Stochastic Calculus

Kellen Kanarios

University of Michigan

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Background

Definition

A real-valued random variable X is a mapping $X : \Omega \to \mathbb{R}$. Here, Ω is the **sample space** and $\mathbb{P}$ is the measure of the **probability space**, such that $\mathbb{P}(\Omega) = 1$.

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3. In $\mathbb{R}^3$: volume.

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2. We cannot see the corresponding $\omega \in \Omega$, but the $X(\omega) \in \mathbb{R}$ gives us partial information about ω.
3. $\mathbb{P}$ tells us how likely subsets $A \subseteq \omega$ are to occur.

Background

Example

Consider the case where you flip a coin. Using our previous definition, this could be described as $\Omega = \{\text{heads}, \text{tails}\}$ and

$$X(\omega) = \begin{cases} 1, & \text{if } \omega = \text{heads} \\ -1, & \text{if } \omega = \text{tails} \end{cases} \text{ where } \omega \in \Omega.$$

This would yield the familiar notation of $\mathbb{P}(X = 1) = .5$ and $\mathbb{P}(X = -1) = .5$ for a fair coin.

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A sequence of coin flips. At each time t your process corresponds to a random variable (aka coinflip) X_t.

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1. $W_0 = 0$
2. The process W has independent increments
3. For $s < t$ the random variable $W_t - W_s$ has the Gaussian distribution $\mathcal{N}(0, t - s)$
4. W has continuous trajectories

Wiener Process

Theorem

A Wiener trajectory is with probability one, nowhere differentiable, and it has locally infinite total variation.

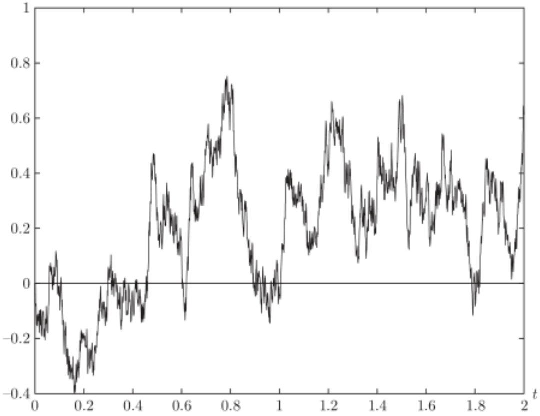

Figure: Wiener trajectory

Quadratic Variation

Definition

Let X be a stochastic process. Suppose P is a partition of $[0, t]$ denoted t_k and let $\|P\|$ be the mesh of the partition then the **quadratic variation** of X on P is defined to be:

$$[X]_t = \lim_{\|P\| \to 0} \sum_{k=1}^{n} (X_{t_k} - X_{t_{k-1}})^2$$

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Comments

Note that Quadratic Variation itself is a stochastic process. An intuitive way to think about quadratic variation is the internal clock of a process, describing how randomness accumulates over time.

Quadratic Variation of Wiener Process

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Proof.

Let $P = \{0 = t_0 \leq t_1 \leq \cdots \leq t_m = t\}$ be a partition of the interval $[0, t]$. Then the quadratic variation on P is

$$[W]^P = \sum_{k=1}^{m} (W_{t_k} - W_{t_{k-1}})^2$$

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Therefore,

$$E\left[[W]^P\right] = \sum_{k=1}^{m} E\left[(W_{t_k} - W_{t_{k-1}})^2\right]$$

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Thus, $Var(W_{t_{k+1}} - W_{t_k}) = E((W_{t_{k+1}} - W_{t_k})^2)$.

Quadratic Variation of Wiener Process

Proof.

