Stochastic Calculus

Kellen Kanarios

Mentor: Ethan Zell

University of Michigan

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Background

Definition

A real-valued random variable X is a mapping $X : \Omega \rightarrow \mathbb{R}$. Here, Ω is the **sample space** and $\mathbb{P}$ is the measure of the **probability space**, such that $\mathbb{P}(\Omega) = 1$.

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Intuition for a **measure**:

1. In $\mathbb{R}$: length,
2. In $\mathbb{R}^2$: area,
3. In $\mathbb{R}^3$: volume.

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2. We cannot see the corresponding $\omega \in \Omega$, but the $X(\omega) \in \mathbb{R}$ gives us partial information about ω.
3. $\mathbb{P}$ tells us how likely subsets $A \subseteq \Omega$ are to occur.

Background

Example

Consider the case where you flip a coin. Using our previous definition, this could be described as $\Omega = \{\text{heads}, \text{tails}\}$ and

$$X(\omega) = \begin{cases} 1, & \text{if } \omega = \text{heads} \\ -1, & \text{if } \omega = \text{tails} \end{cases} \qquad \text{where } \omega \in \Omega.$$

This would yield the familiar notation of $\mathbb{P}(X = 1) = .5$ and $\mathbb{P}(X = -1) = .5$ for a fair coin.

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Example

A sequence of coin flips. At each time t your process corresponds to a random variable (aka coinflip) X_t.

Wiener Process

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A stochastic process W is called a **Wiener process** if the follow conditions hold:

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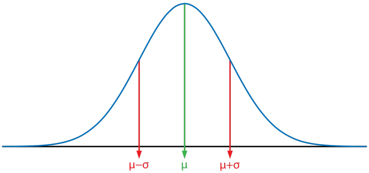
2. The process W has independent increments,
3. For $s < t$ the random variable $W_t - W_s$ has the Gaussian distribution $\mathcal{N}(0, t - s)$,



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Definition

A stochastic process W is called a **Wiener process** if the follow conditions hold:

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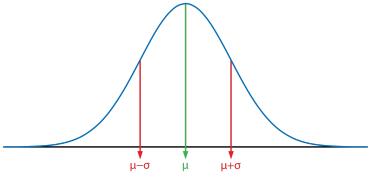

4. W has continuous trajectories.

Wiener Process

Theorem

A Wiener trajectory is with probability one, nowhere differentiable, and it has locally infinite total variation.

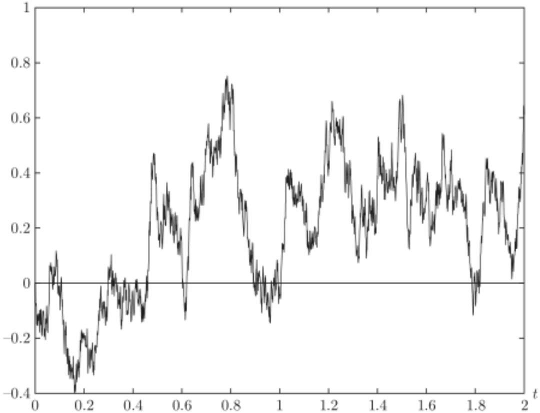

Figure: Wiener trajectory

Quadratic Variation

Definition

Let X be a stochastic process. Suppose P is a partition of $[0, t]$ denoted t_k and let $\|P\|$ be the mesh of the partition then the **quadratic variation** of X is defined to be:

$$[X]_t = \lim_{\|P\| \to 0} \sum_{k=1}^{n} (X_{t_k} - X_{t_{k-1}})^2.$$

Quadratic Variation of Wiener Process

Theorem

Quadratic variation of a Wiener Process on the interval $[0, t]$ *is* t.

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Proof.

Let us fix a point t and subdivide the interval $[0, t]$ into n equally large subintervals of the form $[k\frac{t}{n}, (k+1)\frac{t}{n}]$, where $k = 0, 1, \ldots, n-1$. Then the quadratic variation on P is:

$$[W]_t^P = \sum_{i=1}^{n} (W_{i\frac{t}{n}} - W_{(i-1)(\frac{t}{n})})^2.$$

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$$[W]_t^P = \sum_{i=1}^{n} (W_{i\frac{t}{n}} - W_{(i-1)(\frac{t}{n})})^2.$$

Therefore,

$$\mathbb{E}\left[[W]_t^P\right] = \sum_{i=1}^{n} \mathbb{E}\left[(W_{i(\frac{t}{n})} - W_{(i-1)(\frac{t}{n})})^2\right].$$

Quadratic Variation of Wiener Process

Theorem

$$Var(X) = \mathbb{E}(X^2) - (\mathbb{E}(X))^2.$$

Quadratic Variation of Wiener Process

Proof cont.

