{\tau}$, is the shear stress acting in some direction in the plane of dA.

The rate of work done on the entire control surface by shear stresses is given by

$$\int_{\text{CS}} \vec{\tau}\, dA \cdot \vec{V} = \int_{\text{CS}} \vec{\tau} \cdot \vec{V}\, dA$$

Since the work out across the boundaries of the control volume is the negative of the work done on the control volume, the rate of work out of the control volume due to shear stresses is given by

$$\dot{W}_{\text{shear}} = -\int_{\text{CS}} \vec{\tau} \cdot \vec{V}\, dA$$

This integral is better expressed as three terms

$$\begin{aligned}
\dot{W}_{\text{shear}} &= -\int_{\text{CS}} \vec{\tau} \cdot \vec{V}\, dA \\
&= -\int_{A(\text{shafts})} \vec{\tau} \cdot \vec{V}\, dA - \int_{A(\text{solid surface})} \vec{\tau} \cdot \vec{V}\, dA - \int_{A(\text{ports})} \vec{\tau} \cdot \vec{V}\, dA
\end{aligned}$$

We have already accounted for the first term, since we included $\dot{W}_s$ previously. At solid surfaces, $\vec{V} = 0$, so the second term is zero (for a fixed control volume). Thus,

$$\dot{W}_{\text{shear}} = -\int_{A(\text{ports})} \vec{\tau} \cdot \vec{V}\, dA$$

This last term can be made zero by proper choice of control surfaces. If we choose a control surface that cuts across each port perpendicular to the flow, then $d\vec{A}$ is parallel to $\vec{V}$. Since $\vec{\tau}$ is in the plane of dA, $\vec{\tau}$ is perpendicular to $\vec{V}$. Thus, for a control surface perpendicular to $\vec{V}$,

$$\vec{\tau} \cdot \vec{V} = 0 \qquad \text{and} \qquad \dot{W}_{\text{shear}} = 0$$

4. Other Work

Electrical energy could be added to the control volume. Also electromagnetic energy, e.g., in radar or laser beams, could be absorbed. In most problems, such contributions will be absent, but we should note them in our general formulation.

With all of the terms in $\dot{W}$ evaluated, we obtain

$$\dot{W} = \dot{W}_s - \int_{\text{CS}} \sigma_{nn}\, \vec{V} \cdot d\vec{A} + \dot{W}_{\text{shear}} + \dot{W}_{\text{other}} \tag{4.55}$$

Control Volume Equation

Substituting the expression for $\dot{W}$ from Eq. 4.55 into Eq. 4.54 gives

$$\dot{Q} - \dot{W}_s + \int_{\text{CS}} \sigma_{nn}\, \vec{V} \cdot d\vec{A} - \dot{W}_{\text{shear}} - \dot{W}_{\text{other}} = \frac{\partial}{\partial t}\int_{\text{CV}} e\, \rho\, d\forall + \int_{\text{CS}} e\, \rho \vec{V} \cdot d\vec{A}$$

Rearranging this equation, we obtain

$$\dot{Q} - \dot{W}_s - \dot{W}_{\text{shear}} - \dot{W}_{\text{other}} = \frac{\partial}{\partial t}\int_{\text{CV}} e\,\rho\, d\forall + \int_{\text{CS}} e\,\rho \vec{V}\cdot d\vec{A} - \int_{\text{CS}} \sigma_{nn}\, \vec{V}\cdot d\vec{A}$$

Since $\rho = 1/v$, where v is *specific volume*, then

$$\int_{\text{CS}} \sigma_{nn}\, \vec{V}\cdot d\vec{A} = \int_{\text{CS}} \sigma_{nn}\, v\,\rho \vec{V}\cdot d\vec{A}$$

Hence

$$\dot{Q} - \dot{W}_s - \dot{W}_{\text{shear}} - \dot{W}_{\text{other}} = \frac{\partial}{\partial t}\int_{\text{CV}} e\,\rho\, d\forall + \int_{\text{CS}} (e - \sigma_{nn} v)\,\rho \vec{V}\cdot d\vec{A}$$

