Stochastic Calculus

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Background

Definition

A random variable X_t is a mapping $X_t : \Omega \to \mathbb{R}$. Where Ω is the **sample space** and P is the measure of the **probability space**, such that $P(\Omega) = 1$.

Definition translated into english

1. Unbeknownst to us, someone chooses a random $\omega \in \Omega$. Then we see the $X(\omega) \in \mathbb{R}$.
2. We cannot see the corresponding $\omega \in \Omega$, but the $X(\omega) \in \mathbb{R}$ gives us partial information about ω.

Background

Example

Consider the case where you flip a coin. Using our previous definition, this could be described as $\Omega = \{\text{heads}, \text{tails}\}$ and

$$X(\omega) = \begin{cases} 1, & \text{if } \omega = \text{heads} \\ \text{-}1, & \text{if } \omega = \text{tails} \end{cases} \text{ where } \omega \in \Omega.$$

This would yield the familiar notation of $P(X = 1) = .5$ and $P(X = \text{-}1) = .5$.

Background

Definition

A **stochastic process** is a function that takes a random variable.

Example

Coin tossing. If you consider our random variable from the previous example, a stochastic process would just be some function $f(X(t, \omega))$, where $X(t, \omega)$ is our X, but our mysterious man just picks a new random ω everytime t changes.

Wiener Process

Definition

A stochastic process W is called a **Wiener process** if the follow conditions hold

1. $W_0 = 0$
2. The process W has independent increments
3. For $s < t$ the random variable $W_t - W_s$ has the Gaussian distribution $N(0, t-s)$
4. W has continuous trajectories

Wiener Process

Theorem

A Wiener trajectory is with probability one, nowhere differentiable, and it has locally infinite total variation.

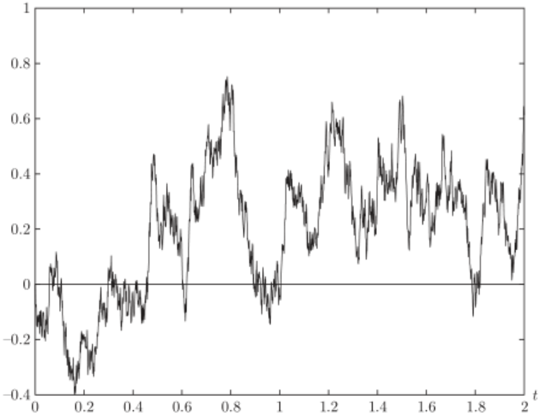

Figure: Wiener trajectory

Quadratic Variation

Motivation

A stochastic process does not have the normal notion of variance.

Definition

Suppose P is a partition of $[0, t]$ denoted t_k and let $\|P\|$ be the mesh of the partition then

$$[X]_t = \lim_{\|P\| \to 0} \sum_{k=1}^{n} (X_{t_k} - X_{t_{k-1}})^2$$

Comments

Note that Quadratic Variation itself is a stochastic process. An intuitive way to think about quadratic variation is the internal clock of a process, describing how randomness accumulates over time.

Quadratic Variation of Wiener Process

Theorem

Quadratic variation of a Wiener Process is t

Proof

Let $P = \{0 = t_0 \leq t_1 \leq \cdots \leq t_m = t\}$ be a partition of the interval $[0, t]$. Then the quadratic variation on P is

$$[W]^P = \sum_{k=1}^{m} (W_{t_k} - W_{t_{k-1}})^2$$

Therefore,

$$E\left[[W]^P\right] = \sum_{k=1}^{m} E\left[(W_{t_k} - W_{t_{k-1}})^2\right]$$

Quadratic Variation of Wiener Process

Theorem

Quadratic variation of a Wiener Process is t

Proof cont.

Note that $E[(W_{t_{i+1}} - W_{t_i})^2] = Var\left[W_{t_{i+1}} - W_{t_i}\right]$ such that

$$E\left[[W]^P\right] = \sum_{k=1}^{m} Var\left[W_{t_k} - W_{t_{k-1}}\right]$$

It follows from the definition of the Wiener process that

$$\begin{aligned} E\left[[W]^P\right] &= \sum_{k=1}^{m} (t_k - t_{k-1}) \\ &= t \end{aligned}$$

Quadratic Variation of Wiener Process

Theorem

Quadratic variation of a Wiener Process is t

Proof end.

Finally, from the definition of discrete expectation we have that $E[[W]^P] := \lim_{\|P\| \to 0} [W]^P := t$ $\square$

Implications

This motivates us to write

$$\int_0^t (dW_t)^2 = t$$

Or equivalently,

$$(dW_t)^2 = dt$$

Everything Itô

The Stochastic Integral

The Problem

Integrals of the form $\int_0^t g_s dW_s$.

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Solution 1

Riemman Integral

1. $\sum_{k=1}^{n} g_s(t_k)(W_{k+1} - W_k)$
2. Not possible due to locally unbounded variation

Geometric Brownian Motion Model

Definition

Geometric Brownian Motion is a stochastic process whose dynamics follow the stochastic differential equation

$$dX_t = \alpha X_t dt + \sigma X_t dW_t$$

We can write the equation as

$$X_t = (\alpha + \sigma W_t) X_t$$

Where W is the time derivative of the Wiener process.

Geometric Brownian Motion Model

Derivation

Assume S_t follows a Geometric Brownian Motion. Using Ito's lemma,

$$d\log(S_t) = \frac{1}{S_t}dS_t - \frac{1}{2S_t^2}(dSt)^2$$

From the definition,

$$d\log(S_t) = \frac{1}{S_t}(\mu S_t dt + \sigma S_t dW_t) - \frac{1}{2S_t^2}(\mu S_t dt + \sigma S_t dW_t)^2$$

Note that

$$(\mu S_t dt + \sigma S_t dW_t)^2 = S_t^2(\mu^2 dt^2 + \sigma^2 dW_t^2 + 2\mu\sigma dt dW_t)$$

From Ito's multiplication table, $dt^2 = 0 = dtdW_t$ and $dW_t^2 = dt$ such that

$$(\mu S_t dt + \sigma S_t dW_t)^2 = S_t^2\sigma^2 dt$$

Geometric Brownian Motion Model

Derivation cont.

Substituting back,

$$
\begin{aligned}
d\log(S_t) &= (\mu dt + \sigma dW_t) - \frac{1}{2S_t^2}(S_t^2\sigma^2 dt) \\
&= (\mu - \frac{\sigma^2}{2})dt + \sigma dW_t
\end{aligned}
$$