Stochastic Calculus

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Background

Definition

A real-valued random variable X is a mapping $X : \Omega \to \mathbb{R}$. Here, Ω is the **sample space** and $\mathbb{P}$ is the measure of the **probability space**, such that $\mathbb{P}(\Omega) = 1$.

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2. In $\mathbb{R}^2$: area,
3. In $\mathbb{R}^3$: volume.

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2. We cannot see the corresponding $\omega \in \Omega$, but the $X(\omega) \in \mathbb{R}$ gives us partial information about ω.
3. $\mathbb{P}$ tells us how likely subsets $A \subseteq \Omega$ are to occur.

Background

Example

Consider the case where you flip a coin. Using our previous definition, this could be described as $\Omega = \{\text{heads}, \text{tails}\}$ and

$$X(\omega) = \begin{cases} 1, & \text{if } \omega = \text{heads} \\ -1, & \text{if } \omega = \text{tails} \end{cases} \qquad \text{where } \omega \in \Omega.$$

This would yield the familiar notation of $\mathbb{P}(X = 1) = .5$ and $\mathbb{P}(X = -1) = .5$ for a fair coin.

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A sequence of coin flips. At each time t your process corresponds to a random variable (aka coinflip) X_t.

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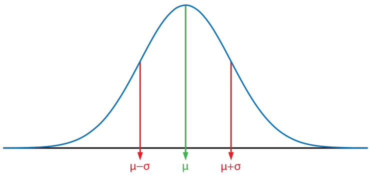

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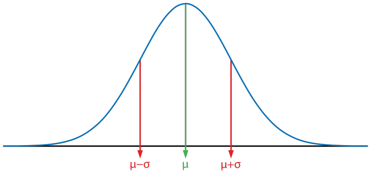

4. W has continuous trajectories.

Wiener Process

Theorem

A Wiener trajectory is with probability one, nowhere differentiable, and it has locally infinite total variation.

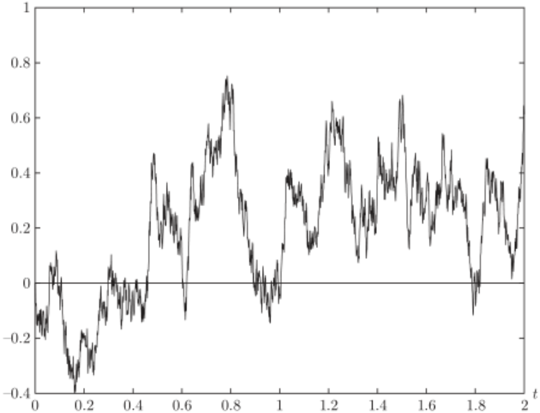

Figure: Wiener trajectory

Quadratic Variation

Definition

Let X be a stochastic process. Suppose P is a partition of $[0, t]$ denoted t_k and let $\|P\|$ be the mesh of the partition then the **quadratic variation** of X is defined to be:

$$[X]_t = \lim_{\|P\|\to 0} \sum_{k=1}^{n} (X_{t_k} - X_{t_{k-1}})^2$$

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Comments

Note that Quadratic Variation itself is a stochastic process. An intuitive way to think about quadratic variation is the internal clock of a process, describing how randomness accumulates over time.

The Stochastic Integral

The Problem

Integrals of the form $\int_0^t g_s dW_s$ for some stochastic process g_s.

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Possible Solution 1

Try to define like the Riemman Integral

1. $\sum_{k=1}^{n} g_s(W_{t_{k+1}} - W_{t_k})$
2. Not possible due to locally unbounded variation of a Wiener Process.

The Stochastic Integral

Definition

A stochastic process is **simple** on $[a, b]$ when there exists deterministic points in time $a = t_0 < t_1 < \cdots < t_n = b$ such that $g_s = g_{t_k}$ for all $s \in [t_k, t_{k+1}]$.

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Real Solution

Now we can define the stochastic integral by

$$\int_a^b g_s dW_s = \sum_{k=0}^{n-1} g_{t_k}[W_{t_{k+1}} - W_{t_k}]$$

For some simple stochastic process g_s.

