ESO204A, Fluid Mechanics and rate Processes

Laminar, incompressible, viscous flow: Exact Solutions

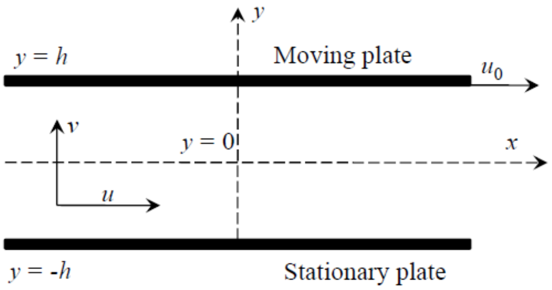
Couette-Poiseuille flow

Chapter 4 of F M White
Chapter 5 of Fox McDonald

Couette-Poiseuille Flow

Laminar, incompressible, steady flow between two infinitely long parallel plates; top plate moving steadily, bottom plate stationary

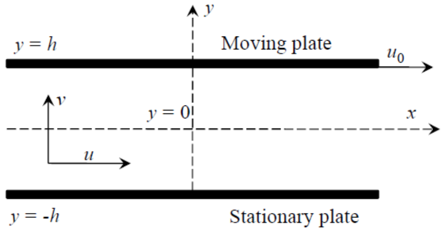
We continue to assume 2-D, fully developed flow



$u = u(x,y), v = (x,y), w = 0$ $\quad \dfrac{\partial \vec{u}}{\partial x} = 0 \quad \dfrac{\partial p}{\partial x}$ may be non-zero

$$u = u(x, y), v = (x, y), w = 0, \frac{\partial \vec{u}}{\partial x} = 0$$

Now, our goal is to find three unknowns (u, v, p) from continuity and momentum Equations



Applying continuity Eq. $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \quad \Rightarrow \frac{\partial v}{\partial y} = 0$

$$v = f(x) \qquad \text{BC: } v(y = h) = 0$$

$$v = 0$$

$$u = u(x, y), v = w = 0; \ \frac{\partial \vec{u}}{\partial x} = 0$$

$$z\text{-mom: } \frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial z} + \nu\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2}\right)$$

$$\frac{\partial p}{\partial z} = 0$$

$$y\text{-mom: } \frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial y} + \nu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right) \Rightarrow \frac{\partial p}{\partial y} = 0$$

$$\Rightarrow p = p(x)$$

$$x\text{-mom: } \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) \qquad \nu = \frac{\mu}{\rho}$$

$$\Rightarrow \frac{d^2 u}{dy^2} = \frac{1}{\mu}\frac{dp}{dx} = \text{constant}$$

$$\Rightarrow \frac{d^2u}{dy^2} = \frac{1}{\mu}\frac{dp}{dx} \quad \Rightarrow \frac{du}{dy} = \frac{1}{\mu}\frac{dp}{dx}y + c_1 \quad \Rightarrow u = \frac{1}{\mu}\frac{dp}{dx}\frac{y^2}{2} + c_1 y + c_2$$

No-slip BCs: $u(x, y = h) = u_0,\ u(x, y = -h) = 0$

$$\Rightarrow u = -\frac{h^2}{2\mu}\frac{dp}{dx}\left(1 - \frac{y^2}{h^2}\right) + \frac{u_0}{2}\left(1 + \frac{y}{h}\right)$$

$$u = u_1\left(1 - \frac{y^2}{h^2}\right) + u_0\frac{1}{2}\left(1 + \frac{y}{h}\right) \quad \text{where} \quad u_1 = -\frac{h^2}{2\mu}\frac{dp}{dx}$$

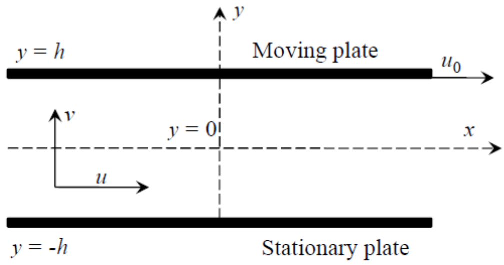
Couette flow : $u_1 = 0 \Rightarrow u = \frac{u_0}{2}\left(1 + \frac{y}{h}\right)$

Poiseuille flow : $u_0 = 0 \Rightarrow u = -\frac{h^2}{2\mu}\frac{dp}{dx}\left(1 - \frac{y^2}{h^2}\right)$

To find pressure , we use

$$\frac{dp}{dx} = a \ (\text{constant})$$

$$\Rightarrow p = ax + b$$



Velocity field

$$v = w = 0$$

$$u = u_1\left(1 - \frac{y^2}{h^2}\right) + u_0\frac{1}{2}\left(1 + \frac{y}{h}\right) \quad \text{where} \quad u_1 = -\frac{h^2}{2\mu}\frac{dp}{dx}$$

$$\frac{u}{u_0} = \frac{u_1}{u_0}\left(1 - \frac{y^2}{h^2}\right) + \frac{1}{2}\left(1 + \frac{y}{h}\right)$$