Viscous effects can make the normal stress, σ_{nn}, different from the negative of the thermodynamic pressure, $-p$. However, for most flows of common engineering interest, $\sigma_{nn} \simeq -p$. Then

$$\dot{Q} - \dot{W}_s - \dot{W}_{\text{shear}} - \dot{W}_{\text{other}} = \frac{\partial}{\partial t}\int_{\text{CV}} e\,\rho\, d\forall + \int_{\text{CS}} (e + pv)\,\rho \vec{V}\cdot d\vec{A}$$

Finally, substituting $e = u + V^2/2 + gz$ into the last term, we obtain the familiar form of the first law for a control volume,

$$\dot{Q} - \dot{W}_s - \dot{W}_{\text{shear}} - \dot{W}_{\text{other}} = \frac{\partial}{\partial t}\int_{\text{CV}} e\,\rho\, d\forall + \int_{\text{CS}} \left(u + pv + \frac{V^2}{2} + gz\right)\rho \vec{V}\cdot d\vec{A} \tag{4.56}$$

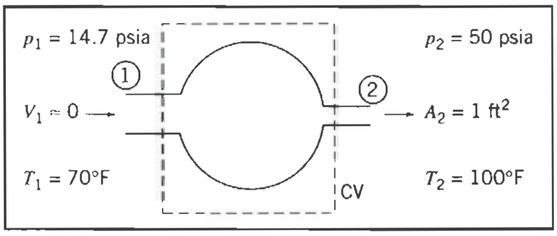
Each work term in Eq. 4.56 represents the rate of work done by the control volume on the surroundings. Note that in thermodynamics, for convenience, the combination $u + pv$ (the fluid internal energy plus what is often called the "flow work") is usually replaced with enthalpy, $h \equiv u + pv$ (this is one of the reasons h was invented).

EXAMPLE 4.16 Compressor: First Law Analysis

Air at 14.7 psia, 70°F, enters a compressor with negligible velocity and is discharged at 50 psia, 100°F through a pipe with 1 ft^2 area. The flow rate is 20 lbm/s. The power input to the compressor is 600 hp. Determine the rate of heat transfer.

EXAMPLE PROBLEM 4.16

GIVEN: Air enters a compressor at ① and leaves at ② with conditions as shown. The air flow rate is 20 lbm/s and the power input to the compressor is 600 hp.



FIND: Rate of heat transfer.

SOLUTION:

Governing equations:

$$\underbrace{\cancel{\frac{\partial}{\partial t}}\int_{CV} \rho\, d\forall}_{=0(1)} + \int_{CS} \rho \vec{V} \cdot d\vec{A} = 0$$

$$\dot{Q} - \dot{W}_s - \underbrace{\cancel{\dot{W}_{shear}}}_{=0(4)} = \underbrace{\cancel{\frac{\partial}{\partial t}}\int_{CV} e\, \rho\, d\forall}_{=0(1)} + \int_{CS}\left(u + pv + \frac{V^2}{2} + gz\right)\rho \vec{V} \cdot d\vec{A}$$

Assumptions:
(1) Steady flow
(2) Properties uniform over inlet and outlet sections.
(3) Treat air as an ideal gas, $p = \rho RT$.
(4) Area of CV at ① and ② perpendicular to velocity, thus $\dot{W}_{shear} = 0$.
(5) $z_1 = z_2$.
(6) Inlet kinetic energy is negligible.

Under the assumptions listed, the first law becomes

$$\dot{Q} - \dot{W}_s = \int_{CV}\left(u + pv + \frac{V^2}{2} + gz\right)\rho \vec{V} \cdot d\vec{A}$$

$$\dot{Q} - \dot{W}_s = \int_{CS}\left(h + \frac{V^2}{2} + gz\right)\rho \vec{V} \cdot d\vec{A}$$

or

$$\dot{Q} = \dot{W}_s + \int_{CS}\left(h + \frac{V^2}{2} + gz\right)\rho \vec{V} \cdot d\vec{A}$$

For uniform properties, assumption (2), we can write

$$\dot{Q} = \dot{W}_s + \left(h_1 + \underbrace{\cancel{\frac{V_1^2}{2}}}_{\approx 0(6)} + gz_1\right)(-\rho_1 V_1 A_1) + \left(h_2 + \frac{V_2^2}{2} + gz_2\right)(\rho_2 V_2 A_2)$$

