597 Notes

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Abstract

This is a compilation of all of my notes for Math 592 this semester. The course is taught by Alexander Perry. We will follow the Hatcher textbook.

Contents

Chapter 1

Abstract Measures

Lecture 4: Fourth Lecture

1.1 Caratheodory's Theorem

Jan. 10:00

Definition 1.1.1 (Outer Measure). $\mu^* : \mathcal{P}(x) \to [0, \infty]$ such that

- $\mu^*(\emptyset) = 0$,
- Monotonicity i.e. $A \subset B \Rightarrow \mu^*(A) \leq \mu^*(B)$
- Countable subadditivity $\mu^*(\bigcup_{i=1} A_i) \leq \sum_{i=1} \mu^*(A_i)$

Proposition 1.1.1. Consider $\mathcal{E} \subset \mathcal{P}(X)$ such that $\emptyset, X \in \mathcal{E}$ and $\rho : \mathcal{E} \to [0, \infty]$ with $\rho(\emptyset) = 0$. Then

$$\mu^* : \mathcal{P}(X) \to [0, \infty]$$

defined by

$$\mu^*(A) := \inf \left\{ \sum_{i=1} \rho(E_j) \mid A \subset \bigcup_{j=1} E_j,\ E_i \in \mathcal{E} \right\}$$

is an outer measure.

Proof. Sufficient to show countable subadditivity (rest DIY). Pick $A_1, A_2, \ldots \subset X$. Want to show

$$\mu^*(\bigcup_{j=1} A_j) \leq \sum_{i=1} \mu^*(A_i).$$

Without loss of generality, $\mu^*(A_i) < \infty$ for every i. Otherwise, LHS = RHS = ∞. Fix $\epsilon > 0$. Then sufficient to show

$$\mu^*(\bigcup_{j=1} A_j) \leq \sum_{i=1} \mu^*(A_i) + \epsilon.$$

For every j, there exists some $E_{j_k} \in \mathcal{E}$ such that $A_j \subset \bigcup_k E_{j_k}$ and

$$\mu^*(A_j) \leq \sum_{k=1} \rho(E_{j_k}) \leq \mu^*(A_j) + \underbrace{\frac{\epsilon}{2^j}}_{\star}.$$

Then

$$\bigcup_j A_j = \bigcup_j \bigcup_k E_{j_k} = \bigcup_{j,k} E_{j_k}$$

and

$$\mu^*\left(\bigcup_j A_j\right) \leq \sum_{(j,k)\in\mathbb{N}^2} \rho(E_{j_k})$$
$$\overset{\text{Tonelli}}{=} \sum_j \sum_k \rho(E_{j_k})$$
$$\leq \sum_j (\mu^*(A_j) + \frac{\epsilon}{2^j})$$
$$= \sum_j \mu^*(A_j) + \epsilon.$$

■

Definition 1.1.2. Let μ^* be an outer measure on X. Say $A \subset X$ is μ^*-measureable if

$$\mu^*(E) = \mu^*(E \cap A) + \mu^*(E \cap A^c), \quad \forall E \subset X.$$

Theorem 1.1.1 (Caratheodory's Theorem). Let μ^* be an outer measure on X, and $\mathcal{A} \subset \mathcal{P}(X)$ the set of μ^*-measureable sets.

(a) $\mathcal{A}$ is a σ-algebra.

(b) $\mu := \mu^*|_A$ is a measure.

(c) (X, A, μ) is a complete measure space.

Proof of (a). Four steps:

(i) $\emptyset \in \mathcal{A}$, $\mathcal{A}$ closed under complements, (DYI)

(ii) Closed under finite unions ($\mathcal{A}$ is an algebra),

(iii) Closed under countable disjoint union,

(iv) Closed under countable general union.

In fact (i) − (iii) ⇒ (iv)[a].

For (ii), it suffices to show $A, B \in \mathcal{A} \Rightarrow A \cup B \in \mathcal{A}$. Pick $E \subset X$.

$$\mu^*(E) \overset{A\in\mathcal{A}}{=} \mu^*(E \cap A) + \mu^*(E \cap A^c)$$
$$\overset{B\in\mathcal{A}}{=} \mu^*(E \cap A \cap B) + \mu^*(E \cap A \cap B^c)$$
$$+ \mu^*(E \cap A^c \cap B) + \mu^*(E \cap A^c \cap B^c).$$

Use $A \cup B = (A \cap B) \cup (A^c \cap B) \cup (A \cap B^c)$. From subadditivity,

$$\mu^*(E \cap (A \cup B)) \leq \mu^*(E \cap A \cap B) + \mu^*(E \cap A^c \cap B) + \mu^*(E \cap A \cap B^c).$$

Therefore,

$$\mu^*(E) \geq \mu^*(E \cap (A \cup B)) + \mu^*(E \cap A^C \cap B^c).$$

However, $\leq$ holds from subadditivity. Thus, $A \cup B \in \mathcal{A}$.

For (iii), pick $A_1, A_2, \ldots \in \mathcal{A}$ disjoint. Then define

$$B := \bigcup_i A_i.$$

Set

$$B_n \coloneqq \bigcup_{j=1}^n A_j = B_{n-1} \sqcup A_n.$$

Then

$$\begin{aligned}\mu^*(E \cap B_n) &\overset{A_n \in \mathcal{A}}{=} \mu^*(E \cap B_n \cap A_n) + \mu^*(E \cap B_n \cap A_n^c) \\ &= \mu^*(E \cap A_n) + \mu^*(E \cap B_{n-1}).\end{aligned}$$

Repeat: $\mu^*(E \cap B_n) = \sum_{j=1}^n \mu^*(E \cap A_j)$. Therefore,

$$\begin{aligned}\mu^*(E \cap B_n) &\overset{B_n \in \mathcal{A}}{=} \mu^*(E \cap B_n) + \mu^*(E \cap B_n^c) \\ &= \sum_{j=1}^n \mu^*(E \cap A_j) + \mu^*(E \cap B_n^c).\end{aligned}$$

Letting $n \to \infty$, we get

$$\begin{aligned}\mu^*(E) &\overset{B_n \subset B}{\geq} \sum_{j=1} \mu^*(E \cap A_j) + \mu^*(E \cap B^c) \\ &\overset{\text{subadd}}{\geq} \mu^*(E \cap \bigcup_{j=1}^\infty A_j) + \mu^*(E \cap B^c) \\ &= \mu^*(E \cap B) + \mu^*(E \cap B^c) \\ &\overset{\text{subadd}}{\geq} \mu^*(E).\end{aligned}$$

Therefore, "=" holds everywhere. Thus, $B \in \mathcal{A}$. ■

[a] Write general in form of countable disjoint

Proof of (b). Clear that $\mu^*(\emptyset) = 0$. Therefore, it is sufficient to show that if $A_1, A_2, \ldots \in \mathcal{A}$ are disjoint, then

$$\mu^*(\bigcup_{j=1}^\infty A_j) = \sum_{j=1}^\infty \mu^*(A_j).$$

In previous inequalities, use $E = B$. Then you get

$$\mu^*(B) = \sum_{j=1}^\infty \mu^*(A_j).$$

■

Proof of (c). Suppose that $A \subset B \in \mathcal{A}$, $\mu^*(B) = 0$. Then we want to show that $A \in \mathcal{A}$, i.e.

$$\mu^*(E) = \mu^*(E \cap A) + \mu^*(E \cap A^c).$$

But,

$$\mu^*(E) \leq \underbrace{\mu^*(\underbrace{E \cap A}_{\subset A \subset B})}_{=0} + \underbrace{\mu^*(\underbrace{E \cap A^c}_{\subset E})}_{\leq \mu^*(E)}.$$

■

Lecture 5: Fifth Lecture

1.2 Hahn-Komogorov Theorem

Jan. 10:00

Remark (Special case). $X = \mathbb{R}$, $\mathcal{E} = \{\text{finite unions of } \underbrace{\text{half open}}_{(a,b],(a,\infty),\ldots} \text{ intervals}\}$. $\rho(E) = $ length of E.

$$(\mathbb{R}, \mathcal{E}, \rho) \xrightarrow{\text{Prop}} (\mathbb{R}, \mathcal{P}(\mathbb{R}), \mu^*) \xrightarrow{\text{Cor.}} (\mathbb{R}, \mathcal{A}, \mu).$$

Would like

$$\mathcal{E} \subset \mathcal{A} \text{ and } \mu|_{\mathcal{E}} = \rho.$$

Features of $(\mathcal{E}, \rho)$

- $\mathcal{E}$ is an algebra,
- $\rho : \mathcal{E} \to [0, \infty]$ is a premeasure i.e.
  - $\rho(\emptyset) = 0$,
  - If $E_1, E_2, \ldots \in \mathcal{E}$ are disjoint such that $\bigcup_{i=1}^{\infty} E_i \in \mathcal{E}$, then $\rho\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} \rho(E_i)$

Theorem. Setup:

- $\mathcal{A}_0 \subset \mathcal{P}(X)$ is an algebra.
- $\mu_0 : \mathcal{A}_0 \to [0, \infty]$ is a premeasure.
- $\mu^* : \mathcal{P}(X) \to [0, \infty]$ outer measure from prop.
- $(X, \mathcal{A}, \mu)$ measure space from Caratheodory's theorem.

Theorem 1.2.1 (Hahn-Kolmogorov). The following things hold:

(a) $\mathcal{A}_0 \subset \mathcal{A}$

(b) $\underbrace{\mu|_{\mathcal{A}_0}}_{\mu^*|_{\mathcal{A}_0}} = \mu_0$

Proof of (a). Pick $A \in \mathcal{A}_0$. Need to show $A \in \mathcal{A}$ i.e.

$$\mu^*(E) = \mu^*(E \cap A) + \mu^*(E \cap A^c), \quad \forall E \subset X.$$

By subadditivity,

$$\mu^*(E) \leq \mu^*(E \cap A) + \mu^*(E \cap A^c), \quad \forall E \subset X.$$

To prove the opposite you may assume $\mu^*(E) < \infty$. Pick $\epsilon > 0$. Then it suffices to show:

$$\mu^*(E) + \epsilon \geq \mu^*(E \cap A) + \mu^*(E \cap A^c).$$

Use definition of $\mu^*(E)$. Then $B_i \in \mathcal{A}_0$ with $i \in \mathbb{N}$ such that

$$E \subset B := \bigcup_{i=1}^{\infty} B_i \text{ and } \underbrace{\sum_{i=1}^{\infty} \mu_0(B_i)}_{\star} \leq \mu^*(E) + \epsilon.$$

Then by finite additivity of μ_0

$$\star = \sum_{i=1}^{\infty} \Big(\mu_0(B_i \cap A) + \mu_0(B_i \cap A^c)\Big).$$

Note

$$B_i \cap A \in \mathcal{A}_0 \text{ and } E \cap A \subset \bigcup_{i=1}^{\infty} B_i \cap A.$$

Therefore,

$$\sum_{i=1}^{\infty} \mu_0(B_i \cap A) \geq \mu^*(E \cap A).$$

Similarly,

$$\sum_{i=1}^{\infty} \mu_0(B_i \cap A^c) \geq \mu^*(E \cap A^c).$$

It follows that

$$\mu^*(E) + \epsilon \geq \mu^*(E \cap A) + \mu^*(E \cap A^c).$$

∎

Proof of (b). Let $A \in \mathcal{A}_0$ ($\Rightarrow A \in \mathcal{A}$). Want to show:

$$\mu(A) \overset{\text{Cor.}}{=} \mu^*(A) = \mu_0(A).$$

A trivial covering of A by elements of $\mathcal{A}_0$:

$$A = A \cup \emptyset \cup \emptyset \cup \cdots .$$

Therefore,

$$\mu^*(A) \leq \mu_0(A) + \mu_0(\emptyset) + \mu_0(\emptyset) + \cdots = \mu_0(A).$$

To prove $\mu^*(A) \geq \mu_0(A)$ look at an arbitrary covering

$$A \subset \bigcup_{i=1}^{\infty} B_i, \quad B_i \in \mathcal{A}_0.$$

Set $A_n = A \cap \left(B_n \setminus \bigcup_{i=1}^{n-1} B_i\right)$. Then

- $A_n \in \mathcal{A}_0$,
- $A = \bigsqcup_{i=1}^{\infty} A_i$ disjoint union.

Since μ_0 is a premeasure,

$$\mu_0(A) = \sum_{i=1}^{\infty} \mu_0(A_n) \leq \sum_{i=1}^{\infty} \mu_0(B_n).$$

Take the inf over all coverings $\{B_i\}_{i\in\mathbb{N}}$ Thus,

$$\mu^*(A) \geq \mu_0(A).$$

■

Definition 1.2.1 (Hahn-Kolmogorov Extension). $(X, \mathcal{A}, \mu)$ is the HK-extension of $(X, \mathcal{A}_0, \mu_0)$.

Theorem 1.2.2 (Uniqueness of HK-extension). Let $(X, \mathcal{A}_0, \mu_0)$ be a premeasure space and $(X, \mathcal{A}, \mu)$ the HK-extension. Suppose $(X, \mathcal{A}', \mu')$ is another extension. Then

(a) $\mu' \leq \mu$ on $\mathcal{A} \cap \mathcal{A}'$.

(b) If μ_0 is σ-finite then $\mu' = \mu$ on $\mathcal{A} \cap \mathcal{A}'$.

