MATH 602 Notes

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Abstract

This is a graduate level functional analysis taught by Dmitry Chelkak at University of Michigan. We will loosely follow Functional Analysis by Peter Lax as the textbook.

Functional analysis is a core subject in mathematics. It has connections to probability and geometry, and is of fundamental importance to the development of analysis, differential equations, quantum mechanics and many other branches in mathematics, physics, engineering and theoretical computer science. The goal of this course is to introduce students to the basic concepts, methods and results in functional analysis. Topics to be covered include linear spaces, normed linear spaces, Banach spaces, Hilbert spaces, linear operators, dual operators, the Riesz representation theorem, the Hahn-Banach theorem, uniform boundedness theorem, open mapping theorem, closed graph theorem, compact operators, Fredholm Theory, reflexive Banach spaces, weak and weak* topologies, spectral theory, and applications to classical analysis and partial differential equations.

Contents

Chapter 1

Linear Spaces and Functionals

Lecture 1: First Lecture

Why the name functional analysis? 30 Aug. 01:00

- Functions on $\mathbb{R}, \mathbb{C}, \mathbb{R}^n = X$.
- Topological spaces $\subset$ Metric spaces $\subset$ Normed linear space $\subset$ Hilbert spaces.
  - Will revisit Hilbert later. Has inner product.

Definition 1.0.1. A metric space has a metric ρ that satisfies the following properties

1. $\rho(x, y) = 0$ iff $x = y$.
2. $\rho(x, y) = \rho(y, x)$.
3. $\rho(x, y) \leq \rho(x, z) + \rho(z, y)$.

Definition 1.0.2. A normed linear space is a vector space X over $\mathbb{R}$ or $\mathbb{C}$ with a norm $\|\cdot\| : X \to \mathbb{R}_+$ that satisfies the following properties

1. $\|x\| = 0$ iff $x = 0$.
2. $\|\alpha x\| = |\alpha|\|x\|$.
3. $\|x + y\| \leq \|x\| + \|y\|$.

Remark. We can define a metric from a norm as $\rho(x, y) = \|x - y\|$.

Notation. $\mathcal{U}$ open set; $B(x, r) \subset U$ open ball of radius r centered at x.

In this course, we will be discussing linear functions in normed linear spaces. In some sense, we will be studying "linear algebra in infinite dimensional spaces". Not entirely correct (will see later).

Remark. All of our spaces will be complete. This means that every Cauchy sequence converges.

Example. Examples of normed linear spaces

1. $\mathbb{R}^n$, $\|x\|_p = \left(\sum_{k=1}^n |x_k|^p\right)^{1/p}$ for $1 \leq p < \infty$.
2. $x = (x_1, x_2, \ldots)$, $\ell^p = \{x \mid \sum_{k=1}^\infty |x_k|^p < \infty\}$, $\|x\|_p = \left(\sum_{k=1}^\infty |x_k|^p\right)^{1/p}$.

3. $L^2([0,1], \mathrm{d}\mu)$, $X = \{f : [0,1] \to \mathbb{R} \mid \int_0^1 |f(x)|^p \mu(dx) < \infty\}$.

4. $X = C([0,1], \mathbb{R})$, $\|f\|_{C([0,1])} := \sup_{x\in[0,1]} |f(x)|$.

5. $C^k([0,1], \mathbb{R})$, $\|f\|_{C([0,1])} := \sum_{j=0}^{k} \sup_{x\in[0,1]} |f^{(j)}(x)|$.

6. $C^\infty([0,1], \mathbb{R})$. Not a normed space!

   - There exists a slightly weaker notion that applies here.

Now why do we call this functional analysis? Rewording our previous statement, we will be studying *linear functionals* in normed linear spaces. Why is studying linear functions non-trivial? From linear algebra, if $A : \mathbb{R}^n \to \mathbb{R}^n$ linear, then there exists some basis such that

$$A = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}$$

Namely, we have a spectrum of A and eigenvectors of A. What happens in infinite dimensions?

Example. Consider $S : \ell^2 \to \ell^2$ defined by $S(x_1, x_2, \ldots) = (0, x_1, x_2, \ldots)$. This is a linear operator.

$$S = \begin{pmatrix} 0 & 1 & & \\ 0 & 0 & 1 & \\ \vdots & \ddots & \ddots & \ddots \end{pmatrix}$$

What is the spectrum of S? $\sigma(S) = \{0\}$. What are the eigenvectors of S? $S(x_1, x_2, \ldots) = \lambda(x_1, x_2, \ldots)$ implies that $x_1 = 0$ and $\lambda x_k = x_{k+1}$ for $k \geq 1$. This implies that $\lambda = 0$ and $x_k = 0$ for $k \geq 1$. Thus, the eigenvectors of S are $(0, x_2, x_3, \ldots)$ for $x_k \in \mathbb{C}$.

Example. $S^* : \ell^2 \to \ell^2$ defined by $S^*(x_1, x_2, \ldots) = (x_2, x_3, \ldots)$. This is a linear operator.

$$S^* = \begin{pmatrix} 0 & & & \\ 1 & 0 & & \\ 0 & 1 & 0 & \\ \vdots & \ddots & \ddots & \ddots \end{pmatrix}$$

Here we have $S^*(x_1, x_2, \ldots) = \lambda(x_1, x_2, \ldots)$ implies that $x_2 = \lambda x_1$, $x_3 = \lambda x_2$, etc. Thus, the eigenvectors of S^* are $(x_1, \lambda x_1, \lambda^2 x_1, \ldots)$ for $x_1 \in \mathbb{C}$ and we have infinitely many eigenvalues.

Remark. Spectral theory of linear operators will tell us when it makes sense to talk about eigenvalues and eigenvectors of linear operators.

Essentially, in these complicated spaces there is no analagous Jordan normal form for infinite dimensional spaces. This is why we need functional analysis, where we study the spaces themselves. This begs the question why are we even interested in eigenvalues and eigenvectors?

Example (Sturm-Liouville). $V : [0,1] \to \mathbb{R}$. Want to find ψ such that $-\psi'' + V\psi = \lambda\psi$ for $\psi : [0,1] \to \mathbb{R}$ and $\psi(0) = \psi(1) = 0$. If no V, then we can use Fourier analysis with $\lambda_n = \pi^2 n^2$, $\psi(x) = \sin(\pi n x)$. However, this is a linear operator in ψ.

Next Lecture: Consider linear mappings $A : X \to X$ such that $\ker A$ is finite dimensional and $\underbrace{\text{codim}}_{\dim X\,/\,\operatorname{Im} A}\ \operatorname{Im} A < \infty$. Denote $\operatorname{ind} A := \dim\ker A - \text{codim}\ \operatorname{Im} A$. Then we have the following

$$\operatorname{ind}(A+G) = \operatorname{ind} A \quad \text{if} \quad \dim \operatorname{Im} G < \infty$$

Exercise. If $\dim X < \infty$, then $\forall A$, $\mathrm{ind} A = 0$.

Lecture 2: Second Lecture

As motivation, suppose we have a linear mapping $M : \mathbb{R}^n \to \mathbb{R}^n$. Is it true that $\forall g \in \mathbb{R}^n$ does there exist some $f \in \mathbb{R}^n$ such that $Mf = g$. In a finite dimensional case, we know that this is the case if and only if $\ker M = \{0\}$. 6 Sep. 01:00

Example. Let us revisit the linear operator $S'x = (0, x_1, x_2, \ldots)$. Then $\exists y$ such that $S'x = y$ has no solutions. However, $\ker S' = \{0\}$. Thus, we do not have the same nice result.

1.1 Index of a Linear Operator

Definition. Linear space X over $\kappa = \mathbb{R}$ or $\mathbb{C}$.

1. $x, y \in X \Rightarrow x + y \in X$
2. $x \in X, \lambda \in \kappa \Rightarrow \lambda x \in X$

Definition 1.1.1. The *dimension* of X denoted $\dim X$ is the number of linearly independent vectors in X.

Remark. $\dim X$ either in $\mathbb{N}$ or ∞. $\dim X = 0 \Leftrightarrow X = \{0\}$.

Definition 1.1.2. $M : X \to Y$ is a *linear operator* if

$$M(\alpha x + \beta y) = \alpha Mx + \beta My.$$

Will use $Mx := M(x)$.

Definition 1.1.3. $\ker M := \{x \in X : Mx = 0\}$.

Definition 1.1.4. Range of M is Ran $M := \{y \in Y : \exists x \in X \text{ such that } Mx = y\}$.

Exercise. Try the following exercises

1. M is a bijection if and only if $\ker M = \{0\}$ and Ran $M = Y$.
2. If M is a bijection then M^{-1} is also linear.

Definition 1.1.5. $X \supset X_0$ is a linear subspace if $x, y \in X_0 \Rightarrow \alpha x + \beta y \in X_0$.

Definition 1.1.6. We define the quotient space X/X_0 as the equivalence classes defined by the equivalence relation

$$x \sim y \Leftrightarrow x - y \in X_0.$$

Exercise. Try the following exercises

1. Show that X/X_0 is correct definition for linear space.
2. Show $M : X \,/\, \ker M \to \text{Ran } M$ is well-defined and a bijection.

Proposition 1.1.1. Given $X_0 \subset X$, $\exists X_0' \subset X$ such that $X = X_0 \oplus X_0'$ i.e. $\forall x \in X$, there exists a unique decomposition

$$x = \underbrace{x_0}_{\in X_0} + \underbrace{x_0'}_{\in X_0'}.$$

The choice of X_0' is not unique.

Proof. Let us consider the family of all subspaces $X' \subset X$ such that

$$X' \cap X_0 = \{0\}. \tag{$\star$}$$

This family has a partial order:

$$X_1' \leq X_2' \quad \text{if} \quad X_1' \subset X_2'.$$

We make use of the well known Zorn's Lemma.

Lemma 1.1.1 (Zorn's Lemma). If $\{X_\alpha'\}_{\alpha \in A}$ is totally ordered then $\bigcup_{\alpha \in A} X_\alpha'$ also satisfies the condition. This element is a maximal element.

Let X_0' be any maximal element by Lemma 1.1.1. If $x \in X$ cannot be decomposed as $x = x_0 + x_0'$ then $\text{span}(X_0', x)$ satisfies ($\star$) and is strictly bigger. Thus, we have a contradiction. ■

Definition 1.1.7. Given $X_0 \subset X$, we define $\text{codim } X_0 = \dim X \,/\, X_0$.

Remark. $\text{codim } X_0 = \dim X_0'$.

Proof. Consider $M : X \to X_0'$ defined by $x_0 + x_0' \to x_0'$. ■

Definition 1.1.8. $G : X \to Y$ (linear) is called *degenerate* if Ran G is finite dimensional.

Definition 1.1.9. $M : X \to Y$, $L : Y \to X$ are called *pseudoinverse* to each other if $\exists G_X : X \to X$ and $G_Y : Y \to Y$ degenerate such that $ML = \text{Id}_Y + G_y$ and $LM = \text{Id}_X + G_X$.

Example. $S : (x_1, x_2, \dots) \mapsto (0, x_1, x_2, \dots)$ and $S^* : (x_1, x_2, \dots) \mapsto (x_2, x_3, \dots)$. Then $S^*S = \text{Id}$. However, $SS^*(x_1, x_2, \dots) = (0, x_2, \dots)$. Therefore, $SS^* = \text{Id} + G$, where $G(x_1, x_2, \dots) = (x_1, 0, \dots)$.

Remark. A few remarks:

- If G - degenerate then AG, GB are also degenerate.
- If G_1, G_2 - degenerate then $G_1 + G_2$ is also degenerate.
- If M_1, M_2 have pseudoinverses L_1, L_2 then $M_1 M_2$ has pseudoinverse $L_2 L_1$.

Theorem 1.1.1. M has a pseudoinverse if and only if $\dim \ker M < \infty$ and $\text{codim Ran } M < \infty$.

Proof. Suppose M has a pseudoinverse L. First consider $M = \text{Id} + G : X \to X$ (pseudoinverse Id). Note that $\ker M \subset \text{Ran } G$. Then $\dim \ker M \leq \dim \text{Ran} G < \infty$. Note that $\text{Ran } M \supset \ker G$.

Therefore,
$$\text{codim Ran } M \leq \text{codim } \ker G = \dim \text{Ran} G < \infty.$$

Now we consider the general case $LM = \text{Id}_X + G_X$. Then $\ker M \subset \ker LM$ - finite dimensional. Similary, $ML = \text{Id}_Y + G_Y$, so that $\text{Ran } M \supset \text{Ran } ML$. It follows that $\text{codim Ran } M \leq \text{codim Ran } ML < \infty$.

Now suppose $\dim \ker M < \infty$ and $\text{codim Ran } M < \infty$. Let us construct an appropriate L. We can decompose $X = \ker M \oplus X'$ and $Y = Y' \oplus \text{Ran } M$. We know that $M : X' \to \text{Ran } M$ is a bijection. Therefore, define $L|_{\text{Ran } M} := M^{-1}$ and $L|_{Y'} := 0$. Thus, $LM|_{X'} = \text{Id}$ and $ML|_{\text{Ran } M} = \text{Id}$.

Exercise. Understand why we are done.

■

Definition 1.1.10. In this case, $\text{ind } M = \dim \ker M - \text{codim Ran } M$ is called the *index* of M.

Exercise. Try the following exercises

- If $M : \mathbb{R}^n \to \mathbb{R}^m$ is then $\text{ind } M = m - n$.
- $\text{ind } S = -1$, $\text{ind } S^* = 1$.

Theorem 1.1.2. If both M_1, M_2 have pseudoinverses then $\text{ind } M_1 M_2 = \text{ind } M_1 + \text{ind } M_2$.

Theorem 1.1.3. If M has a pseudoinverse then so do $M + G$ (G - degenerate) and $\text{ind } (M + G) = \text{ind } M$.

Corollary 1.1.1. If $G : X \to X$ is degenerate then $\text{ind } (\text{Id} + G) = \text{ind Id} = 0$. In particular, if $\ker(\text{Id} + G) = \{0\}$ then $\text{Ran } (\text{Id} + G) = X$.

Remark. Identity can be replaced with any invertible operator.

Lecture 3: Third Lecture

Reminder: Last time, we covered the following 11 Sep. 01:00

- Defined G - degenerate as operators such that $\dim \text{Ran } G < \infty$.
- $M : X \to Y$ has a pseudoinverse $L : Y \to X$ if both LM and ML are Id +degenerate.
- Introduced theorem: M has a pseudoinverse iff $\dim \ker M < \infty$ and $\text{codim Ran } M = \dim Y \,/\, \text{Ran } M < \infty$.
- Defined $\text{ind } M := \dim \ker M - \text{codim Ran } M$.

Proof of Theorem 1.1.2. First we introduce a lemma.

Lemma 1.1.2. Let $V_0, V_1, \ldots, V_{n+1}$ be finite dimensional and $T_k : V_k \to V_{k+1}$ be linear such that $\ker T_{k+1} = \text{Ran } T_k$ (Exact sequence). If $\dim V_0 = \dim V_{n+1} = 0$, then $\sum_{k=1}^{n} (-1)^k \dim V_k = 0$.

Proof of lemma. We have the following bijection

$$V_k \big/ \ker T_k \longleftrightarrow \operatorname{Ran} T_k \subset V_{k+1}$$

Therefore, $\dim V_k = \dim\ker T_k + \dim\operatorname{Ran} T_k$. Since our sequence is exact, $\dim\operatorname{Ran} T_k = \dim\ker T_{k+1}$. Therefore, $\dim V_k = \dim\ker T_k + \dim\ker T_{k+1}$. We then just have a telescoping sum. ■

We introduce the following SES

$$0 \to \ker M_1 \overset{\mathrm{Id}}{\hookrightarrow} \ker LM \xrightarrow{M_1} \underbrace{\ker L}_{\subset Y} \xrightarrow{y\mapsto y \bmod \operatorname{Ran} M} Y\big/\operatorname{Ran} M \xrightarrow{L} Z\big/\operatorname{Ran} LM \to Z\big/\operatorname{Ran} L \to 0$$

From the lemma, we have that

$$0 = \dim\ker M - \dim\ker LM + \dim\ker L - \operatorname{codim}\operatorname{Ran} M + \operatorname{codim}\operatorname{Ran} LM - \operatorname{codim}\operatorname{Ran} L$$

Reordering,

$$\begin{aligned} 0 &= (\dim\ker M - \operatorname{codim}\operatorname{Ran} M) - (\dim\ker LM - \operatorname{codim}\operatorname{Ran} LM) + (\dim\ker L - \operatorname{codim}\operatorname{Ran} L) \\ &= \operatorname{ind} M - \operatorname{ind} LM + \operatorname{ind} L \end{aligned}$$

Thus, $\operatorname{ind} M + \operatorname{ind} L = \operatorname{ind} LM$ as desired. ■

Proof of Theorem 1.1.3. First consider $M = \mathrm{Id} : X \to X$. Then

$$\ker G \overset{\mathrm{Id}}{\hookrightarrow} X \xrightarrow{\mathrm{Id}+G} X$$

By Theorem 1.1.2,

$$\operatorname{ind}(\mathrm{Id}_{\ker G\to X}) + \operatorname{ind}(\mathrm{Id}_X + G) = \operatorname{ind}(\mathrm{Id}_{\ker G\to X})$$

Here equality follows from the fact that the composition of the maps in our exact sequence is the same as the first mapping. Thus, $\operatorname{ind}(\mathrm{Id}_X + G) = 0$.

Now consider general M. If L is a pseudoinverse for M then L is a pseudoinverse for $M + G$ as well. From the first case, we have that $\operatorname{ind}(LM) = 0$ and $\operatorname{ind}(L(M+G)) = 0$. Thus,

$$\operatorname{ind} M = -\operatorname{ind} L = \operatorname{ind}(M+G)$$

as desired. ■

1.2 Banach Spaces and bounded linear operators

Definition 1.2.1. A norm on a linear space X (over $\mathbb{R}$ or $\mathbb{C}$) is a function $\|\cdot\| : X \to \mathbb{R}$ such that

(i) $\|x\| = 0 \Leftrightarrow x = 0$.

(ii) $\|\lambda x\| = |\lambda|\|x\|$.

(iii) $\|x+y\| \le \|x\| + \|y\|$.

Remark. $\rho(x,y) = \|x-y\|$ is a metric on X.

Definition 1.2.2. A normed vector space $(X, \|\cdot\|)$ is called a *Banach space* if it is a complete metric space.

Example. We have the following examples

- $(E, \mathrm{d}\mu)$, $1 \leq p < \infty$. Then
$$L^p(E, \mathrm{d}\mu) = \{f : E \to \mathbb{C} \mid \|f\|_p := \left(\int_E |f|^p\right)^{1/p} \mathrm{d}\mu < \infty\}$$
is complete.
- E - compact metric space. Then
$$C(E) = \{f : E \to \mathbb{C} \mid f \text{ - continuous}\}$$
with $\|f\|_{C(E)} := \sup_{x \in E} |f(x)|$ is complete.

1.2.1 Subspaces and quotient spaces

Exercise. Suppose we have some $X_0 \subset X$, then $(X_0, \|\cdot\|_X)$ is Banach if and only if X_0 is closed in X.

Answer. Consider Cauchy sequences. ⊛

Definition. Consider $X \,/\, X_0 = \{X \bmod X_0 \mid x \in X\}$ as a linear space.

Definition 1.2.3. We define the *quotient norm* on $X \,/\, X_0$ as $\|x \bmod X_0\| := \inf_{y \in X_0} \|y - x\|$.

Proposition 1.2.1. Definition 1.2.3 defines a norm on $X \,/\, X_0$ if X_0 is closed.

Proof. We must satisfy the three properties of a norm.

- Scalar multiplication is trivial.
- Triangle inequality is relatively simple. There exists $y_1 \in X_0$ such that
$$\|y_1 - x_1\| \leq \inf\{\|y_1 - x_1\| \mid y_1 \in X_0\} + \epsilon$$
Therefore,
$$\|x_1 + x_2 \bmod X_0\| \leq \|x_1 \bmod X_0\| + \|x_2 \bmod X_0\| + 2\epsilon$$
- $\|x \bmod X_0\| = 0 \Leftrightarrow x = 0$. Is non-trivial. For example, if X_0 is any dense subset then this norm will always be 0. However, closedness does give us this property.

  Consider a sequence $\{x_n\}$ such that $\forall n$, $\exists x_0 \in X_0$ such that $\left\|x - x_0^{(n)}\right\| \leq \frac{1}{2^n}$. This sequence is Cauchy i.e. $\left\|x_0^{(m)} - x_0^{(n)}\right\| \leq \frac{1}{2^{m-n}}$. Since X_0 is closed, this sequence must converge to some $x_0^* \in X_0$. However, this sequence converges to x. Thus, $x \in X_0$.

■

Definition 1.2.4. Let $\|\cdot\|_1$ and $\|\cdot\|_2$ be norms on the same X. They are equivalent if $\exists C_1, C_2 > 0$ such that $\forall x \in X$, $C_1\|x\|_1 \leq \|x\|_2 \leq C_2\|x\|_1$.

Example. $X = \mathbb{R}^n$, $\|x\|p = \left(\sum_{k=1}^n |x_k|^p\right)^{1/p}$ for $1 \leq p < \infty$. Then all are equivalent.

Theorem 1.2.1. If $\dim X < \infty$, then all norms on X are equivalent.