The Stochastic Integral

How do we generalize this?

Quadratic Variation of Wiener Process

Theorem

Quadratic variation of a Wiener Process on the interval $[0, t]$ *is* t.

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Proof.

Let $P = \{0 = t_0 \leq t_1 \leq \cdots \leq t_m = t\}$ be a partition of the interval $[0, t]$. Then the quadratic variation on P is

$$[W]_t^P = \sum_{k=1}^{m} (W_{t_k} - W_{t_{k-1}})^2$$

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$$[W]_t^P = \sum_{k=1}^{m} (W_{t_k} - W_{t_{k-1}})^2$$

Therefore,

$$\mathbb{E}\left[[W]_t^P\right] = \sum_{k=1}^{m} \mathbb{E}\left[(W_{t_k} - W_{t_{k-1}})^2\right]$$

Quadratic Variation of Wiener Process

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$$\mathbb{E}(W_{t_k} - W_{t_{k-1}}) = 0$$

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Thus, $Var(W_{t_k} - W_{t_{k-1}}) = E((W_{t_k} - W_{t_{k-1}})^2)$.

Quadratic Variation of Wiener Process

Proof.

Since $\mathbb{E}[(W_{t_k} - W_{t_{k-1}})^2] = Var\left[W_{t_k} - W_{t_{k-1}}\right]$, we can write:

$$\mathbb{E}\left[[W]_t^P\right] = \sum_{k=1}^{m} Var\left[W_{t_k} - W_{t_{k-1}}\right]$$

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Using $W_{t_k} - W_{t_{k-1}} \in \mathcal{N}(0, t_k - t_{k-1})$:

$$\begin{aligned}\mathbb{E}\left[[W]_t^P\right] &= \sum_{k=1}^{m}(t_k - t_{k-1}) \\ &= t\end{aligned}$$

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$$\begin{aligned}\mathbb{E}\left[[W]_t^P\right] &= \sum_{k=1}^{m}(t_k - t_{k-1}) \\ &= t\end{aligned}$$

Now we must show $\mathbb{E}([W]_t^P) = [W]_t = t$. $\square$

Quadratic Variation of Wiener Process

Implications

This motivates us to write

$$\int_0^t (dW_t)^2 = t$$

Or equivalently,

$$(dW_t)^2 = dt$$

This is one third of something called Ito's multiplication table.

Ito's Multiplication Table

Definition

Ito's multiplication table is:

	dt	dW_t
dt	0	0
dW_t	0	dt

We have already shown the only interesting result: $(dW_t)^2 = dt$.

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Quick sketch.

Suppose you are integrating on the interval $[a, b]$. Partition [a,b] into N increments. Then $dt \sim \frac{1}{N}$ and $(dt)^2 \sim \frac{1}{N^2}$. It follows that:

$$\sum_{i=1}^{N} \frac{1}{N^2} = \frac{1}{N}$$

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Similarly, suppose you are integrating on the interval $[a, b]$. Partition [a,b] into N increments. Since $dW_t = \sqrt{dt}$, $dt \sim \frac{1}{N}$ and $(dt)(dW_t) \sim \frac{1}{N^{3/2}}$. It follows that:

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This completes the multiplication table and provides the foundation for introducing Ito's lemma.

Ito's Lemma

Theorem (Ito's formula)

Assume that process X has a stochastic differential given by

$$dX_t = \mu_t dt + \sigma_t dW_t$$

Define the process Z by $Z(t) = f(t, X_t)$. Then Z has a stochastic differential given by

$$df(t, X_t) = \left\{ \frac{\partial f}{\partial t}(t, X_t) + \mu_t \frac{\partial f}{\partial x}(t, X_t) + \frac{1}{2}\sigma_t^2 \frac{\partial^2 f}{\partial x^2}(t, X_t) \right\} dt + \sigma \frac{\partial f}{\partial x}(t, X_t) dW_t$$

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Heuristic proof.