For steady flow, from conservation of mass,

$$\int_{CS} \rho \vec{V} \cdot d\vec{A} = 0$$

Therefore, $-(\rho_1 V_1 A_1) + (\rho_2 V_2 A_2) = 0$, or $\rho_1 V_1 A_1 = \rho_2 V_2 A_2 = \dot{m}$. Hence we can write

$$\dot{Q} = \dot{W}_s + \dot{m}\left[(h_2 - h_1) + \frac{V_2^2}{2} + g\underbrace{\cancel{(z_2 - z_1)}}_{=0(5)}\right]$$

Assume that air behaves as an ideal gas with constant c_p. Then $h_2 - h_1 = c_p(T_2 - T_1)$, and

$$\dot{Q} = \dot{W}_s + \dot{m}\left[c_p(T_2 - T_1) + \frac{V_2^2}{2}\right]$$

From continuity $V_2 = \dot{m}/\rho_2 A_2$. Since $p_2 = \rho_2 RT_2$,

$$V_2 = \frac{\dot{m}}{A_2}\frac{RT_2}{p_2} = \frac{20 \text{ lbm}}{\text{s}} \times \frac{1}{1 \text{ ft}^2} \times \frac{53.3 \text{ ft} \cdot \text{lbf}}{\text{lbm} \cdot {}^\circ\text{R}} \times 560^\circ\text{R} \times \frac{\text{in.}^2}{50 \text{ lbf}} \times \frac{\text{ft}^2}{144 \text{ in.}^2}$$

$V_2 = 82.9$ ft/s

$$\dot{Q} = \dot{W}_s + \dot{m}c_p(T_2 - T_1) + \dot{m}\frac{V_2^2}{2}$$

Note that power input is *to* the CV, so $\dot{W}_s = -600$ hp, and

$$\dot{Q} = -\frac{600\ \text{hp}}{} \times \frac{550\ \text{ft} \cdot \text{lbf}}{\text{hp} \cdot \text{s}} \times \frac{\text{Btu}}{778\ \text{ft} \cdot \text{lbf}} + \frac{20\ \text{lbm}}{\text{s}} \times 0.24\ \frac{\text{Btu}}{\text{lbm} \cdot {}^\circ\text{R}} \times 30^\circ\text{R}$$

$$+\frac{20\ \text{lbm}}{\text{s}} \times \frac{(82.9)^2}{2}\ \frac{\text{ft}^2}{\text{s}^2} \times \frac{\text{slug}}{32.2\ \text{lbm}} \times \frac{\text{Btu}}{778\ \text{ft} \cdot \text{lbf}} \times \frac{\text{lbf} \cdot \text{s}^2}{\text{slug} \cdot \text{ft}}$$

$$\dot{Q} = -277\ \text{Btu/s} \longleftarrow$$

{heat rejection} $\dot{Q}$

> This problem illustrates use of the first law of thermodynamics for a control volume. It is also an example of the care that must be taken with unit conversions for mass, energy, and power.

EXAMPLE 4.17 Tank Filling: First Law Analysis

A tank of 0.1 m^3 volume is connected to a high-pressure air line; both line and tank are initially at a uniform temperature of 20°C. The initial tank gage pressure is 100 kPa. The absolute line pressure is 2.0 MPa; the line is large enough so that its temperature and pressure may be assumed constant. The tank temperature is monitored by a fast-response thermocouple. At the instant after the valve is opened, the tank temperature rises at the rate of 0.05°C/s. Determine the instantaneous flow rate of air into the tank if heat transfer is neglected.

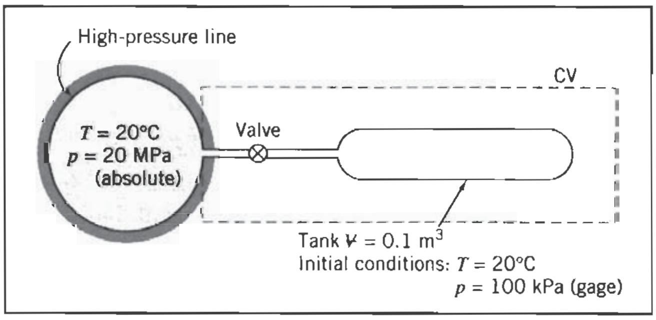
EXAMPLE PROBLEM 4.17

GIVEN: Air supply pipe and tank as shown. At $t = 0^+$, $\partial T/\partial t = 0.05$°C/s.