Nondimensional form

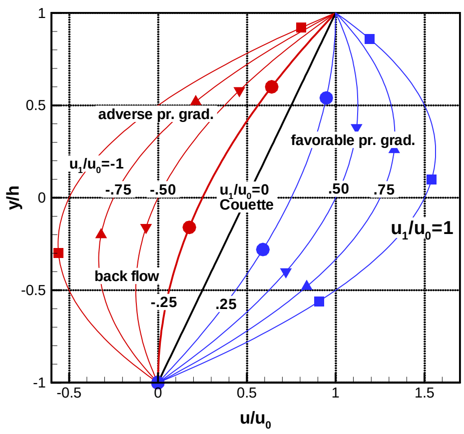
Couette-Poiseuille Flow

$$\frac{u}{u_0}=\frac{u_1}{u_0}\left(1-\frac{y^2}{h^2}\right)+\frac{1}{2}\left(1+\frac{y}{h}\right)$$

where $\dfrac{u_1}{u_0}=-\dfrac{h^2}{2\mu u_0}\dfrac{dp}{dx}$

$$\frac{du}{dy}=-\frac{2y}{h^2}u_1+\frac{1}{2h}u_0$$

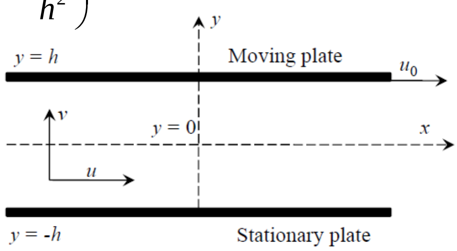
$$\left[\frac{du}{dy}\right]_{y=-h}=\frac{2}{h}\left(u_1+\frac{u_0}{4}\right)$$



$$\left[\frac{du}{dy}\right]_{y=-h}>0 \Rightarrow \frac{u_1}{u_0}>-\frac{1}{4} \qquad \left[\frac{du}{dy}\right]_{y=-h}\leq 0 \Rightarrow \frac{u_1}{u_0}\leq-\frac{1}{4} \Rightarrow \frac{dp}{dx}\geq\frac{\mu u_0}{2h^2}$$

Adverse pr. grad. Is a necessary, but not sufficient condition for backflow

Poiseuille Flow $u=u_1\left(1-\dfrac{y^2}{h^2}\right)$



where $u_1=-\dfrac{h^2}{2\mu}\dfrac{dp}{dx}$

$$u_{av}=\frac{1}{2h}\int_{-h}^{h}u\,dy$$

$$=\frac{1}{2h}\int_{-h}^{h}u_1\left(1-\frac{y^2}{h^2}\right)dy=\frac{2}{3}u_1$$

$$\frac{du}{dy}=-\frac{2y}{h^2}u_1=-\frac{3y}{h^2}u_{av} \qquad \Rightarrow \tau_w=\mu\left[\frac{du}{dy}\right]_{y=-h}=\frac{3\mu u_{av}}{h}$$

$$\frac{\tau_w}{\frac{1}{2}\rho u_{av}^{\ 2}}=\frac{6\mu}{\rho u_{av}h} \Rightarrow C_f=\frac{6}{\mathrm{Re}_h} \qquad \mathrm{Po}=C_f\,\mathrm{Re}=6$$

Couette Flow

$$u = \frac{u_0}{2}\left(1+\frac{y}{h}\right) = u_{\rm av}\left(1+\frac{y}{h}\right)$$

$$\tau_w = \mu\left[\frac{du}{dy}\right]_{y=-h} = \frac{\mu u_0}{2h} = \frac{\mu u_{\rm av}}{h}$$

$$\frac{\tau_w}{\frac{1}{2}\rho u_{\rm av}{}^2} = \frac{2\mu}{\rho u_{\rm av} h} \quad C_f = \frac{2}{\mathrm{Re_h}}$$

$$\mathrm{Po} = C_f\,\mathrm{Re} = 2$$

Poiseuille Flow: Summary

$$u = u_1\left(1-\frac{y^2}{h^2}\right)$$

$$u_{\rm av} = \frac{1}{2h}\int_{-h}^{h} u\,dy = \frac{1}{2h}\int_{-h}^{h} u_1\left(1-\frac{y^2}{h^2}\right)dy = \frac{2}{3}u_1$$

$$\frac{du}{dy} = -\frac{2y}{h^2}u_1 = -\frac{3y}{h^2}u_{\rm av} \Rightarrow \tau_w = \mu\left[\frac{du}{dy}\right]_{y=-h} = \frac{3\mu u_{\rm av}}{h}$$

$$\frac{\tau_w}{\frac{1}{2}\rho u_{\rm av}{}^2} = \frac{6\mu}{\rho u_{\rm av} h} \Rightarrow C_f = \frac{6}{\mathrm{Re_h}} \Rightarrow \mathrm{Po} = C_f\,\mathrm{Re} = 6$$