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Using f from the theorem, we will consider the second order Taylor expansion:

$$df = \frac{\partial f}{\partial t}dt + \frac{\partial f}{\partial x}dX_t + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(dX_t)^2 + \frac{1}{2}\frac{\partial^2 f}{\partial t^2}(dt)^2 + \frac{\partial^2 f}{\partial t \partial x}dtdX_t.$$

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so, we obtain

$$(dX_t)^2 = \mu_t^2 (dt)^2 + 2\mu_t \sigma_t (dt)(dW_t) + \sigma_t^2 (dW_t)^2,$$

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From Ito's multiplication table we have:

$$(dX_t)^2 = \sigma_t^2 dt$$

Ito's Lemma

Heuristic proof cont.

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Substituting back for $(dX_t)^2$,

$$df = \frac{\partial f}{\partial t}dt + \frac{\partial f}{\partial x}dX_t + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(\sigma_t^2)(dt) + \frac{1}{2}\frac{\partial^2 f}{\partial t^2}(dt)^2 + \frac{\partial^2 f}{\partial t \partial x}dtdX_t.$$

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Applying Ito's multiplication table one more time,

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Now substituing in (dX_t),

$$\begin{aligned} df &= \frac{\partial f}{\partial t}dt + \frac{\partial f}{\partial x}(\mu_t dt + \sigma_t dW_t) + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(\sigma_t^2)(dt) + \frac{\partial^2 f}{\partial t \partial x}(\mu_t dt + \sigma_t dW_t)dt. \\ &= \left\{\frac{\partial f}{\partial t}(t, X_t) + \mu_t \frac{\partial f}{\partial x}(t, X_t) + \frac{1}{2}\sigma_t^2 \frac{\partial^2 f}{\partial x^2}(t, X_t)\right\} dt + \sigma \frac{\partial f}{\partial x}(t, X_t)dW_t \end{aligned}$$

$\square$

Geometric Brownian Motion Model

Definition

Geometric Brownian Motion is a stochastic process whose dynamics follow the stochastic differential equation

$$dX_t = \alpha X_t dt + \sigma X_t dW_t \text{ for some } \alpha, \beta \in \mathbb{R}$$

Where dW_t is the infinitesimal increment of the Wiener process.

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Claim:

The closed form of Geometric Brownian Motion is:

$$dX_t = X_0 e^{(\mu - \frac{\sigma^2}{2})dt + \sigma dW_t}$$

Geometric Brownian Motion Model

Proof.

Assume X_t follows a Geometric Brownian Motion. Using Ito's lemma,

$$d\ln(X_t) = \frac{1}{X_t}dX_t - \frac{1}{2X_t^2}(dX_t)^2$$

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$$d\ln(X_t) = \frac{1}{X_t}(\mu X_t dt + \sigma X_t dW_t) - \frac{1}{2X_t^2}(\mu X_t dt + \sigma X_t dW_t)^2$$

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From Ito's multiplication table, $(dt)^2 = 0 = (dt)(dW_t)$ and $(dW_t)^2 = dt$ such that

$$(\mu X_t dt + \sigma X_t dW_t)^2 = X_t^2 \sigma^2 dt$$

Geometric Brownian Motion Model

Proof.

Substituting back,

$$\begin{aligned} d\ln(X_t) &= (\mu dt + \sigma dW_t) - \frac{1}{2X_t^2}(X_t^2\sigma^2 dt) \\ &= (\mu - \frac{\sigma^2}{2})dt + \sigma dW_t. \end{aligned}$$

$\square$

Geometric Brownian Motion Model

Proof.

Substituting back,

$$\begin{aligned} d\ln(X_t) &= (\mu dt + \sigma dW_t) - \frac{1}{2X_t^2}(X_t^2\sigma^2 dt) \\ &= (\mu - \frac{\sigma^2}{2})dt + \sigma dW_t. \end{aligned}$$

It follows that $dX_t = e^{(\mu - \frac{\sigma^2}{2})dt + \sigma dW_t}$ $\square$