Since $W_{(i)(\frac{t}{n})} - W_{(i-1)(\frac{t}{n})} \in \mathcal{N}(0, i(\frac{t}{n}) - (i-1)(\frac{t}{n}))$,

$$\mathbb{E}(W_{i(\frac{t}{n})} - W_{(i-1)(\frac{t}{n})}) = 0.$$

Quadratic Variation of Wiener Process

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$$\mathbb{E}(W_{i(\frac{t}{n})} - W_{(i-1)(\frac{t}{n})}) = 0.$$

It follows that:

$$(\mathbb{E}\left[W_{i(\frac{t}{n})} - W_{(i-1)(\frac{t}{n})}\right])^2 = 0.$$

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Proof cont.

Since $W_{(i)(\frac{t}{n})} - W_{(i-1)(\frac{t}{n})} \in \mathcal{N}(0, i(\frac{t}{n}) - (i-1)(\frac{t}{n}))$,

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It follows that:

$$(\mathbb{E}\left[W_{i(\frac{t}{n})} - W_{(i-1)(\frac{t}{n})}\right])^2 = 0.$$

$$\text{Thus, } Var(W_{i(\frac{t}{n})} - W_{(i-1)(\frac{t}{n})}) = E(\left[W_{i(\frac{t}{n})} - W_{(i-1)(\frac{t}{n})}\right]^2).$$

Quadratic Variation of Wiener Process

Proof cont.

Since $W_{(i)(\frac{t}{n})} - W_{(i-1)(\frac{t}{n})} \in \mathcal{N}(0, i(\frac{t}{n}) - (i-1)(\frac{t}{n}))$,

$$\mathbb{E}(W_{i(\frac{t}{n})} - W_{(i-1)(\frac{t}{n})}) = 0.$$

It follows that:

$$(\mathbb{E}\left[W_{i(\frac{t}{n})} - W_{(i-1)(\frac{t}{n})}\right])^2 = 0.$$

Thus, $Var(W_{i(\frac{t}{n})} - W_{(i-1)(\frac{t}{n})}) = E(\left[W_{i(\frac{t}{n})} - W_{(i-1)(\frac{t}{n})}\right]^2)$.

Since $\mathbb{E}[(W_{i(\frac{t}{n})} - W_{(i-1)(\frac{t}{n})})^2] = Var\left[W_{i(\frac{t}{n})} - W_{(i-1)(\frac{t}{n})}\right]$, we can write:

$$\mathbb{E}\left[[W]_t^P\right] = \mathbb{E}\left[\sum_{k=1}^{n}(W_{i(\frac{t}{n})} - W_{(i-1)(\frac{t}{n})})^2\right] = Var\left[\sum_{k=1}^{n}\left[W_{i(\frac{t}{n})} - W_{(i-1)(\frac{t}{n})}\right]\right].$$

Quadratic Variation of Wiener Process

Stats fact

$$Var\left[\sum_{k=1}^{n}\left[W_{i(\frac{t}{n})}-W_{(i-1)(\frac{t}{n})}\right]\right]=\sum_{k=1}^{n}Var\left[W_{i(\frac{t}{n})}-W_{(i-1)(\frac{t}{n})}\right]$$

$$+\sum_{i\neq\ell}^{n}Cov\left(\left[W_{i(\frac{t}{n})}-W_{(i-1)(\frac{t}{n})}\right],\left[W_{\ell(\frac{t}{n})}-W_{(\ell-1)(\frac{t}{n})}\right]\right)$$

Quadratic Variation of Wiener Process

Proof cont.