Proof of (a). Pick $A \in \mathcal{A} \cap \mathcal{A}'$ and $B_i \in \mathcal{A}_0$ such that

$$A \subset B := \bigcup_{i=1}^{\infty} B_i.$$

Then

$$\mu'(A) \leq \sum_{i=1}^{\infty} \mu'(B_i) = \sum_{i=1}^{\infty} \mu_0(B_i).$$

Take inf over $\{B_i\}_{i\in\mathbb{N}}$. Then we get

$$\mu'(A) \leq \mu^*(A) = \mu(A).$$

Note that

$$\mu'(B) \overset{cont.}{=} \lim_{N\to\infty} \mu'\Big(\underbrace{\bigcup_{i=1}^{N} B_i}_{\in\mathcal{A}_0}\Big) = \lim_{N\to\infty} \mu\left(\bigcup_{i=1}^{N} B_i\right) \overset{cont.}{=} \mu(B).$$

■

Proof of (b). Assume μ_0 is σ-finite. Pick $A \in \mathcal{A} \cap \mathcal{A}'$. It is then sufficient to show that

$$\mu'(A) \geq \mu^*(A) = \mu(A).$$

(i) First assume $\mu(A) < \infty$. Pick $\epsilon > 0$. Pick $B_i \in \mathcal{A}_0$ such that $A \subset B = \bigcup_{i=1}^{\infty} B_i$.

$$\mu(B) \leq \sum_{i=1}^{\infty} \mu(B_i) = \sum_{i=1}^{\infty} \mu_0(B_i) \leq \mu^*(A) + \epsilon = \mu(A) + \epsilon.$$

Since $\mu(A) < \infty$,

$$\mu(B \setminus A) \leq \epsilon.$$

Then

$$\begin{aligned} \mu(A) \leq \mu(B) = \mu'(B) &= \mu'(A) + \mu'(B \setminus A) \\ &\leq \mu'(A) + \mu(B \setminus A) \leq \mu'(A) + \epsilon. \end{aligned}$$

As $\epsilon \to 0$,

$$\mu(A) \leq \mu'(A).$$

(ii) In general, write $X = \bigcup_{i=1}^{\infty} X_i$ with $X_i \in \mathcal{A}_0$ and $\mu_0(X_n) < \infty$. Without loss of generality[a], X_i are disjoint. Then $A = \bigcup_{i=1}^{\infty} A \cap X_n$ disjoint. Therefore,

$$\mu(A) = \sum_{i=1}^{\infty} \underbrace{\mu(A \cap X_n)}_{<\infty} \overset{(i)}{=} \sum_{i=1}^{\infty} \mu'(A \cap X_n).$$

■

[a] Can write any covering as disjoint covering

Corollary 1.2.1. Let $(X, \mathcal{A}_0, \mu_0)$ be a σ-finite pre-measure space, and $(X, \mathcal{A}, \mu)$ the HK-extension. Then

(i) $\mu|_{\langle\mathcal{A}_0\rangle}$ is the unique extension of μ_0 to $\langle\mathcal{A}_0\rangle$.

(ii) $(X, \mathcal{A}, \mu)$ is the completion of $(X, \langle\mathcal{A}_0\rangle, \mu)$.

Proof. (i) follows from theorem. (ii) is a HW question. ■

Lecture 6: Sixth Lecture

1.3 Borel Measure on the Real Line

Jan. 18:00

Definition 1.3.1. A function $F : \mathbb{R} \to \mathbb{R}$ is a distribution function if F is increasing and right continuous i.e.

$$\lim_{x\to a} = \lim_{x\to a^+} F(x) = F(a).$$

$$\begin{cases} F(\infty) := \lim_{x\to\infty} F(x) \in (-\infty,\infty] \\ F(-\infty) := \lim_{x\to-\infty} F(x) \in [-\infty,\infty) \end{cases} \quad \text{exist.}$$

Remark. cdf = distribution function F with $F(-\infty) = 0$ and $F(\infty) = 1$.

Example. $F(x) = x$, $F(x) = e^x$, ...

Example. $F(x) = H(x) = \begin{cases} 1, & \text{if } x \geq 0 \\ 0, & \text{if } x < 0 \end{cases}$

Example. $\mathbb{Q} = \{r_1, r_2, \ldots\}$, $F(x) = \sum_{n=1}^{\infty} 2^{-n} H(x - r_n)$

Lemma 1.3.1. Let μ be a Borel measure on $\mathbb{R}$ ($\mu : B(\mathbb{R}) \to [0,\infty]$) such that

$$\mu([-N,N]) < \infty, \quad \forall N > 0 \qquad (\mu \text{ locally finite}).$$

Then the function

$$F(x) = F_\mu(x) = \begin{cases} \mu((0,x]), & \text{if } x > 0 \\ 0, & \text{if } x = 0 \\ -\mu((x,0]), & \text{if } x < 0 \end{cases}$$

is a distribution function.

Proof. DYI using continuity from above/below.

> **Note.** F right continuous at a iff for every sequence x_n with $x_n > a$, $x_n \to a$ we have
>
> $$\lim_{n\to\infty} F(x_n) = F(a).$$

■

Answer. First consider $a > 0$. Let $x_n = a + \frac{1}{n}$. Then

$$F(x) = \mu\big((0, a + 1/n]\big).$$

Let $A_n = (0, x_n]$. Note that $\bigcup_{i=1}^{\infty} A_i = (0,a]$ and $A_1 \supset A_2 \supset \cdots$. Since μ is a measure, we have continuitiy from above such that

$$\lim_{n\to\infty} F(x_n) = \mu(\bigcap_{i=1}^{\infty} A_n) = \lim_{n\to\infty} \mu(A_n) = \mu((0,a]) = F(a).$$

For $a < 0$, do same thing but with unions. Works because $\mu([-N,N]) < \infty$, $\forall N > 0$.

⊛

Definition 1.3.2. The following are h-intervals: $(a,b], (-\infty,b], (a,\infty), \emptyset, \mathbb{R}$.

Lemma 1.3.2. If $A_0 = \{$finite disjoint unions of h-intervals$\}$, then

(1) A_0 is an algebra.

(2) $\langle A_0 \rangle = B(\mathbb{R})$.

Proof of (1). DIY. ■

Proof of (2). We did this in Lecture 2, Prop 7. ■

Definition 1.3.3. Given a distribution function F, h-interval I, set

$$\ell_F(I) = \begin{cases} F(b) - F(a), & \text{if } I = (a, b] \\ F(b) - F(\infty), & \text{if } I = (-\infty, b] \\ F(\infty) - F(a), & \text{if } I = (a, \infty] \\ F(\infty) - F(-\infty), & \text{if } I = \mathbb{R} \\ 0, & \text{if } I = \emptyset \end{cases}.$$

Proposition 1.3.1. Given a distribution function F, define $\mu_0 = \mu_{0,F} : \mathcal{A}_0 \to [0, \infty]$ as follows. If $A = \bigcup_{i=1}^n I_i$ is a finite disjoint union of h-intervals, then

$$\mu_0(A) := \sum_{i=1}^N \ell_F(I_i).$$

Then μ_0 is a well-defined σ-finite premeasure on $\mathcal{A}_0$.

Proof that μ_0 is well defined. Want to show that if $\bigcup_{i=1}^m I_i = \bigcup_{k=1}^n J_k$ then

$$\sum_{i=1}^n \ell_F(I_i) = \sum_{i=1}^m \ell_F(J_i).$$

Idea is that you can find a common refinement through intersections. ■

Proof that μ_0 is a premeasure. We must show

(b) $\mu_0(\emptyset) = 0$

(c) Finite additivity

(d) Countable additivity within $\mathcal{A}_0$ i.e.

$$A_1, A_2, \ldots \in \mathcal{A}_0, \quad A := \bigcup_{i=1}^\infty A_i \in \mathcal{A}_0$$
$$\Rightarrow \mu_0(A) = \sum_{i=1}^\infty \mu_0(A_i).$$

(e) μ_0 is σ-finite.

We will only show (d). For (c), it is enough to check the case when A_i are h-intervals.

$$A \supset \bigcup_{i=1}^\infty A_i \overset{(c)}{\Rightarrow} \mu_0(A) \geq \mu_0(\bigcup_{i=1}^n A_i) \overset{(c)}{=} \sum_{i=1}^n \mu_0(A_i).$$

As $n \to \infty$,

$$\mu_0(A) \geq \sum_{i=1}^\infty \mu_0(A_i).$$

To prove the other direction suppose $A = (a, b]$ and $\epsilon > 0$. Write $A_j = (a_j, b_j]$. Since F is right continuous, $\exists \delta > 0$ such that

$$F(a) \leq F(a+\delta) < F(a) + \epsilon$$

and $\forall j, \exists \delta_j > 0$ such that

$$F(b_j) \leq F(b_j + \delta_j) < F(b_j) + \epsilon \cdot 2^{-j}.$$

Now we have that $[a+\delta, b] \subset (a, b]$ and $[a+\delta, b] \subset \bigcup_{j=1}^{\infty}(a_j, b_j + \delta_j)$. By compactness, $\exists N < \infty$ such that $[a+\delta, b] \subset \bigcup_{j=1}^{N}(a_j, b_j)$. Without loss of generality, $I_j \not\subset I_k$ when $j \neq k$. Relabel $b_j + \delta_j \in (a_{j+1}, b_{j+1} + \delta_{j+1})$. Note $a_1 \leq a + \delta$. Then

$$\begin{aligned}
\mu_0(A) &= F(b) - F(a) \\
&< F(b) - F(a+\delta) + \epsilon \\
&\leq \underbrace{F(b_N + \delta_N) - F(a_N)}_{< F(b_N) - F(a_N) + \epsilon \cdot 2^{-N}} + \sum_{j=1}^{N-1} \underbrace{\big(F(a_{j+1}) - F(a_j)\big)}_{\leq F(b_j+\delta_j) - F(a_j) \leq F(b_j) - F(a_j) + \epsilon \cdot 2^{-j}} + \epsilon \\
&\leq \sum_{j=1}^{N} \left(F(b_j) - F(a_j) + \epsilon 2^{-j}\right) + \epsilon \\
&< \sum_{j=1}^{N} \mu_0(A_j) + 2\epsilon, \qquad \epsilon \to 0.
\end{aligned}$$

For (e),

$$\mathbb{R} = \bigcup_{-\infty}^{\infty} (n, n+1].$$

Therefore,

$$\mu_0\big((n, n+1]\big) = F(n+1) - F(n) < \infty.$$

■

Theorem 1.3.1. For every distribution function F there exists a unique Borel Measure μ_F on $\mathbb{R}$ such that

$$\mu_F\big((a, b]\big) = F(b) - F(a), \qquad \forall a < b.$$

Moreover, $\mu_F = \mu_G \Leftrightarrow F = G +$ constant

Proof. Follows from HK. 2nd part DIY. ■

Remark. In fact, HK gives a complete measure extending μ_F. Also denote this μ_F.

Definition 1.3.4. μ_F is called the Lebesgue-Stieltjes measure associated to F.

Remark (Special case)**.** When $F(x) = x$, this is called the Lebesgue measure.

Lecture 7: Seventh Lecture

1.4 Properties of Lebesgue-Stieltjes measures

Jan. 20:00

Notation. $F(a-) := \lim_{x\to a^-} F(x)$.

Lemma 1.4.1. We have the following properties:

(1) $\mu_F\big((a,b)\big) = F(b-) - F(a)$

(2) $\mu_F([a,b]) = F(b) - F(a-)$

(3) $\mu_F(\{a\}) = F(a) - F(a-)$

(4) $\mu_F\big([a,b)\big) = F(a) - F(a-)$

We have similar formulas for unbounded intervals.

Proof of (1). Write $(a,b) = \bigcup_{n=1}^{\infty}(a, b-\frac{1}{n}]$. By continuity from below,

$$\mu_F\big((a,b)\big) = \lim_{n\to\infty} \mu\big((a, b-\frac{1}{n})\big) = \lim_{n\to\infty} F(b-\frac{1}{n}) - F(a) = F(b-) - F(a).$$

■

Proof of (2)-(4). DIY! ■

Example. $F(x) = H(x) = \begin{cases} 1, & \text{if } x \geq 0 \\ 0, & \text{if } x < 0 \end{cases}$. Then $\mu_F(x) = \delta_0$ is Dirac mass at $x = 0$.

$$\delta_0(A) = \begin{cases} 1, & \text{if } 0 \in A \ 1, \\ \text{if } 0 \notin A \end{cases}.$$

Example. Pick $\{x_n\}_{i=1}^{\infty}$, countable (eg $\mathbb{Q}$) and $c_n > 0$ such that $\sum_{i=1}^{\infty} c_n < \infty$. Set

$$F(x) = \sum_{n=1}^{\infty} c_n H(x - x_n).$$

Then

$$\mu_F = \sum_{n=1}^{\infty} c_n \delta_{x_n}, \quad \mu_F(A) = \sum_{n | x_n \in A} c_n.$$

This is the discrete measure.

Example. F continuous iff μ_F is a continuous measure i.e. $\mu_F(\{a\}) = 0$ for every a.

Example. $F(x) = x$, then μ_F is the Lebesgue measure.

Example (Middle thirds Cantor set). Let $C := \bigcap_{n=1}^{\infty} K_n$, where we have

$$
\begin{aligned}
K_0 &:= [0,1] \\
K_1 &:= K_0 \setminus \left(\frac{1}{3}, \frac{2}{3}\right) \\
K_2 &:= K_1 \setminus \left(\frac{1}{9}, \frac{2}{9}\right) \cup \left(\frac{7}{9}, \frac{8}{9}\right) \\
&\ \vdots \\
K_n &:= K_{n-1} \setminus \bigcup_{k=1}^{3^n - 1} \left(\frac{3k+1}{3^{n+1}}, \frac{3^{k+2}}{3^{n+1}}\right).
\end{aligned}
$$

We see that C is uncountable and with $m(C) = 0$. And observe that $x \in C$ if and only if $x = \sum_{n=1}^{\infty} \frac{a_n}{3}$ for some $a_n \in \{0, 2\}$. Hence, we can instead formulate K_n by

$$
K_n = \bigcup_{\substack{a_i \in \{0,2\} \\ 1 \leq i \leq n}} \left[\sum_{i=1}^{\infty} \frac{a_i}{3^i}, \sum_{i=1}^{\infty} \frac{a_i}{3^i} + \frac{1}{3^n}\right].
$$

Remark. Can get interesting measure associated to C. Pick F = Cantor function. See Devil's staircase.