Proof. Let $X = \mathbb{R}^n$ as a linear space and $\|\cdot\|_1$ be the Euclidean norm. Let $\{e_k\}$ be a standard basis. Then

$$\begin{aligned}\left\|\sum_{k=1}^{n} c_k e_k\right\|_2 &\leq \max_{k=1,\dots,n} |c_k| \cdot \sum_{k=1}^{n} \|e_k\|_2 \\ &\leq \max_{k=1,\dots,n} |c_k| \cdot n \max_{k=1,\dots,n} \|e_k\|_2\end{aligned}$$

Note that

$$\max_{k=1,\dots,n} |c_k| \leq \left\|\sum_{k=1}^{n} c_k e_k\right\|_1$$

Therefore, we can take $C_2 := n \max_{k=1,\dots,n} \|e_k\|_2$. Now consider the mapping defined by $x \mapsto \|\cdot\|_2$ for some unknown norm. From above, this must be a continuous mapping. Namely,

$$\|x - y\|_2 \leq C_2 \|x - y\|_1$$

This function has > 0 minimum on the whole sphere. Therefore, $\forall x \in X$ such that $\|x\|_1 = 1$, we have that

$$M := \min_x \|x\|_2 > 0$$

Thus,

$$1 = \|x\|_1 \leq \frac{1}{M} \|x\|_2$$

By linearity, this holds for all $x \in X$. ■

Theorem 1.2.2. $\dim X < \infty$ if and only if the closed unit ball $\overline{B}(0, 1)$ is compact.

Lecture 4: Fourth Lecture

Last time: We covered the following 13 Sep. 01:00

- Banach spaces = complete normed linear spaces.
- If $X_0 \subset X$ - closed subspace then X_0 is a Banach space, $X \,/\, X_0$ well-defined normed space.
- $\|\cdot\|_1$, $\|\cdot\|_2$ on X are equivalent if $\exists C_1, C_2$ such that

$$\|x\|_1 \leq C_1 \|x\|_2 \quad \text{and} \quad \|x\|_2 \leq C_2 \|x\|_1$$

- If X finite dimensional then all norms are equivalent.
  - All norms in $\mathbb{R}^n$ are equivalent to $\|\cdot\|_{\text{Eucl}}$.
  - $\forall \|\cdot\|$, $\exists C'^{:\|\cdot\|' \leq C'\|\cdot\|_{\text{Eucl}}}$.
  - By compactness of unit ball, we have the other direction.
  - $\dim X = \infty$ then not all norms are equivalent (even if we require complete wrt to each norm).

Remark. A deep and important fact (to be discussed later): If $\|\cdot\|' \leq C\|\cdot\|$ and X is complete with respect toeach of them, then $\exists C'$ such that $\|\cdot\| \leq C'\|\cdot\|'$.

Definition 1.2.5. K is compact if for $K \subset \bigcup_\alpha U_\alpha$, there exists some $\alpha_1, \dots, \alpha_k$ such that $K \subset \bigcup_{i=1}^k U_{\alpha_i}$.

Definition 1.2.6. K is precompact if

(i) $\overline{K}$ is compact.

(ii) $\forall \{x_n\}_{n=1}^\infty$, there exists a Cauchy subsequence.

(iii) $\forall \varepsilon > 0$, $\exists \{x_n\}_{n=1}^N$ such that $K \subset \bigcup_{n=1}^N B(x_n, \varepsilon)$.

Proof of Theorem 1.2.2. For the $\Leftarrow$ direction, we have that all norms are equivalent and $\overline{B}(0,1)$ wrt Euclidean is compact. For the $\Rightarrow$ direction, we want to show $\dim X = \infty \Rightarrow \overline{B}_x(0,1)$ is not compact. Construct $x_n \in X$ such that $\|x_n\| = 1$ and $\|x_n - x_m\| \geq \frac{1}{2}$ if $n \neq m$. By induction, $X_n = \operatorname{span}(x_1, \dots, x_n)$. Take $z \in X$ such that

$$\operatorname{dist}(z, X_n) \coloneqq \inf_{yX_n} \|z - y\| = 1.$$

We can do this because $z \notin X_n \Rightarrow \operatorname{dist}(z, X_n) > 0$. We then just rescale this to have norm 1. Since X_n is finite dimensional, it is closed. Find $y_n \in X_n$ such that $\|z - y_n\| \leq 2$. Take $x_{n+1} \coloneqq \frac{z - y_n}{\|z - y_n\|}$. ■

Exercise. If X - Banach then one can find y_n such that $\|z - y_n\| = 1$.

1.2.2 Linear Operators

Definition. Let X, Y be Banach spaces and $M : X \to Y$ a linear operator.

Definition 1.2.7. M is called *bounded* if $\exists C > 0$ such that $\forall x X$, $\|Mx\| \leq C\|x\|$.

Definition 1.2.8. $\|M\| \coloneqq \inf\{C > 0 \mid \|Mx\| \leq C\|x\| \ \forall x \in X\}$.

Definition 1.2.9. $\mathcal{L}(X, Y) \coloneqq \{M : X \to Y \mid M \text{ bounded}\}$ equipped with $\|\cdot\|_{\mathcal{L}(X,Y)}$ from Definition 1.2.8.

Note. $\|(M_1 + M_2)x\| \leq \|M_1 x\| + \|M_2 x\| \leq (C_1 + C_2)\|x\|$

Exercise. Try the following exercises:

- Show that in Definition 1.2.8, the infimum is a minimum.
- Show that $\|M_1 M_2\| \leq \|M_1\| \cdot \|M_2\|$.

Remark. $\dim X < \infty \Rightarrow$ all linear operators are bounded.

Proposition 1.2.2. If Y - complete then $\mathcal{L}(X, Y)$ is complete.

Proof. Let M_n be a Cauchy sequence in $\mathcal{L}(X, Y)$. Then $\forall \epsilon > 0$, $\exists N$ such that $\forall m, n \geq N$

$$\|M_n - M_m\|_{\mathcal{L}(X,Y)} \leq \epsilon.$$

Therefore, for all $x \in X$

$$\|M_n x - M_m x\| \leq \epsilon \|x\|_X.$$

Denote $Mx = \lim_{n\to\infty} M_n x$. $M_n x$ is cauchy so $\exists$ a limit. We need to show that M is linear and bounded. This is straightforward using

$$M(\alpha x + \beta y) = \alpha Mx + \beta My$$
$$\|M - M_n\| \leq \epsilon, \quad \text{if } n \geq N$$

■

Remark. Clearly, M is bounded if and only if M is Lipschitz. However, this can be weakened to M is continous at 0.

Proof. $\exists \delta > 0$ such that $\|x\| \leq \delta \Rightarrow \|Mx\| \leq 1$. Take $C = \frac{1}{\delta}$. ■

Definition 1.2.10. $C \in \mathcal{L}(X, Y)$ is called *compact* if $C(\overline{B}_X(0,1))$ is precompact in Y.

Remark. A few remarks:

(a) If G is degenerate then G is compact.

(b) If C is compact then MC, CM is compact.

Later in the class, we will see that this class of operators is important because $\operatorname{ind}(\mathrm{Id} + C) = 0$ and is well defined.

Chapter 2

Hahn-Banach Theorem

2.1 Motivation

Definition 2.1.1. Given a Banach space X, a *dual space* $X' := \mathcal{L}(X, \mathbb{R}/\mathbb{C})$. An element $f \in X'$ is called a *functional*.

Example. If $X = L^p$, $1 \leq p < \infty$ then $X' = L^q$, where $\frac{1}{p} + \frac{1}{q} = 1$.

How do we know that we have non-trivial functionals? This is the motivation for Hahn-Banach theorem.

Theorem 2.1.1 (Hahn-Banach)**.** Let $X_0 \subset X$ be a subspace and $f_0 \in X_0'$ such that $\|f_0\|_{X_0'} \leq 1$. Then $\exists f \in X'$ such that $f|_{X_0} = f_0$ and $\|f\|_{X'} \leq 1$.

Proof. Zorn's lemma.

$$\{(Y, f) \text{ such that } X_0 \subset Y \subset X,\ f : Y \to \mathbb{R} \text{ such that } f|_{X_0} = f_0,\ \|f\| \leq 1\}$$

Say $(Y_1, f_1) \prec (Y_2, f_2)$ if $Y_1 \subset Y_2$ and $f_2|_{Y_1} = f_1$. Note $\{(Y_\alpha, f_\alpha)\}$ is totally ordered, then $Y := \bigcup_\alpha Y_\alpha$, $f|_{Y_\alpha} := f_\alpha$. By Zorn's lemma, there exists a maximal element. Now we must show that $Y = X$. Suppose this is not the case. Let $z \notin Y$. We want to extend f to $\operatorname{span}(Y, z)$. Sufficient to show that $\exists \alpha \in \mathbb{R}$ such that $\|y + z\| \geq |f(y) + \alpha|$. To see this, define $f(y + z) = \alpha + f(y)$. For all $y \in Y$, consider $\alpha \in [-f(y) - \|y + z\|, -f(y) + \|y + z\|]$. Such an α exists if $\forall y_1, y_2$

$$-f(y_1) - \|y_1 + z\| \leq -f(y_2) + \|y_2 + z\|$$

Note

$$f(y_2) - f(y_1) = f(y_2 - y_1) \leq \|y_2 - y_1\| \leq \|y_2 + z\| + \|y_1 + z\|$$

■

Corollary 2.1.1. $\forall x \in X$, $\exists f_x \in X'$ such that $\|f_x\| = 1$ and $f_x(x) = \|x\|$.

Proof. Let $X_0 = \operatorname{span}(x_0)$ and $f_0(\lambda x_0) := |\lambda| \|x_0\|$. Then $\|f_0\| = 1$. Extend to $f : X \to \mathbb{R}$. ■

Note. Always $|f_x(c)| \leq \|f_x\| \cdot \|x\|)$

Lecture 5: Fifth Lecture

Recap: 18 Sep. 01:00

1. $\mathcal{L}(X, Y) := \{M : X \to Y \mid M \text{ is linear such that } \exists C \geq 0,\ \forall x \in X,\ \|Mx\|_Y \leq C\|x\|_X\}$

- $\|M\|_{\mathcal{L}(X,Y)} := \min\{C \geq 0 \mid \forall x \in X,\ \|Mx\|_Y \leq C\|x\|_X\}$

2. Dual space: $X' := \mathcal{L}(X, \mathbb{R})$.

- Can be $\mathbb{R}$ or $\mathbb{C}$.
- Hahn-Banach theorem: $X_0 \subset X$, $f_0 \in X_0'$ with $\|f_0\| \leq 1$ then $\exists f \in X'$ such that $f|_{X_0} = f_0$ and $\|f\| \leq 1$.

Corollary 2.1.2. Moreover, for all $x \in X$, $\|x\| = \max_{f \in X',\ \|f\|=1} |f(x)|$.

Proof. For every $f \in X'$, $|f(x)| \leq \underbrace{\|f\|}_{=1} \|x\|$. Equality holds if $f = f_x$. ■

Corollary 2.1.3. $X \hookrightarrow X''$ isometrically.

Proof. $x \mapsto L_x : f \in X' \mapsto f(x)$. Then

$$|L_x(f)| = |f(x)| \leq \|x\|_X \cdot \|f\|_{X'}.$$

Therefore, $\|L_x\|_{X''} \leq \|x\|_X$. On the other hand,

$$|L_x(f_x)| = |f_x(x)| = \|x\| = \|x\| \cdot \|f(x)\|.$$

Thus, $\|L_x\|_{X''} \geq \|x\|_X$. ■

Definition 2.1.2. X is called *reflexive* if $X = X''$.

Example. Recall the example from 597.

1. $X = L^p(E, \mathrm{d}\mu)$, $1 < p < \infty$.
2. $X' = L^q(E, \mathrm{d}\mu)$, $\frac{1}{p} + \frac{1}{q} = 1$.

- $g \in L^q$, $f \in L^p$ then $f \mapsto \int f\bar{g}\mathrm{d}\mu \in \mathbb{R}$ is a functional.
- $p = 1 \Rightarrow q = \infty$, where $L^\infty(E, \mathrm{d}\mu) = \{f : E \to \mathbb{R} \mid \operatorname{esssup} |f| < \infty\}$.

In particular, L^p with $1 < p < \infty$ is reflexive. However L^1 is not because $(L^\infty)' \not\supset L^1$.

Example. Let us consider the previous example in the case of sequences i.e. ℓ^∞.

$$\ell^\infty = \{(a_1, a_2, \dots \mid \|(a_n)\|_{\ell^\infty} := \sup_{n\in\mathbb{N}} |a_n| < \infty\}.$$

Let $g = (g_k)_{k=1}^\infty \in \ell^1$. Recall

$$\|g\|_{\ell^1} = \sum_{k=1}^\infty |g_k| < \infty.$$

Consider $a \in \ell^\infty \mapsto \sum_{k=1}^\infty a_k \bar{g}_k$. Let us show there exists a functional on ℓ^∞ which is not of this form. Define

$$\ell_*^\infty := \{a \in \ell^\infty \mid \exists \lim_{n\to\infty} a_n\}.$$

Consider a function "lim" on ℓ_*^∞ ($\|$"lim"$\| \leq 1$). By Hahn-Banach, there exists LIM $\in (\ell^\infty)'$ such that $\|\mathrm{LIM}\|_{(\ell^\infty)'} \leq 1$ and $\mathrm{LIM}(A) = \lim_{n\to\infty} a_n$ if this limit exists. LIM is sometimes called "Banach limit of bounded sequences".

Exercise. Show that $\forall a \in \ell^\infty$, $\liminf_{n\to\infty} a_n \le \mathrm{LIM}(a) \le \limsup_{n\to\infty} a_n$.

Answer. Start with $\liminf = -\limsup$. Then use linearity. ⊛

Example. $C([-1,1])$ is not reflexive. Recall

$$C' = \{\text{finite signed Borel measures on } [-1,1]\}.$$

Proof. Assume that $C'' = C$. Then for each $\ell \in C'$, $\|f\|_{C'} = \max_{\substack{f\in C''=C \\ |f|_C=1}} |\ell(f)|$. Take

$$\ell : f \in C \mapsto \int_{-1}^{0} f(t)\mathrm{d}t - \int_{-1}^{0} f(t)\mathrm{d}t.$$

Clearly, $\|\ell\|_{C'} \le 2$. On the other hand, $\forall \epsilon$, $\exists f \in C$ such that $\max_{[-1,1]} |f| = 1$ and $\ell(f) \ge 2 - \epsilon$. Therefore, $\|\ell\|_{C'} = 2$. However, $\nexists f$ such that $\|f\| = 1$ and $|\ell(f)| = 2$. ■

2.1.1 Other Versions/Extensions of Hahn-Banach Theorem

Theorem 2.1.2 (HB for $\mathbb{C}$-linear functionals)**.** $X_0 \subset X$ - $\mathbb{C}$-linear, normed. $f_0 \in X_0'$ with $\|f_0\| \le 1$. Then $\exists f \in X'$ with $\|f\| \le 1$ such that $f_{X_0} = f_0$.

Proof. First we introduce a useful lemma

Lemma 2.1.1. If $f : X \to \mathbb{C}$ - complex linear then $\exists T : X \to \mathbb{R}$ - real linear, such that $f(x) = T(x) - iT(x)$ for every $x \in X$. Vice versa, if T is $\mathbb{R}$-linear, then f is $\mathbb{C}$-linear. Moreover, if T is bounded, then $\|f\| = \|T\|$.

Proof. For the first part, write $f(x) = f_1(x) + if_2(x)$, where f_1 and f_2 are the real and imaginary components of f respectively. Then

$$if(x) = f(ix) = f_1(ix) + if_2(ix).$$

Therefore, $f_1(x) = f_2(ix)$ and $f_2(x) = -f_1(ix)$. Thus, we have an $\mathbb{R}$-linear T. Now suppose that T is $\mathbb{R}$ - linear. Then

$$\begin{aligned} f(e^{i\alpha}x) &= f(x)\cos\alpha + f(ix)\sin\alpha \\ &= T(x)\cos\alpha - iT(-x)\sin\alpha - iT(ix)\cos\alpha + T(ix)\sin\alpha \\ &= e^{i\alpha}f(x) \end{aligned}$$

Lastly, suppose that T is bounded. It is clear that

$$|T(x)| \le |f(x)| \le \|f\|\|x\|.$$

Therefore, $\|T\| \le \|f\|$. Vice versa, $\forall x \in X$, $\exists \alpha \in \mathbb{R}$ such that $f(e^{i\alpha}x) = e^{i\alpha}f(x) \in \mathbb{R}$. Thus,

$$|f(x)| = |f(e^{i\alpha}x)| = |T(e^{i\alpha}x)| \le \|T\|\|e^{i\alpha}x\| = \|T\|\|x\|.$$

■

Take f_0, consider $T_0 \coloneqq \operatorname{Re} f_0$ (note $\|T_0\| \le 1$). Apply HB for $\mathbb{R}$ to "construct" $T : X \to \mathbb{R}$ that extends T_0. Now define $f(x) = T(x) - iT(ix)$. It is $\mathbb{C}$-linear by the second part of the lemma and $\|f\| \le 1$ due to the last part of the lemma. ■

Theorem 2.1.3 (HB with symmetries)**.** Let $A_\nu : X \to X$ that commute with each other i.e. $A_\nu A_\mu = A_\mu A_\nu$ for all ν, μ. Assume $\|A_\nu x\| = \|x\|$. Let $X_0 \subset X$ be a subspace and $f_0 \in X_0'$ such that $\|f_0\| \le 1$ and $f_0(A_\nu x) = f_0(x)$ for all ν. Then $\exists f \in X'$ such that $f(A_\nu x) = f(x)$.

Proof. Proof can be found in Lax, section 33. ■

Example. Recall Banach limit LIM. Then $S' : \ell^\infty \to \ell^\infty$, $(a_1, a_2, \ldots)(a_2, a_3, \ldots)$ on ℓ^∞_* respects S'. $\exists \mathrm{LIM} \in (\ell^\infty)'$ such that $\mathrm{LIM}(S'a) = \mathrm{LIM}(a)$.

Lecture 6: Sixth Lecture

2.2 Hahn-Banach Theorem and Convex Sets

20 Sep. 01:00

Theorem 2.2.1. Let X - $\mathbb{R}$ linear vector space (not necessarily normed), for some $X_0 \subset X$ we have $\ell_0 : X_0 \to \mathbb{R}$ linear. Assume $p : X \to \mathbb{R}$ satisfies

(i) $p(\lambda x) = \lambda p(x)$ if $\lambda > 0$,

(ii) $p(x + y) \leq p(x) + p(y)$.

Assume that $X_0 \subset X$ is such that for all $x \in X_0$, $\ell_0(x) \leq p(x)$. Then $\exists \ell : X \to \mathbb{R}$ such that $\ell(x) \leq p(x)$ for all $x \in X$ and $\ell(x) = \ell_0(x)$ for all $x \in X_0$.

Proof. The same as with $p(x) = \|x\|$. ■

Definition 2.2.1. $C \subset X$ convex if $\forall x_1, x_2 \in C$ and $\forall \lambda \in [0, 1]$ we have $\lambda x_1 + (1 - \lambda)x_2 \in C$.

Definition 2.2.2. $x \in C$ is called *interior* if $\forall y \in X$, $\exists \epsilon > 0$ such that $\forall t \in [0, \epsilon)$ we have that $x + ty \in C$.

Remark. If X is Banach this does not imply that $\exists \epsilon > 0$ such that $B(x, \epsilon) \subset C$.

Theorem 2.2.2. We have the following

(a) Let C - convex, with 0 as an interior point of C. Then define $p_C(x) \coloneqq \inf\{\lambda > 0 \mid \frac{x}{\lambda} \in C\}$. Then p_C satisfies (i) and (ii). Moreover, if all points of C are interior, then $C = \{x \in X \mid p_C(x) < 1\}$.

(b) Vice versa, given p satisfying (i) and (ii), the set $\{x \in X \mid p(x) < 1\}$ is convex. p is a norm on X iff $p(x) = p(-x)$ and C does not contain a line through 0.

Proof of (a). To show p_C satisfies (i) is trivial. For (ii), note that $\frac{x_1}{p_C(x_1)+\epsilon} \in C$. By convexity, $\frac{x_1+x_2}{p_C(x_1)+p_C(x_2)+2\epsilon} \in C$. Thus, $p_C(x_1 + x_2) \leq p_C(x_1) + p_C(x_2) + 2\epsilon$. Now if $x \in C$ is interior then there exists $\epsilon > 0$ such that $(1 + \epsilon)x \in C$. Therefore, $p_C(x) \leq \frac{1}{1+\epsilon} < 1$. If $p_C(x) < 1$, then $\exists \epsilon > 0$ such that $(1 + \epsilon)x \in C$. Thus, $x \in C$ and $0 \in C$. ■

Proof of (b). If $p(x_1) < 1$, $p(x_2) < 1$ then $p(\alpha x_1 + (1 - \alpha)x_2) \leq \alpha p(x_1) + (1 - \alpha)p(x_2) < 1$. Thus, all points interior. ■

Exercise. Prove the remainder of (b).

Theorem 2.2.3 (Geometric HB / Hyperplane Separation Theorem)**.** Let X - $\mathbb{R}$ linear space.