Since Wiener increments are independent of eachother,

$$Cov\left(\left[W_{i(\frac{t}{n})} - W_{(i-1)(\frac{t}{n})}\right], \left[W_{\ell(\frac{t}{n})} - W_{(\ell-1)(\frac{t}{n})}\right]\right) = 0.$$

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Therefore,

$$E\left[[W]_t^P\right] = \sum_{k=1}^{n} Var\left[W_{i(\frac{t}{n})} - W_{(i-1)(\frac{t}{n})}\right]$$

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Using $W_{(i)(\frac{t}{n})} - W_{(i-1)(\frac{t}{n})} \in \mathcal{N}(0, i(\frac{t}{n}) - (i-1)(\frac{t}{n}))$:

$$\begin{aligned}\mathbb{E}\left[[W]_t^P\right] &= \sum_{k=1}^{n}(i(\frac{t}{n}) - (i-1)(\frac{t}{n})) \\ &= t\end{aligned}$$

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Now we must show $\mathbb{E}([W]_t^P) = [W]_t = t$.

Quadratic Variation of Wiener Process

Stats fact

If $Var(X) = 0$ then $X = \mathbb{E}[X]$.

Quadratic Variation of Wiener Process

Proof.

Therefore, we must show that :

$$Var(\lim_{\|P\|\to 0} [W]_t^P) = 0.$$

Or equivalently,

$$Var(\lim_{n\to\infty} \sum_{i=1}^{n} \left[W_{i(\frac{t}{n})} - W_{(i-1)(\frac{t}{n})}\right]^2) = 0,$$

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and we showed earlier:

$$Var(\lim_{n\to\infty}\sum_{i=1}^{n}\left[W_{i(\frac{t}{n})} - W_{(i-1)(\frac{t}{n})}\right]^2) = \lim_{n\to\infty}\sum_{i=1}^{n} Var(\left[W_{i(\frac{t}{n})} - W_{(i-1)(\frac{t}{n})}\right]^2).$$

Quadratic Variation of Wiener Process

Proof.

Also from earlier:

$$Var(\lim_{n\to\infty}\sum_{i=1}^{n}\left[W_{i(\frac{t}{n})}-W_{(i-1)(\frac{t}{n})}\right]^2)=\lim_{n\to\infty}\sum_{i=1}^{n}\mathbb{E}(\left[W_{i(\frac{t}{n})}-W_{(i-1)(\frac{t}{n})}\right]^4)$$
$$-\mathbb{E}\left(\left[W_{i(\frac{t}{n})}-W_{(i-1)(\frac{t}{n})}\right]^2\right)^2.$$

Quadratic Variation of Wiener Process

Stats fact

$$\mathsf{E}\left[(X-\mu)^p\right] = \begin{cases} 0, & \text{if } p \text{ is odd} \\ \sigma^p (p-1)!!, & \text{if } p \text{ is even.} \end{cases}$$

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Also from earlier:

$$Var(\lim_{n\to\infty}\sum_{i=1}^{n}\left[W_{i(\frac{t}{n})}-W_{(i-1)(\frac{t}{n})}\right]^2)=\lim_{n\to\infty}\sum_{i=1}^{n}\mathbb{E}(\left[W_{i(\frac{t}{n})}-W_{(i-1)(\frac{t}{n})}\right]^4)$$
$$-\mathbb{E}\left(\left[W_{i(\frac{t}{n})}-W_{(i-1)(\frac{t}{n})}\right]^2\right)^2.$$

Therefore,

$$Var(\lim_{n\to\infty}\sum_{i=1}^{n}\left[W_{i(\frac{t}{n})}-W_{(i-1)(\frac{t}{n})}\right]^2)=\lim_{n\to\infty}\sum_{i=1}^{n}3\left(i(\frac{t}{n})-(i-1)(\frac{t}{n})\right)^2$$
$$-\left(i(\frac{t}{n})-(i-1)(\frac{t}{n})\right)^2.$$

Quadratic Variation of Wiener Process

Proof.

Simplifying,

$$
\begin{aligned}
Var\left(\lim_{n\to\infty}\sum_{i=1}^{n}\left[W_{i(\frac{t}{n})} - W_{(i-1)(\frac{t}{n})}\right]^2\right) &= \lim_{n\to\infty}\sum_{i=1}^{n} 2\frac{t^2}{n^2}, \\
&= \lim_{n\to\infty} 2\frac{t^2}{n}, \\
&= 0.
\end{aligned}
$$

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$$

Thus,

$$
[W]_t = \mathbb{E}\left([W]_t\right) = \mathbb{E}\left(\lim_{\|P\|\to 0}[W]_t^P\right).
$$

Quadratic Variation of Wiener Process

Proof.