Example (μ_F vs. m). The following:

1. μ_F, m continuous : $\mu_F(\{a\}) = m(\{a\}) = 0$
2. μ_F, m mutually singular:
$$
m(C) = 0, \quad \mu_F(\underbrace{\mathbb{R} \setminus C}_{\text{open}}) = 0.
$$

Problem 1.4.1. How to work with LS measures?

Answer. $\mu = \mu_F$ LS measure, $A \in \mathcal{A}_\mu \supset \mathcal{B}(\mathbb{R})$. By construction,

$$
\mu(A) = \inf\{\sum_{j=1}^{\infty} \underbrace{\mu(I_j)}_{F(b)-F(a)} \mid I_j \text{ h -interval}, A \subset \bigcup_{i=1}^{\infty} I_j\}. \tag{$\star$}
$$

$I_j = (a_j, b_j]$, $-\infty \leq a_j < b_j < \infty$ or $I_j = (a_j, \infty)$ with $-\infty \leq a_j \leq \infty$. ⊛

Lemma 1.4.2. If μ is a LS measure and $A \in \mathcal{A}_\mu$, then

$$
\mu(A) = \inf\left\{\sum_{j=1}^{\infty} \mu\big((a_j, b_j)\big) \mid A \subset \bigcup_{j=1}^{\infty} (a_j, b_j)\right\}, \tag{$\star\star$}
$$

where $-\infty \leq a_j \leq b_j \leq \infty$.

Proof. "$\leq$ " direction follows from subadditity. To prove "$\geq$" way we may assume $\mu(A) < \infty$. Pick

$\epsilon > 0$. Then ($\star$) $\Rightarrow$ can make $A \subset \bigcup_{i=1}^{\infty} I_j, I_j$ h-interval such that

$$\sum_{j=1}^{\infty} \mu(I_j) \leq \mu'(A) + \epsilon.$$

Want $A \subset \bigcup_{j=1}^{\infty} I'_j$ open such that

$$\sum_{j=1}^{\infty} \mu(I'_j) \leq \mu(A) + \epsilon.$$

If $I_j = (a_j, \infty)$, set $I'_j = I_j$ open interval. If $I_f = (a_j, b_j]$, set $I'_j = (a_j, b_j + \delta_j)$ where $\delta_j > 0$ is such that

$$F(b_j + \delta_j) - F(b_j) \leq \frac{\epsilon}{2^j}.$$

Check that this works! ■

Corollary 1.4.1 (Outer regularity). If we have μ, A as above then

$$\mu(A) = \inf \{\mu(U) \mid U \supset A,\ U \text{ open}\}.$$

Proof. "$\leq$ " clear by monotonicity. For "$\geq$", without loss of generality assume $\mu(A) < \infty$. From lemma,

$$A \subset \bigcup_{j=1}^{\infty} (a_j, b_j) =: U$$

such that

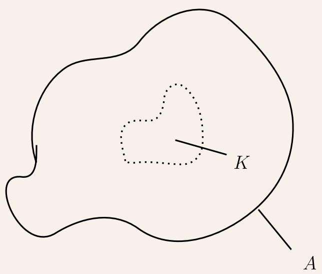
$$\sum_{j=1}^{\infty} \mu\big((a_j, b_j)\big) \leq \mu(A) + \epsilon.$$

By subadditivity,

$$\mu(U) \leq \sum_{j=1}^{\infty} \mu\big((a_j, b_j)\big) \leq \mu(A) + \epsilon.$$

■

Theorem 1.4.1 (Inner regularity). Let μ, A as above.



$$\mu(A) = \sup \{\mu(K) \mid K \subset A \text{ compact}\}.$$

Proof. Set $S := \text{RHS}$. By monotinicity,

$$\mu(A) \geq S.$$

For "$\leq$ " it suffices to show:

1. If $\mu(A) < \infty$, $\forall \epsilon$, $\exists K \subset A$ such that $\mu(K) \geq \mu(A) - \epsilon$.
2. If $\mu(A) = \infty$, $\forall t > 0$, $\exists K \subset A$ such that $\mu(K) \geq t$.

We next need to consider three cases

Case 1: A is bounded say $A \subset [-N, N]$

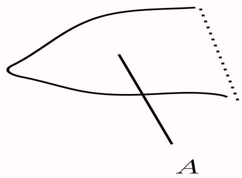


Figure 1.1: bounded-A

$\overline{A}$ closed. Then

$$\begin{cases} \overline{A} \in \mathcal{B}(\mathbb{R}) \subset \mathcal{A}_\mu \\ \overline{A} \subset [-N, N] \text{ so } \mu(\overline{A}) < \infty \end{cases}.$$

Fix $\epsilon > 0$. By outer regularity, $\exists U$ open such that $U \supset \overline{A} \setminus A$ and $\mu(U) \leq \mu(\overline{A} \setminus A) + \epsilon$. Then

$$\begin{aligned} \mu(K) &= \mu(A \setminus U) \\ &= \mu(A) - \mu(A \cap U) \\ &= \mu(A) - \Big(\mu(u) - \mu(U \setminus A)\Big) \\ &\geq \mu(A) - \mu(U) + \mu(\overline{A} \setminus A) \\ &\geq \mu(A) - \epsilon. \end{aligned}$$

Case 2a: A is unbounded but $\mu(A) < \infty$

Pick $\epsilon > 0$. Pick $N > 0$ such that $\mu(A \setminus [-N, N]) < \frac{\epsilon}{2}$. From Case 1, $\exists K \subset A \cap [-N, N]$ such that $\mu(K) \geq \mu(A \cap [-N, N]) - \frac{\epsilon}{2}$. Then $\mu(K) \geq \cdots > \mu(A) - \epsilon$

Case 2b: A is unbounded but $\mu(A) = \infty$
Given $t > 0$ pick N such that $\mu(A \cap [-N, N]) \geq t + 1$. From Case 1, $K \subset A \cap [-N, N] \subset A$ such that $\mu(K) > t$.

■

Lecture 8: Eighth Lecture

Jan. 23:00

Definition. If X is a topological space, then

Definition 1.4.1 (G_δ set)**.** G_δ set is a countable intersection of open sets.

Definition 1.4.2 (F_δ-set)**.** F_δ-set is a countable union of closed sets.

Corollary 1.4.2. If μ is a Lebesgue-Stieltjes measure and $A \in \mathcal{A}_\mu$, then

$$\exists \begin{cases} \text{a } G_\delta - setV \\ \text{a } F_\delta - setH \end{cases}$$

such that

$$\begin{cases} H \subset A \subset V, & \text{and} \\ \mu(V \setminus H) = 0 \end{cases}.$$

Up to null sets and measurable sets is an F_δ and G_δ.

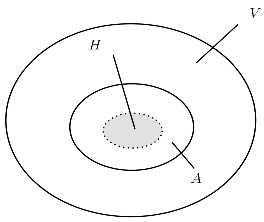


Figure 1.2: corr-sets

Proof. We will proceed by cases:

Case 1: $\mu(A) < \infty$

By outer regularity, $\forall n \geq 1$, $\exists U_n$ open such that $A \subset U_n$ and $\mu(U_n \setminus A) < \frac{1}{n}$. By inner regularity, $\forall n$, $\exists K_n$ compact such that $K_n \subset A$ and $\mu(A \setminus K_n) < \frac{1}{n}$. Set

$$V := \bigcap_{i=1}^{\infty} U_i \qquad (G_\delta)$$
$$H := \bigcup_{i=1}^{\infty} K_i \qquad (F_\delta).$$

Then $V \setminus H \subset U_n \setminus K_n$ for all n. Therefore,

$$\mu(V \setminus H) \leq \mu(U_n \setminus K_n) < \frac{2}{n} \to 0.$$

Case 2: $\mu(A) = \infty$

DYI. Look at $A \cap [-N, N]$.

■

1.5 Lebesgue Measure

Remark (Lebesgue Measure). $m = \mu_F$, $F(x) = x$,

$$m((a,b]) = b - a = m((a,b)) = m([a,b]).$$

Set $\mathcal{L} := \mathcal{A}_m = \{\text{Lebesgue measurable sets}\}$.

1.5.1 Invariant properties

Theorem. Given $E \in \mathcal{L}$ and $r, s \in \mathbb{R}$, set

$$E + s := \{x + s \mid x \in E\} \subset \mathbb{R}$$
$$rE := \{rx \mid x \in E\} \subset \mathbb{R}.$$

Theorem 1.5.1. If $E \in \mathcal{L}$, then $E + s, reE \in \mathcal{L}$, and

$$m(E + s) = m(E), \quad m(rE) = |r| \cdot m(E). \tag{$\star$}$$

Proof idea. Go from intervals to Borel sets to $\mathcal{L}$. From homework 1, if $E \in \mathcal{B}(\mathbb{R})$, then $E + s \in \mathcal{B}(\mathbb{R})$ and $rE \in \mathcal{B}(\mathbb{R})$. Let $\mathcal{A}_0 := \{\text{finite unions of intervals}\} \subset \mathcal{B}(\mathbb{R})$. Then

$$E \in \mathcal{A}_0 \Rightarrow rE, E + s \in \mathcal{A}_0 \quad \text{and} \quad (\star) \text{ holds.}$$

Now use uniqueness in HK-extension in a clever way (HW). ■

1.5.2 Interesting subsets of $\mathbb{R}$

Example. The middle-third Cantor set C is uncountable but $m(C) = 0$.

Example. Vitali set $E \subset [-1, 1]$ (HW3). E not Lebesgue measurable. From HW3, every $A \in \mathcal{L}$ with $m(A) > 0$ contains a non-measurable subset.

Example. Write $\mathbb{Q} \cap [0, 1] = \{r_1, r_2, \ldots\}$ and set

$$U = \bigcup_{i=1}^{\infty} (r_n - \pi^{-n}, r_n + \pi^{-n}) \subset \mathbb{R}.$$

Then U is open, $U \supset \mathbb{Q} \cap [0, 1]$ (which is dense in $[0, 1]$). Set

$$K := [0, 1] \setminus U.$$

Then K is compact and nowhere dense / has no interior (contains no open set). However, we have $m(K) > 0$ (so $K \neq 0$). We can see this by subadditivity i.e.

$$\begin{aligned} m(U) &\leq \sum_{i=1}^{\infty} m\big((r_n - \pi^{-n}, r_n + \pi^{-n})\big) \\ &= \sum_{i=1}^{\infty} 2\pi^{-n} = \frac{2}{\pi - 1} < 1. \end{aligned}$$

Note

$$\begin{aligned}\mu(K) &\geq \mu([0,1]) - m(U)\\ &\geq 1 - m(u) > 0.\end{aligned}$$

Chapter 2

Integration

2.1 Measurable Functions

Lecture 9: Ninth Lecture

Jan. 25:00

Definition 2.1.1. Suppose $(X, \mathcal{A})$, $(Y, \mathcal{B})$ are measurable spaces. Then $f : X \to Y$ is $(\mathcal{A}, \mathcal{B})$-measurable if $B \in \mathcal{B} \Rightarrow f^{-1}(B) \in \mathcal{A}$.

Lemma 2.1.1. If $\mathcal{B} = \langle \mathcal{E} \rangle$, $\mathcal{E} \subset \mathcal{P}(Y)$, then $f : X \to Y$ is $(\mathcal{A}, \mathcal{B})$-measurable iff

$$f^{-1}(B) \in \mathcal{A}, \quad \forall B \in \mathcal{E}.$$

Proof. "$\Rightarrow$" trivial. For the other direction. Let

$$D := \{E \subset Y \mid f^{-1}(E) \in \mathcal{A}\}.$$

Then D is a σ-algebra with $D \supset \mathcal{E}$. Thus,

$$D \supset \langle \mathcal{E} \rangle = \mathcal{B}.$$

■

Definition 2.1.2. Let $(X, \mathcal{A})$ be a measurable space:

(1) $f : X \to \mathbb{R}$ is $\mathcal{A}$-measurable iff it is $(\mathcal{A}, \mathcal{B}(\mathbb{R}))$-measurable.

(2) $f : X \to \mathbb{C}$ is $\mathcal{A}$-measurable iff $\mathrm{Re}(f)$, $\mathrm{Im}(f)$ are $(\mathcal{A}, \mathcal{B}(\mathbb{R}))$-measurable.

(3) $f : X \to \overline{\mathbb{R}} = [-\infty, \infty]$ is $\mathcal{A}$-measurable iff f is $(A, \mathcal{B}(\overline{\mathbb{R}}))$-measurable where

$$\mathcal{B}(\overline{\mathbb{R}}) = \{E \subset \overline{\mathbb{R}} \mid E \cap R \in \mathcal{B}(\mathbb{R})\}.$$

Example. $\mathcal{A} = \mathcal{P}(X)$ then all functions are $\mathcal{A}$-measurable.

Example. $\mathcal{A} = \{\emptyset, X\}$ then only constant functions are $\mathcal{A}$-measurable.

Lemma 2.1.2. If $f : X \to \mathbb{R}$, then the following are equivalent

(1) f is $\mathcal{A}$-measurable.