(1) C - convex with interior point and $y \notin C$. Then $\exists \ell : X \to \mathbb{R}$ such that $\forall x \in C$ $\ell(x) \leq \ell(y)$.

(2) C_1, C_2 - convex, where at least one has an interior point and $C_1 \cap C_2 = \emptyset$. Then $\exists \ell : X \to \mathbb{R}$ such that $\sup_{x_1 \in C_1} f(x_1) \leq \inf_{x_2 \in C_2} f(x_2)$.

Proof of (1). Without loss of generality, 0 - interior point of C with p_C. Consider $X_0 = \operatorname{span}(y)$. Define $\ell_0 : X_0 \to \mathbb{R}$ by $\ell_0(y) = 1$. Then $\ell_0(y) \le p_C(y)$. Extend to $\ell \le p_C$ then for every $x \in C$ $\ell(x) \le p_C(x) < 1$. ■

Proof of (2). Denote $C \coloneqq C_1 - C_2 = \{x_1 - x_2 \mid x_1 \in C_1,\ x_2 \in C_2\}$. Since $C_1 \cap C_2 = \emptyset$, $0 \notin C_1 - C_2$. By (1), $\exists \ell : X \to \mathbb{R}$ such that $\ell(x_1 - x_2) \le \ell(0) = 0$ for every $x_1 \in C_1$, $x_2 \in C_2$. ■

Definition 2.2.3. T - any set. Let $B = \{f : T \to \mathbb{R} \mid \exists C > 0,\ \forall t \in T,\ |f(t)| \le C\}$. Linear $\ell : B \to \mathbb{R}$ is called *positive* if $\forall f$ such that $f(t) \ge 0$ for all t we have $\ell(f) \ge 0$.

Remark. By linearity if $\forall t,\ f(t) \le g(t)$ then $\ell(f) \le \ell(g)$.

Theorem 2.2.4. Let $B_0 \subset B$ be a subspace such that $\exists f_0 \in B_0$ such that $f_0 \ge 1$. Assume $\ell_0 : b_0 \to \mathbb{R}$ is a positive linear functional. Then $\exists f : B \to \mathbb{R}$ positive such that $f|_{B_0} = f_0$.

Proof. Define $p(f) \coloneqq \inf \ell_0(f_0)$ (for all $f \in B$), where the infimum is over $f_0 \in B_0$ such that $f \le f_0$. Then $\forall \epsilon > 0$, $\exists f_0, g_0$ such that $f \le f_0$ and $g \le g_0$ such that

$$\ell_0(f_0) \le p(f) + \epsilon \quad \text{and} \quad \ell_0(g_0) \le p(g) + \epsilon.$$

Therefore,

$$f + g \le f_0 + g_0 \Rightarrow p(f+g) \le \ell_0(f_0 + g_0) \le p(f) + p(g) + 2\epsilon$$

Moreover, if $f \in B_0$ $p(f) = \ell_0(f)$ due to positivity of ℓ_0 $(\forall f_0 \in B_0, \ell_0(f) \le \ell_0(f_0))$. Use HB to extend ℓ_0 to $\ell : B \to \mathbb{R}$ such that $\ell(f) \le p(f)$ $\forall f \in B$. To see that f is positive, suppose $f \le 0$. Then $\ell(f) \le p(f) \le 0$. Thus, f is positive. ■

Example. $T = \mathbb{R} / 2\pi\mathbb{Z}$. Then $\exists m : \mathcal{P}(T) \to \mathbb{R}$ such that $m(S) = \operatorname{Leb}(S)$ if S is measurable and is finite additive.

Proof. Let $B_0 : \{f \mid T \to \mathbb{R} \mid f$ bounded and Lebesgue-measurable$\}$. Then $\ell_0(f) \coloneqq \int_\pi f(t)\mathrm{d}t$. Extend ℓ_0 to all functions on T keeping the positivity $f \ge 0 \Rightarrow \ell(f) \ge 0$. Denote $m(S) \coloneqq f(\mathbf{1}_S)$. ■

Example. Consider $A_\alpha f = (t \mapsto f(t + \alpha)) \in B$ (rotations of the circle), so that $A_\alpha A_\beta = A_\beta A_\alpha$. From HB with symmetries, one can also require that $m(S + \alpha) = m(S)$ i.e. m is invariant under rotations.

Remark. This cannot be possible in higher dimension due to Banach-Tarski.

Lecture 7: Seventh Lecture

We will develop the following hierarchy 25 Sep. 01:00

$$\underset{\langle\cdot\rangle : X \times X \to \mathbb{R}}{\text{Hilbert space(s)}} \subset \underset{\|\cdot\| : X \to \mathbb{R}}{\text{Banach spaces}} \subset \underset{C^\infty}{\text{locally convex topological spaces}}$$

Definition 2.2.4. X is called a *locally convex topological* (LCT) space if X is a linear space with a topology such that

(i) $x, y \mapsto x + y$ and $\lambda, x \mapsto \lambda x$ are continuous operations.

(ii) $\exists$ a local base of the topology formed by convex sets.

Definition 2.2.5. A *local base* of x, $\{U_\alpha\}$ is a collection of open sets such that $\forall\alpha$ $x \in U_\alpha$ and $\forall U \ni x$ open $\exists\alpha$ such that $U_\alpha \subset U$.

Remark. A norm can be thought of as a convex set containing the origin.

Discussion of LCT definition:

(a) One can always assume that U_α is symmetric with respect to 0.

   – Replace U_α by $U_\alpha \cap (-U_\alpha)$.

(b) $\forall\alpha$ p_{U_α} is a semi-norm - does not require $p(x) = 0 \Rightarrow x = 0$.

Remark. Recall $p_k(x) \coloneqq \inf\{\lambda > 0 : \frac{x}{\lambda} \in \lambda U_k\}$. Does NOT satisfy $p_k(x) = 0 \Rightarrow x = 0$.

Example (LCT space). Consider

$$C^\infty([0,1]) \coloneqq \{f : [0,1] \to \mathbb{R} \mid \forall k,\ f^{(k)} \in C([0,1])\}.$$

Define $p_k(f) \coloneqq \max_{t\in[0,1]} |f^{(k)}(t)|$ (or $\widetilde{p}_k(f) \coloneqq \sum_{j=0}^{k} \max_{t\in[0,1]} |f^j(t)|$). We define open sets to be sets of the form

$$U_{k,\epsilon} \coloneqq \{f \in C^\infty([0,1]) : \widetilde{p}_k(f) < \epsilon\}.$$

Now what does it mean that $f_n \to f$ in $C^\infty([0,1])$? Note $f_n - f \to 0$. Therefore, $\forall k, \epsilon > 0$, $\exists n$ such that $f_n - f \in U_{k,\epsilon}$ i.e.

$$\sum_{j=0}^{k} \max |f_n^{(j)} - f^{(j)}| < \epsilon.$$

Continuing in this manner, we have that $\forall k \in \mathbb{N}$, $f_n^{(k)} \to f^{(k)}$ in $C([0,1])$ (i.e. $\max |f_n^{(k)} - f^{(k)}| \to 0$).

Definition 2.2.6. X-Banach is called *uniformly convex* if $\exists\epsilon : (0,2] \to (0,+\infty)$ continuous such that $\lim_{t\to 0} \epsilon(t) = 0$ such that $\forall x, y \in X$ with $\|x\| = \|y\| = 1$ then

$$\left\|\frac{x+y}{2}\right\| \leq 1 - \epsilon(\|x-y\|).$$

Example. ℓ_1, ℓ_∞ norm on $\mathbb{R}^2$ is not uniformly convex (draw picture).

Theorem 2.2.5 (Milman (1940s)). If X is uniformly convex then X is reflexive.

Theorem 2.2.6. If X is a uniformly convex Banach space then there is a set F that is closed and convex, $z \notin F$ such that $\exists! x_* \in F$ such that $\|z - x_*\| = \inf_{x\in F} \|z - x\|$.

Proof of existence. Let x_n[a] $\in F$ be such that

$$\|z - x_n\| \to \inf_{x\in F} \|z - x\| =: d.$$

Without loss of generality, assume $z = 0$. Define $y_n \coloneqq \frac{x_n}{\|x_n\|}$. Note that

$$\frac{y_n + y_m}{2} = \left(\frac{1}{2\|x_n\|} + \frac{1}{2\|x_m\|}\right) \underbrace{(\alpha x_n + (1-\alpha)x_m)}_{\in K}$$

Therefore,

$$\left\|\frac{y_n + y_m}{2}\right\| \geq \left(\frac{1}{2\|x_n\|} + \frac{1}{2\|x_m\|}\right) \cdot d \xrightarrow{n,m\to\infty} 1$$

By uniform convexity, we have that $\|y_n - y_m\| \to 0$ as $n, m \to \infty$. Therefore, $\{y_n\}$ is a Cauchy sequence. Since X is complete, $\exists y_* \in X$ such that $y_n \to y_*$. Thus, $x_n \to dy_*$. ■

[a] In finite dimensions, this is a compact set and therefore the infimum is attained.

Proof of uniqueness. Similar argument as existence (Exercise). ■

Example. Consider $X = C([-1,1])$. Even if F is closed and convex, $z \notin F$, we do not have a unique closest point. Define

$$F := \left\{ f \in C([-1,1]) \mid \int_{-1}^{0} f(t)\mathrm{d}t = 0 = \int_{0}^{1} f(t)\mathrm{d}t \right\}$$

Let $g(t) = \sin \pi t$, so that $\int_{-1}^{0} g(t)\mathrm{d}t = \frac{2}{\pi} = -\int_0^1 g(t)\mathrm{d}t$. Then $\forall f \in F$, $\max_{t\in[-1,1]} |f - g| \geq \frac{2}{\pi}$. Equality holds if and only if

$$f(t) = \begin{cases} g(t) + \frac{2}{\pi}, & \text{if } t \in [-1, 0] \\ g(t) - \frac{2}{\pi}, & \text{if } t \in [0, 1] \end{cases}$$

This is not continuous. Therefore, inequality must be strict.

Exercise. Show that OK for $\frac{2}{\pi} + \epsilon$.

Chapter 3

Hilbert Space(s)

Definition 3.0.1. X-linear over $\mathbb{R}$ or $\mathbb{C}$. $\langle \cdot, \cdot \rangle : X \times X \to \mathbb{R}$ or $\mathbb{C}$ is called a scalar product if

(i) $x \mapsto \langle x, y \rangle$ is linear, $y \mapsto \langle x, y \rangle$ is anti linear ($\langle x, \lambda y \rangle = \overline{\lambda} \langle x, y \rangle$).

(ii) $\langle x, y \rangle = \overline{\langle y, x \rangle}$.

(iii) $\langle x, x \rangle > 0$ if $x \neq 0$.

Proposition 3.0.1. $\|x\| := \sqrt{\langle x, x \rangle}$ is a norm on X.

Definition 3.0.2. X is called Hilbert if it is complete with respect to $\|\cdot\|$.

Lecture 8: Eighth Lecture

25 Sep. 01:00

Proof of proposition. First, we have that

$$\|\lambda x\| = \sqrt{\lambda \overline{\lambda}} \cdot \|x\| = |\lambda| \cdot \|x\|.$$

Therefore, we must check the triangle inequality. Note that

$$\langle x + y, x + y \rangle = \langle x, x \rangle + \langle y, y \rangle + 2 \operatorname{Re} \langle x, y \rangle$$

By Cauchy Schwartz, we have that

$$\langle x + y, x + y \rangle \leq \langle x, x \rangle + \langle y, y \rangle + 2 \|x\| \|y\|.$$

Thus, $\|x + y\|^2 \leq (\|x\| + \|y\|)^2$ and the triangle inequality follows. ■

Example. $\ell^2 = \{x = (x_1, x_2, \ldots) \mid \|x\|^2 := \sum_{k=1}^{\infty} x_k^2 < \infty\}$ is a Hilbert space.

- Scalar product is $\langle x, y \rangle = \sum_{k=1}^{\infty} x_k \overline{y_k}$.

Example. $L^2(\Omega, \mu) = \{f : \Omega \to \mathbb{C} \mid \|x\|^2 := \int_\Omega |f|^2 \,\mathrm{d}m < \infty\}$ is a Hilbert space.

- Scalar product is $\langle f, g \rangle = \int f \overline{g} \,\mathrm{d}m$.

Definition 3.0.3. X is called *separable* if there exists a countable everywhere dense set of points in X.

Lemma 3.0.1 (Parallelogram Identity). If X - Hilbert space then $\|x+y\|^2+\|x-y\|^2 = 2(\|x\|^2+\|y\|^2)$.

Proof. Straightforward. ■

Corollary 3.0.1. If X - Hilbert then $\forall x, y$ such that $\|x\| = \|y\| = 1$, we have that

$$\left\|\frac{x+y}{2}\right\|^2 \leq 1 - \left\|\frac{x-y}{2}\right\|^2$$

In particular, X is uniformly convex.

Definition 3.0.4. x, y are called *orthogonal* if $\langle x, y\rangle = 0$.

Proposition 3.0.2. X - Hilbert. $Y \subset X$ - closed linear subspace. Define $Y^\perp := \{x \in X \mid \forall y \in Y,\ \langle x, y\rangle = 0\}$. Then

(i) $Y^\perp$ is a closed subspace.

(ii) For each $z \in X$, $\exists! x \in Y^\perp$ and $y \in Y$ such that $z = x + y$ (and $\|z\|^2 = \|x\|^2 + \|y\|^2$).

(iii) $(Y^\perp)^\perp = Y$.

Remark. $\exists$ Banach spaces X and closed subspaces of Y of those such that $\nexists$ closed subspace Y' such that $\forall z \in X$, $\exists! y, y'$ with $z = y + y'$.

Proof of Proposition (i). Let $x_n \in Y^\perp$ such that $x_n \to x_* \in X$. Then $\forall y \in Y$,

$$|\langle x_n - x_*, y\rangle| \leq \|x_n - x_*\|^{1/2} \cdot \|y\|^{1/2} \xrightarrow{n\to\infty} 0$$

Therefore, $\langle x_n, y\rangle \to \langle x_*, y\rangle$ and $\langle x_*, y\rangle = 0$ for every $y \in Y$. Thus, $x_* \in Y^\perp$. ■

Proof of Proposition (ii). Given z, let y be the best approximation to z in Y (Remember there exists a unique such y). Note that $\forall y' \in Y$,

$$\|z - y - \lambda y'\|^2 \geq \|z - y\|^2$$

As an exercise, it can be show that $\langle z-y, y'\rangle = 0$. Additionally, it is an exercise to show uniqueness i.e. if $z = x + y = x' + y'$ then $x - x' = y - y' = 0$. ■

Proof of Proposition (iii). Clearly, $Y \subset (Y^\perp)^\perp$. Let $z \in (Y^\perp)^\perp$. From (ii), consider $z = x + y$ for $x \in Y^\perp$ and $y \in Y$. Then

$$0 = \langle z, x\rangle = \langle x, x\rangle + \langle y, x\rangle$$

Since $\langle y, x\rangle = 0$, we must have that $\langle x, x\rangle = 0$. Thus, $x = 0$ and $z \in Y$. ■

3.1 Dual of Hilbert Space

Recall

$$X' = \{\ell : X \to \mathbb{R} \mid \exists C \geq 0,\ |\ell(x)| \leq C\|x\|\}$$

Given $y \in X$, let $\ell_y(x) := \langle x, \overline{y}\rangle$. This is bounded by Cauchy-Schwartz $\|\ell_y\|_{X'} \leq \|y\|_X$ i.e. $X \hookrightarrow X'$. Moreover, $x = y \Rightarrow \|\ell_y\| \geq \|y\|$.

Theorem 3.1.1 (Riesz-Frechet). X - Hilbert space. Then $X \cong X'$, where $\cong$ means isometric under $y \mapsto \ell_y$.

Proof. Let $\ell \in X'$. Define $Y := \ker \ell$. Y is closed as ℓ is bounded (bounded iff continuous and inverse image of 0). Then we know that $\dim Y^\perp \leq 1$. To see this, suppose that $y_1, y_2 \in Y^\perp$. Then $\exists \alpha, \beta$ such that $\alpha y_1 + \beta y_2 = 0$. Therefore, $\alpha y_1 = -\beta y_2$. If $\dim Y^\perp = 0$, then $Y = X$ and $\ell = 0$. If $\dim Y^\perp = 1$. Find $y \in Y^\perp$ such that $\ell(y) = \langle y, \overline{y} \rangle$. Scale it. Then $\ker(\ell - \langle \cdot, \overline{y} \rangle) = X$. ∎

3.1.1 Application: There exists Radon-Nykodim derivatives.

Definition 3.1.1. ν is called *absolutely continuous* with respect to μ if $\mu(F) = 0 \Rightarrow \nu(F) = 0$.

Theorem 3.1.2. $E = [0, 1]$, μ, ν - Borel measures (let finite for simplicity) such that $\forall F$ - measurable, $\exists h : E \to [0, \infty)$ measurable with $\nu(F) = \int_F h \mathrm{d}\mu$.

Proof. Consider $X = L^2(\mathrm{d}\mu + \mathrm{d}\nu)$. It is complete. Consider $\ell : X \to \mathbb{R}$ defined by $f \mapsto \int f \mathrm{d}\mu$. Then $\exists g \in X$ such that

$$\ell(f) = \langle f, g \rangle = \int fg(\mathrm{d}\mu + \mathrm{d}\nu)$$

Then $\forall f$,

$$\int f(\mathbf{1} - g)\mathrm{d}\mu = \int fg \mathrm{d}\nu$$

∎

Exercise. Prove $0 \leq g \leq 1$. Explain that $h = \frac{1-g}{g}$ is what we need (Take $fg = \mathbf{1}_F$).

Lecture 9: Ninth Lecture

Remark. On complementary subspaces in Banach spaces. For Banach space X, do we have that $\forall Y$ - closed subspace of X, $\exists Z \subset X$ such that $\forall x \in X$, $\exists! x = y + z$? 2 Oct. 01:00

- We know that this is true for Hilbert spaces.

Theorem 3.1.3. The above remark is false for all spaces that are not Hilbert ALWAYS.

3.2 Hilbert Space(s) cont.

Goal: We want to prove the following theorem

Theorem 3.2.1. X - Hilbert. Then $\exists \{x_\alpha\}$ pairwise orthogonal with $\|x_\alpha\| = 1$ i.e. *orthonormal* such that for each $x \in X$, $x = \sum_\alpha a_\alpha x_\alpha$ where

- $a_\alpha = \langle x, x_\alpha \rangle$,
- $\|x\|^2 = \sum_\alpha |a_\alpha|^2$.
  - $\langle x, y \rangle = \sum_\alpha a_\alpha b_\alpha$, $b_\alpha = \langle y, x_\alpha \rangle$.

Remark. A part of the statement $\Rightarrow$ at most countably many $a_\alpha \neq 0$, the series converges and the result does not depend on the order.

Remark. Norm implication can be proven using *polarization identity* i.e. for $\mathbb{R}$

$$4\langle x, y\rangle = \|x+y\|^2 - \|x-y\|^2$$

or in $\mathbb{C}$,

$$8\langle x, y\rangle = \|x+y\|^2 + i\|x+iy\|^2 - \|x-y\|^2 - i\|x-iy\|^2$$

Proposition 3.2.1. X - Hilbert. Let $\{x_\alpha\}$ - orthonormal. Then

(1) $\forall x \in X$, $\|x\|^2 \geq \sum_\alpha |\langle x, x_\alpha\rangle|^2$

(2) If $\sum_\alpha |a_\alpha|^2 < \infty$, then $\sum_\alpha a_\alpha x_\alpha$ converges in X, $\|x\|^2 = \sum_\alpha |a_\alpha|^2$, and $a_\alpha = \langle x, x_\alpha\rangle$

(3) The set of $x \in X$ that satisfy (2) is $\overline{\operatorname{span}(\{x_\alpha\})}$.

Proof of (1). Enough to prove for finite collections. Note that

$$\begin{aligned}\|x - \sum_{k=1}^n \langle x, x_{\alpha_k}\rangle x_{\alpha_k}\|^2 &= \|x\|^2 + \sum_{k=1}^n |\langle x, x_{\alpha_k}\rangle|^2\|x_{\alpha_k}\|^2 - \sum_{k=1}^n \langle x, x_{\alpha_k}\rangle\overline{\langle x, x_{\alpha_k}\rangle} - \sum_{k=1}^n \langle x_{\alpha_k}, x\rangle\langle x, x_{\alpha_k}\rangle \\ &= \|x\|^2 - \sum_{k=1}^n |\langle x, x_{\alpha_k}\rangle|^2 \geq 0\end{aligned}$$

Showing ≥ 0 is an exercise. ■

Proof of (2). $\sum_\alpha a_\alpha x_\alpha$ is a countable (at most) sum. Define $\mathcal{S}_n := \sum_{k=1}^n a_{\alpha_k} x_{\alpha_k}$. Then

$$\|\mathcal{S}_n - \mathcal{S}_m\|^2 = \sum_{k=n+1}^m |a_{\alpha_k}|^2 \to 0$$

Thus, $\mathcal{S}_n$ is Cauchy and $\mathcal{S}_n \to x \in X$. Since $\|\cdot\|$ is continuous and

$$\|S_n\|^2 = \sum_{k=1}^N |a_{\alpha_k}|^2,$$

It follows that as $n \to \infty$ $\|x\| = \sum_{k=1}^\infty |a_{\alpha_k}|^2$. Lastly, we have for sufficiently large n that $\langle S_n, x_\alpha\rangle = a_\alpha$. Thus, $\langle x, x_\alpha\rangle = a_\alpha$. ■

Proof of (3). The first direction

$$\{x = \sum_\alpha a_\alpha x_\alpha \mid \sum_\alpha |a_\alpha|^2 < \infty\} \subset \overline{\operatorname{span}(x_\alpha)}$$

is trivial. For the other direction, consider $(\overline{\operatorname{span}(x_\alpha)})^\perp$. Namely,

$$(\overline{\operatorname{span}(x_\alpha)})^\perp = \{y \in X \mid \forall \alpha\ \langle y, x_\alpha\rangle = 0\} =: Y$$

The $\subset$ direction follows from definition. For $\supset$,

$$\begin{aligned} y \perp x_\alpha &\Rightarrow \perp \text{ all finite linear combinations} \\ &\Rightarrow \perp \text{ each } x \in \overline{\operatorname{span}(x_\alpha)}\end{aligned}$$

where the last implication follows from continuity of $\langle \cdot, \cdot\rangle$. Thus, $\overline{\operatorname{span}(x_\alpha)} = Y^\perp$.