By dominating convergence theorem (outside scope),

$$\begin{aligned}
\mathbb{E}\left(\lim_{\|P\|\to 0} [W]_t^P\right) &= \lim_{\|P\|\to 0} \mathbb{E}\left([W]_t^P\right), \\
&= \lim_{\|P\|\to 0} t, \\
&= t.
\end{aligned}$$

□

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&= t.
\end{aligned}$$

Thus, $[W]_t = t$. $\square$

Quadratic Variation of Wiener Process

Implications

This motivates us to write:

$$\int_0^t (dW_t)^2 = t.$$

Or equivalently,

$$(dW_t)^2 = dt.$$

This will come up frequently later as we transitition into Ito calculus.

The Stochastic Integral

The Problem

Integrals of the form $\int_0^t g_s dW_s$ for some stochastic process g_s.

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1. $\sum_{k=1}^{n} g_s(W_{t_{k+1}} - W_{t_k})$,
2. Not possible due to locally unbounded variation of a Wiener Process.

The Stochastic Integral

Definition

A stochastic process is **simple** on $[a, b]$ when there exists deterministic points in time $a = t_0 < t_1 < \cdots < t_n = b$ such that $g_s = g_{t_k}$ for all $s \in [t_k, t_{k+1}]$.

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Real Solution

Now we can define the stochastic integral by

$$\int_a^b g_s dW_s = \sum_{k=0}^{n-1} g_{t_k}[W_{t_{k+1}} - W_{t_k}],$$

for some simple stochastic process g_s.

The Stochastic Integral

Generalized Version

By the **simple approximation theorem** (outside of scope), for some stochastic process g_s there exists a sequence of simple functions g_s^n, such that $g_s^n \to g_s$. Therefore, we define our integral for non-simple functions as:

$$\int_a^b g_s dW_t = \lim_{n\to\infty} \int_a^b g_s^n dW_t.$$

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Something to think about

How might we evaluate $\displaystyle\int_0^t W_s dW_s$?

Stochastic Differential Equations

Definition

An ito process, X_t, is a process that can be represented as:

$$X_t = X_0 + \int_0^t \mu_s ds + \int_0^t \sigma_s dW_s.$$

Stochastic Differential Equations

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We often use the notation:

$$dX_t = \mu_t dt + \sigma_t dW_t.$$

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This is known as a stochastic differential equation, and a useful tool for solving these is Ito's lemma, which we will see shortly.

Ito's Multiplication Table

Definition

Ito's multiplication table is:

	dt	dW_t
dt	0	0
dW_t	0	dt

We have already shown the only interesting result: $(dW_t)^2 = dt$.

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Quick sketch.

Suppose you are integrating on the interval $[a, b]$. Partition [a,b] into N increments. Then $dt \sim \frac{1}{N}$ and $(dt)^2 \sim \frac{1}{N^2}$. It follows that:

$$\sum_{i=1}^{N} \frac{1}{N^2} = \frac{1}{N},$$

$$\lim_{N\to\infty} \sum_{i=1}^{N} \frac{1}{N^2} = \lim_{N\to\infty} \frac{1}{N} = \int_a^b (dt)^2 = 0.$$

Ito's Multiplication Table

Quick sketch.

Similarly, suppose you are integrating on the interval $[a, b]$. Partition [a,b] into N increments. Since $dW_t = \sqrt{dt}$, $dt \sim \frac{1}{N}$ and $(dt)(dW_t) \sim \frac{1}{N^{3/2}}$. It follows that:

$$\sum_{i=1}^{N} \frac{1}{N^{3/2}} = \frac{1}{N^{1/2}},$$

$$\lim_{N\to\infty} \sum_{i=1}^{N} \frac{1}{N^{3/2}} = \lim_{N\to\infty} \frac{1}{N^{1/2}} = \int_a^b (dt)(dW_t) = 0.$$

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□

This completes the multiplication table and provides the foundation for introducing Ito's lemma.

Ito's Lemma

Theorem (Ito's formula)

Assume that process X has a stochastic differential given by

$$dX_t = \mu_t dt + \sigma_t dW_t.$$

Define the process Z by $Z(t) = f(t, X_t)$. Then Z has a stochastic differential given by

$$df(t, X_t) = \left\{ \frac{\partial f}{\partial t}(t, X_t) + \mu_t \frac{\partial f}{\partial x}(t, X_t) + \frac{1}{2}\sigma_t^2 \frac{\partial^2 f}{\partial x^2}(t, X_t) \right\} dt + \sigma \frac{\partial f}{\partial x}(t, X_t) dW_t.$$

Ito's Lemma

Heuristic proof.