FIND: $\dot{m}$ at $t = 0^+$.

SOLUTION:

Choose CV shown, apply energy equation.



Governing equation:
$$\overset{=0(1)}{\cancel{\dot{Q}}} - \overset{=0(2)}{\cancel{\dot{W}_s}} - \overset{=0(3)}{\cancel{\dot{W}_{\text{shear}}}} - \overset{=0(4)}{\cancel{\dot{W}_{\text{other}}}} = \frac{\partial}{\partial t}\int_{\text{CV}} e\,\rho\, d\mathcal{V} + \int_{\text{CS}} (e + pv)\rho\vec{V}\cdot d\vec{A}$$

$$e = u + \overset{\simeq 0(5)}{\cancel{\frac{V^2}{2}}} + \overset{\simeq 0(6)}{\cancel{gz}}$$

Assumptions:
(1) $\dot{Q} = 0$ (given).
(2) $\dot{W}_s = 0$.
(3) $\dot{W}_{\text{shear}} = 0$.
(4) $\dot{W}_{\text{other}} = 0$.
(5) Velocities in line and tank are small.
(6) Neglect potential energy.
(7) Uniform flow at tank inlet.
(8) Properties uniform in tank.
(9) Ideal gas, $p = \rho RT$, $du = c_v\, dT$.

Then

$$\frac{\partial}{\partial t}\int_{\text{CV}} u_{\text{tank}}\,\rho\, d\mathcal{V} + (u + pv)\big|_{\text{line}}(-\rho VA) = 0$$

This expresses the fact that the gain in energy in the tank is due to influx of fluid energy (in the form of enthalpy $h = u + pv$) from the line. We are interested in the initial instant, when T is uniform at 20°C, so $u_{\text{tank}} = u_{\text{line}} = u$, the internal energy at T; also, $pv_{\text{line}} = RT_{\text{line}} = RT$, and

$$\frac{\partial}{\partial t}\int_{\text{CV}} u\,\rho\, d\mathcal{V} + (u + RT)(-\rho VA) = 0$$

Since tank properties are uniform, $\partial/\partial t$ may be replaced by d/dt, and

$$\frac{d}{dt}(uM) = (u + RT)\dot{m}$$

(where M is the instantaneous mass in the tank and $\dot{m} = \rho VA$ is the mass flow rate), or

$$u\frac{dM}{dt} + M\frac{du}{dt} = u\dot{m} + RT\dot{m} \tag{1}$$

The term dM/dt may be evaluated from continuity:

Governing equation: $$\frac{\partial}{\partial t}\int_{\text{CV}} \rho d\mathcal{V} + \int_{\text{CS}} \rho\vec{V}\cdot d\vec{A} = 0$$

$$\frac{dM}{dt} + (-\rho VA) = 0 \quad \text{or} \quad \frac{dM}{dt} = \dot{m}$$

Substituting in Eq. 1 gives

$$u\dot{m} + Mc_v\frac{dT}{dt} = u\dot{m} + RT\dot{m}$$

or

$$\dot{m} = \frac{Mc_v(dT/dt)}{RT} = \frac{\rho\mathcal{V}c_v(dT/dt)}{RT} \tag{2}$$

But at $t = 0$, $p_{\text{tank}} = 100$ kPa (gage), and

$$\rho = \rho_{\text{tank}} = \frac{p_{\text{tank}}}{RT} = \frac{(1.00 + 1.01)10^5}{} \frac{\text{N}}{\text{m}^2} \times \frac{\text{kg}\cdot\text{K}}{287\ \text{N}\cdot\text{m}} \times \frac{1}{293\ \text{K}} = 2.39\ \text{kg/m}^3$$

Substituting into Eq. 2, we obtain

$$\dot{m} = 2.39\,\frac{\text{kg}}{\text{m}^3} \times 0.1\,\text{m}^3 \times 717\,\frac{\text{N}\cdot\text{m}}{\text{kg}\cdot\text{K}} \times 0.05\,\frac{\text{K}}{\text{s}} \times \frac{\text{kg}\cdot\text{K}}{287\,\text{N}\cdot\text{m}} \times \frac{1}{293\,\text{K}} \times 1000\,\frac{\text{g}}{\text{kg}}$$

$$\dot{m} = 0.102\ \text{g/s} \qquad \dot{m}$$

This problem illustrates use of the first law of thermodynamics for a control volume. It is also an example of the care that must be taken with unit conversions for mass, energy, and power.