Now take $x \in Y^\perp$. Define $a_\alpha := \langle x, x_\alpha\rangle$. Consider $\tilde{x} := \sum_\alpha a_\alpha x_\alpha$, which exists due to (1) and (2). Then for each α, $\langle x - \tilde{x}, x_\alpha\rangle = 0$. Therefore, $x - \tilde{x} \in Y$ and $\tilde{x} \in Y^\perp$. Thus, $x - \tilde{x} \in Y$ and $x - \tilde{x} \in Y^\perp$, so it must be the case that $x = \tilde{x}$. ■

Proof of Theorem 3.2.1. Zorn's lemma for all possible orthonormal families in X i.e. $\{x_\alpha\} \prec \{y_\beta\}$ if $\subset$. If $\{x_\alpha\}$ is a maximal element, then $\overline{\operatorname{span}(x_\alpha)} = X$. If not, take $y \in \overline{\operatorname{span}(x_\alpha)}^\perp$ such that $\|y\| = 1$ and add it to $\{x_\alpha\}$. ∎

Corollary 3.2.1. If X is a separable infinite-dimensional Hilbert space, then $X \cong \ell^2$.

Proof. Let $x_1, x_2, \ldots$ be a countable everywhere dense set. Gram-Schmidt orthogonalization i.e.

$$\begin{aligned} x_1 &\mapsto \frac{x_1}{\|x_1\|} =: \tilde{x}_1, \\ x_2 &\mapsto \frac{x_2 - \langle x_2, x_1\rangle \tilde{x}_1}{\|x_2 - \langle x_2, x_1\rangle \tilde{x}_1\|} =: \tilde{x}_2, \\ &\vdots \end{aligned}$$

Then $\{\tilde{x}_k\}$ - countable orthonormal set such that $\operatorname{span}(\{\vec{x}_k\}) \supset \{x_k\}$. Since $\{x_k\}$ is dense $\overline{\operatorname{span}(\{x_k\})} = X$. Now consider a map

$$x = \sum_{k=1}^{\infty} a_k \tilde{x}_k \in X \mapsto a = (a_k)_{k=1}^{\infty} \in \ell^2$$

This is an isometry. ∎

Example. $X = L^2([0,1]) \cong \ell^2$ i.e. $x_n(t) = e^{2\pi i n t}$ and the isometry is the Fourier transform.

Theorem 3.2.2 (Riesz Representation). If X - Hilbert, $\ell : X \to \mathbb{R}$ bounded linear function then $\exists y \in X$ such that $\ell(x) = \langle x, y\rangle$

Theorem 3.2.3 (Lax-Milgram's Lemma). $B : X \times X \to \mathbb{R}$ - linear in first (anti)-linear in second such that

$$|B(x, y)| \le C \cdot \|x\| \cdot \|y\|$$

and

$$|B(y, y)| \ge b\|y\|^2$$

for some $b > 0$. Then for each $\ell \in X'$, $\exists y \in X$ such that $\ell(x) = B(x, y)$.

Proof. By Riesz representation theorem, $\forall y \in X$, $\exists z \in X$ such that $B(x, y) = \langle x, z\rangle$ for every $x \in X$. This gives a mapping $L : y \mapsto z$ linear. Note that

$$b\|y\|^2 \le B(y, y) = \langle y, z\rangle \le \|y\| \cdot \|z\|$$

Therefore, $b\|y\| \le \|z\|$ so that $\operatorname{Ran} L$ is closed. To see this, suppose $z_n \in \operatorname{Ran} L$, then, by the inequality, $y_n \to y$ and $z = L(y)$. Moreover, $\operatorname{Ran} L = X$. Otherwise, $\exists x \neq 0$, such that $x \perp \operatorname{Ran} L$. Then $\forall y$

$$B(x, y) = \langle x, z\rangle = 0$$

This is a contradiction because if $y = x$ then $B(x, y) \ge b\|y\|^2 > 0$. ∎

Lecture 10: Tenth Lecture

4 Oct. 01:00

3.3 Comments/examples on the duality

Reminder: In Hilbert spaces $X = X'$. Each bounded linear functional is of the form $x \mapsto \langle x, y\rangle$ for some $y \in H$.

Given a subspace $Y \subset X$, $Y^\perp = \{y \mid \forall x \in Y \ \langle y, x\rangle = 0\}$.

Remark. If $Y \neq \overline{Y}$, then $Y^\perp = \overline{Y}^\perp$.

Definition 3.3.1. X - Banach. $Y \subset X$. Define $Y^\perp = \{\ell \in X' \mid \forall x \in Y \ \ell(x) = 0\}$.

Remark. A few remarks:

(a) $Y^\perp = \overline{Y}^\perp$.

(b) $Y^\perp$ is a closed subspace of X'.

3.3.1 Example in Complex Analysis

Proposition 3.3.1. $Y \subset X$ - Banach. Then, for each $z \in X$,

$$d(z; Y) := \inf_{y \in Y} \|z - y\| = \max_{\substack{\ell \in Y^\perp \subset X' \\ \|\ell\|_{X'} \leq 1}} |\ell(z)| =: M(z)$$

Proof. Since $\|\ell\| \leq 1$,

$$|\ell(z)| = |\ell(z - y)| \leq \|z - y\|,$$

where the first equality follows from $\ell \in Y^\perp$. Thus, $M(z) \leq d(z; Y)$. Vice versa, let

$$\begin{aligned} \ell : \operatorname{span}(Y, z) &\longrightarrow \mathbb{R} \\ y + \lambda z &\longmapsto \lambda d(z) \end{aligned}$$

Note that

$$\|y + \lambda z\| \geq |\lambda| \operatorname{dist}(z; Y) \Rightarrow \|\ell\| \leq 1$$

Can extend to X by Hahn-Banach. ■

Corollary 3.3.1. $z \in \overline{Y} \Leftrightarrow d(z) = 0$ iff $\ell \in Y^\perp \Rightarrow \ell(z) = 0$.

Theorem 3.3.1 (Runge)**.** $\Omega \subset \mathbb{C}$ open, simply connected, bounded. $K \subset \Omega$ compact. For each $f \in \operatorname{Hol}(\Omega, \mathbb{C})$, $\forall \epsilon > 0$, $\exists P \in \mathbb{C}[z]$ polynomial such that $\max_{z \in K} |f(z) - P(z)| \leq \epsilon$.

Proof. Consider $X = C(K)$ (complex-valued). $\forall \ell \in (C(K))'$ such that $\ell(z^k) = 0$ for every $k = 0, 1, \ldots$ does $\ell(f) = 0$? Note

$$f(z) = \frac{1}{2\pi i} \int \frac{f(\zeta)}{\zeta - z} \, d\zeta$$

can be approximated uniformly on $z \in K$ by a linear combination of $\frac{1}{\zeta_m - z}$. Therefore, it is enough to show that $\ell\left(\frac{1}{\zeta - z}\right) = 0$ for each $\zeta \notin K$. If $\Omega = \mathbb{D}$, $K = r\mathbb{D}$ then write

$$\frac{1}{\zeta - z} = \frac{1}{\zeta} + \frac{z}{\zeta^2} + \frac{z^2}{\zeta^3} + \cdots$$

we can truncate. For $K \neq \mathbb{D}$, note that this still holds if $|\zeta| > \max_{z \in K} |z|$ (expand $\frac{1}{\zeta - z}$) truncate.

The key observation is that

$$\underbrace{\zeta}_{\in\mathbb{C}\setminus K} \longmapsto \ell\left(\frac{1}{\zeta - z}\right) \in \mathbb{C}$$

is an analytic function. In summary, $|\zeta|$ - big $\Rightarrow F(\zeta) = 0$. By analyticity, $\forall \zeta \notin K$ $F(\zeta) = 0$. ∎

3.3.2 Solving PDEs

$\Omega \subset \mathbb{R}^d$ bounded domain with smooth boundary. Given $g : \Omega \to \mathbb{R}$. Find $f : \Omega \to \mathbb{R}$ such that $-\Delta f = -\sum_{k=1}^{d} \frac{\partial^2 f}{\partial x_k^2} = g$.

Problem 3.3.1. What are the "right" function spaces for f and g?

Consider $H^0 = L^2(\Omega)$. $\forall \phi \in C_0^\infty(\Omega)$,

$$\langle -\Delta f, \phi\rangle_{L^2} = \langle g, \phi\rangle_{L^2}$$

Introduce another Hilbert space with the scalar product

$$\langle f, \phi\rangle = \int_\Omega \nabla f \cdot \nabla \phi$$

Definition 3.3.2. Consider the completion of $C_0^\infty(\Omega)$ with respect to the scalar product $\langle \phi, \psi\rangle$ defined as

$$\langle \phi, \psi\rangle = \int_\Omega \nabla \phi \cdot \nabla \psi.$$

This is called a *Sobolev space* denoted $H_0^1 = W_0^{1,2}$.

Lemma 3.3.1. $\phi \in C_0^\infty \Rightarrow \|\phi\|_{L^2} \leq \operatorname{diam}(\Omega) \cdot \|\phi\|_{H_0^1}$.

Corollary 3.3.2. We have the following:

(1) $H_0^1 \subset L^2$. Moreover, for each $f \in H_0^1$ ($\frac{\partial f}{\partial x_j} \in L^2$) and $\psi \in C_0^\infty$,

$$\int_\Omega \frac{\partial f}{\partial x_j}\psi = -\int_\Omega f \frac{\partial \psi}{\partial x_j}$$

This is the *weak formulation* of the derivative.

(2) However, $H_0^1 \not\subset C(\Omega)$.

(3) Zero boundary conditions survive.

Example (Ex of (2)). If $d \geq 3$, then

$$f(x) = \frac{1}{|x|^{-d/2+1+\epsilon}} \in H^1$$

Example (Ex of (3)). If $f \in H_0^1$ then $\forall x \in \partial\Omega$,

$$\frac{1}{|B(x,r)|}\int f \to 0$$

Now we return to understanding the "right" setup.

- $-\Delta f = g$

- "Dual formulation"
$$\langle f, \phi \rangle_{H_0^1} = \langle g, \phi \rangle_{L^2}$$
Since $\langle g, \phi \rangle_{L^2}$ is a bounded linear functional, by Riesz representation theorem $\exists f \in H_0^1$ that satisfies this equality.

- The "right" set of inputs then must be the ones that make $\langle g, \phi \rangle_{L^2}$ a bounded linear functional.

Note that if $g \in L^2(\Omega)$ then
$$\int_{L^2} g\phi \leq \|g\|_{L^2} \|\phi\|_{L^2} \leq \|g\|_{L^2} \cdot \|\phi\|_{H_0^1}$$
which means that Riesz representation theorem can be applied. (In fact, one can go further, $g \in H^{-1}$)

> **Remark.** For more in depth treatment of examples, see Lax Section 7.

Chapter 4

Bounded Linear Operators

Lecture 11: Eleventh Lecture

Reminder: X, Y - Banach, $M : X \to Y$ linear then 9 Oct. 01:00

$$\mathcal{L}(X,Y) \coloneqq \{M : X \to Y \mid \exists C > 0,\ \|Mx\|_Y \leq C\|x\|_X\}$$

is a Banach space.

4.1 Baire's Theorem

Definition 4.1.1. $A \subset X$ - metric (or topological space) is called *everywhere dense* if $\overline{A} = X$. Equivalently, $\forall \mathcal{U} \neq \emptyset$ open, $U \cap A \neq \emptyset$.

Definition 4.1.2. $A \subset X$ - metric (or topological space) is called *nowhere dense* if $\operatorname{Int} \overline{A} = \emptyset$. Equivalently, $\forall \mathcal{U} \neq \emptyset$ open, $\exists \emptyset \neq V \subset \mathcal{U}$ such that $V \cap A = \emptyset$.

Definition 4.1.3. $A \subset X$ is *meager* (or 1st category) if $\exists F_1, F_2, \dots$ nowhere dense such that $A \subset \bigcup_{n=1}^{\infty} F_n$.

Theorem 4.1.1 (Baire)**.** Let X - complete metric space and $A \subset X$ meager. Then $X \setminus A$ is everywhere dense.

Remark. In particular, $A \neq X$.

Proof of Theorem 4.1.1. Suppose that $X\ A$ is not everywhere dense. Then $\exists \emptyset \neq \mathcal{U}$ such that $X \setminus A \cap \mathcal{U} = \emptyset$. Therefore, $\mathcal{U} \subset A$. We can assume that $\exists x_0, r_0$ such that $B(x_0, r_0) \subset A \subset \bigcup_{k=1}^{\infty} F_k$. Since F_1 not everywhere dense, we can find x_1, r_1, $0 < r_1 \leq \frac{1}{2}r_0$, such that $\overline{B}(x_1, r_1) \subset B(x_0, r_0)$ with $\overline{B}(x_1, r_1) \cap F_1 = \emptyset$. Similarly, we can find x_2, r_2, $0 < r_2 \leq \frac{1}{2}r_1$ such that $\overline{B}(x_2, r_2) \subset B(x_1, r_1)$ and $\overline{B}(x_2, r_2) \cap F_2 = \emptyset$. Take $x \in \cap_{k=1}^{\infty} \overline{B}(x_k, r_k)$. Then we have that $x \in A$ but $x \notin F_k$. Thus, we have a contradiction. ∎

4.2 Open Mapping Theorem

Theorem 4.2.1 (Open Mapping Theorem)**.** Let $M \in \mathcal{L}(X,Y)$, X, Y - Banach, such that M is surjective. Then M is an open mapping i.e.

$$M(B_X(0,1)) \supset B_Y(0,r), \quad \text{for some } r > 0.$$

Corollary 4.2.1. If $M \in \mathcal{L}(X,Y)$ is a bijection, then $M^{-1} \in \mathcal{L}(Y,X)$.

Proof. By Theorem 4.2.1, M^{-1} is continuous. Thus, $M^{-1} \in \mathcal{L}(Y,X)$. ■

Corollary 4.2.2. Let $\|\cdot\|$ and $\|\cdot\|'$ be norms in X such that X is complete with respect to each of them. If $\exists C > 0$ such that $\forall x \in X$ $\|x\| \le C\|x\|'$, then $\exists C' > 0$ such that $\forall x \in X$ $\|x\|' \le C\|x\|$.

Proof. Define $M : (X, \|\cdot\|) \to (X, \|\cdot\|')$ by $Mx = x$. Then $M \in \mathcal{L}(X,X)$ and M is a bijection. Thus, $M^{-1} \in \mathcal{L}(X,X)$. Therefore, $\exists C' > 0$ such that $\forall x \in X$ $\|x\|' \le C'\|x\|$. ■

Corollary 4.2.3 (Closed Graph Theorem). Let X, Y - Banach, $M : X \to Y$ linear. Then $M \in \mathcal{L}(X,Y)$ if and only if $\operatorname{Gr} M := \{(x, Mx) \mid x \in X\}$ is a closed subspace in $X \times Y$.

Remark. e.g. $\|(x,y)\|_{X\times Y} := \|x\|_X + \|y\|_Y$.

Exercise. Show $X \times Y$ is complete.

Proof of Corollary 4.2.3. The forward direction is trivial. If $(x_n, Mx_n) \to (x,y)$ then $x_n \to x \Rightarrow Mx_n \to Mx = y$. For the reverse direction, introduce another norm on X

$$\|x\|'_X := \|x\|_X + \|Mx\|_Y.$$

We want to apply Corollary 4.2.2. If we can, then we will have that $\|x\|'_X \le C'\|x\|_X$. Let x_n be Cauchy with respect to $\|\cdot\|'$ and Mx_n is Cauchy in Y. Then $x_n \to x$ and $Mx_n \to y$. Since GrM is closed, $y = Mx$. Therefore, $\|x_n - x\|' \to 0$. ■

4.2.1 Informal Discussion

Let $f : [0,1] \to \mathbb{R}$. Consider the mapping $f \mapsto -f''$ with boundary conditions $f(0) = f(1) = 0$. f has eigenvalues $\lambda_n = \pi n$ and eigenfunctions $f_n = \sqrt{2}\sin \pi n$. Note that this is not a bounded operator. Namely, $f \in L^2 \not\Rightarrow f'' \in L^2$. Want to find $\mathcal{D}(A) \subset L^2$ such that if $f \in \mathcal{D}(A)$ then $f'' \in L^2$. What gives us this property is that G_A is closed in $L^2 \times L^2$.

4.2.2 Proof of Open Mapping Theorem

Proof. Note that $\bigcup_{k=1}^{\infty} \overline{M(B_x(0,k))} = \operatorname{Ran} M = Y$. From Theorem 4.1.1, every $\overline{M(B_X(0,k))}$ cannot be nowhere dense. Therefore, $\exists y_0 \in Y$, $r > 0$, such that $\overline{M(B_X(0,1))} \supset B_Y(y_0, r)$. Therefore, $\overline{M(B_X(0,1))} \supset B_Y(-y_0, r)$. By convexity, $\overline{M(B_X(0,1))} \supset B_Y(0,r)$.

Now we must show that we can get rid of the closure. To do this, we will show that $M(B_X(0,2)) \supset B_Y(0,r)$. Take $y_0 \in B_Y(0,r)$. Find $x_1 \in B_X(0,1)$ such that $\|y_0 - Mx_1\| < \frac{r}{2}$. Define $y_1 := y_0 - Mx_1$. Find $x_2 \in B_X(0, \frac{1}{2})$ such that $\|y_1 - Mx_2\| < \frac{r}{4}$. Similarly, can define $y_2 := y_1 - Mx_2$ and iterate. Then $\|x_k\| \le \frac{1}{2^k} \Rightarrow \sum_{k=1}^{\infty} x_k$ converges. We know

$$\|y_0 - M(x_1 + \cdots + x_n)\| < \frac{r}{2^n}$$

Since $\sum_{k=1}^{n} x_k \in B_X(0,2)$, we have that $y_0 \in M(B_X(0,2))$. Thus, $M(B_X(0,2)) \supset B_Y(0,r)$. ■

Lecture 12: Twelfth Lecture

9 Oct. 01:00

4.3 Weak and Weak* Topologies

Theorem 4.3.1 (Uniform Boundedness Principle). Let X, Y be Banach spaces. Let $\{M_\alpha\}_{\alpha \in A}$ with each $M_\alpha \in \mathcal{L}(X, Y)$.

(a) Assume for all $x \in X$, $\exists c(x)$ such that $\forall \alpha$ $M_\alpha x < c(x)$. Then there exists some $C > 0$ such that $\forall \alpha$ $\|M_\alpha\| \le C$.

(b) Assume $\forall x \in X$, $\forall \ell \in Y'$, $\exists c(x, \ell)$ such that $\forall \alpha$ $|\ell(M_\alpha x)| \le c(x, \ell)$. Then $\exists C > 0$ such that $\forall \alpha \|M_\alpha\| \le C$.

Proof of (a). Consider the following sets:

$$F_N \coloneqq \{x \in X \mid \forall \alpha\ \|M_\alpha x\| \le N\|x\|\}$$

Note that each F_N is closed and $\bigcup_{N=1}^{\infty} F_N = X$ ($N \ge C(x) \Rightarrow x \in F_N$). By Theorem 4.1.1, there exists some N, $x_0, r_0 > 0$ such that $F_N \supset B(x_0, r_0)$. Therefore, $\forall x$ such that $\|x\| < r$, for every α we have that

$$\|M_\alpha(x_0 + x)\| \le N\|x_0 + x\|$$
$$\|M_\alpha(x_0 - x)\| \le N\|x_0 - x\|$$

Therefore,

$$\|M_\alpha x\| \le \frac{N}{2}\left(\|x_0 + x\| + \|x_0 - x\|\right)$$

Assume that $\|M_\alpha\|$ are unbounded. Then $\exists x_\alpha$ with $\|x_\alpha\| = \frac{r}{2}$ such that $\|M_\alpha x_\alpha\|$ is arbitrarily big. This is a contradiction. ■

Proof of (b). Fix $x \in X$. Consider

$$F'_N \coloneqq \{\ell \in Y' \mid \forall \alpha\ |\ell(M_\alpha x)| \le N\|x\|\|\ell\|\} \Rightarrow \|M_\alpha x\| \le N\|x\|$$

Note that F'_N is closed and $\bigcup_{N=1}^{\infty} F'_N = Y'$. By Theorem 4.1.1, there exists some N such that $\operatorname{Int} F'_N \ne 0$. Argue similarly as in (a). ■

Definition 4.3.1 (Weak Topology). X - Banach. The *weak topology* on X is the coarsest topology such that all $\ell \in X'$ are continuous.