Since $E[(W_{t_{i+1}} - W_{t_i})^2] = Var\left[W_{t_{i+1}} - W_{t_i}\right]$, we can write:

$$E\left[[W]^P\right] = \sum_{k=1}^{m} Var\left[W_{t_k} - W_{t_{k-1}}\right]$$

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$$E\left[[W]^P\right] = \sum_{k=1}^{m} Var\left[W_{t_k} - W_{t_{k-1}}\right]$$

Using $W_{t_k} - W_{t_{k-1}} \in \mathcal{N}(0, t_k - t_{k-1})$:

$$\begin{aligned} E\left[[W]^P\right] &= \sum_{k=1}^{m} (t_k - t_{k-1}) \\ &= t \end{aligned}$$

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$$\begin{aligned} E\left[[W]^P\right] &= \sum_{k=1}^{m}(t_k - t_{k-1}) \\ &= t \end{aligned}$$

Finally, from the definition of discrete expectation we have that $E[[W]^P] := \lim_{\|P\| \to 0} [W]^P := t$ $\square$

Quadratic Variation of Wiener Process

Implications

This motivates us to write

$$\int_0^t (dW_t)^2 = t$$

Or equivalently,

$$(dW_t)^2 = dt$$

The Stochastic Integral

The Problem

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Possible Solution 1

Try to define like the Riemman Integral

1. $\sum_{k=1}^{n} g_s (W_{t_{k+1}} - W_{t_k})$

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Integrals of the form $\int_0^t g_s dW_s$.

Possible Solution 1

Try to define like the Riemman Integral

1. $\sum_{k=1}^{n} g_s(W_{t_{k+1}} - W_{t_k})$
2. Not possible due to locally unbounded variation of a Wiener Process.

The Stochastic Integral

Definition

A function is **simple** on $[a, b]$ when there exists deterministic points in time $a = t_0 < t_1 < \cdots < t_n = b$ such that $g_s = g_{t_k}$ for all $s \in [t_k, t_{k+1}]$.

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Real Solution

Now we can define the stochastic integral by

$$\int_a^b g_s dW_s = \sum_{k=0}^{n-1} g_{t_k} [W_{t_{k+1}} - W_{t_k}]$$

For some simple function g_s.

Geometric Brownian Motion Model

Definition

Geometric Brownian Motion is a stochastic process whose dynamics follow the stochastic differential equation

$$dX_t = \alpha X_t dt + \sigma X_t dW_t \text{ for some } \alpha, \beta \in \mathbb{R}$$

Where dW_t is the infinitesimal increment of the Wiener process.

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Claim:

The closed form of Geometric Brownian Motion is:

$$X_t = X_0 e^{(\mu - \frac{\sigma^2}{2})dt + \sigma dW_t}$$

Geometric Brownian Motion Model

Proof.

Assume X_t follows a Geometric Brownian Motion. Using Ito's lemma,

$$d\ln(X_t) = \frac{1}{X_t}dX_t - \frac{1}{2X_t^2}(dX_t)^2$$

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From our definition of Geometric Brownian Motion,

$$d\ln(X_t) = \frac{1}{X_t}(\mu X_t dt + \sigma X_t dW_t) - \frac{1}{2X_t^2}(\mu X_t dt + \sigma X_t dW_t)^2$$

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Note that:

$$(\mu X_t dt + \sigma X_t dW_t)^2 = X_t^2(\mu^2 dt^2 + \sigma^2 dW_t^2 + 2\mu\sigma dt dW_t)$$

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From Ito's multiplication table, $(dt)^2 = 0 = (dt)(dW_t)$ and $(dW_t)^2 = dt$ such that

$$(\mu X_t dt + \sigma X_t dW_t)^2 = X_t^2 \sigma^2 dt$$

Geometric Brownian Motion Model

Proof.

Substituting back,

$$
\begin{aligned}
d\ln(X_t) &= (\mu dt + \sigma dW_t) - \frac{1}{2X_t^2}(X_t^2\sigma^2 dt) \\
&= (\mu - \frac{\sigma^2}{2})dt + \sigma dW_t.
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It follows that $X_t = e^{(\mu - \frac{\sigma^2}{2})dt + \sigma dW_t}$ $\square$