(2) $\forall a, b \in \mathbb{R}$,

$$\begin{aligned} f^{-1}((a,\infty)) &\in \mathcal{A} \\ f^{-1}([a,\infty)) &\in \mathcal{A} \\ f^{-1}((-\infty,b)) &\in \mathcal{A} \\ f^{-1}((-\infty,b]) &\in \mathcal{A}. \end{aligned}$$

Proof. Use above lemma and note that each of these generate $\mathcal{B}(\mathbb{R})$. DYI. ■

Remark. If X is a topological space, then

$$f : X \to \mathbb{R} \text{ continuous} \Rightarrow f \text{ Borel measurable} \Rightarrow f \text{ Lebesgue measurable}.$$

where Borel measurable is $(\mathcal{B}(X), \mathcal{B}(\mathbb{R}))$-measurable and Lebesgue measurable is $(\mathcal{B}(X), \mathcal{L})$-measurable

Lemma 2.1.3. Suppose $f, g : X \to \mathbb{R}$ are $\mathcal{A}$-measurable.

(a) If $\phi : \mathbb{R} \to \mathbb{R}$ is Borel measurable (e.g. continuous) then $\phi \circ f$ is Borel measurable i.e.

$$f^3, \sin f, \frac{1}{f} \text{ for } f(x) \neq 0.$$

(b) $f + g$, $f \cdot g$ are $\mathcal{A}$-measurable.

Proof of (a). Composition of measurable is measurable. DYI. ■

Proof of $f + g$. By lemma, it suffices to show that $(f+g)^{-1}((a,\infty)) \in \mathcal{A}$. Note

$$(f+g)^{-1}((a,\infty)) = \bigcup_{r\in\mathbb{Q}} \underbrace{f^{-1}((r,\infty)) \cap g^{-1}((a-r,\infty))}_{\mathcal{A}\text{-measurable}}.$$

■

Proof of fg . Follows from the trick:

$$fg = \frac{1}{2}((f+g)^2 - f^2 - g^2).$$

■

Lemma 2.1.4. Suppose $f_n : X \to \overline{\mathbb{R}}$ is $\mathcal{A}$-measurable for $n = 1, 2, \ldots$ Then

(1) $\sup_n f_n$, $\inf_n f_n$, $\limsup_n f_n$, $\liminf_n f_n$ are $\mathcal{A}$-measurable.

(2) If $f(x) = \lim_{n\to\infty} f_n(x)$ exists for every x, then f is $\mathcal{A}$-measurable.

Proof of (1). Use lemma again. Let $a \in \mathbb{R}$. Then

$$\begin{aligned} (\sup_n f_n)^{-1}((a,\infty)) &= \bigcup_{n=1}^{\infty} f_n^{-1}((a,\infty)) \in \mathcal{A} \\ (\inf_n f_n)^{-1}((a,\infty)) &= \bigcap_{n=1}^{\infty} f_n^{-1}((a,\infty)) \in \mathcal{A} \\ \limsup_n f_n &= \inf_n \sup_{m\geq n} f_m \ \mathcal{A}\text{-measurable} \\ \liminf_n f_n &= \sup_n \inf_{m\geq n} f_m \ \mathcal{A}\text{-measurable}. \end{aligned}$$

■

Proof of (2). If f exists then $f = \limsup_n f_n$ and it follows from (1). ■

Corollary 2.1.1. If f, g are $\mathcal{A}$-measurable then $\max\{f, g\}$, $\min\{f, g\}$ are $\mathcal{A}$-measurable.

Definition 2.1.3. Given $f : X \to \overline{\mathbb{R}}$ define

$$f^+(x) := \max\{f(x), 0\}, \qquad f^-(x) := \max\{-f(x), 0\}.$$

Then for every x either $f^+(x) = 0$ or $f^-(x) = 0$ and

$$\begin{aligned} f &= f^+ - f^- \\ |f| &= f^+ + f^-. \end{aligned}$$

Additionally, $f^+, f^-, |f|$ are $\mathcal{A}$-measurable.

Definition 2.1.4. Given $E \subset X$ define

$$\mathbf{1}_E = \chi_E(x) := \begin{cases} 1, & \text{if } x \in E \\ 0, & \text{if } x \notin E \end{cases}.$$

This is the characteristic function or indicator function of E.

Remark. $E \in \mathcal{A} \Leftrightarrow \mathbf{1}_E$ is $\mathcal{A}$-measurable.

Definition 2.1.5 (Simple function)**.** A simple function on $(X, \mathcal{A})$ is a $\mathcal{A}$-measurable function $\phi : X \to \mathbb{C}$ with finite range ($\mathbb{R}$ not $\overline{\mathbb{R}}$). If $\phi(X) = \{c_1, c_2, \ldots, c_n\} \subset \mathbb{C}$ then

$$\phi = \sum_{n=1}^{N} c_n \cdot \mathbf{1}_{E_n},$$

where $E_n = \{x \mid \phi(x) = c_n\} \subset X$.

Lemma 2.1.5. If $\phi, \psi : X \to \mathbb{C}$ are simple then

(a) $\phi + \psi$, $\phi \cdot \psi$, $c\phi$ for $c \in \mathbb{C}$ are simple functions.

(b) $\max\{\phi, \psi\}$ and $\min\{\phi, \psi\}$ are simple.

Proof. DYI. ■

Theorem 2.1.1. Let $(X, \mathcal{A})$ be a measurable space, and $f : X \to [0, \infty]$ any function. The following are equivalent:

(a) f is $\mathcal{A}$-measurable.

(b) f is an increasing limit of simple functions i.e.

$$\exists f_n : X \to [0, \infty) \text{ simple}$$

such that $f_n(x) \leq f_{n+1}(x)$ and $f_n(x) \to f(x)$ for every x.

Sketch. $(b) \Rightarrow (a)$ follows from previous lemma. For $(a) \Rightarrow (b)$ picture: ■

Lecture 10: Tenth Lecture

Jan. 25:00

Theorem 2.1.2. Let $(X, \mathcal{A})$ be a measurable space. Then any $\mathcal{A}$-measurable function $f : X \to [0, \infty]$ is the pointwise limit of an increasing sequence $(\phi_n)_{n=1}$ of simple functions such that

$$\phi_n(x) \leq \phi_{n+1}(x) \text{ and } \phi_n(x) \to f(x) \text{ for every } x.$$

Proof. Set $F_n := f^{-1}([2^n, \infty])$ and $E_{n,k} := f^{-1}((k \cdot 2^{-n}, (k+1)2^{-n}))$ for $0 \leq k \leq 2^{2n} - 1$ and

$$\phi_n := 2^n \mathbf{1}_{F_n} + \sum_{k=0}^{2^{2n}-1} k \cdot 2^{-n} \mathbf{1}_{E_{n,k}}.$$

Check (HW):

$$\begin{cases} 0 \leq \phi_n \leq \phi_{n+1}, \\ 0 \leq f - \phi_n \leq 2^{-n} \text{ on } X \setminus \underbrace{F_n}_{f \leq 2^n} \end{cases}. \tag{$\star$}$$

■

Remark. ($\star$) implies that $\phi_n \to f$ uniformly on any set where f is bounded i.e.

$$f \leq C \text{ on } A \Rightarrow \sup_A |f - \phi_n| \to 0.$$

Corollary 2.1.2. If $f : X \to \overline{\mathbb{R}}$ or ($f : X \to \mathbb{C}$) is $\mathcal{A}$-measurable, then there is a sequence of simple functions $\phi_n : X \to \mathbb{R}$ ($X \to \mathbb{C}$) that converges pointwise to f, and

(a) $|\phi_n(x)| \leq |\phi_{n+1}(x)| \leq |f(x)|$ for $x \in X$,

(b) $\phi_n \to f$ uniformly on any set where f is bounded.

Sketch. If $f : X \to \overline{\mathbb{R}}$, write $f = f^+ - f^-$. Apply theorem to f^+, f^- i.e.

$$\phi_n^+ \to f^+ \text{ and } \phi_n^- \to f^-$$

then $\phi = \phi_n^+ - \phi_n^- \to f$. If $f : X \to \mathbb{C}$, write $f = \operatorname{Re} f + i \operatorname{Im} f$ and approximate $\operatorname{Re} f$ and $\operatorname{Im} f$. ■

2.2 Integration of nonnegative functions

Definition. Let $(X, \mathcal{A}, \mu)$ be a measure space.

Definition 2.2.1. $L^+ = L^+(X, \mathcal{A}) = \{\mathcal{A}\text{-measurable functions } f : [0, \infty]\}$. Want to define $\int f d\mu = \int f \in [0, +\infty]$.

Definition 2.2.2. If $\phi = \sum_{i=1}^{n} c_i \mathbf{1}_{E_i}$ is a simple function in L^+, set

$$\int \phi := \sum_{i=1}^{n} c_i \mu(E_i).$$

Note. If we get $0 \cdot \infty$, then $0 \cdot \infty = 0$.

If $A \in \mathcal{A}$ set $\int_A \phi := \int \underbrace{\phi \cdot \mathbf{1}_A}_{\text{simple fn}}$

Lemma 2.2.1. $\int \phi$, $\int_A \phi$ do not depend on representation of ϕ as a simple function.

Proof. DIY. ∎

Proposition 2.2.1. If $\phi, \psi \in L^+$ are simple functions, then:

(a) $\int c \cdot \phi = c \cdot \int \phi$ for $c \geq 0$.

(b) $\int (\phi + \psi) = \int \phi + \int \psi$.

(c) $\phi \leq \psi \Rightarrow \int \phi \leq \int \psi$.

(d) $A \mapsto \int_A \phi$ is a measure on $(X, \mathcal{A})$

Proof. DIY. ∎

Definition 2.2.3. If $f \in L^+$, then we set

$$\int f = \sup \left\{ \int \phi \;\middle|\; \phi \text{ simple}, \phi \leq f, \phi \geq 0 \right\}.$$

Proposition 2.2.2. $\int f$ is unambiguously defined when $f \in L^+$ is simple.

Proof. It is clear that if $f, g \in L^+$ and $f \leq g$ then $\int f \leq \int g$ and for $f \in L^+$ with $c \geq 0$ then $\int cf = c \int f$. It is less clear that

$$\int (f + g) = \int f + \int g.$$

∎

Theorem 2.2.1 (Monotone convergence theorem)**.** If $f_n \in L^+$ satisfies $f_n(x) \leq f_{n+1}(x)$ for every x and $f(x) := \lim_{n\to\infty} f_n(x) \in [0, \infty]$, then $f \in L^+$, and

$$\int f = \lim_{n\to\infty} \int f_n.$$

Proof. We know that $f \in L^+$. Since $f_n \leq f$ for every n,

$$\int f_n \leq \int f \Rightarrow \lim_{n\to\infty} \int f_n \leq \int f.$$

For the reverse inequality, it suffices to show that for some simple function $0 \leq \phi \leq f$ then

$\lim_{n\to\infty}\int f_n \geq \int \phi$. Fix ϕ. It suffices to show that $\forall \alpha \in (0,1)$

$$\lim_{n\to\infty}\int f_n \geq \alpha \cdot \int \phi.$$

Fix α. Set

$$E_n := \{x \mid f_n(x) \geq \alpha \cdot \phi(x)\} \subset X.$$

Then $E_n \in \mathcal{A}$ (superlevel set of measurable function). We have that $E_n \subset E_{n+1}$ and $\bigcup_{n=1}^{\infty} E_n = X$. CHECK. Also:

$$\int f_n \geq \int f_n \cdot \mathbf{1}_{E_n} \geq \int \alpha\phi \cdot \mathbf{1}_{E_n} = \alpha \int_{E_n} \phi.$$

Now $A \mapsto \int_A \phi$ is a measure on $(X, \mathcal{A})$. By continuity from below,

$$\lim_{n\to\infty}\int_{E_n} \phi = \int_X \phi = \int \phi.$$

Then

$$\lim_{n\to\infty}\int f_n \geq \lim_{n\to\infty} \alpha \int_{E_n} \phi = \alpha \int \phi.$$

■

Example. $X = \mathbb{N}$, $\mathcal{A} = \mathcal{P}(\mathbb{N})$, $\mu =$ counting measure. Then

$$L^+ = \{f : \mathbb{N} \to [0,\infty]\}$$
$$\int f = \sum_{j=1}^{\infty} f(j) \in [0,\infty].$$

By MCT, $f_n(j) \to f(j)$ for every j. Therefore,

$$\sum_{j=}^{\infty} f_n(j) \to \sum_{j=1}^{\infty} f(j).$$

Corollary 2.2.1. If $f, g \in L^+$, then $\int (f+g) = \int f + \int g$.