Definition 4.3.2 (Weak-$\star$ Topology). The *weak-$\star$ topology* on X' is the coarsest topology such that $\forall x \in X$, $\ell \mapsto \ell(x)$ is continuous.

Remark. Weak topology on X' is finer than weak-$\star$ topology on X' if $X \subset X''$.

Remark. Local base of weak topology at x_0 is given by

$$\mathcal{U}_{\ell_1, \ell_2, \ldots, \ell_n; \epsilon}(x_0) \coloneqq \{x \mid |\ell_i(x - x_0)| < \epsilon,\ 1 \le i \le n\}$$

Similarly, for weak-$\star$ on X' a local base is given by

$$\mathcal{U}_{x_1, x_2, \ldots, x_n; \epsilon}(\ell_0) \coloneqq \{\ell \mid |\ell(x_i) - \ell_0(x_i))| < \epsilon,\ 1 \le i \le n\}$$

Must check

- If $y \in \mathcal{U}_{\ell_1, \ldots, \ell_n; \epsilon}(x) \cap \mathcal{U}_{\ell'_1, \ldots, \ell'_n; \epsilon}$ then $\exists \epsilon'' > 0$ such that

$$\mathcal{U}_{\ell_1, \ldots, \ell_n, \ell'_1, \ldots, \ell'_n; \epsilon''} \subset \mathcal{U}_{\ell_1, \ldots, \ell_n; \epsilon}(x) \cap \mathcal{U}_{\ell'_1, \ldots, \ell'_n; \epsilon}$$

Theorem 4.3.2 (Alaoglu). The closed unit ball in X'

$$\overline{B}' := \{\ell \in X' \mid \|\ell\|_{X'} \leq 1\}$$

is compact in weak-$\star$ topology.

Definition 4.3.3. K is called sequentially compact if $\forall x_n \in K$, $\exists$ subsequence and $x \in K$ such that $x_{n_k} \to x$ as $k \to \infty$.

Exercise. If the topology is *metrizable* then compactness and sequential compactness are equivalent.

Later: If X is separable, then weak-$\star$ topology is metrizable.

Example. $X = C(E)$ where E is a separable complete metric space i.e. $[0,1]$. Then

$$X' = M(E) := \{\mu = \mu^+ - \mu^- \mid \mu^\pm \text{ Borel measures on } E,\ \mu^+ \perp \mu^-\},$$

where $\|\mu\| = \mu^+(E) + \mu^-(E)$. Consider the closed unit ball in X'. By Theorem 4.3.2, the closed unit ball is compact. Thus, if μ_n are probability measures on E, then $\exists \mu_{n_k} \to \mu$ in weak-$\star$ topology for some μ (not necessarily a PM).

Definition 4.3.4. $x_n \xrightarrow{\text{weak}} x$ if $\forall \ell \in X'$, $\ell(x_n) \to \ell(x)$.

Definition 4.3.5. $x_n \xrightarrow{\text{weak-}\star} x$ if $\forall x \in X$, $\ell_n(x) \to \ell(x)$.

Proposition 4.3.1. We have the following:

(a) Let $X' \ni \ell_n \xrightarrow{\text{weak-}\star} \ell$. Then $\|\ell_n\|$ are bounded.

(b) Let $X \ni x_n \xrightarrow{\text{weak}} x$. Then $\|x_n\|$ are bounded.

Proof of (a). $\forall x,\ \ell_n(x) \to \ell(x) \Rightarrow \forall x\ \exists c(x)$ such that $|\ell_n(x)| \leq c(x)$. By Theorem 4.3.1, $\|\ell_n\|$ are uniformly bounded. ∎

Proof of (b). $M_n : \mathbb{R} \to X$ defined by $t \mapsto tx_n$. Then $\forall t, \ell$ we have $|\ell(tx_n)| \to |\ell(tx)|$ is bounded. By Theorem 4.3.1, $\|M_n\| = \|x_n\|$ are bounded. ∎

Lecture 13: Thirteenth Lecture

HW1 Comments: In a Banach space we have 18 Oct. 01:00

- Linear space
- Norm
  - Induces a topology

In Hilbert spaces,

- Scalar product

Remark. The local base constructed in the previous lecture forms a locally convex topological space.

From Proposition 4.3.1, we only need to consider bounded sets. Otherwise, we will not have any sort of convergence.

Lemma 4.3.1. Assume $x_n \xrightarrow{\text{weak}} x$ then $\|x\| \leq \liminf_{n\to\infty} \|x_n\|$ and $x \in \overline{\text{conv}(x_n)}$.

Proof. First, we find (HB) some $\ell \in X'$ such that $\|\ell\| = 1$ and $\ell(x) = \|x\|$. As $\ell(x_n) \to \ell(x)$, then $\ell(x) = \|x\| \leq \liminf_{n\to\infty} |\ell(x_n)| \leq \liminf_{n\to\infty} \|x_n\|$. For the second part, define

$$C := \overline{\text{conv}(x_1, \ldots, x_n, \ldots)},$$

so that C is closed in the normed topology and convex. If $x \notin C$, then by Geometric Hahn-Banach, $\exists \ell$ such that $\sup_{y\in C} \ell(y) < \ell(x)$. This is a contradiction because $x_n \xrightarrow{weak} x$. ■

Corollary 4.3.1 (Corollary of Theorem 4.3.2)**.** If X is reflexive then $\overline{B} = \{x \in X \mid \|x\|_X \leq 1\}$ is compact in weak topology.

Proof. Since $X = (X')'$, weak-$\star$ on X defined from X'' is the weak topology. ■

Remark. A few remarks:

(a) Also always weak sequentially-compact (even if X' is not separable).

(b) Moreover, Corollary is if and only if X is reflexive.

Remark. X-reflexive if and only if X' is reflexive.

Proof. For the first direction, $X''' = (X'')' = X'$. For the reverse direction, note that $X \subset X''$ is a closed subspace (X is complete and embedding is isometric). If $X \neq X''$, then $\exists \ell \neq 0$ such that $\ell|_X = 0$. Since $X''' = X'$, this is a contradiction. ■

Theorem 4.3.3. We have the following:

(a) If X' is separable then weak topology on $\overline{B} \subset X$ is metrizable.

(b) If X is separable then weak-$\star$ topology on $\overline{B'} \subset X'$ is metrizable.

Proof. We will proceed in steps

Step 1: Let $L \subset X'$ be countable everywhere dense. Then for each $\ell_1, \ldots, \ell_n, \epsilon$ and $x_0 \in \overline{B}$, there exists $\widetilde{\ell}_1, \ldots, \widetilde{\ell}_n \in L$ such that

$$\mathcal{U}_{\widetilde{\ell}_1,\ldots,\widetilde{\ell}_n;\frac{\epsilon}{2}}(x_0) \cap \overline{B} \subset \mathcal{U}_{\ell_1,\ldots,\ell_n;\epsilon}(x_0) \cap \overline{B}$$

Find $\widetilde{\ell}_k$ such that $\|\widetilde{\ell}_k - \ell_k\| < \epsilon/4$.

Step 2: Let $L = \{\widetilde{\ell}_1, \widetilde{\ell}_2, \ldots\}$. For $x, y \in \overline{B}$, denote

$$d(x,y) := \sum_{k=1}^{\infty} \frac{|\widetilde{\ell}_k(x) - \widetilde{\ell}_k(y)|}{|\widetilde{\ell}_k(x) - \widetilde{\ell}_k(y)| + 1} \cdot 2^{-k}$$

Now we must show that the topology defined by this metric is the same. First we will show that $\exists r > 0$ such that $\forall y$, $d(x_0, y) < r \Rightarrow y \in \mathcal{U}_{\widetilde{\ell}_1,\ldots,\widetilde{\ell}_n;\epsilon}(x_0)$. Take $r := \frac{\epsilon}{1+\epsilon} \cdot 2^{-n}$.

Now we must show that given $r > 0$, $\exists \widetilde{\ell}_1, \ldots, \widetilde{\ell}_n, \epsilon$ such that $y \in \mathcal{U}_{\widetilde{\ell}_1,\ldots,\widetilde{\ell}_n;\epsilon} \Rightarrow d(x_0, y) < r$.

Find N such that $2^{-N} < \frac{r}{2}$, then pick ϵ small enough so that $\sum_{k=1}^{N} \frac{|\widetilde{\ell}-\widetilde{\ell}|}{1+|\widetilde{\ell}-\widetilde{\ell}|} \cdot 2^{-k} < \frac{r}{2}$.

■

Remark. If so, compactness if and only sequential compactness.

4.3.1 Proof of Alaoglu's Theorem

Theorem 4.3.4 (Tychonoff's Theorem)**.** If each K_α is compact then $\prod_{\alpha \in A} K_\alpha$ is compact.

Remark. Base of topology is $\prod_{\alpha \in A} \mathcal{U}_\alpha$. All but finitely many $\mathcal{U}_\alpha = K_\alpha$.

Proof of Theorem 4.3.2. Define

$$\begin{aligned} \Phi : \overline{B'} &\longrightarrow \prod_{x \in X} [-\|x\|, \|x\|] \\ \ell &\longmapsto \ell(x) \end{aligned}$$

We have:

(a) Φ is a bijection onto its image.

(b) Both Φ and ϕ^{-1} are continuous.

It remains to prove that $\phi(\overline{B}')$ is closed in $\prod_{x \in X}[-\|x\|, \|x\|]$. Let $p \in \overline{\phi(\overline{B}')}$. Want

$$p_{x+y} = p_x + p_y \tag{4.1}$$
$$p_{\lambda x} = \lambda p_x \tag{4.2}$$
$$|p_x| \leq \|x\| \tag{4.3}$$

(4.1) is a statement about only three coordinates in $\prod_{x \in X}[\cdots]$, which is preserved by taking the closure. ■

Lecture 14: Fourteenth Lecture

23 Oct. 01:00

Example. $L^p(K)$ metric (compact) with measure, e.g. $K = [0,1]$. $1 < p < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$. $X' = L^q(K)$ both reflexive, separable.

Example. $X = L^1$, $X' = L^\infty$. Not reflexive. X separable but X' is not (consider $(\pm 1, \pm 1, \ldots)$).

Example. $X = C(K)$, $X' = M(K)$. Not reflexive. X is separable but X' is not.

Remark. If X' is separable, then X is separable.

Proof. Let $\{\ell_k\}_{k=1}^\infty \subset X'$ be dense. Find $x_k \in X$ such that $|\ell(x_k)| > \frac{1}{2}\|\ell_k\|$. We want to show $\overline{\operatorname{span}(\{x_k\})} = X$. Suppose that this is not the case. Then $\exists x \notin \operatorname{span}(\{x_k\})$ and $\ell \in X'$ such that $\ell(x) \neq 0$ and $\ell(x_k) = 0$ for every k. Find n such that

$$\|\ell - \ell_n\| < \frac{1}{3}\|\ell\|$$

Note that

$$|(\ell - \ell_n)x_n| \leq \frac{1}{3}\|\ell\|\|x_n\|$$

However,

$$\|\ell - \ell_n\| = |\ell_n(x_n)| > \frac{1}{2}\|\ell_n\| > \frac{1}{3}\|\ell_n\|$$

Now consider $\{q_1x_1 + q_2x_2 + \cdots + q_nx_n \mid q \in \mathbb{Q}\}$. ■

4.4 Klein-Milman Theorem

Reminder: X - locally convex topological space if it is a topological, linear, Haursdorff space i.e.

$$x, y \mapsto x + y$$
$$\lambda, x \mapsto \lambda x$$

are continuous operators and such that there exists a local base of the topology formed by convex sets.

Example. X with weak topology, X' with weak-$\star$ topology.

Fact on LCT:

(a) $x \neq y \Rightarrow \exists \ell \in X'$ continuous linear functional such that $\ell(x) \neq \ell(y)$.

(b) If E-closed, convex and $y \notin E$ then there exists a continuous linear functional ℓ such that $\sup_{x \in E} \ell(x) < \ell(y)$.

Definition 4.4.1. Let $K \subset X$ convex. A point $x \in K$ is called extreme if $\nexists x_1, x_2 \in K \setminus \{x\}$ and $\alpha \in (0,1)$ such that $x = \alpha x_1 + (1-\alpha)x_2$.

Definition 4.4.2. A subset $E \subset K$ is called extreme if $E \ni x = \alpha x_1 + (1-\alpha)x_2$ with $x_1, x_2 \in K$, $\alpha \in (0,1)$ then $x_1, x_2 \in E$.

Theorem 4.4.1. X - LCT space, K - convex, compact in X.

(a) $K_{\text{ext}} := \{x \in K \mid x \text{ extreme point of } K\} \neq \emptyset$.

(b) $K = \overline{\text{conv}(K_{\text{ext}})}$.

Proof of (a). Consider all possible closed, convex, extreme subsets of K ($\neq \emptyset$). If $(E_\alpha)_\alpha$ is totally ordered by inclusion then $\bigcap_{\alpha \in A} E_\alpha$ is again the same. We must show that this intersection is non-empty. If $\bigcap_{\alpha \in A} E_\alpha = \emptyset$ then $K \subset \bigcap_{\alpha \in A}(X \setminus E_\alpha)$ is an open cover. Therefore,

$$\bigcap_{\alpha \in A} E_\alpha = E_{\alpha_1} \cap E_{\alpha_2} \cap \cdots \cap E_{\alpha_n} = E_{\alpha_n} \neq \emptyset$$

By Zorn's lemma, there exists some E closed, convex, extreme $\neq \emptyset$ that does not contain any other. We want that $E = \{x\}$. If not $\exists x, y \in E$ with $x \neq y$. Find $\ell \in X'$ such that $\ell(x) \neq \ell(y)$. If $M := \max_{x \in E} \ell(x)$ then $E \cap \{x \in X \mid \ell(x) = M\}$ is closed, convex and extreme. Note that M is attained because E is compact and ℓ is continuous. Thus, we have a strictly smaller non-empty set, which is a contradiction. ■

Proof of (b). Assume $\exists x \in K$ such that $x \notin \overline{\text{conv}(K_{\text{ext}})}$. Find $\ell \in X'$ such that $\ell(x) > \max_{y \in \overline{\text{conv}(K_{\text{ext}})}} \ell(y)$. Consider $E := K \cap \ell^{-1}(\max_{y \in K} \ell(y)) \in \mathcal{E}$. Similar to (a), $\exists z \in E$ such that $z \in K_{\text{ext}}$. Contradiction since $\ell(z) \geq \ell(x) > \ell(y)$ for each $y \in K_{\text{ext}}$. ■

Example. $K = \overline{B} \subset L^1([0,1])$ closed unit ball. There are no extreme points! Suppose there is an extreme point $f \in \overline{B}$, with $\|f\|_1 = \int_0^1 |f| \mathrm{d}x \leq 1$. Find t_0 such that

$$\int_0^{t_0} |f| \mathrm{d}x = \int_{t_0}^1 |f| \mathrm{d}x = \frac{1}{2}\|f\|$$

Define

$$f_1 := 2f \cdot \mathbf{1}_{[0,t_0]}$$
$$f_2 := 2f \cdot \mathbf{1}_{[t_0,1]}$$

Then $f_1, f_2 \in \overline{B}$ and $f = \frac{1}{2}(f_1 + f_2)$.

Corollary 4.4.1. In particular, $\nexists X$ - Banach such that $X' = L^1([0,1])$.

Exercise. $C([0,1])$. Two extreme points $f \equiv 1$, $f \equiv -1$. $\overline{\mathrm{conv}(\overline{B}_{\mathrm{ext}})} = \{f \text{ constant}\}$.

Lecture 15: Fifteenth Lecture

25 Oct. 01:00

Theorem 4.4.2 (Azola-Ascoli). E - metric compact. $\mathcal{F} \subset C(E)$, $\mathcal{F}$ - precompact (in the norm topology) if and only if

(a) $\forall x \in E$, $\exists M > 0$ such that $\forall f \in \mathcal{F}$, $|f(x)| \leq M$.

(b) $\mathcal{F}$ is equicontinuous. $\forall \epsilon > 0$, $\exists \delta > 0$ such that $\forall f \in \mathcal{F}$, $d(x,y) < \delta \Rightarrow |f(x) - f(y)| < \epsilon$.

Proof of $\Rightarrow$. For (a), if $\{f(x)\}_{f \in \mathcal{F}}$ is unbounded, then $\exists f_n \in \overline{\mathcal{F}}$ such that no subsequence converges. For (b), given $x \in E$ and $\epsilon > 0$ denote $\mathcal{U}^{(\epsilon)}_{x,r} := \{f \in C(E) \mid \sup_{y \in B(x,r)} |f(y) - f(x)| < \epsilon\}$. This is an open set in $C(E)$ (exercise). Note $\bigcup_{r>0} \mathcal{U}^{(\epsilon)}_{x,r} = C(E) \supset \overline{\mathcal{F}}$ - compact. Therefore, there exists a finite subcover such that

$$\overline{\mathcal{F}} \subset \bigcup_{i=1}^{n} \mathcal{U}^{(\epsilon)}_{x,r_i} = \mathcal{U}^{(\epsilon)}_{x,r_x},$$

where $r_x = \min\{r_1, \ldots, r_n\}$. Now consider $E \subset \bigcup_{x \in E} B(x, \frac{1}{2}r_x)$. Since E compact, $E \subset B(x_1, \frac{1}{2}r_{x_1}) \cup \cdots \cup B(x_n, \frac{1}{2}r_{x_n})$. Denote $r := \min\{r_{x_1}, \ldots, r_{x_n}\}$. Then if $d(x,y) < \frac{1}{2}r$, $\exists k$ such that both $x, y \in B(x_k, r_{x_k})$ and $|f(x) - f(y)| < 2\epsilon$. ■

Proof of $\Leftarrow$. Let $f_n \in \mathcal{F}$. Consider $\{x_m\}_{m=1}^{\infty}$ - countable everywhere dense (exists b/c E-metric, compact). Find a subsequence f_{n_k} such that $\forall m$, $\exists \lim_{k\to\infty} f_{n_k}(x_m)$ (Possible for each m due to (a) then apply diagonal process). Denote $f(x_m) := \lim_{k\to\infty} f_{n_k}(x_m)$. Use (b) to prove $d(x_m, x_{m'}) < \epsilon \Rightarrow |f(x_m) - f(x_{m'})| < \delta$. This then allows us to define $f \in C(E)$ from values $f(X_m)$. It remains to show that $f_{n_k} \to f$ in $C(E)$. Note

$$|f_{n_k}(x) - f(x)| \leq |f_{n_k}(x) - f_{n_k}(x_m)| + |f(x) - f(x_m)| + |f_{n_k}(x_m) - f(x_m)|$$

Since E is compact, finitely many x_m such that $d(x, x_m) < \delta$. Choose k to be the max such that $|f_{n_k}(x) - f(x_m)| < \epsilon$. Then

$$|f_{n_k}(x) - f(x)| \leq |f_{n_k}(x) - f_{n_k}(x_m)| + |f(x) - f(x_m)| + |f_{n_k}(x_m) - f(x_m)| \leq 3\epsilon$$

■

Theorem 4.4.3 (Stone-Weierstrass, $C(E) = C(E, \mathbb{R})$). Let $\mathcal{A} \subset C(E)$ be a sub-algebra such that $1 \in A$. Then $\mathcal{A}$ is dense in $C(E)$ if (and only if) $\forall x, y \in E$ with $x \neq y$, $\exists p \in \mathcal{A}$ such that $p(x) \neq p(y)$.

Lemma 4.4.1. $\exists P_n \in \mathbb{R}[x]$ such that $P_n(x) \to |x|$ uniformly on $[-1,1]$.

Proof. Take for granted. Idea: $P_0 := 0$, $P_{n+1}(x) = P_n(x) + \frac{1}{2}\left[x^2 - (P_n(x))^2\right]$. ■

Proof of Theorem 4.4.3. Denote $\mathcal{L} := \overline{\mathcal{A}}$. This is still a sub-algebra (exercise). Note that $\mathcal{L}$ is a lattice i.e. $f, g \in \mathcal{L} \Rightarrow \max\{f, g\}, \min\{f, g\} \in \mathcal{L}$. To see this,

$$\max\{f, g\} = \frac{1}{2}\left[f + g + |f - g|\right]$$

where $|f - g|$ is a uniform limit of $f - g$.

Step 1: Fix $y \in E$. For each $x \in E$ find $f_x^{(y)}$ such that $|f_x^{(y)}(y) - f(y)| < \epsilon$ and $|f_x^{(y)}(x) - f(x)| < \epsilon|$.

Define $\mathcal{U}_x^{(y)} := \{x' \mid f_x^{(y)}\{x' \in E \mid f_x^{(y)}(x') - f(x') < \epsilon\}$. This is an open cover of E. Therefore, $\exists$ finite sub-cover $\mathcal{U}_{x_1}^{(y)}, \ldots, \mathcal{U}_{x_n}^{(y)}$. Take $f^{(y)} := \min\{f_{x_1}^{(y)}, \ldots, f_{x_n}^{(y)}\} \in \mathcal{L}$. We have $|f^{(y)}(y) - f(y)| < \epsilon$ and $\forall x \in E$, $f^{(y)}(x) - f(x) < \epsilon$

Step 2: Denote $V^{(y)} := \{y' \in E \mid f^{(y)}(y') - f(y') > -\epsilon\}$.