4-9 THE SECOND LAW OF THERMODYNAMICS

Recall that the system formulation of the second law is

$$\left.\frac{dS}{dt}\right)_{\text{system}} \geq \frac{1}{T}\dot{Q} \tag{4.5a}$$

where the total entropy of the system is given by

$$S_{\text{system}} = \int_{M(\text{system})} s\,dm = \int_{\forall(\text{system})} s\,\rho\,d\forall \tag{4.5b}$$

To derive the control volume formulation of the second law of thermodynamics, we set

$$N = S \qquad \text{and} \qquad \eta = s$$

in Eq. 4.10 and obtain

$$\left.\frac{dS}{dt}\right)_{\text{system}} = \frac{\partial}{\partial t}\int_{\text{CV}} s\,\rho\,d\forall + \int_{\text{CS}} s\,\rho\vec{V}\cdot d\vec{A} \tag{4.57}$$

The system and the control volume coincide at t_0; thus in Eq. 4.5a,

$$\left.\frac{1}{T}\dot{Q}\right)_{\text{system}} = \left.\frac{1}{T}\dot{Q}\right)_{\text{CV}} = \int_{\text{CS}} \frac{1}{T}\left(\frac{\dot{Q}}{A}\right)dA$$

In light of this, Eqs. 4.5a and 4.57 yield the control volume formulation of the second law of thermodynamics

$$\frac{\partial}{\partial t}\int_{\text{CV}} s\,\rho\,d\forall + \int_{\text{CS}} s\,\rho\,\vec{V}\cdot d\vec{A} \geq \int_{\text{CS}} \frac{1}{T}\left(\frac{\dot{Q}}{A}\right)dA \tag{4.58}$$

In Eq. 4.58, the factor $(\dot{Q}/A)$ represents the heat flux per unit area into the control volume through the area element dA. To evaluate the term

$$\int_{\text{CS}} \frac{1}{T}\left(\frac{\dot{Q}}{A}\right)dA$$

both the local heat flux, $(\dot{Q}/A)$, and local temperature, T, must be known for each area element of the control surface.

4-10 SUMMARY

In this chapter we wrote the basic laws for a system: mass conservation (or continuity), Newton's second law, the angular-momentum equation, the first law of thermodynamics, and the second law of thermodynamics. We then developed an equation (sometimes called the Reynolds Transport Theorem) for relating system formulations to control volume formulations. Using this we derived control volume forms of:

- ✓ The mass conservation equation (sometimes called the continuity equation).
- ✓ Newton's second law (in other words, a momentum equation) for:
  - ○ An inertial control volume.
  - ○ A control volume with rectilinear acceleration.
  - ○*A control volume with arbitrary acceleration (on the CD).
- ✓ *The angular-momentum equation for:
  - ○ A fixed control volume.
  - ○ A rotating control volume (on the CD).
- ✓ The first law of thermodynamics (or energy equation).
- ✓ The second law of thermodynamics.

We discussed the physical meaning of each term appearing in these control volume equations, and used the equations for the solution of a variety of flow problems. In particular, we used a differential control volume* to derive a famous equation in fluid mechanics—the Bernoulli equation—and while doing so learned about the restrictions on its use in solving problems.

PROBLEMS

4.1 In order to cool a six-pack as quickly as possible, it is placed in a freezer for a period of 1 hr. If the room temperature is 25°C and the cooled beverage is at a final temperature of 5°C, determine the change in specific entropy of the beverage.

4.2 A mass of 3 kg falls freely a distance of 5 m before contacting a spring attached to the ground. If the spring stiffness is 400 N/m, what is the maximum spring compression?

4.3 A fully loaded Boeing 777-200 jet transport aircraft weighs 715,000 lbf. The pilot brings the 2 engines to full takeoff thrust of 102,000 lbf each before releasing the brakes. Neglecting aerodynamic and rolling resistance, estimate the minimum runway length and time needed to reach a takeoff speed of 140 mph. Assume engine thrust remains constant during ground roll.

4.4 A police investigation of tire marks showed that a car traveling along a straight level street had skidded to a stop for a total distance of 50 m after the brakes were applied.

* These topics apply to a section that may be omitted without loss of continuity in the text material.