Proof. Pick ϕ_n, ψ_n simple such that $\phi_n \to f$ and $\psi_n \to g$. Then

$$\underbrace{\phi_n + \psi_n}_{\text{simple}} \to f + g.$$

Therefore,

$$\int (f+g) \overset{\text{MCT}}{=} \lim_n \int (\phi_n + \psi_n) = \lim_n \int \phi_n + \lim_n \int \psi_n \overset{\text{MCT}}{=} \int f + \int g.$$

■

Corollary 2.2.2 (Tonelli for sums and integrals)**.** If $f_j \in L^+$ for $j \in \mathbb{N}$, then

$$\int \underbrace{\sum_{j=1}^{\infty} f_j}_{\in L^+} = \sum_{j=1}^{\infty} \int f_j.$$

Proof. Apply MCT to $\sum_{j=1}^{n} f_j$ for $n \in \mathbb{N}$. ∎

Lecture 11: Eleventh Lecture

Jan. 25:00

Corollary 2.2.3. $L^+ \supset f \mapsto \int f$ is $\mathbb{R}_{\geq 0}$-linear:

$$f, g \in L^+, a, b > 0 \Rightarrow \int (af + bg) = a \int f + b \int g.$$

Theorem 2.2.2 (Fatou's lemma)**.** If $f_n \in L^+$, $n \in \mathbb{N}$, then

$$\int \liminf_n f_n \leq \liminf_n \int f_n.$$

Proof. Set $g_m := \inf_{m \geq n} f_m$. Then $g_n \in L^+$ and $g_n \to \liminf_n f_n$. By MCT,

$$\int \liminf_n f_n = \lim_n \int g_n.$$

Since $g_n \leq f_n$,

$$\underbrace{\liminf_n \int g_n}_{=\lim \int g_n} \leq \liminf_n \int f_n.$$

∎

Example. $(\mathbb{R}, \mathcal{L}, m)$. Can have: $f_n \geq 0$, $f_n \to 0$ pointwise, but $\int f_n = 1$ for every n.

a) (Escape to horizontal infinity). $f_n = \mathbf{1}_{(n, n+1)}$

b) (Escape to width infinity). $f_n = \frac{1}{n} \cdot \mathbf{1}_{(0,n)}$

c) (Escape to vertical infinity). $f_n = n \cdot \mathbf{1}_{(0, \frac{1}{n})}$

Remark. Next we will replcae pointwise by almost everywhere.

Lemma 2.2.2 (Chebyshev's inequality)**.** If $f \in L^+$ and $c > 0$, then

$$\mu(\{x \in X \mid f(x) \geq c\}) \leq \frac{1}{c} \int f.$$

Proof. Define $E := \{x \in X \mid f(x) \geq c\}$. Then

$$\int f \geq \int f \cdot \mathbf{1}_E \geq \int c \cdot \mathbf{1}_E = c \cdot \mu(E).$$

∎

Proposition 2.2.3. If $f \in L^+$, then the following are equivalent:

(i) $\int f = 0$.

(ii) $f = 0$ almost everywhere i.e. $\mu(\{x \in X \mid f(x) > 0\}) = 0$.

Proof.

(i) $\Rightarrow$ (ii) . Set $A_n := \{x \in X \mid f(x) \geq \frac{1}{n}\}$. By Chebyshev,

$$\mu(A_n) \leq n \int f.$$

Since $A_n \subset A_{n+1}$ and $\bigcup A_n = A$ so

$$\lim_{n\to\infty} \mu(A_n) = \mu(\bigcup_{n=1}^{\infty} A_n) = \mu(A) = 0.$$

(ii) $\Rightarrow$ (i) . First assume $f = \sum_{i=1}^{n} c_i \mathbf{1}_{E_i}$. Without loss of generality, assume that $f \neq 0$, $0 < c_1 < \cdots < c_n$, and $A = \bigcup_{i=1}^{\infty} E_i$ disjoint. Then

$$f = 0 \text{ a.e.} \Rightarrow \mu(E_i) = 0 \Rightarrow \int f = \sum_{i=1}^{n} c_i \mu(E_i) = 0.$$

In general, pick any ϕ simple with $0 \leq \phi \leq f$. Then $\{\phi = 0\} \supset \{f = 0\}$, so $\phi = 0$ almost everywhere. Therefore, $\int \phi = 0$ from above. Then

$$\int f = \sup\left\{\int \phi \;\middle|\; 0 \leq \phi \leq f,\ \phi \text{ simple}\right\} = 0.$$

■

Corollary 2.2.4. If $f, g \in L^+$ and $f = g$ almost everywhere, then $\int f = \int g$.

Proof. Set $A := \{f \neq g\}$ (measurable). Then $\mu(A) = 0$ and $f \cdot \mathbf{1}_{A^c} = g \cdot \mathbf{1}_{A^c}$. Therefore,

$$\int f = \int f \cdot \mathbf{1}_A + \int f \cdot \mathbf{1}_{A^c}.$$

Since $f \cdot \mathbf{1}_A = g \cdot \mathbf{1}_A = 0$ almost everywhere, $\int f \cdot \mathbf{1}_A = \int g \cdot \mathbf{1}_A = 0$. Therefore,

$$\int f = \int f \cdot \mathbf{1}_{A^c} = \int g \cdot \mathbf{1}_{A^c} = \int g.$$

■

Corollary 2.2.5 (MCT, FL a.e. versions)**.** Suppose $f_n, f \in L^+$. Then

(a) If $f_n(x) \to f(x)$ for almost every x, then $\lim \int f_n = \int f$.

(b) If $f_n(x) \to f(x)$ for almost every x, then $\liminf_{n\to\infty} \int f_n \geq \int f$.

Proof. DIY. Set "bad sets" to $+\infty$. ■

Proposition 2.2.4. If $f \in L^+$ and $\int f < \infty$ then

(a) $\{f = \infty\} = \{x \in X \mid f(x) = \infty\}$ is a null set.

(b) $\{f > 0\} = \{x \in X f(x) > 0\}$ is σ-finite.

Proof. Set $A_t = \{f > t\} \subset X$ for $0 < t < \infty$. By Chebyshev,

$$\mu(A_t) \leq \frac{1}{t} \int f.$$

(a) $\{f = \infty\} = \bigcap_{n=1}^{\infty} A_n$ decreasing. Therefore,

$$\mu(\{f=\infty\}) = \lim_{n\to\infty} \underbrace{\mu(A_n)}_{\le \frac{1}{n}\int f} = 0.$$

(b) $\{f > 0\} = \bigcup_{n=1}^{\infty} A_{\frac{1}{n}}$ and

$$\mu(A_{\frac{1}{n}}) \le n \int f < \infty.$$

■

2.3 Integration of $\mathbb{R}$ and $\mathbb{C}$ valued functions

Remark. $(X, \mathcal{A}, \mu)$ measure space. $f : X \to \mathbb{R}$ measurable. We want to define

$$\int f = \int f d\mu.$$

Remark. We write $f = f^+ - f^-$ for $f^+, f^- \in L^+$. Set

$$\int f = \int f^+ - \int f^-.$$

Need to avoid $\infty - \infty$.

Lecture 12: Twelth Lecture

Feb. 1:00

Definition 2.3.1. If $X \to \overline{\mathbb{R}}$ is measurable, set

$$\int f := \int f^+ - \int f^-$$

assuming $\int f^+ < \infty$ or $\int f^- < \infty$. This holds if

$$\int |f| = \int f^+ + f^- < \infty.$$

Definition 2.3.2 (Integrable). Say $f : X \to \mathbb{R}$ is measurable. Then f is integrable if $\int |f| < \infty$.

Notation. Temporary notation is $f \in \tilde{L}(X, \mu; \mathbb{R})$.

Proposition 2.3.1. $\tilde{L}(X, \mu, \mathbb{R})$ is an $\mathbb{R}$-vector space and $f \in \tilde{L}(X, \mu; \mathbb{R}) \mapsto \int f \in \mathbb{R}$ is $\mathbb{R}$-linear.

Proof. DIY. (Prop 2.21 in Folland) ■

Definition 2.3.3. $f : X \to \mathbb{C}$ is integrable if $\int |f| < \infty$. Write $f \in \tilde{L}(X, \mu; \mathbb{C})$.

$$\max\{\operatorname{Re} f, \operatorname{Im} f\} \le |f| \le |\operatorname{Re} f| + |\operatorname{Im} f|.$$

Therefore, f is integrable if and only iff $\operatorname{Re} f$, $\operatorname{Im} f$ integrable. Set

$$\int f := \int \operatorname{Re} f + i \int \operatorname{Im} f.$$

Corollary 2.3.1. $\tilde{L}(X,\mu,\mathbb{R})$ is an $\mathbb{R}$-vector space and $f \in \tilde{L}(X,\mu;\mathbb{C}) \mapsto \int f \in \mathbb{C}$ is $\mathbb{C}$-linear.

Proof. DIY. ∎

Proposition 2.3.2. If $f \in \tilde{L}(X,\mu;\mathbb{C})$, then

$$\left|\int f\right| \leq \int |f|.$$

Proof. First suppose that f is real valued. Then

$$\left|\int f\right| = \left|\int f^+ - \int f^-\right| \leq \left|\int f^+\right| + \left|\int f^-\right| = \int f^+ + \int f^- = \int |f|.$$

In general, may assume $\int f \neq 0$.

$$\alpha := \overline{\int f \Big/ \int |f|} \in \mathbb{C}, \quad \text{where } |\alpha| = 1.$$

Then

$$\left|\int f\right| = \alpha \int f = \int \alpha f,$$

so $\int \alpha f > 0$. Therefore,

$$\left|\int f\right| = \operatorname{Re} \int \alpha f = \int \operatorname{Re}(\alpha f) \underbrace{\leq}_{\mathbb{R}\text{-case}} \int \underbrace{|\operatorname{Re} \alpha f|}_{\leq |f|} \leq \int |f|.$$

∎

Proposition 2.3.3. If $f \in \tilde{L}(X,\mu;\mathbb{C})$ then $X \supset \{f \neq 0\}$ is σ-finite.

Proof. Did this when $f \geq 0$. In general, use $\{f \neq 0\} = \{|f| \neq 0\}$. ∎

Proposition 2.3.4. If $f, g \in \tilde{L}(X,\mu;\mathbb{C})$ then the following are equivalent:

(i) $f = g$ almost everywhere.

(ii) $\int |f - g| = 0$.

(iii) $\int_E f = \int_E g$ for every $E \in \mathcal{A}$.

Proof. From last time, (i) ⇔ (ii) (apply to $|f - g| \geq 0$). For (ii) ⇒ (iii),

$$\begin{aligned} \left|\int_E f - \int_E g\right| &= \left|\int \mathbf{1}_E (f - g)\right| \\ &\leq \int |\mathbf{1}_E (f - g)| \\ &\leq \int |f - g| = 0. \end{aligned}$$

Now for (iii) $\Rightarrow$ (i), write $f - g = u + iv = (u^+ - u^-) + i(v^+ - v^-)$. Set

$$E \coloneqq \{u^+ > 0\}.$$

Then

$$\begin{cases} u^+ = 0 & \text{on } E^c \\ u^- = 0 & \text{on } E \end{cases}.$$

Therefore,

$$0 = \operatorname{Re}\left(\int_E f - \int_E g\right) = \int_E u^+ = \int u^+.$$

By last time, if $\int u^+ = 0$ then $u^+ = 0$ almost everywhere. Similarly,

$$u^- = v^+ = v^- = 0 \text{ almost everywhere.}$$

■

Remark. Can integrate almost everywhere defined functions i.e. f is defined on E^c, where E is a null set. Exten f to $\tilde{f} : X \to \mathbb{C}$ by $\tilde{f}|_E \equiv 0$. Then define

$$\int f \coloneqq \int \tilde{f}.$$

This allows us to integrate $\overline{\mathbb{R}}$-valued functions as long as $\mu\{f = \pm\infty\} = 0$.

Proof. (HW). ■

Definition 2.3.4. Define $L^1 \coloneqq L^1(X, \mu; \mathbb{C}) \coloneqq \tilde{L}(X, \mu; \mathbb{C})/\sim$, where

$$f \sim g \Leftrightarrow f = g \text{ a.e.}$$

Lemma 2.3.1. Suppose that f_n for $n \in \mathbb{N}$ are almost everywhere defined functions measurable functions on X such that $f(x) \coloneqq \lim_{n\to\infty} f_n(x)$ exists for almost every x. Then f is measurable (up to redefinition on a null set).

Proof. See HW for the proof. ■

Theorem 2.3.1 (Dominated Convergence Theorem)**.** Let $(f_n)_n$ be a sequence in L^1 such that

(a) $f_n \to f$ almost everywhere.

(b) $\exists g \in L^1$ with $g \geq 0$ such that $|f_n| \leq g$ almost everywhere for every n.

Then $f \in L^1$ and $\int f = \lim \int f_n$.

Proof. From Lemma 2.3.1, f is measurable. Also, $|f_n| \leq g$ almost everywhere implies that $|f| \leq g$ almost everywhere. To prove $\int f_n \to \int f$, we may assume that f_n, f are $\mathbb{R}$-valued. Since $|f_n| \leq g$ almost everywhere, $g \pm f_n \geq 0$ almost everywhere. Apply Fatou's lemma,

$$\begin{aligned} \int g + \int f &= \int \lim(g + f_n) \\ &\leq \liminf \int (g + f_n) \\ &= \int g + \liminf \int f_n. \end{aligned}$$

Similarly,

$$\begin{aligned}\int g - \int f = \int \lim(g - f_n) \\ &\leq \liminf \int (g - f_n) \\ &= \int g - \limsup \int f_n.\end{aligned}$$

Since $\int g < \infty$,

$$\limsup \int f_n \leq \int f \leq \liminf \int f_n.$$

Thus,

$$\lim \int f_n = \limsup \int f_n = \int f = \liminf \int f_n.$$

■

Lecture 13: Thirteenth Lecture

3 Feb. 11:00

Example. Compute $I = \lim_{n\to\infty} \int_0^1 \underbrace{\frac{1 + nx^2}{(1 + x^2)^n}}_{f_n} dx$. Then

$$f_n(x) \to \begin{cases} 0, & \text{if } x > 0 \\ 1, & \text{if } x = 0 \end{cases}.$$

Because,

$$\left(1 + x^2\right)^n \geq 1 + nx^2 + \binom{n}{2} x^4.$$

By DCT, we need some g such that $|f_n| \leq g$ and $\int_0^1 g < \infty$. Can use $g = 1$.

Remark. In the "escape to ∞" examples, condition in DCT not satisfied.

Corollary 2.3.2 (Fubini's Theorem for integrals and series). Suppose $f_n \in L^1$ for $n \in \mathbb{N}$ such that $\sum_{n=1}^{\infty} |f_n| < \infty$. Then

(i) $\sum_{n=1}^{\infty} f_n(x)$ converges almost everywhere to a function in L^1.