Find a finite sub-cover and consider $f_y := \max\{f^{(x_1)}, \ldots, f^{x_n}\} \in \mathcal{L}$. Then for each x $|f_y(x) - f(x)| < \epsilon$.

■

Corollary 4.4.2 (Stone-Weierstrass, $C(E) = C(E, \mathbb{C})$)**.** Let $\mathcal{A} \subset C(E)$ be a sub-algebra such that $1 \in A$. Then $\mathcal{A}$ is dense in $C(E)$ if (and only if) $\forall x, y \in E$ with $x \neq y$, $\exists p \in \mathcal{A}$ such that $p(x) \neq p(y)$ and $p \in \mathcal{A} \Rightarrow \bar{p} \in \mathcal{A}$.

Proof. Exercise: Consider $\mathcal{A}_{\mathbb{R}} := \{\mathrm{Re}(p) \mid p \in \mathcal{A}\}$. Note $\mathrm{Im}(p) = \mathrm{Re}(-ip)$. Therefore, must only check that $\mathcal{A}_{\mathbb{R}}$ is an algebra. ■

Chapter 5

Spectral Theory of Linear Operators

Lecture 16: Sixteenth Lecture

5.1 Adjoint Operators & Compact Operators

30 Oct. 01:00

Definition 5.1.1. $C \in \mathcal{L}(X,Y)$ is a *compact operator* if $C(\overline{B}_X)$ is pre-compact in Y (norm topology).

Proposition 5.1.1. We have the following:

(a) If C_1, C_2 are compact then $C_1 + C_2$ are compact.

(b) Suppose that C is compact, A is bounded. Then both AC and CA are compact (when it makes sense).

(c) C_k are compact and $\|C_k - C\| \to 0$ then C is compact.

Proof of (a). Note that

$$(C_1 + C_2)(\overline{B}_X) \subset C_1(\overline{B}_X) + C_2(\overline{B}_X)$$

where the RHS is the *Minkowski sum* i.e. $F_1 + F_2 := \{y_1 + y_2 \mid y_1 \in F_1,\ y_2 \in F_2\}$. $\forall \epsilon > 0$ both these sets admit ϵ-nets i.e. $\{y_1^{(1)}, \ldots, y_m^{(1)}\}$ such that $\bigcup_{k=1}^m \overline{B}(y_k^{(1)}, \epsilon) \supset C_1(\overline{B}_X)$.

Exercise. $\{y_k^{(1)} + y_\ell^{(2)}\}$ is a 2ϵ-net for $C_1(\overline{B}_X) + C_2(\overline{B}_X)$.

Note that this is an equivalent formulation for pre-compactness. Thus, we are done. ■

Proof of (b). Note that $[CA](\overline{B}_X) \subset C(\|A\| \cdot \overline{B}) = \|A\| \cdot C(\overline{B})$ - pre-compact. Similarly, $[AC](\overline{B}) \subset A\left[\overline{C(\overline{B})}\right]$. Since $\overline{C(\overline{B})}$ is compact and A is continuous, $A\left[\overline{C(\overline{B})}\right]$ is compact. ■

Proof of (c). Let $\epsilon > 0$. Find k such that $\|C_k - C\| < \epsilon$. Then for every $x \in \overline{B}_X$

$$\|C_k x - Cx\| < \epsilon$$

By assumption, $C_k(\overline{B})$ is pre-compact. Therefore, it admits a finite ϵ-net $\{y_1, \ldots, y_m\}$. Thus, we have that $\{y_1, \ldots, y_m\}$ is a 2ϵ-net for $C(\overline{B}_X)$. ■

Remark. In $\ell(H)$ for H-Hilbert, the only closed ideal is the ideal of compact operators.

Definition 5.1.2. Let $A \in \mathcal{L}(X,Y)$. The *transpose* or *Banach adjoint* of A is $A' \in \mathcal{L}(Y', X')$ is defined as follows:

$$[A'\ell](x) := \ell(Ax)$$

Remark. This generalizes the transpose of a matrix.

Proposition 5.1.2. We have the following:

(a) $(A' + B') = A' + B'$ and $(\lambda A)' = \lambda A'$.

(b) $(AB)' = B'A'$.

(c) $\|A'\|_{\mathcal{L}(Y',X')} = \|A\|_{\mathcal{L}(X,Y)}$.

(d) $\ker A' = (\operatorname{Ran} A)^{\perp} := \{\ell \in Y' \mid \ell|_{\operatorname{Ran} A} = 0\}$.

(e) $\ker A = (\operatorname{Ran} A')^{\perp} := \{x \in X \mid \forall \ell \in \operatorname{Ran} A',\ \ell(x) = 0\}$.

(f) $X = Y$, $\exists A^{-1} \in \mathcal{L}(X) \Leftrightarrow \exists (A')^{-1} \in \mathcal{L}(X')$.

Proof of (a). Note

$$[(A' + B')\ell](x) = \ell(Ax + Bx) = \ell(Ax) + \ell(Bx) = [A'\ell](x) + [B'\ell](x)$$

and

$$[(\lambda A')\ell](x) = \ell(\lambda Ax) = \lambda \ell(Ax) = \lambda[A'\ell](x)$$

■

Proof of (b). Note

$$[(AB)'\ell](x) = \ell(ABx) = [A'\ell](Bx) = [B'A'\ell](x)$$

■

Proof of (c). Note that

$$\|A'\| = \sup_{\|\ell\|=1} \|A'\ell\| = \sup_{\|\ell\|=1} \sup_{\|x\|=1} |(A'\ell)(x)| = \sup_{\|x\|=1} \|Ax\| = \|A\|$$

■

Proof of (d). Note that

$$\ker A' = \{\ell \in Y' \mid A'\ell = 0\}$$

Since $A'\ell = 0 \Leftrightarrow \ell(Ax) = 0\ \forall x \in X$, $\ker A' = (\operatorname{Ran} A)^{\perp}$. ■

Proof of (e). Note that

$$\ker A = \{x \in X \mid Ax = 0\}$$

Since $Ax = 0 \Leftrightarrow \ell(Ax) = 0\ \forall \ell \in Y'$, $\ker A = (\operatorname{Ran} A')^{\perp}$. ■

Proof of (f). By Open Mapping Theorem, we must only show that $\ker A = \{0\}$ and $\operatorname{Ran} A = X$. For the forward direction, we have that $AA^{-1} = \mathrm{Id}$. Therefore, $(A')^{-1}A' = \mathrm{Id}$ and A' is invertible. For the reverse direction, we have that $\ker A = (\operatorname{Ran} A')^{\perp} = \{0\}$. Similarly, we have that $\{0\} = \ker A' = (\operatorname{Ran} A)^{\perp}$. Therefore, it is sufficient to show that $\operatorname{Ran} A$ is closed. Let Ax_n Cauchy.

Consider

$$\begin{aligned}\|x_n - x_m\| &= \sup_{\|\ell\|=1} |\ell(x_n) - \ell(x_m)| \\ &= \sup_{\|\ell\|=1} \left|A'(A')^{-1}\ell(x_n - x_m)\right| \\ &= \sup_{\|\ell\|=1} \left|[(A')^{-1}\ell]A(x_n - x_m)\right| \\ &\leq \|(A')^{-1}\|\|Ax_n - Ax_m\|\end{aligned}$$

Therefore, x_n is Cauchy. Since X is closed, $\exists x = \lim x_n$ and $Ax_n \to Ax$. ∎

Remark. $A''|_X = A$.

$$[A''L_X](\ell) = L_X(A'\ell) = (A'\ell)(x) = \ell(Ax) = L_{Ax}[\ell]$$

Lecture 17: Seventeenth Lecture

1 Nov. 01:00

Proposition 5.1.3. $C \in \mathcal{L}(X)$ compact if and only if $C' \in \mathcal{L}(X')$ is compact.

Proof. We want to show that $C'(\overline{B}_{X'})$ is pre-compact. Recall that each $m \in C'(\overline{B}_{X'})$ is a linear function on X. Define $K := \overline{C}(\overline{B}_{X'})$ so that K is compact in X. Note that if $\ell \in \overline{B}_{X'}$ then

$$|(C'\ell)(x) - (C'\ell)(y)| = |\ell(C'(x-y))| \leq \|C'\|\|x - y\|$$

Therefore, $C'\ell$ with $\ell \in \overline{B}_{X'}$ is equicontinuous on K (and uniformly bounded). By Arzela-Ascoli, $C'\ell_n$ contains a convergent subsequence in the space of continuous functional on K i.e. $C'\ell_{n_k} \to m \in X'$. Then, m is also linear and $\forall x \in X$

$$\|m(x)\| \leq \limsup_{k\to\infty} \|(C'\ell_{n_k})(x)\| \leq \|C'\|$$

Thus, we are done. For the reverse direction, C'' is compact and therefore $C''|_X = C$ is compact. ∎

5.2 Hilbert Spaces

Assume $X = H$ - Hilbert space and $A \in \mathcal{L}(H)$. Recall that we have the isometry

$$\begin{aligned}\mathcal{J} : H &\longrightarrow H' \\ y &\longmapsto (\cdot \mapsto \langle \cdot, y\rangle)\end{aligned}$$

i.e. $\mathcal{J}_y x = \langle x, y\rangle$. Note that $\mathcal{J}(y_1 + y_2) = \mathcal{J}y_1 + \mathcal{J}y_2$ but $\mathcal{J}(\lambda y) = \overline{\lambda}\mathcal{J}_y$, i.e. $\mathcal{J}$ is *anti-linear*. For each $A : H \to H$ we have $A' : H' \to H'$ by composing with $\mathcal{J}$. Define A^* by $\mathcal{J}_{A^* y} = A^*\mathcal{J}_y$ i.e. $A^* = \mathcal{J}^{-1}(A^*\mathcal{J})$

Lemma 5.2.1. $\forall x, y \in H$, $\langle Ax, y\rangle = \langle x, A^* y\rangle$.

Proof. Note

$$\langle Ax, y\rangle = \mathcal{J}_y(Ax) = (A'\mathcal{J}_y)(x) = \mathcal{J}_{A^* y}(x)$$

∎

Remark. A few remarks:

(a) $\mathcal{J}^{-1}(A'\mathcal{J})$ is a linear operator,

(b) $A \mapsto \mathcal{J}^{-1}(A'\mathcal{J})$ is anti-linear.

Definition. $A \in \mathcal{L}(H)$ - Hilbert (over $\mathbb{C}$)

Definition 5.2.1. A is called *self-adjoint* if $A^* = A$.

Definition 5.2.2. A is called *normal* if $AA^* = A^*A$.

Definition 5.2.3. A is called *unitary* if $AA^* = \mathrm{Id} \Rightarrow \langle x_1, x_2 \rangle = \langle AA^* x_1, x_2 \rangle = \langle A^* x_1, A^* x_2 \rangle$.

5.3 Spectrum and Resolvent of Bounded Operators

Remark. We will use X to denote Banach space and H to be Hilbert space.

Definition. Let $A \in \mathcal{L}(X)$.

Definition 5.3.1. The *spectrum* is defined as $\sigma(A) := \left\{\lambda \in \mathbb{C} \mid \nexists (\lambda \,\mathrm{Id} - A)^{-1} \in \mathcal{L}(X)\right\}$.

Definition 5.3.2. The *resolvent set* is defined as $\rho(A) := \mathbb{C} \setminus \sigma(A)$.

Definition 5.3.3. $R(\lambda) := (\lambda I - A)^{-1}$ is called the *resolvent*.

Remark. $\sigma(A') = \sigma(A)$ b/c $(\lambda I_X - A)' = \lambda I_{X'} - A'$.

Theorem 5.3.1. $\sigma(A)$ is closed, $\sigma(A) \neq \emptyset$ and $\max_{\lambda \in \sigma(A)} |\lambda| = \lim_{n\to\infty} \|A^n\|^{1/n} \leq \|A\|$.

Lemma 5.3.1. Let $A \in \mathcal{L}(X)$ such that $\exists A^{-1} \in \mathcal{L}(X)$. If $B \in \mathcal{L}(X)$ satisfies $\|B\| < \frac{1}{\|A^{-1}\|}$ then $\exists (A+B)^{-1} \in \mathcal{L}(X)$.

Proof. If $A = \mathrm{Id}$, then $(I - B + B^2 - B^3 + \cdots)(I + B) = I$. In general, $(A+B)^{-1} := (I + A^{-1}B)^{-1}A^{-1}$. ∎

Proof of Theorem 5.3.1. From the Lemma, we know that $\rho(A)$ is an open set so $\sigma(A)$ is closed. Suppose that $|\lambda| > \|A\|$. Then

$$(\lambda I - A)^{-1} = \lambda^{-1} I + \lambda^{-2} A + \lambda^{-3} A^2 + \cdots \Rightarrow \lambda \in \rho(A) \qquad (\star)$$

In particular, $\sigma(A) \subset \overline{B}(0; \|A\|)$. Similarly, now suppose that $|\lambda| > \lim_{n\to\infty} \|A^n\|^{1/n}$. Then $\exists N$ such that $|\lambda| - \epsilon > \|A^N\|^{1/N}$ and $\|A^N\| < (|\lambda| - \epsilon)^N$. For every $k = Nm + \ell$, we have that

$$\|A^k\| \leq (|\lambda| - \epsilon)^{Nm} \cdot \|A\|^{\ell} \leq C(N)(|\lambda| - \epsilon)^k$$

where $C(N)$ is a constant depending on N. Thus,

$$\|\lambda^{-k+1} A^k\| \leq C(N) \cdot \left(\frac{|\lambda| - \epsilon}{|\lambda|}\right)^k \to 0$$

and $\lambda \in \rho(A)$. Next we must show that $\sigma(A) \neq \emptyset$. Consider $R(\lambda) = (\lambda I - A)^{-1}$ as a function $R : \rho(A) \to \mathcal{L}(X)$. This is a holomorphic function because $(\star)$ provides a Taylor expansion near λ.

If $\sigma(A) = \emptyset$, then there is no singularity and

$$\lambda > \|A\| \Rightarrow \|R(\lambda)\| < \frac{1}{|\lambda|} \cdot \frac{1}{1 - \frac{\|A\|}{|\lambda|}} = \frac{1}{|\lambda| - \|A\|} \to 0.$$

By max principle, $\|R(\lambda)\| = 0$. Thus, we have a contradiction. Finally, we must show that we cannot have

$$\rho' := \max_{\lambda \in \sigma(A)} |\lambda| < \lim_{n\to\infty} \|A^n\|^{1/n} =: \rho.$$

Note that

$$A^n = \frac{1}{2\pi i} \int_{|\lambda| = \rho' + \epsilon} \lambda^n R(\lambda)\, d\lambda$$

Therefore,

$$\|A^n\| \leq (\rho' + \epsilon)^{n+1} \cdot \max_{|\lambda| = \rho' + \epsilon} \|R(\lambda)\| \Rightarrow \rho = \lim_{n\to\infty} \|A^n\|^{1/n} \leq \rho' + \epsilon$$

■

Remark. $\lim_{n\to\infty} \|A^n\|^{1/n}$ is called *spectral radius* and the limit always exists b/c $\|A^{n+m}\| \leq \|A^n\|\|A^m\|$. Then $\log \|A^n\| \leq \log \|A^n\| + \log \|A^m\|$. Since sub-additive sequence has limit, $\exists \lim_{n\to\infty} \frac{\log \|A^n\|}{n}$.

Example. $X = \mathbb{R}^n$ (with any norm e.g. Euclidean). $A \in \mathbb{R}^{n\times n}$ matrix. Then there exists a basis (Jordan Normal Form) such that A is block diagonal with blocks of the form

$$\begin{pmatrix} \lambda_1 & 1 & & \\ & \lambda_2 & 1 & \\ & & \ddots & \\ & & & \lambda_m \end{pmatrix}$$

Then $R(\lambda) = (\lambda I - A)^{-1}$ will be block diagonal of the form

$$\begin{pmatrix} (\lambda - \lambda_1)^{-1} & (\lambda - \lambda_1)^{-1} & \\ & \ddots & \\ & & (\lambda - \lambda_1)^{-1} \end{pmatrix}$$

Poles at eigenvalues.

Example. $X = \ell^2$ with $Sx := (0, x_1, x_2, \ldots)$, $S^*x = (x_2, x_3, \ldots)$. For S^* each λ such that $|\lambda| < 1$ works for $x_k = \lambda^k$. Then $\ker(\lambda I - S^*) \neq \{0\}$ for every λ with $|\lambda| < 1$. It follows that $\sigma(S^*) \supset \mathbb{D} := \{\lambda \mid |\lambda| < 1\}$. Thus, $\sigma(S) = \sigma(S^*) = \overline{\mathbb{D}}$ by theorem.

Exercise. S has no eigenvalues.

Lecture 18: Eighteenth Lecture

Definition 5.3.4. If $\ker(\lambda I - A) \neq \{0\}$ then λ is called an *eigenvalue* and $x \in \ker(\lambda I - A)$ is called an *eigenfunction* or *eigenvector*. 6 Nov. 01:00

5.4 Spectral Theory of Compact Operators

Example. Consider $A = \begin{bmatrix} 0 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 1 & 0 & \cdots \\ 0 & 0 & 0 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$. Then $\ker A = \operatorname{span}\{e_1\} \subset \ker A^2 = \operatorname{span}\{e_1, e_2\} \cdots$.

Lemma 5.4.1. $Y \subset X$ closed. Then $\exists x \in X$ with $\|x\| = 1$ and $\operatorname{dist}(x, Y) \geq \frac{1}{2}$.

Proof. Take $z \notin Y$, define $d := \operatorname{dist}(z, Y)$. Then $\exists y \in Y$ such that $\|z - y\| \leq 2d$. Now let $x := \frac{z-y}{\|z-y\|}$. ■

Theorem 5.4.1. Let $C \in \mathcal{L}(X)$ be compact. Define $T := I - C$. Then,

(a) $\ker T$ is finite dimensional.

(b) $\exists n < +\infty$ such that $\ker T^{n+1} = \ker T^n$.

(c) $\operatorname{Ran} T$ is closed.

Proof of (a). $\ker T = \{x \in X \mid x = Cx\}$. If $\dim X = \infty$, then $E := \{x \in \ker T \mid \|x\| \leq 1\}$ is not compact. However, $E \subset C(\overline{B}_X(0,1))$ is pre-compact because C is compact. Thus, we have a contradiction. ■

Proof of (b). Suppose for each n, $\ker T^n \subsetneq T^{n+1}$. Define $Y_n := (T^n)^{-1}(\{0\})$ closed subspace. From the Lemma, there exists some $x_n \in X_n$ such that $\|x_n\| = 1$ and $\operatorname{dist}(x_n, X_{n-1}) \geq \frac{1}{2}$. Note

$$Cx_n - Cx_m = x_n - x_m - Tx_n + Tx_m$$

WIthout loss of generality, $n > m$. Then $x_n - x_m - Tx_n + Tx_m \in X_{n-1}$. Therefore, $\|Cx_n - Cx_m\| \geq \frac{1}{2}$. This contradicts compactness of $C(\overline{B}_X)$. ■

Proof of (c). Let $y_n = Tx_n \to y$. Denote $d_n := \operatorname{dist}(x_n, \ker T)$. Assume $\{d_n\}_{n=1}^{\infty}$ is bounded. Find $z_n \in \ker T$ such that $\|x_n - z_n\| \leq 2d_n$. Note that

$$T(x_n - z_n) = (x_n - z_n) - C(x_n - z_n).$$

Since C is compact, there is a convergent subsequence of $C(x_n - z_n)$. Since $\|x_n - z_n\| \leq 2d_n$ and d_n is bounded, there is also a convergent subsequence of $x_n - z_n$. Then,

$$Tx_* = \lim_{n_k \to \infty} T(x_{n_k} - z_{n_k}) = \lim_{n_k \to \infty} y_{n_k} = y.$$

If d_n is not bounded, then $\exists d_{n_k} \to \infty$. Consider $u_{n_k} := \frac{x_{n_k} - z_{n_k}}{d_{n_k}}$. Then $\|u_{n_k}\| \leq 2$ and

$$\operatorname{dist}(u_{n_k}, \ker T) = 1. \tag{$\star$}$$

Additionally, we have that $Tu_{n_k} \to 0$. Therefore, there exists some subsubsequence $u_{n_{k_j}}$ such that $Tu_{n_{k_j}} \to 0$ and $Cu_{n_{k_j}} \to u$, where $u_{n_k} \to u$. This means that $u \in \ker T$. However, from $(\star)$, we know that this cannot be the case. Thus, we have a contradiction and d_n must be bounded. ■

Remark. $\ker(I - C)^n = I - nC + \binom{n}{2}C^2 + \cdots + C^n$ is compact, so that $\ker T^n$ is finite dimensional.

Lecture 19: Nineteenth Lecture

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Lemma 5.4.2. Let $C : X \to X$ be compact and $Y \subset X$ be a closed invariant subspace i.e. $x \in Y \Rightarrow Cx \in Y$. Then both

(a) $C|_Y$

(b) $\widetilde{C} : X \,/\, Y \to X \,/\, Y$ defined by $\widetilde{C}([x]_Y) = [Cx]_Y$

are compact operators.