Using f from the theorem, we will consider the second order Taylor expansion:

$$df = \frac{\partial f}{\partial t}dt + \frac{\partial f}{\partial x}dX_t + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(dX_t)^2 + \frac{1}{2}\frac{\partial^2 f}{\partial t^2}(dt)^2 + \frac{\partial^2 f}{\partial t \partial x}dtdX_t.$$

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By definition we have:

$$dX_t = \mu_t dt + \sigma_t dW_t.$$

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$$df = \frac{\partial f}{\partial t}dt + \frac{\partial f}{\partial x}dX_t + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(dX_t)^2 + \frac{1}{2}\frac{\partial^2 f}{\partial t^2}(dt)^2 + \frac{\partial^2 f}{\partial t \partial x}dtdX_t.$$

By definition we have:

$$dX_t = \mu_t dt + \sigma_t dW_t.$$

so, we obtain

$$(dX_t)^2 = \mu_t^2(dt)^2 + 2\mu_t\sigma_t(dt)(dW_t) + \sigma_t^2(dW_t)^2.$$

Ito's Lemma

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From Ito's multiplication table we have:

$$(dX_t)^2 = \sigma_t^2 dt.$$

Ito's Lemma

Heuristic proof cont.

Substituting back for $(dX_t)^2$,

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Now substituing in (dX_t),

$$\begin{aligned} df &= \frac{\partial f}{\partial t}dt + \frac{\partial f}{\partial x}(\mu_t dt + \sigma_t dW_t) + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(\sigma_t^2)(dt) + \frac{\partial^2 f}{\partial t \partial x}(\mu_t dt + \sigma_t dW_t)dt, \\ &= \left\{ \frac{\partial f}{\partial t}(t, X_t) + \mu_t \frac{\partial f}{\partial x}(t, X_t) + \frac{1}{2}\sigma_t^2 \frac{\partial^2 f}{\partial x^2}(t, X_t) \right\} dt + \sigma \frac{\partial f}{\partial x}(t, X_t)dW_t. \end{aligned}$$

$\square$

Ito's Lemma

Example

Find $d(W_t^2)$.

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Solution

Define X by $X_t = W_t$ and set $f(t,x) = x^2$. In terms of Ito's formula we have

$$\frac{\partial f}{\partial t}(t,x) = 0, \ \frac{\partial f}{\partial x}(t,x) = 2x, \ \frac{\partial^2 f}{\partial x^2}(t,x) = 2.$$

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$$d(W_t^2) = \left\{0 + 0 \cdot 2X_t + \frac{1}{2} \cdot 1 \cdot 2\right\} dt + 1 \cdot 2X_t dW_t.$$

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Thus,

$$d(W_t^2) = dt + 2W_t dW_t.$$

Ito's Lemma

Revisiting

Now if we integrate both side such that:

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Now if we integrate both side such that:

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This gives us:

$$\int_0^t W_t dW_t = \frac{W_t^2}{2} - \frac{t}{2}.$$

Real world application

Definition

Geometric Brownian Motion is a stochastic process whose dynamics follow the stochastic differential equation

$$dX_t = \alpha X_t dt + \sigma X_t dW_t \text{ for some } \alpha, \beta \in \mathbb{R},$$

where dW_t is the infinitesimal increment of the Wiener process.

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Geometric Brownian Motion is one of the building blocks for the modeling of asset prices, and turns up naturally in many other places.

Real world application

Closed form

With our toolbox of Ito calculus and the ability to solve linear ODE's, it can be easily shown that the closed form of Geometric Brownian Motion is:

$$dX_t = X_0 e^{(\mu - \frac{\sigma^2}{2})dt + \sigma dW_t}.$$

Real world application

Use in Finance

Black Scholes Model is one of the most widely used models of stock price behaviour and is built on Geometric Brownian motion.

Why GBM?

Some of the arguments for GBM are:

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Why GBM?

Some of the arguments for GBM are:

- The expected returns are independent of the value of the process, which is how stocks behave in reality,
- A GBM process only assumes positive values, just like real stock prices.

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This is why GBM is just a building block for more advanced models. For example, one that models volatility stochastically is a stochastic volatility model, which is built off GBM.