(ii) $\int \sum_{n=1}^{\infty} f_n = \sum_{n=1}^{\infty} \int f_n$

Proof. Set $F_n := \sum_{i=1}^{n} f_i$, $G := \sum_{i=1}^{\infty} |f_i|$ (measurable and $\int G < \infty$ by Tonelli). Therefore, $G(x) < \infty$ for almost every x. It follows than $F_n(x)$ converges almost everywhere to $F(x) := \sum_{i=1}^{\infty} f_i(x)$. By construction, $F_n \leq G$. Thus, by DCT

$$\int F_n \to \int F$$

as desired. ■

Corollary 2.3.3. Suppose $f : X \times (a, b) \to \mathbb{C}$ satisfies $f(\cdot, t) \in L^1(\mu)$ for every t. Set

$$F(t) := \int f(x,t) d\mu(x).$$

(a) Suppose $t \mapsto f(x,t)$ continuous for every (x,t) and $\exists g \in L^1(\mu)$ such that $|f(x,t)| \leq g$ for every t, x. Then F is continuous.

(b) Suppose $\frac{\partial f}{\partial t}(x,t)$ exists for every (x,t) and $\exists g \in L^1(\mu)$ such that $|\frac{\partial f}{\partial t}(x,t)| \leq g(x)$ for every x, t. Then F is differentiable and

$$F'(t) = \int \frac{\partial f}{\partial t}(x,t) d\mu(x).$$

Proof of (a). Consider any $t \in (a, b)$ and any sequence $b_n \to t$. Sufficient to show that $F(t_n) \to F(t)$. Apply DCT with

$$\begin{cases} f_n(x) = f(x, t_n), \\ f(x) = f(x,t) \end{cases}.$$

By continuity of $t \mapsto f(x,t)$, we know that $f_n \to f$. Since $|f_n| \leq g$, by DCT

$$F(t_n) = \int f_n(x) \to \int f(x) = F(t).$$

■

Proof of (b). Pick $t_n \to t$ as above. Apply DCT to

$$\begin{cases} h_n(x) = \frac{f(x,t_n) - f(x,t)}{t_n - t}, \\ h(x) = \frac{\partial f}{\partial t}(x,t) \end{cases}.$$

We know that $h_n(x) \to h(x)$ for every x. Therefore, $x \mapsto \frac{\partial f}{\partial t}(x,t)$ is measurable. By Mean Value Theorem,

$$|h_n(x)| \leq \sup_s \left| \frac{\partial f}{\partial t}(x,s) \right| \leq g(x).$$

By DCT,

$$\frac{F(t_n) - F(t)}{t_n - t} \to \int \frac{\partial f}{\partial t}(x,t) d\mu(x).$$

■

Remark. L^1 is a $\mathbb{C}$-vector space. For $f \in L^1$ set $\|f\| := \int |f|$.

Proposition 2.3.5. $\|\cdot\| : L^1 \to \mathbb{R}_{\geq 0}$ is well-defined, and

(i) $\|af\| = |a| \cdot \|f\|$ for $a \in \mathbb{C}$, $f \in L^1$,

(ii) $\|f + g\| \leq \|f\| + \|g\|$, for $f, g \in L^1$,

(iii) $\|f\| = 0 \Leftrightarrow f = 0$ almost everywhere.

Proof. DIY. ■

Remark. Say $(L^1, \|\cdot\|)$ is a normed vector space.

Remark. Define $\rho(f, g) \coloneqq \|f - g\| = \int |f - g|$ defines a metric on L^1.

Corollary 2.3.4. $(L^1, \|\cdot\|)$ is complete i.e. every Cauchy sequence converges.

Remark. We say that $(L^1, \|\cdot\|)$ is a Banach space.

Proof sketch. Let $(f_n)^\infty$ be a Cauchy sequence in L^1 i.e. $\forall \epsilon$, $\exists N$ such that $\forall m, n \geq n$, $\|f_m - f_n\| < \epsilon$. Therefore, we can find $n_1 < n_2 < n_3 < \cdots$ such that

$$m, n \geq n_j \Rightarrow \|f_m - f_n\| \leq 2^{-j}.$$

Set $g_j \coloneqq f_{n_j} - f_{n_{j-1}}$ for $j \geq 2$, $g_1 = f_{n_1}$. Then

$$\sum_{j=1}^{\infty} \int \underbrace{|g_j|}_{\|f_{n_j} - f_{n_{j-1}}\|} < \infty.$$

By Fubini for series, $f \coloneqq \lim_{j\to\infty} \sum_{i=1}^{j} g_i = f_{n_j}$ exists almost everywhere and $f \in L^1$. Can show that $f_n \to f$ in L^1 i.e. $\int |f_n - f| \to 0$. ■

Theorem 2.3.2 (Density of simple functions). If $f \in L^1$ and $\epsilon > 0$ then $\exists \phi : X \to \mathbb{C}$ simple such that

$$\int |f - \phi| < \epsilon.$$

Proof. Have seen $\exists \phi_n$ simple such that

$$|\phi_n| \leq |\phi_{n+1}| \leq |f| \quad \text{and} \quad \phi_n \to f \text{ a.e.}$$

Then $|\phi_n - f| \leq 2|f|$ and $\int 2|f| < \infty$. By DCT,

$$\int |\phi_n - f| \to 0.$$

■

Theorem. Special case $X = \mathbb{R}$.

Theorem 2.3.3. If μ is a Lebesgue-stieltjes measure on $\mathbb{R}$ and $f \in L^1(\mu)$, $\epsilon > 0$, then $\exists$ a step function

$$\phi = \sum_{i=1}^{n} c_i \cdot \mathbf{1}_{I_i}, \quad I_i \text{ given intervals,}$$

and $\int |f - \phi| < \epsilon$.

Lecture 14: Fourteenth Lecture

2.4 L^1 Space

7 Feb. 11:00

We now introduce another space called L^p *spaces*, which are function spaces defined using a natural generalization of the p-norm for finite-dimensional vector spaces. We sometimes call it Lebesgue spaces also.

Before we start, we need to define a *norm*.

Note. Recall that we say f is integrable means

$$\int |f| < \infty,$$

and if $f = g$ a.e., then

$$\int f = \int g$$

Remark. We have

- With ??, $L^1(X, \mathcal{A}, \mu)$ is a normed vector space.
- We say that the L^1-*metric* $\rho(f, g)$ is simply

$$\rho(f, g) = \int |f - g| \, .$$

Dense Subsets of L^1

Note. Recall the definition of a *dense set*[a].

[a] https://en.wikipedia.org/wiki/Dense_set

Definition 2.4.1 (Step function). A *step function* on $\mathbb{R}$ is

$$\psi = \sum_{i=1}^{N} c_i \mathbb{1}_{I_i},$$

where I_i is an interval.

Notation. We denote the collection of continuous functions with compact support by $C_c(\mathbb{R})$.

Theorem 2.4.1. We have the following.

(1) {integrable simple functions} is dense in $L^1(X, \mathcal{A}, \mu)$ (with respect to L^1-metric).

(2) $(X, \mathcal{A}, \mu) = (\mathbb{R}, \mathcal{A}_\mu, \mu)$, where μ is a Lebesgue-Stieltjes-measure. Then the set of integrable simple functions is dense in $L^1(\mathbb{R}, \mathcal{A}_\mu, \mu)$.

(3) $C_c(\mathbb{R})$ is dense in $L^1(\mathbb{R}, \mathcal{L}, m)$.

Proof. We prove this one by one.

(1) Since there exists simple functions $0 \leq |\phi_1| \leq |\phi_2| \leq \ldots \leq |f|$, where $\phi_n \to f$ pointwise. Then by ??, we have

$$\lim_{n \to \infty} \int \underbrace{|f_n - f|}_{\leq |\phi_n| + |f| \leq 2|f|} = 0$$

where $2|f|$ is in L^1.

(2) Let $\mathbb{1}_E$ approximate by $\sum_{i=1}^{\infty} c_i \mathbb{1}_{I_i}$. From ?? for Lebesgue-Stieltjes-measure,

$$\forall \epsilon' > 0 \; \exists I = \bigcup_{i=1}^{N} I_i \text{ such that } \mu(E \triangle I) \leq \epsilon'.$$

(3) To approximate $\mathbb{1}_{(a,b)}$, we simply consider function $g \in C_c(\mathbb{R})$ such that

$$\int \left|\mathbb{1}_{(a,b)} - g\right| \leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

■

2.5 Riemann Integrability

We are now working in $(\mathbb{R}, \mathcal{L}, m)$. Let's first revisit the definition of Riemann Integral. Let P be a partition of $[a, b]$ as

$$P = \{a = t_0 < t_1 < \ldots < t_k = b\}.$$

Then the *lower Riemann sum* of f using P is equal to L_P, which is defined as

$$L_P = \sum_{i=1}^{K} \left(\inf_{[t_{i-1}, t_i]} f\right)(t_i - t_{i-1}),$$

and the *upper Riemann sum* of f using P is equal to U_P, which is defined as

$$U_P = \sum_{i=1}^{K} \left(\sup_{[t_{i-1}, t_i]} f\right)(t_i - t_{i-1}).$$

Then we call

- *Lower Riemann integral* of $f = \underline{I} = \sup_P L_P$
- *Upper Riemann integral* of $f = \overline{I} = \inf_P U_P$

Definition 2.5.1 (Riemann (Darboux) integrable)**.** A <u>bounded</u> function $f\colon [a, b] \to \mathbb{R}$ is called *Riemann (Darboux) integrable* if $\underline{I} = \overline{I}$. If so, then we write

$$\underline{I} = \overline{I} = \int_a^b f(x)\,\mathrm{d}x.$$

Note. We see that

- If $P \subset P'$, then
$$L_P \leq L_{P'}, \quad U_{P'} \leq U_P.$$
- Recall that continuous functions on $[a, b]$ are Reimann integrable on $[a, b]$.

Theorem 2.5.1. Let $f\colon [a, b] \to \mathbb{R}$ be a <u>bounded</u> function. Then

(1) If f is Reimann integrable, then f is Lebesgue measurable, thus Lebesgue integrable. Further,

$$\int_a^b f(x)\,\mathrm{d}x = \int_{[a,b]} f\,\mathrm{d}m.$$

(2) If f is Reimann integrable $\Leftrightarrow$ f is continuous <u>Lebesgue a.e.</u>[a]

[a] Here, we mean that the set where f is discontinuous is a null set under Lebesgue measure.

Proof. There exists $P_1 \subset P_2 \subset \ldots$ such that $L_{P_n} \nearrow \underline{I}$ and $U_{P_n} \searrow \overline{I}$.

Note. Here, we took refinements of P_n if needed.

Now, define simple (step) functions

- $\phi_n = \sum_{i=1}^{K} \left(\inf_{[t_{i-1}, t_i]} f \right) \mathbb{1}_{(t_{i-1}, t_i]}$
- $\psi_n = \sum_{i=1}^{K} \left(\sup_{[t_{i-1}, t_i]} f \right) \mathbb{1}_{(t_{i-1}, t_i]}$

if $P_n = \{a = t_0 < t_1 < \ldots < t_K\}$. Let $\phi \coloneqq \sup_n \phi_n$ and $\psi \coloneqq \inf_n \psi_n$. We then see that ϕ, ψ are Lebesgue (Borel) measurable function.

Note. Note that

- $\exists M > 0$ such that $\forall_{n \in \mathbb{N}} \ |\phi_n|, |\psi_n| \leq M \mathbb{1}_{[a,b]}$
- $\int \phi_n \, \mathrm{d}m = L_{P_n}, \int \psi_n \, \mathrm{d}m = U_{P_n}$

By **??** and the fact that $M \mathbb{1}_{[a,b]} \in L^1(\mathbb{R}, \mathcal{L}, m)$, we have

$$\underline{I} = \lim_{n \to \infty} \int \phi_n \, \mathrm{d}m = \int \phi \, \mathrm{d}m, \quad \overline{I} = \lim_{n \to \infty} \int \psi_n \, \mathrm{d}m = \int \psi \, \mathrm{d}m.$$

Thus,

$$f \text{ is Riemann integrable} \Leftrightarrow \int \phi = \int \psi \Leftrightarrow \int (\psi - \phi) = 0 \Leftrightarrow \psi = \phi \text{ Lebesgue a.e.}$$

■

2.6 Modes of Convergence

As we should already see, there are different *modes* of convergence. Let's formalize them.

Definition. Let $f_n, f \colon X \to \mathbb{C}$, and $S \subset X$. Then we have the following definitions.

Definition 2.6.1 (Pointwise convergence). $f_n \to f$ *pointwise* on S if

$$\forall_{x \in S} \forall_{\epsilon > 0} \exists_{N \in \mathbb{N}} \forall_{n \geq N} \ |f_n(x) - f(x)| < \epsilon.$$

Definition 2.6.2 (Uniformly convergence). $f_n \to f$ *uniformly* on S if

$$\forall_{\epsilon > 0} \exists_{N \in \mathbb{N}} \forall_{x \in S} \forall_{n \geq N} \ |f_n(x) - f(x)| < \epsilon.$$

Remark. We see that we can replace $\forall \epsilon > 0$ by $\forall k \in \mathbb{N}$ with ϵ changing to $\frac{1}{k}$.