Proof. Proof of (a) is trivial. For (b), we can construct the sequence of map

$$B_{X\,/\,Y} \longrightarrow 2B_X \longrightarrow \text{precompact set in } X \longrightarrow {}^{X}\big/_{Y}$$

For the first mapping, find a representative sufficiently close in norm. For the second mapping, use the compactness of C. For the last mapping, use $x \mapsto [x]$. This is a contraction. ∎

Theorem 5.4.2. Let C be a compact operator, $T = I - C$. Then $\operatorname{codim} \operatorname{Ran} T = \dim \ker T < \infty$.

Proof. First, assume that $\ker T = \{0\}$. Then we want to show that $\operatorname{Ran} T = X$. Assume $X \supsetneq \operatorname{Ran} T \supsetneq \operatorname{Ran} T^2 \supsetneq \cdots$.

> **Note.** $\operatorname{Ran} T^n = \operatorname{Ran}(I + \text{comp})$ is closed.

Find $x_n \in \operatorname{Ran} T^n$ such that $\|x_n\| = 1$ and $\operatorname{dist}(x_n, \operatorname{Ran} T^{n+1}) \geq \frac{1}{2}$. Without loss of generality, suppose that $m > n$. Then $Cx_n - Cx_m = x_n - x_m - Tx_n + Tx_m \in \operatorname{Ran} T^{n+1}$. Therefore,

$$\|Cx_n - Cx_m\| \geq \frac{1}{2}$$

This is a contradiction.
In general, note that $C : \ker T \to \ker T$ defined by $Tx = 0 \Rightarrow Cx = x$. Find N so that $\ker T^{n+1} = \ker T^N$. Use lemma and consider

$$\widetilde{T} := I - \widetilde{C} : {}^{X}\big/_{\ker T^N} \longrightarrow {}^{X}\big/_{\ker T^N}$$

Then $\ker \widetilde{T} = \{0\}$. From above, we know that $\operatorname{Ran} \widetilde{T} = X \,/\, \ker T^N$. Take $y \in X$ then $\exists [x]$ such that $\widetilde{T}[x] = [y]$. Equivalently, $Tx = y + n$ with $n \in \ker T^N$. Therefore, $X = \operatorname{Ran} T + \ker T^N$. It remains to show that $\dim(\operatorname{Ran} T \cap \ker T^N) = \ker T^N - \ker T$. This is because

$$\dim \ker T^N - \operatorname{codim} \operatorname{Ran} T = \dim(\operatorname{Ran} T \cap \ker T^N) = \ker T^N - \ker T$$

Let $y \in \operatorname{Ran} T \cap \ker T^N$, where $y = Tx$ with $x \in \ker T^{N+1} = \ker T^N$. This means that

$$\operatorname{Ran} T \cap \ker T^N = \operatorname{Ran} T|_{\ker T^N}.$$

Since $\ker T^N$ is finite dimensional,

$$\operatorname{Ran} T \cap \ker T^N = \operatorname{Ran} T|_{\ker T^N} = \dim \ker T^N - \dim \ker T.$$

∎

Corollary 5.4.1 (Fredholm's alternative)**.** Let C be a compact operator. $I - C$ is invertible if and only if $\ker T = \{0\}$, which happens if and only if $\ker T^* = \{0\}$, which happens if and only if $\operatorname{Ran} T = X$ and $\operatorname{Ran} T^* = X'$.

Proof. $\ker T = \{0\}$ if and only if $\operatorname{Ran} T^* = X'$. $\operatorname{Ran} T = X$ if and only if $\ker T^* = \{0\}$. $\operatorname{Ran} T^* = X' \Leftrightarrow \ker T^* = \{0\}$. Finally, from the Theorem $\ker T = \{0\} \Leftrightarrow \operatorname{Ran} T = X$. ∎

Lecture 20: Twentieth Lecture

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Theorem 5.4.3. Let $C \in \mathcal{L}(X)$ be compact with $\dim X = \infty$.

(a) Then $\sigma(C) = \{0\} \cup \{\lambda_k\}_{k=1}^N$ with $0 \le N \le +\infty$, $|\lambda_k| \ne 0$, do not have other accumulation points.

(b) Each λ_k is an eigenvalue. $1 \le \dim\ker(I\lambda_k - C) < \infty$ and $\exists n_k < \infty$ such that $\ker(\lambda_k I - C)^{n_k+1} = \ker(\lambda_k I - C)^{n_k}$

(c) $\|R(\lambda)\| = O(|\lambda - \lambda_k|^{-n_k})$ as $\lambda \to \lambda_k$.

Proof of (a). Note that if there are no other accumulation points then there are at most countably many $\lambda \in \sigma(C)$. Assume $\lambda_k \to \lambda \ne 0$. Let $x_k \in X$ be such that $\|x_k\| = 1$ and $Cx_k = \lambda_k x_k$. Denote $X_n \coloneqq \operatorname{span}(x_1, \ldots, x_n)$. Find $y_n \in X_n$ such that $\|y_n\| = 1$ and $\operatorname{dist}(y_n, X_{n-1}) \ge \frac{1}{2}$. Then for $n > m$

$$\begin{aligned}
Cy_n - C(y_m) &= C(\alpha_n x_n) + \underbrace{\cdots}_{\in X_{n-1}} \\
&= \lambda_n \alpha_n x_n + \underbrace{\cdots}_{\in X_{n-1}} \\
&= \lambda_n y_n + \underbrace{\cdots}_{\in X_{n-1}}
\end{aligned}$$

By construction, $|\alpha_n| \ge \frac{1}{2}$. Hence,

$$\|Cy_n - Cy_m\| \ge |\lambda_n| \operatorname{dist}(y_n, X_{n-1}) \ge \frac{|\lambda_n|}{2}$$

If $|\lambda_n| \not\to 0$, then we have a contradiction with the compactness of $C(\overline{B}_X)$. ■

Proof of (b). We already know this. $\operatorname{ind}\dim\ker(\lambda_k I - C) \ge 1$ by Fredholm's alternative. ■

Proof of (c). Consider $Y \coloneqq \ker(\lambda_k I - C)^{n_k} = \ker(\lambda_k I - C)^{n_k+1}$. This is an invariant subspace for C. Let

$$\widetilde{T} = \lambda_k \widetilde{I} - \widetilde{C} : {}^{X}\!/_{Y} \longrightarrow {}^{X}\!/_{Y}.$$

Then $\ker\widetilde{T} = \{0\}$. Therefore, for $\lambda_k \in \rho(\widetilde{T})$, we have $(\lambda\widetilde{I} - \widetilde{C})^{-1}$ is continuous near λ_k. It follows that $\|(\lambda\widetilde{I} - \widetilde{C})^{-1}\| = O(1)$ as $\lambda \to \lambda_k$. Now let $\lambda \ne \lambda_k$ be close to λ_k with $\lambda \in \rho(\lambda I - C)$. Take $z \in X$, define $x \coloneqq (\lambda I - C)^{-1} z$. We will first solve $(\lambda I - C)x = z$ in $X \,/\, Y$. This means that there exists $[x]_Y$ such that

$$(\lambda I - \widetilde{C})[x]_Y = [z]_Y$$

and $\|[x]_Y\| \le O(1) \cdot \|z\|$. Lifting back to the original space, we know that there exists some $x \in X$ such that

$$(\lambda I - C)x = z + y \tag{$\star$}$$

for some $y \in Y$ and $\|x\| \le 2\|[x]_Y\| = O(1) \cdot \|z\|$. Thus, $\|y\| = O(\|z\|)$. It remains to find $x_1 \in Y$ such that $(\lambda I - C)x_1 = y$. Since $\dim Y < \infty$, this is a finite dimensional question. Therefore, $\exists x_1$ solving ($\star$) with $\|x_1\| = O(|\lambda - \lambda_k|^{n_k}) \cdot \|y\|$. ■

Remark. There are compact operators that do not have eigenvectors at all (even w/ $\lambda = 0$).

Remark. How $(\lambda I - C)$ with $\lambda = 0$ looks may be complicated.

Example. $X = C([0,1])$, $C : f \mapsto F(x) = \int_0^x K(x,t) f(t)\mathrm{d}t$, where K is a continuous function called the *kernel* of C. Then $\sigma(C) = \{0\}$. No eigenfunctions! Note that this example is still relatively nice because we have invariant subspaces. Namely, $\forall x_0$ we have that

$$\{f \in C([0,1]) \mid f|_{[0,x_0]} \equiv 0\}$$

is an invariant subspace. This is a "continuous chain of invariant subspaces".

5.5 Compact self-adjoint operators in a Hilbert space

Definition. $A : H \to H$, H - Hilbert

Definition 5.5.1. A is called *self-adjoint* if $A = A^*$ i.e. $\langle Ax, y\rangle = \langle x, Ay\rangle$.

Definition 5.5.2. A is called *normal* if $AA^* = A^*A$.

Definition 5.5.3. A is called *unitary* if $A^* = A^{-1}$.

Remark. If $\dim H < \infty$ and $A^*A = AA^*$ then there is no non-trivial Jordan blocks. Moreover, eigenvectors are orthogonal.

Theorem 5.5.1. Let $C = C^* : H \to H$ be a compact, self-adjoint operator. Then

(a) For each $0 \neq \lambda_k \in \sigma(X)$, $\ker(\lambda_k I - C)^2 = \ker(\lambda_k I - C)$ and $\lambda_k \in \mathbb{R}$.

(b) If $Cx_n = \lambda_n x_n$ and $Cx_m = \lambda_m x_m$ with $\lambda_n \neq \lambda_m$, then $\langle x_n, x_m\rangle = 0$.

(c) Moreover, $\exists$ an orthonormal basis of H formed by eigenvectors of C (including maybe x's with $Cx = 0$).

Lecture 21: Twenty-First Lecture

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Proof of (a). If $Cf_k = \lambda_k f_k$ then

$$\langle Cf_k, f_k\rangle = \langle f_k, Cf_k\rangle = \overline{\lambda_k}\langle f_k, f_k\rangle$$

Therefore, $\lambda_k = \overline{\lambda_k}$, so $\lambda_k \in \mathbb{R}$. Now note that

$$\langle (C - \lambda_k)^2 x, x\rangle = \|(C - \lambda_k)x\|^2$$

Thus, $\ker(C - \lambda_k)^2 = \ker(C - \lambda_k)$. ∎

Proof of (b). If $\lambda_n \neq \lambda_m$ then

$$\langle Cf_n, f_m\rangle = \lambda_n\langle f_n, f_m\rangle = \lambda_m\langle f_n, f_m\rangle$$

Thus, $\langle f_n, f_m\rangle$ must be zero. ∎

Proof of (c). For each $\lambda_k \neq 0$, $\dim\ker(C - \lambda_k) < \infty$, find an orthonormal basis, take the union of those. Then consider $Y := (\operatorname{span} f_k)^\perp$. Note

$$\langle Cy, f_k\rangle = \langle y, C^* f_k\rangle = \overline{\lambda_k}\langle y, f_k\rangle = 0$$

for $y \in Y$. Consider $C|_Y$, which is compact, normal, with no $\lambda_k \neq 0$. Therefore, $\sigma(C|_Y) = \{0\}$. By Lemma 5.5.1, $C|_Y = 0$ and $Y = \ker C$. ∎

Theorem 5.5.2. Let $C : H \to H$ be a compact, normal operator. Then

(a) For each $0 \neq \lambda_k \in \sigma(X)$, $\ker(\lambda_k I - C)^2 = \ker(\lambda_k I - C)$.

(b) If $Cx_n = \lambda_n x_n$ and $Cx_m = \lambda_m x_m$ with $\lambda_n \neq \lambda_m$, then $\langle x_n, x_m \rangle = 0$.

(c) Moreover, $\exists$ an orthonormal basis of H formed by eigenvectors of C (including maybe x's with $Cx = 0$).

Proof of (a). Note that

$$\|(C-\lambda)x\|^2 = \langle (C^* - \overline{\lambda})(C-\lambda)x, x \rangle = \|(C^* - \overline{\lambda})x\|^2$$

Assume $x \in \ker(C-\lambda)^2 \setminus \ker(C-\lambda)$. Then $y := (C-\lambda)x \in \ker(C-\lambda) \cap \operatorname{Ran}(C-\lambda)$. Note

$$\|y\|^2 = \|(C-\lambda)x\|^2 = \langle (C^* - \overline{\lambda})x, (C-\lambda)x \rangle = 0$$

where the last equality follows from the fact that $y \in \ker(C-\lambda) = \ker(C^* - \overline{\lambda})$. Thus, $\ker(C-\lambda)^2 = \ker(C-\lambda)$ ∎

Proof of (b). Same as the case of self-adjoint operators. ∎

Lemma 5.5.1. Let $A : H \to H$ be a normal operator. Then $\max_{\lambda \in \sigma(A)} |\lambda| = \|A\|$.

Remark. We know $\max_{\lambda \in \sigma(A)} |\lambda| = \lim_{n \to \infty} \|A^n\|^{1/n} \leq \|A\|$.

Proof. For $A^* = A$, note that

$$\|Ax\|^2 = \langle A^2 x, x \rangle \leq \|A^2\| \cdot \|x\|^2$$

Since A^{2m} is self adjoint, we have that $\|A\|^{2m} \leq \|A^{2m}\|$. ∎

Proof of Lemma 5.5.1. Since A is normal, we have that

$$B := A^* A = AA^* = B^*$$

so that B is self-adjoint. Recall that $\|A^*\| = \|A\|$. Note

$$\|Ax\|^2 = \langle Bx, x \rangle \leq \|B\| \cdot \|x\|^2$$

Therefore, $\|B\| = \|A\|^2$. We already know that

$$\|B^{2^n}\| = \|B\|^{2^n} = \|A\|^{2^{n+1}}$$

Since A is normal,

$$\|B^{2^n}\| = \|(A^*)^{2^n} A^{2^n}\| = \|A^{2^n}\|^2.$$

∎

Corollary 5.5.1. $C = C^* : H \to H$ compact, self-adjoint. Denote $\lambda_1^{(+)} \geq \lambda_2^{(+)} \geq \cdots \geq 0 \geq \cdots \geq \lambda_2^{(-)} \geq \lambda_1^{(-)}$. Then

$$\lambda_k^{(+)} = \max_{\substack{Y \subset H \\ \dim X = k}} \min_{x \in X} \frac{\langle Cx, x \rangle}{\|x\|^2} = \min_{\substack{X \subset H \\ \dim X = k-1}} \max_{x \perp X} \frac{Cx, x\rangle}{\|x\|^2}.$$

and similarly for $\lambda_k^{(-)}$.

Proof. Consider (f_k) - eigenbasis. Let $x = \sum_k x_k f_k$. Then

$$\frac{\langle Cx, x \rangle}{\|x\|^2} = \frac{\sum_k \lambda_k |x_k|^2}{\sum_k |x_k|^2}$$

To prove the first equality, pick $X = \operatorname{span}(f_1, \dots, f_k)$ then

$$\min_X \frac{\langle Cx, x\rangle}{\|x\|^2} \geq \lambda_k^{(+)}.$$

For the reverse direction, $\exists x$ such that $\langle x, f_j^{(+)}\rangle = 0$ for every $j \leq k-1$. Then $\frac{\langle Cx,x\rangle}{\|x\|^2} = \lambda_k^{(+)}$. ■

5.6 Polar decomposition, singular values, trace & Hilbert-Schmidt classes

Reminder: (Linear Algebra)
$\exists A = US$, with $U^*U = \mathrm{Id}$ and $S^* = S$.

Definition 5.6.1. Let $C : H \to H$ - compact. Define $(C^*C)^{1/2}x := \sum_k \lambda_k^{1/2}(C^*C)x_k f_k$ for $\sum_k x_k f_k$ where (f_k) is the eigenbasis.

Remark. A few remarks:

(a) $(C^*C)^{1/2}$ is well-defined, bounded, and compact. (approximate by $x \mapsto \sum_{k=1}^N \lambda^{1/2} x_k f_k$)

Notation. $|C| := (C^*C)^{1/2}$.

Note. $\|Cx\|^2 = \langle C^*Cx, x\rangle = \langle |C|x, x\rangle = \||C|x\|^2$.

Proposition 5.6.1. $C : H \to H$ - compact, $|C| := (C^*C)^{1/2}$. Then $\exists U$ such that $U^*U|_{\operatorname{Ran} S} = \mathrm{Id}$.

Proof. Since $\|Cx\| = \||C|x\|$, this allows us to define an isometry $|C|x \mapsto Cx$. This is a good definition on $\operatorname{Ran}|C|$. U - isometry lifts onto $\overline{\operatorname{Ran}|C|}$ as an isometry. Define $U|_{(\operatorname{Ran}|C|)^\perp} := 0$. ■

Remark. $\|x\|^2 = \langle x, x\rangle = \langle U^*Ux, x\rangle = \|Ux\|^2$ i.e. U is an isometry on $\operatorname{Ran} S$.

Definition 5.6.2. Singular values of C are $s_j(C) := \lambda_j(|C|)$.

Definition 5.6.3. C is called a trace class if $\sum s_j(C) < \infty$.

Definition 5.6.4. C is called a Hilbert-Schmidt class if $\sum s_j(C)^2 < \infty$.

Definition 5.6.5. $C \in S_p$ (Schlatten's class) if $\sum s_j(C)^p < \infty$ for $1 \leq p < \infty$.

Lecture 22: Twenty-Second Lecture

Facts on Schlatten's classes: 20 Nov. 01:00

(a) S_p is an ideal: If $C \in S_p$, B - bounded, then $BC \in S_p$ and $CB \in S_p$.

(b) $C \in S_p$, $B \in S_q$. Then $BC, CB \in S_r$ where $\frac{1}{r} = \frac{1}{p} + \frac{1}{q}$ for $p, q, r \in [1, \infty)$.

Remark. $C \in S^p \Leftrightarrow C \in S^q$. Moreover, $s_j(C^*) = s_j(C)$. To see this, note that

$$s_j(C^*) = s_j(C) \Leftrightarrow \lambda_j(C^*C) = \lambda_j(CC^*)$$

Let x so that $C^*Cx = \lambda_j x$. Then $CC^*Cx = \lambda_j Cx$. Thus, λ_j is also an eigenvalue of CC^*.

Facts about Hilbert-Schmidt operators:

(a) (see HW)

$$\|C\|_{\text{HS}}^2 = \sum_{k=1}^{\infty} \|Ce_k\|^2$$

If f_k is the eigenbasis of $|C|$ then

$$\sum_{k=1}^{\infty} \|Cf_k\|^2 = \sum_{k=1}^{\infty} (\lambda_k(|C|))^2$$

(b) B, C - Hilbert-Schmidt, then BC is trace class and $\|BC\|_{T_r} \le \|B\|_{\text{HS}} \|C\|_{\text{HS}}$.

Proof. Note that

$$\|BC\|_{\text{tr}} = \sum_{k=1}^{\infty} \lambda_k(|BC|) = \sum_{k=1}^{\infty} \langle |BC| f_k, f_k \rangle$$

where f_k is an orthonormal eigenbasis of $|BC|$. From polar decomposition, $A = U|A|$ and $|A| = U^*A$. Therefore,

$$\|BC\|_{\text{tr}} = \sum_{k=1}^{\infty} \langle U^*BCf_k, f_k \rangle = \sum_{k=1}^{\infty} \langle Cf_k, B^*Uf_k \rangle$$

Define $g_k := Uf_k$, so that g_k is orthonormal. By Cauchy Schwartz,

$$\|BC\|_{\text{tr}} \le \sum_{k=1}^{\infty} \|Cf_k\| \|B^*g_k\| \le \left(\sum_{k=1}^{\infty} \|Cf_k\|^2 \right)^{1/2} \left(\sum_{k=1}^{\infty} \|B^*g_k\|^2 \right)^{1/2} \le \|C\|_{\text{HS}} \cdot \|B^*\|_{\text{HS}}$$

■

Facts about trace class:

(a) Let C - trace class and $(e_k)_{k=1}^{\infty}$ be an orthonormal basis. Then $\text{tr}(C) := \sum_{k=1}^{\infty} \langle Ce_k, e_k \rangle$ converges and does not depend on choice of basis.

Proof. Let f_k be the eigenbasis of $|C|$. We have $e_k = \sum_{j=1}^{\infty} \langle e_k, f_j \rangle f_j$ and

$$\begin{aligned} Ce_k &= \sum_{j=1}^{\infty} \langle e_k, f_j \rangle \cdot U|C| f_j \\ &= \sum_{j=1}^{\infty} s_j \langle e_k, f_j \rangle \cdot g_j \end{aligned}$$

for $g_j = Uf_j$. Then

$$\langle Ce_k, e_k \rangle = \sum_{j=1}^{\infty} s_j \langle e_k, f_j \rangle \cdot \langle g_j, e_k \rangle$$

Note

$$\sum_{k=1}^{\infty}\langle Ce_k, e_k\rangle \le \sum_{j=1}^{\infty} s_j \sum_{k=1}^{\infty} |\langle e_k, f_j\rangle| \cdot |\langle g_j, e_k\rangle| \le \sum_{j=1}^{\infty} s_j \|f_j\|\|g_j\| = \sum_{j=1}^{\infty} s_j = \|C\|_{\mathrm{tr}}$$

Now note that

$$\sum_{k=1}^{\infty}\langle e_k, f_j\rangle\langle g_j, e_k\rangle = \sum_{j=1}^{\infty} S_j\langle f_j, g_j\rangle$$

which does not depend on e_k. ■

(b) (Without proof) C - trace class, then $(\lambda_j(C))_{j=1}^{\infty} \in \ell^1$ and $\sum_{j=1}^{\infty}{}_j(C) = \mathrm{tr}(C)$.