Lemma 2.6.1. Let $B_{n,k}$ be

$$B_{n,k} \coloneqq \left\{ x \in X \mid |f_n(x) - f(x)| < \frac{1}{k} \right\}.$$

Then

(1) $f_n \to f$ pointwise on S if and only if

$$S \subset \bigcap_{k=1}^{\infty} \bigcup_{N=1}^{\infty} \bigcap_{n=N}^{\infty} B_{n,k}.$$

(2) $f_n \to f$ uniformly on S if and only if $\exists N_1, N_2, \ldots \in \mathbb{N}$ such that

$$S \subset \bigcap_{k=1}^{\infty} \bigcap_{n=N_k}^{\infty} B_{n,k}.$$

Proof. This essentially follows from Definition 2.6.1. ■

Definition. Let $(X, \mathcal{A}, \mu)$ be a measure space. Assuming that f_n, f are measurable functions, then we have the following.

Definition 2.6.3 (Converge almost everywhere). $f_n \to f$ *almost everywhere* means

$$\exists \text{ null set } E \text{ such that } f_n \to f \text{ pointwise on } E^c.$$

Definition 2.6.4 (Converge in L^1). $f_n \to f$ *in* L^1 means

$$\lim_{n\to\infty} \|f_n - f\| = 0.$$

Proof. We see that different function sequences converge in different senses.

Exercise. Classify in what senses do (1), (2), (3) and the **type write** function converge.

⊛

Lecture 14: Fifteenth Lecture

8 Feb. 11:00

Example. $f_n = \frac{1}{n}\mathbf{1}_{(0,n)}$.

- $f_n \to 0$ uniformly, not in L^1

Example. $f_n = \mathbf{1}_{(n,n+1)}$.

- $f_n \to 0$ pointwise, not uniformly, not in L^1

Example. $f_n = \mathbf{1}_{[0,\frac{1}{n})}$.

- $f_n \to 0$ almost everywhere
- Not pointwise, not L^1

Example. $f_n =$ "typewriter".

- $f_n \to 0$ in L^1
- $\forall x$ $f_n(x)$ does not converge.

Remark. Set $B_{n,k} := \{x \in X \mid |F_n(x) - f(x)| < \frac{1}{k}\}$. Then

$$\{x \in X \mid f_n(x) \to f(x)\} = \bigcap_{k=1}^{\infty} \bigcup_{N=1}^{\infty} \bigcap_{n\geq N} B_{n,k}.$$

Recall Chebyshev,

$$\mu\{|f| \geq c\} \leq \frac{1}{c}\int |f|d\mu.$$

Proposition 2.6.1. $(X, \mathcal{A}, \mu)$ measurable space. $f_n, f \in L^1$. Assume $\sum_{n=1}^{\infty} |f_n - f| < \infty$. Then $f_n \to f$ almost everywhere.

Proof. Set

$$E := \bigcup_{k=1}^{\infty} \bigcap_{N=1}^{\infty} \bigcup_{n \geq N}^{\infty} B_{n,k}^c = \{f_n \not\to f\}.$$

By Chebyshev,

$$\mu(B_{n,k}^c) \leq k \int |f_n - f|.$$

Fix k. Then

$$\mu(\bigcup_{n \geq N}^{\infty} B_{n,k}^c) \leq k \sum_{n \geq N} |f_n - f| \to 0 \text{ as } N \to \infty.$$

By continuity from above,

$$\mu(\bigcap_{N=1}^{\infty} \bigcup_{n \geq N}^{\infty} B_{n,k}^c) = 0.$$

By continuity from below, $\mu(E) = 0$. ■

Corollary 2.6.1. If $f_n \to f$ in L^1, then $\exists$ a subsequence f_{n_j} such that $f_{n_j} \to f$ almost everywhere.

Proof. Pick n_j such that

$$\begin{cases} \int |f_{n_j} - f| < 2^{-j} \\ n_j < n_{j+1} \end{cases}.$$

Then $\sum |f_{n_j} - f| \to 0$. From proposition, $f_{n_j} \to f$ almost everywhere. ■

Definition 2.6.5. We say that $f_n \to f$ in measure if $\forall \epsilon > 0$ then

$$\mu\{|f_n - f| \geq \epsilon\} \to 0 \text{ as } n \to \infty.$$

2.6.1 Partial Summary

- $f_n \to f$ in fast $L^1 \Rightarrow f_n \to f$ in L^1
- $f_n \to f$ in fast $L^1 \Rightarrow f_n \to f$ almost everywhere
- $f_n \to f$ almost everywhere $\not\Rightarrow f_n \to f$ in L^1
- $f_n \to f$ in $L^1 \not\Rightarrow f_n \to f$ almost everywhere

Remark. Read $f_n \to f$ in measure $\Rightarrow f_{n_j} \to f$ almost everywhere

2.6.2 Egoroff's Theorem

Definition 2.6.6. f_n, f measurable on $(X, \mathcal{A}, \mu)$. $f_n \to f$ almost uniformly if and only if $\forall \epsilon > 0$, then $\exists E \in \mathcal{A}$ such that $\mu(E) < \epsilon$ and $f_n \to f$ uniformly on E^c.

Theorem 2.6.1 (Egoroff's Theorem)**.** Let $(X, \mathcal{A}, \mu)$ be a measurable space. Assume that $\mu(X) < \infty$ and suppose that $f_n \to f$ almost everywhere. Then $f_n \to f$ almost uniformly.

Proof. Fix $\epsilon > 0$. Since $f_n \to f$ almost everywhere,

$$\mu(\bigcup_{k=1}^{\infty} \bigcap_{N=1}^{\infty} \bigcup_{n \geq N}^{\infty} B_{n,k}^c) = 0.$$

By monotinicity, for every k

$$\mu(\bigcap_{N=1}^{\infty} \bigcup_{n \geq N}^{\infty} B_{n,k}^c) = 0.$$

By continuity from above (use $\mu(X) < \infty$ here), for every k

$$\lim_{N \to \infty} \mu(\bigcup_{n \geq N} B_{n,k}^c) = 0.$$

For every k, $\exists N_k$ such that

$$\mu(\bigcup_{n \geq N_k}^{\infty} B_{n,k}^c) < \frac{\epsilon}{2^k}.$$

Set

$$E := \bigcup_{k=1}^{\infty} \bigcup_{n \geq N_k}^{\infty} B_{n,k}^c.$$

Check that E works such that $f_n \to f$ uniformly on E^c. ■

Example. Consider $(\mathbb{R}, \mathcal{L}, m)$, $f_n = \mathbf{1}_{(n,n+1)} \to 0$ almost everywhere. But $f_n \not\to f$ almost uniformly.

Corollary 2.6.2. If $f : [a, b] \to \mathbb{C}$ is Lebesgue measurable, then $\forall \epsilon > 0$, $\exists K \subset [a, b]$ compact such that $m(K^c) < \epsilon$ and f is continuous on K.

Proof. $C^0([a, b]) \subset L^1$ dense. Therefore, $\exists f_n \in C^0$ such that

$$\int |f_n - f| \to 0.$$

Set

$$B_{n,k} := \{x \in X \mid |f_n(x) - f(x)| < \frac{1}{k}\}.$$

Without loss of generality, pass to subsequence $\sum_{n=1}^{\infty} |f_n - f| < \infty$. Then $f_n \to f$ almost everywhere. By Egoroff, $\exists F \subset [a, b]$ such that $m(F) < \frac{\epsilon}{2}$ and $f_n \to f$ uniform on F^c. By inner regularity, $\exists K \subset F^c$ compact such that $m(F^c \setminus K) < \frac{\epsilon}{2}$. ■

2.7 Product Measures

Problem 2.7.1. Suppose we have 2 Lebesgue-Stieltjes measures μ_1, μ_2 on $\mathbb{R}$. Can we define a measure μ on $\mathbb{R}^2$ such that

$$\mu(I_1 \times I_2) = \mu(I_1) \cdot \mu(I_2)?$$

What about more generally $(X_i, \mathcal{A}_i, \mu_i)$ can we define a measure μ on $(X_1 \times \cdot \times X_n, \mathcal{A})$? Need to construct $\mathcal{A}$ using tensor product.

Lecture 15: Sixteenth Lecture

What we want is if we have $E_i \in \mathcal{A}_i$ then 10 Feb. 11:00

$$\begin{cases} E_1 \times E_2 \in \mathcal{A} \\ \mu(E_1 \times E_2) = \mu_1(E_1) \cdot \mu_2(E_2) \end{cases}.$$

2.7.1 Product σ-algebras

Definition. We have a product space $\prod_{\alpha \in I} X_\alpha$, with $x = (x_\alpha)_\alpha$, for $x_\alpha \in X_\alpha$. Additionally, we have a projection map $\pi_\alpha : X \to X_\alpha$.

Definition 2.7.1. Suppose $(X_\alpha, \mathcal{A}_\alpha)$ is a measurable space. Then the product σ-algebra $\bigotimes_\alpha \mathcal{A}_\alpha$ on $X = \prod_\alpha X_\alpha$ is the smallest σ-algebra such that $\pi_\alpha : X \to X_\alpha$ are $(\mathcal{A}, \mathcal{A}_\alpha)$-measurable for every α.

Proposition 2.7.1. In the case, $X_1 \times \cdots \times X_n$.

- $\mathcal{A}_1 \otimes \cdots \otimes \mathcal{A}_n = \langle \{\underbrace{E_1 \times \cdots \times E_n}_{\text{rectangle}}\} \rangle$
- If $\mathcal{A}_i = \langle \mathcal{E}_j \rangle$ then can pick $E_j \in \mathcal{E}_j$.

Proof. DIY (Prop 1.4 in Folland) ∎

Proposition 2.7.2. If $X_1, \ldots, X_n$ are metric spaces and $X = X_1 \times \cdots \times X_n$, then $\bigotimes_{j=1}^n \mathcal{B}_{x_j} \subset \mathcal{B}_x$. We have equality if X is separable[a].

[a]contains a countable dense subset

Sketch. By proposition,

$$\bigotimes_{j=1}^{n} \mathcal{B}_{X_j} = \langle \{\underbrace{U_1 \times \cdots \times U_n}_{\text{open in } X} \mid U_j \subset X_j \text{ open}\} \rangle \subset \mathcal{B}_X.$$

Now suppose X_j separable. Pick $C_j \subset X_j$ as a countable dense sequence. Let

$$\mathcal{E}_j := \{B_{X_j}(x, r) \mid x \in C_j, r \in \mathbb{Q}_{\geq 0}\}.$$

Then every open set is a (countable) union of balls $B_1 \times \cdots \times B_n$, $B_j \in \mathcal{E}_j$. Therefore,

$$\mathcal{B}_X \subset \bigotimes_{j=1}^{n} B_{X_j}.$$

∎

Example. Define metric on X as

$$d_X((x_1, \ldots, x_n), (y_1, \ldots, y_n)) = \max_j d_{X_j}(x_j, y_j).$$

Corollary 2.7.1. $\mathcal{B}_{\mathbb{R}_n} = \mathcal{B}_{\mathbb{R}} \otimes \cdots \otimes \mathcal{B}_{\mathbb{R}}$.

Corollary 2.7.2. If $(X, \mathcal{A})$ is a measurable space, then a function $f : X \to \mathbb{C}$ is $(\mathcal{A}, \mathcal{B}_{\mathbb{C}})$-measurable if and only if $\operatorname{Re} f$, $\operatorname{Im} f$ are $(\mathcal{A}, \mathcal{B}_{\mathbb{R}})$-measurable.

Proof. DIY. ∎

2.7.2 Back to Product Measures

We will proceed with the following setup:

- $(X_j, \mathcal{A}_j, \mu_j)$ for $1 \leq j \leq n$ are measure spaces.
- $X = X_1 \times \cdots \times X_n$, $\mathcal{A} = \mathcal{A}_1 \otimes \cdots \otimes \mathcal{A}_n$.

We then want to use HK to construct a measure μ on $(X, \mathcal{A})$. Therefore, we need an algebra $\mathcal{A}'$ on X and a premeasure μ' on $(X, \mathcal{A}')$. Set

$$\mathcal{A}' = \{\text{finite disjoint unions of rectangles } E_1 \times \cdots \times E_n \mid E_j \in \mathcal{A}_j\}.$$

From an above proposition, we know that $\mathcal{A} = \langle \mathcal{A}' \rangle$.

Proposition 2.7.3. $\mathcal{A}'$ is an algebra.

Sketch. **Step 1:** The set

$$\mathcal{E} = \{\text{rectangles} \mid E_j \in \mathcal{A}_j\} \subset \mathcal{P}(X)$$

satisfies

(i) $\emptyset \in \mathcal{E}$

(ii) $E, F \in \mathcal{E} \Rightarrow E \cap F \in \mathcal{E}$

> **Note.** Intersection of rectangles is a rectangle

(iii) $E \in \mathcal{E} \Rightarrow E^c$ is a finite disjoint union of elements of $\mathcal{E}$

> **Note.** Complement of rectangle is union of rectangles.

Step 2: (i) - (iii) $\Rightarrow \mathcal{A}' = \{$finite disjoint union of elements of $\mathcal{E}$ is an algebra$\}$ from Prop 1.7. Now define $\mu' : \mathcal{A}' \to [0, \infty]$ by if $A \in \mathcal{A}'$ is a finite disjoint union of rectangles $E_1^{(k)} \times \cdots \times E_n^{(k)}$ for $1 \leq k \leq m$, then

$$\mu'(A) = \sum_{k=1}^{m} \prod_{j=1}^{n} \mu_j(E_j^{(k)}).$$

∎

Proposition 2.7.4. Using μ' from above, the following hold

(1) μ' is a well defined premeasure on $(X, \mathcal{A}')$

(2) If each μ_j is σ-finite for $1 \le j \le n$ then so is μ'.