5.7 Spectral Theory of Bounded Self-Adjoint Operators

We want to extend the spectral theory of compact self-adjoint operators to bounded self-adjoint operators that are not necessarily compact. Recall that if $C = C^*$ is compact then $C = \sum_{j\ge 0}\lambda_j P_j + 0 \cdot P_{\ker C}$, where P_j is an orthogonal projector onto the eigenspace of λ_j.

If we do not know compactness, it can be the case that there are no eigenfunctions at all. Consider $H = L^2([0,1])$ with $[Af](\lambda) := \lambda f(\lambda)$. We can nevertheless write, $A = \int_0^1 \lambda \mathrm{d}P_\lambda$, where $\mathrm{d}P_\lambda$ is a projector-valued measure $E \subset [0,1] \mapsto P_E = P_E^* = P_E^2$ i.e. $P_E f := f|_E$.

Lecture 23: Twenty-Third Lecture

Goal: Given H - Hilbert and $A = A^* \in \mathcal{L}(H)$ then there is some projector-valued measure such that $A = \int \lambda P(\mathrm{d}\lambda)$. 27 Nov. 01:00

Illustration: Finite matrices or compact self-adjoint operators - there exists an orthonormal basis such that

$$A \sim \begin{pmatrix} \lambda_1 & & \\ & \lambda_2 & \\ & & \ddots \end{pmatrix}$$

i.e. $A = \sum_{\lambda\in\sigma(A)} \lambda P(\{\lambda\})$.

Lemma 5.7.1. $A = A^*$. Then the following holds:

(a) $\|A\| = \sup\limits_{x\ne 0}\left|\frac{\langle Ax,x\rangle}{\|x\|^2}\right| =: \widetilde{M}$

(b) $\sigma(A) \subset [m, M]$ where $m = \inf\limits_{x\ne 0}\frac{\langle Ax,x\rangle}{\|x\|^2}$ and $M = \sup\limits_{x\ne 0}\frac{\langle Ax,x\rangle}{\|x\|^2}$

(c) Moreover, $m, M \in \sigma(A)$.

Proof of (a). $\|A\| \ge \frac{\langle Ax,x\rangle}{\|x\|^2}$. Therefore, we must show the reverse direction. Note that

$$\|A\| = \sup_{\|x\|=1}\|Ax\| = \sup_{\|x\|=\|y\|=1}|\langle Ax, y\rangle| = \sup_{\|x\|=\|y\|=1}\mathrm{Re}\langle Ax, y\rangle$$

By polarization identity,

$$\mathrm{Re}\langle Ax, y\rangle = \frac{1}{4}\left(\langle A(x+y), x+y\rangle - \langle A(x-y), x-y\rangle\right)$$

Therefore,

$$\operatorname{Re}\langle Ax, y\rangle \leq \frac{1}{4} \cdot \widetilde{M}\left(\|x+y\|^2 + \|x-y\|^2\right) = \frac{\widetilde{M}}{2}\left(\|x\|^2 + \|y\|^2\right) = \widetilde{M}$$

■

Proof of (b). First we will show that $\sigma(A) \subset \mathbb{R}$. Let $\lambda \notin \mathbb{R}$. Note that

$$\begin{aligned}\|(\lambda I - A)x\|^2 &= \langle(\lambda I - A)x, (\lambda I - A)x\rangle \\ &= \langle(\operatorname{Re}\lambda I - A)x, (\operatorname{Re}\lambda I - A)x\rangle + \langle i \operatorname{Im}\lambda - x, i \operatorname{Im}\lambda x\rangle \qquad (\star)\end{aligned}$$

To see this, we can examine the cross terms

$$\begin{aligned}b &:= \langle(\operatorname{Re}\lambda I - A)x, i \operatorname{Im}\lambda x\rangle \in i\mathbb{R} \\ &= i \operatorname{Im}\lambda\langle(\operatorname{Re}\lambda I - A)x, x\rangle\end{aligned}$$

Thus, we have that $b + \bar{b} = 0$. Now we have that $(\star) \geq |\operatorname{Im}\lambda|^2 \cdot \|x\|^2 > 0$, so that λ cannot be an eigenvalue. Additionally, if $\{(\lambda I - A)x_n\}$ is Cauchy then $\{x_n\}$ is Cauchy and $\operatorname{Ran}(\lambda I - A)$ is closed.

Now we must show that $\sigma(A) \subset [m, M]$. We know that $\sigma(A) \subset [-\|A\|.\|A\|]$ because $\sigma(A) \subset \mathbb{R}$ from above. Moreover, at least one of $\pm\|A\| \in \sigma(A)$. Apply this to $A + tI$. Then $\sigma(A + tI) = \sigma(A) + t \subset [-\max(|m+t|, |M+t|), \max(|m+t|, |M+t|)]$. Then taking $t \gg 1$ and $t \ll 1$, we get that $\sigma(A) \subset [m, \infty)$ and $\sigma(A) \subset (-\infty, M]$. ■

Proof of (c). If $M \notin \sigma(A)$ then $\|A + t\| < M + t$, which is a contradiction. ■

Remark. A few remarks:

- $A^* = A \Rightarrow \langle Ax, x\rangle = \langle x, Ax\rangle = 2\overline{\langle Ax, x\rangle}$ is real.
- One cannot write max / min instead of sup / inf. Consider $f(\lambda) \mapsto \lambda f(\lambda)$.

Chapter 6

Functional Calculus

6.1 Polynomials

Important idea: We want to consider f so that if $A \sim \begin{bmatrix} \lambda_1 & & \\ & \lambda_2 & \\ & & \ddots \end{bmatrix}$ then $f(A) \sim \begin{bmatrix} f(\lambda_1) & & \\ & f(\lambda_2) & \\ & & \ddots \end{bmatrix}$.

Observation: We know what to do if $f \in \mathbb{R}[z]$. $q = c_0 + c_1 z + \cdots + c_n z^n$. Analogously $q(A) = c_0 + c_1 A + \cdots + c_n A^n$. Most likely, f will not be a polynomial. We will proceed by extending to

1. f that are polynomials,
2. f that are continuous on $\sigma(A)$,
3. f that are bounded, measurable.

Proposition 6.1.1. $A = A^*$, $q \in \mathbb{R}[z]$. Then $\sigma(q(A)) = q(\sigma(A))$. In particular,

$$\|q(A)\| = \|q\|_{C(\sigma(A))} = \max_{\lambda \in \sigma(A)} |q(\lambda)|$$

Proof. It is enough to prove that $0 \in \sigma(q(A)) \Leftrightarrow \exists \lambda \in \sigma(A)$ such that $q(\lambda) = 0$. In general, consider $q - \xi$ instead of q. This works because $\xi \in \sigma(q(A)) \Leftrightarrow 0 \in (q-\xi)(A) \Leftrightarrow \exists \lambda \in \sigma(A) : q(\lambda) - \xi = 0$. Write

$$q(z) = c(z - z_1)\cdots(z - z_n), \quad z_1, \ldots, z_n \in \mathbb{C}$$

Then

$$q(A) = c(A - z_1 I)\cdots(A - z_n I)$$

$q(A)$ is invertible if and only if each $(A - z_k I)$ is invertible. Thus, $0 \in \sigma(q(A)) \Leftrightarrow \exists k : z_k \in \sigma(A)$. ■

Now we must go from polynomials to continuous functions.

Definition 6.1.1. Let $f \in C(\sigma(A))$. Find $q_n \to f$ uniformly on $\sigma(A)$. Define

$$f(A) := \lim_{n \to \infty} q_n(A)$$

Theorem 6.1.1. The limit from the definition exists in the norm topology. The result does not depend on the choice of q_n. Moreover,

$$f(A) = f(A)^* \quad \text{and} \quad \|f(A)\|_{\mathcal{L}(H)} = \|f\|_{C(\sigma(A))}$$

Proof. Nothing to prove :) $C(\sigma(A)) \ni q \mapsto q(A) \in \mathcal{L}(H)$ is an isometry. For self-adjointness,

$$A_n = A_n^*, \quad \|A - A_n\| = \|A^* - A_n^*\| \to 0 \Rightarrow A = A^*$$

■

Lecture 24: Twenty-Fourth Lecture

29 Nov. 01:00

Problem 6.1.1. Do we know that $\sigma(f(A)) = f(\sigma(A))$ for continuous functions?

Answer. Let $\xi \notin f(\sigma(A))$. Then $f - \xi \neq 0$ on $\sigma(A)$, so $f - \xi$ is invertible on $\sigma(A)$, so $(f-\xi)^{-1}$ is continuous on $\sigma(A)$, so $(f-\xi)^{-1} \in C(\sigma(A))$. Then $(f(A) - \xi I)^{-1} \in \mathcal{L}(H)$ and this is an inverse to $(f(A) - \xi I)$ because $(\star)$ is an algebra homomorphism. For polynomials, this is trivial. Otherwise, if $q_n \to f$ and $p_n \to g$ uniformly then $p_n(A)q_n(A)x \to g(A)f(A)x$. Thus, $\xi \notin \sigma(f(A))$. This gives us that $\sigma(f(A)) \subset f(\sigma(A))$. ⊛

Proposition 6.1.2. $A = A^*$, $B = B^* \in \mathcal{L}(H)$. Then $\operatorname{dist}(\sigma(A), \sigma(B)) \leq \|A - B\|$.

Proof. Let $\lambda \in \sigma(A)$ and $\operatorname{dist}(\lambda, \sigma(B)) > \|A - B\|$. Consider $(\lambda I - B)^{-1}$. Then $\sigma((\lambda I - B)^{-1}) \subset [-\|A-B\|^{-1} + \epsilon, \|A-B\|^{-1} - \epsilon]$ for some positive ϵ. Therefore, $\|(\lambda I - B)^{-1}\| < \|A - B\|^{-1}$. Now write

$$\lambda I - A = (\lambda I - B)(I + \underbrace{(\lambda I - B)^{-1}(B - A)}_{\|\cdot\|<1})$$

Since $\lambda I - B$ is invertible, this means that $\lambda I - A$ must be invertible. Thus, $\lambda \notin \sigma(A)$, a contradiction. ■

Remark. Given the proposition, approximate f by polynomials $q_n \to f$ uniformly. If $\xi \in f(\sigma(A))$, then $\exists \lambda \in \sigma(A)$ such that $\xi = f(\lambda) = \lim_{n\to\infty} q_n(\lambda) \in \sigma(q_n(A))$. Since $\|q_n(A) - f(A)\| \to 0$, from the proposition we have that $\xi \in \sigma(f(A))$. Thus, we have the reverse direction $\sigma(f(A)) \supset f(\sigma(A))$.

Theorem 6.1.2 (Functional calculus for measurable functions)**.** $A = A^* \in \mathcal{L}(H)$. One can extend $f \mapsto f(A) = f(A)^* \in \mathcal{L}(H)$ to all bounded Borel-measurable functions f on $\sigma(A)$ such that the following holds:

(a) $\|f(A)\|_{\mathcal{L}(H)} \leq \|f\|_\infty$.

(b) If $f_n \to f$ pointwise and f_n are uniformly bounded, then $\forall x \in H$, $f_n(A)x \to f(A)x$.

(c) $f \mapsto f(A)$ is an algebra homomorphism.

Extension of $f(A)$. Consider $C(\sigma(A)) \ni f \mapsto \langle f(A)x, y\rangle$. Note that

$$|\langle f(A)x, y\rangle| \leq \|f(A)x\|\|y\| \leq \|f\|_{C(\sigma(A))} \cdot \|x\| \cdot \|y\|$$

This is an element of $(C(\sigma(A)))'$. By Riesz's representation theorem, there exists some Borel-regular signed measure $\mu_{x,y}$ on $\sigma(A)$ such that

$$\langle f(A)x, y\rangle = \int_m^M f\mu_{x,y}(d\lambda)$$

and $\|\mu_{x,y}\|_{\mathrm{TV}} \leq \|x\| \cdot \|y\|$. Define

$$B_f(x,y) := \int_m^M f\mu_{x,y}(d\lambda)$$

for bounded measurable f's. This is linear in x, anti-linear in y, and $\|B_f\| \leq \|f\|_\infty$. Therefore, given x there exists $f(A)x \in \mathcal{L}(H)$ such that for every y, $B_f(x,y) = \langle f(A)x, y\rangle$. ■

Proof of (b). Suppose $f_n \to f$ pointwise and f_n uniformly bounded. Note that

$$\langle f_n(A)x, y\rangle - \langle f(A)x, y\rangle = \int_m^M (f_n - f)\mu_{x,y}(d\lambda) \to 0$$

where the last step follows from the dominated convergence theorem. Thus, $f_n(A) \to f(A)$ in the weak operator topology. Similarly,

$$\|f_n(A)x - f(A)x\|^2 = \langle (f_n(A) - f(A))^2 x, x\rangle = \int_m^M |f_n - f|^2 \mu_{x,x}(d\lambda) \to 0$$

Thus, $f_n(A) \to f(A)$ in strong operator topology. ■

Proof of (c). $f \mapsto f(A)$ is a homomorphism. Follows from approximation by continuous functions: $\forall f$ - bounded measurable, $\exists f_n$ - continuous such that $f_n \to f$ pointwise (Lusin's Theorem). We only must show that if $f_n \to f$ and $g_n \to g$ pointwise then $f_n(A)g_n(A)x = (f_n g_n)(A)x \to (fg)(A)x$. We know that $g_n(A)x \to g(A)x$. Since $f_n(A)$ is uniformly bounded, $f_n(A)g_n(A)x \to f_n(A)g(A)x \to f(A)g(A)x$. ■

Remark. Convergence as in (b), is called *strong convergence* or *convergence in the strong operator topology*. There is also weak operator topology: $\langle Bx_n, y\rangle \to \langle Bx, y\rangle\ \forall x, y$.

Corollary 6.1.1. Given $E \subset [m, M]$ Borel measurable, denote $P_E \coloneqq \mathbf{1}_E(A)$. Then $P_E = P_E^* = P_E^2$, i.e. P_E is an orthogonal projector and

$$P_{E_1}P_{E_2} = P_{E_2}P_{E_1} = P_{E_1 \cap E_2}$$

In particular, $E_1 \cap E_2 = \emptyset \Rightarrow P_{E_1}P_{E_2} = 0$ and $P_{E_1 \cap E_2} = P_{E_1} + P_{E_2}$.

Lecture 25: Twenty-Fifth Lecture

Proposition 6.1.3. Let $(E_k)_{k=1}^\infty$ be disjoint and $E \coloneqq \bigcup_{k=1}^\infty E_k$. Then $P_E = \sum_{k=1}^\infty P_{E_k}$, where $\sum_{k=1}^\infty P_{E_k}$ converges in the strong operator topology. 4 Dec. 01:00

Proof. Denote $P^{(n)} \coloneqq \sum_{k=1}^n P_{E_k} = P_{E_1 \cup \cdots \cup E_n}$. Then we have that

$$0 \leq P^{(1)} \leq P^{(2)} \leq \cdots \leq P^{(n)} \leq \cdots \leq I$$

Note that $y_n \coloneqq \langle P^{(n)}x, x\rangle$ is a monotone sequence that is bounded above by $\|x\|^2$. Thus, y_n has a limit. Denote $\langle Px, x\rangle \coloneqq \lim_{N\to\infty}\langle P_n x, x\rangle$. We can then define $\langle Px, y\rangle$ by polarization i.e.

$$\langle Px, y\rangle = \lim_{N\to\infty} \frac{1}{4}\left(\langle P_N(x+y), x+y\rangle - \langle P_N(x-y), x-y\rangle + i\langle P_N(x+iy), x+iy\rangle - i\langle P_N(x-iy), x-iy\rangle\right)$$

Therefore, $\langle Px, y\rangle$ is a bounded sesqui-linear form. From this, we get a bounded linear operator $P \in \mathcal{L}(H)$. Clearly, $\langle Px, y\rangle = \langle x, Py\rangle$ and $P \leq I$. We now want to show that $P_n x \to Px$ Note that

$$\begin{aligned}\|(P - P^{(n)})x\| &\leq \|\sqrt{P - P^{(n)}}\| \cdot \|\sqrt{P - P^{(n)}}x\| \\ &\leq \|\sqrt{P - P^{(n)}}x\| \\ &= \langle (P - P^{(n)})x, x\rangle \to 0\end{aligned}$$

Finally, we must show that P is a projector. Namely, $P = P_E$ for $E = E_1 \cup \cdots \cup E_k$. If $x \in \operatorname{Ran} P_{E_k}$

and $N \geq k$ then $P^{(N)}x = x$. If $x \perp \bigoplus_{k=1}^{\infty} \operatorname{Ran} P_{E_k}$, then $P^{(N)}x = 0$. Thus, P is a projector onto $\overline{\operatorname{span}(\operatorname{Ran} P_{E_k}, k \geq 1)}$. ■

Definition 6.1.2. A countable additive map (in the strong operator topology) $E \mapsto P_E$, such that $P_\emptyset = 0$, $P_{\sigma(A)} = I$, and $P_E = P_E^* = P_E^2$ is a *projector-valued measure*.

Theorem 6.1.3. Given $A = A^* \in \mathcal{L}(H)$, $\exists$ a projector-valued measure on $\sigma(A)$ such that $A = \int \lambda P(\mathrm{d}\lambda)$, where the integral converges in the norm topology. Moreover, $f(A) = \int_{\sigma(A)} f(\lambda) P(\mathrm{d}\lambda)$.

Proof. Using the notation $P_{[a,b]} := P_{[a,b] \cap \sigma(A)}$, we have that

$$\left\| \sum_{k=1}^{n} \lambda_k P_{[a_k, a_{k+1})} - \sum_{k=1}^{m} \mu_k P_{[b_k, b_{k+1})} \right\|$$

Here the difference is a finite linear combination of $\perp$ projectors with coefficients given by $\lambda - \mu$. To prove $A = \int_{\sigma(A)} \lambda P(\mathrm{d}\lambda)$, it is enough to prove $\langle Ax, y \rangle = \langle \int_{\sigma(A)} \lambda P(\mathrm{d}\lambda) x, y \rangle$. By construction, $Ax = \lambda \mu_{x,y}(\mathrm{d}\lambda)$. Additionally, we have that

$$\int_{\sigma(A)} \lambda P(\mathrm{d}\lambda) = \lim_{n \to \infty} \sum_{k=1}^{n} \lambda_k \underbrace{\langle P_{[a_k, a_{k+1})} x, y \rangle}_{\mu_{x,y}([a_k, a_{k+1}))}$$

Convergence is now trivial. ■

Remark. A few remarks:

(a) projector-valued measure P is unique,

(b) If $AB = BA$, $B \in \mathcal{L}(H)$, then all $P_E B = B P_E$.

Definition 6.1.3. A is *unitary equivalent* to Λ if

$$\exists U : H \to \widetilde{H} \text{ isometry such that } H = U^{-1} \Lambda U$$

Definition 6.1.4. x is called a *cyclic vector* if $\operatorname{span}(A^k x)_{k=1}^{\infty} = H$.

Exercise. $A = A^* \in \mathbb{C}^{N \times N}$ and $\exists x$ cyclic then $\nexists$ multiple eigenvalues.

Theorem 6.1.4. Let $A = A^* \in \mathcal{L}(H)$, H-separable. Then $\exists 1 \leq N \leq \infty$ and finite measures $\mu^{(1)}, \mu^{(2)}, \ldots, \mu^{(k)}, \ldots$ $k \leq N$, $k \neq \infty$ supported on $[m, M]$ such that $A \in \mathcal{L}(H)$ is unitary equivalent to $\Lambda \in \mathcal{L}(\widetilde{H})$ defined by $(x_1, x_2, \ldots) \mapsto (\Lambda_1 x_1, \Lambda_2 x_2, \ldots)$ with $\Lambda_k x_k(\lambda) = \lambda x_k(\lambda)$ and $\widetilde{H} = L^2(\mathrm{d}\mu^{(1)}) \oplus L^2(\mathrm{d}\mu^{(2)}) \oplus \cdots$.

Proof. First, we will consider the case where there exists some cyclic vector x. Then we want a single space $L^2(\mathrm{d}\mu)$ as H. Denote $\mu := \mu_{x,x}$ $\Big(\langle f(A)x, x \rangle = \int_{\sigma(A)} f(\lambda) \mu(\mathrm{d}\lambda)\Big)$. Note that

$$\|f(A)x\|^2 = \langle f(A)x, f(A)x \rangle = \int_{\sigma(A)} |f(\lambda)|^2 \mu_{x,x}(\mathrm{d}\lambda)$$

Consider (say f polynomial) $H \ni f(A)x \mapsto f \in L^2(\mathrm{d}\mu_{x,x}) =: \widetilde{H}$. This is an isometry, which extends to the whole H if x is cyclic. ■

Appendix

Appendix A

Additional Proofs

A.1 Proof of ??

We can now prove **??**.

Proof of ??. See here. ■