Proof of (1). It suffices to show (check this) if $E = E_1 \times \cdots \times E_n$ for $E_j \in \mathcal{A}_j$ is a (finite or) countable disjoint union of rectangles

$$E^{(k)} = E_1^{(k)} \times \cdots E_n^{(k)}, \quad E_j^{(k)} \in \mathcal{A}_j.$$

then

$$\prod_{j=1}^{n} \mu_j(E_j) = \sum_{k=1}^{m} \prod_{j=1}^{n} \mu_j(E_j^{(k)}).$$

For this, use Tonelli for series and integrals

$$\begin{aligned}\prod_{j=1}^{n} \mathbf{1}_{E_j}(x_j) = \mathbf{1}_E(x_1, \ldots, x_n) \\ &= \sum_{k=1}^{m} \mathbf{1}_{E^{(k)}}(x_1, \ldots, x_n) \\ &= \sum_{k=1}^{m} \prod_{j=1}^{n} \mathbf{1}_{E_j^{(k)}}\end{aligned}$$

for every $x = (x_1, \ldots x_n) \in X$. Integrate with respect to x_1 (check!)

$$\mu_1(E_1) \prod_{j=2}^{n} \mathbf{1}_{E_j}(x_j) = \sum_{k=1}^{m} \mu_1(E_1^{(k)}) \prod_{j=2}^{n} \mathbf{1}_{E_j^{(k)}}(x_j).$$

Successively integrate with respect to $x_2, \ldots, x_n$ then

$$\prod_{j=1}^{n} \mu_j(E_j) = \sum_{n} \prod_{j=1}^{n} \mu_j(E_j^{(k)}).$$

■

Proof of (2). DIY. ■

Corollary 2.7.3. From the HK extension we get

(i) $\exists$ a measure $\mu = \mu_1 \times \cdots \times \mu_n$ on $\underbrace{\mathcal{A}}_{=\langle \mathcal{A}' \rangle} = \mathcal{A}_1 \otimes \cdots \otimes \mathcal{A}_n$ that extends μ'.

(ii) If $\mu_1, \ldots, \mu_n$ are σ-finite, then μ is the unique such extension.

Corollary 2.7.4 (Associativity). The following properties hold:

(1) $\mathcal{A}_1 \otimes \mathcal{A}_2 \otimes \mathcal{A}_3 = (\mathcal{A}_1 \otimes \mathcal{A}_2) \otimes \mathcal{A}_3 = \mathcal{A}_1 \otimes (\mathcal{A}_2 \otimes \mathcal{A}_3)$

(2) If μ_1, μ_2, μ_3 are σ-finite, then

$$\mu_1 \times \mu_2 \times \mu_3 = (\mu_1 \times \mu_2) \times \mu_3 = \mu_1 \times (\mu_2 \times \mu_3).$$

Lecture 17: Seventeenth Lecture

For $n = 2$, $(X, \mathcal{A}, \mu)$, $(Y, \mathcal{B}, \nu)$. 13 Feb. 11:00

Given $E \subset X \times Y$ set

$$E_x := \{y \in Y \mid (x,y) \in E\} \subset Y$$
$$E^y := \{x \in X \mid (x,y) \in E\} \subset X.$$

If $f : X \times Y \to \mathbb{C}$, set

$$f_x(y) := f(x,y)$$
$$f^y(x) := f(x,y).$$

Proposition 2.7.5. Using the sets above,

(a) $E \in \mathcal{A} \otimes \mathcal{B} \Rightarrow \begin{cases} E_x \in \mathcal{B}, \ \forall x \in X \\ E^y \in \mathcal{A}, \ \forall y \in Y \end{cases}$

(b) If f is $\mathcal{A} \otimes \mathcal{B}$-measurable, then $\begin{cases} f_x, \mathcal{B}\text{-measurable} \\ f^y, \mathcal{A}\text{-measurable} \end{cases}$

Proof of (a). Define $\mathcal{E} \subset \mathcal{P}(X \times Y)$ by $E \in \mathcal{E} \Leftrightarrow$ condition in (a) holds. Note that

- Rectangles $\in \mathcal{E}$.
- $\mathcal{E}$ is a σ-algebra.

CHECK these. However, from these it follows that $\mathcal{E} \supset \mathcal{A} \otimes \mathcal{B}$. ■

Proof of (b). Follows from (a) if $V \subset \mathbb{C}$ then

$$f_x^{-1}(V) = f^{-1}(V)_x$$
$$(f^y)^{-1}(V) = f^{-1}(V)^y.$$

■

Theorem 2.7.1 (Tonelli for sets)**.** Let $(X, \mathcal{A}, \mu)$, $(Y, \mathcal{B}, \nu)$ be σ-finite. Pick $E \in \mathcal{A} \otimes \mathcal{B}$. Then

(a) The functions

$$\begin{cases} x \mapsto \nu(E_x), \\ y \mapsto \mu(E_y) \end{cases}$$

are measurable.

(b) $\mu \times \nu(E) = \int \nu(E_x) d\mu(x) = \int \mu(E^y) d\nu(y)$

Proof idea. Prove that the collection of E for which this is true is a σ-algebra containing all rectangles. To do this, we have the following tools:

- Monotone Convergence Theorem
- Dominating Convergence Theorem

■

Definition 2.7.2. Given a set X, a collection $\mathscr{C} \subset \mathcal{P}(X)$ is called a monotone class if it is closed under countable increasing unions and countable decreasing intersections.

Note. σ-algebra $\Rightarrow$ monotone class

Lemma 2.7.1 (Monotone Class Lemma)**.** If $\mathcal{A} \subset \mathcal{P}(X)$ is an algebra, then $\langle A \rangle$ equals the smallest monotone class containing $\mathcal{A}$.

Proof. DIY (p. 66) ∎

Remark. Back to Tonelli for sets

Proof of Tonelli for sets. Let $\mathscr{C} \subset \mathcal{P}(X \times Y)$ be the collection of sets $E \subset X \times Y$ for which (1)-(2) hold. Then we want to show that $\mathscr{C} = \mathcal{A} \otimes \mathcal{B}$.

Step 1: $\mathscr{C}$ contains all rectangles. (Check)

Step 1': $\mathscr{C}$ contains all finite disjoint unions of rectangles (by additivity), these form an algebra. (Check)

By MCL, it now sufficies to show that $\mathscr{C}$ is a monotone class.

Step 2: $\mathscr{C}$ is closed under countable increasing unions i.e.

$$E_j \nearrow E, \ E_j \in \mathscr{C} \Rightarrow E \in \mathscr{C}.$$

Use MCT.

Step 3: If $\mu(X), \nu(Y) < \infty$, then $\mathscr{C}$ is closed under countable decreasing limits i.e.

$$\mathscr{C} \ni E_j \searrow E \Rightarrow E \in \mathscr{C}.$$

Use DCT.

Step 4: Can write

$$\begin{cases} X = \bigcup_{j=1}^{\infty} X_j, \ X_j \nearrow X \\ Y = \bigcup_{j=1}^{\infty} Y_j, \ Y_j \nearrow Y \end{cases}.$$

Since X, Y are σ-finite, we can assume $\mu(X_j) < \infty$ and $\nu(Y_j) < \infty$. Get (1)-(2) on $X_j \times Y_j$, with

$$E_j := E \cap (X_j \times Y_j).$$

As in Step 2, use MCT.

∎

Theorem 2.7.2 (Tonelli's Theorem)**.** Let $(X, \mathcal{A}, \mu)$, $(Y, \mathcal{B}, \nu)$ be σ-finite measurable spaces. If $f \in L^+(X \times Y, \mu \times \nu)$, then the functions

$$g(x) := \int \underbrace{f_x}_{\text{measurable}} d\nu$$

$$h(y) := \int \underbrace{f^y}_{\text{measurable}} d\mu$$

lie in $L^+(X, \mu)$, $L^+(Y, \nu)$ respectively, and

$$\int f d(\mu \times \nu) = \int g d\mu = \int h d\nu. \tag{$\star$}$$

Notation. We will write:

$$\int fd(\mu \times \nu) = \int \left[\int f(x,y)d\nu(y) \; d\mu(x) = \int \left[\int f(x,y)d\mu(x) \right] d\nu(y) \right.$$

or

$$= \int fd\nu d\mu = \int gd\mu d\nu.$$

Proof. If $f = \mathbf{1}_E$, $E \in \mathcal{A} \otimes \mathcal{B}$, done by previous theorem. Next suppose f is simple such that

$$f = \sum_{j=1}^{n} c_j \mathbf{1}_{E_j}, \quad c_j \in (0, \infty).$$

This is OK by linearity (check!). In general, $\exists (\phi_j)_{j=1}^{\infty}$ of simple functions such that $\phi_j \nearrow f$. Then $g_j \nearrow g$ and $h_j \nearrow h$, where g_j, h_j are the corresponding g, h to ϕ_j. Since each of the g_j, h_j are measurable, g, h are measurable as limits of measurable functions. Thus, ($\star$) holds by MCT. ■

Lecture 18: Eighteenth Lecture

13 Feb. 11:00

Corollary 2.7.5. If $0 \leq f \in L^1(X \times Y)$, then $g(x) < \infty$ for μ-a.e. $x \in X$ and $h(y) < \infty$ for ν-a.e. $y \in Y$.

Proof. By Tonelli,

$$\int gd\mu < \infty.$$

Therefore, $g(x) < \infty$ almost everywhere. ■

Corollary 2.7.6. Suppose f is measurable on $X \times Y$ and

$$g^\star(x) = \int |f|_x d\nu$$

satisfies

$$\int g^\star(x) d\mu < \infty$$

then $f \in L^1(X \times Y)$.

Proof. By Tonelli,

$$\int |f| = \int g^*(x) d\mu < \infty.$$

■

Theorem 2.7.3 (Fubini). If $f \in L^1(X \times Y)$ then

(1) $f_x \in L^1(Y, \nu)$ for almost every $x \in X$,
$f^y \in L^1(X, \mu)$ for almost every $y \in Y$

(2) $g(x) = \int f_x d\nu \in L^1(X, \mu)$

$h(y) = \int f^y d\mu \in L^1(Y, \nu)$

(3) $\int f d(\mu \times \nu) = \int g d\mu = \int h d\nu$

Remark. (3) is typically written as:

$$\int f d(\mu \times \nu) = \int \int f(x,y) d\nu d\mu = \int \int f(x,y) d\mu d\nu.$$

Remark. This all works for f almost everywhere defined.

Proof. Write $f = \operatorname{Re} f + i \operatorname{Im} f$. It suffices to take f real-valued. Then

$$f = f^+ - f^-.$$

We can then apply Tonelli to f^+, f^-. By first corollary,

$$m(\{f^+ = \infty\}) = 0 = m(\{f^- = \infty\}).$$

■

Example. Apply Fubini to $(X, \mathcal{A}, \mu) = (Y, \mathcal{B}, \nu) = (\mathbb{N}, \mathcal{P}(\mathbb{N}), \text{counting})$. Consider $a_{m,n} \in \mathbb{C}$, $(m,n) \in \mathbb{N}^2$. Then

$$\sum_{(m,n)\in\mathbb{N}^2}^{\infty} |a_{m,n}| = \sup_{\substack{F\subseteq\mathbb{N}^2 \\ F\text{finite}}} \sum_{(m,n)\in F} |a_{m,n}| < \infty.$$

Therefore,

$$\begin{cases} \sum_{n=1}^{\infty} a_{m,n} \text{ converges absolutely for every } m \in \mathbb{N} \text{ to a } b_m \in \mathbb{C}, \\ \sum_{m=1}^{\infty} a_{m,n} \text{ converges absolutely for every } m \in \mathbb{N} \text{ to a } c_n \in \mathbb{C} \end{cases}.$$

Thus,

$$\sum_{m=1}^{\infty} b_m = \sum_{n=1}^{\infty} c_n = \sum_{(m,n)\in\mathbb{N}^2} a_{m,n}.$$

Example. Suppose $x, y = [0,1]$, $\mu = \nu = m$. Take

$$0 = \delta_1 < \delta_2 < \delta_3 < \cdots \text{ w/ } \lim_{n\to\infty} \delta_n = 1.$$

Let g_n be smooth, supported in $[\delta_n, \delta_{n+1}]$, non-negative and

$$\int g_n(x) dm = 1.$$

Consider

$$f(x,y) = \sum_{n=1}^{\infty} [g_n(x) - g_{n+1}(x)]\, g_n(y).$$

Note that $\int f(x,y) dm(x) = 0$ for every $x \in [0,1]$ while

$$\begin{cases} \int f(x,y) dm(y) = 1, & \text{if } x \in [\delta_1, \delta_2] \\ \int f(x,y) dm(y) = 0, & \text{otherwise} \end{cases}.$$

Thus, both iterated integrals converge but are not equal.

2.7.3 Complete Products

$(X, \mathcal{A}, \mu)$, $(Y, \mathcal{B}, \nu)$ complete σ-finite measure spaces. However, $\mu \times \nu$ is usually not complete.

Example. $X = Y = \mathbb{R}$, $\mu = \nu = m$. $E \subset X$ which is not measurable. Take $F \subset Y$, $m(F) = 0$. Then $E \times F$ is not $\mu \times \nu$-measurable. However, $E \times F \subset \mathbb{R} \times F$, with $(\mu \times \nu)(\mathbb{R} \times F) = 0$.

Theorem 2.7.4 (Complete Fubini-Tonelli). Keep the same assumptions as before. Suppose f is $\overline{(\mu \times \nu)}$-measurable and either

$$\begin{cases} f \geq 0, \\ f \in L^1 \end{cases}.$$

Then

$$\begin{cases} \text{For almost every } x \in X,\ f_x \text{ is } \mathcal{B}\text{-measurable}, \\ \text{For almost every } y \in Y,\ f^y \text{ is } \mathcal{A}\text{-measurable} \end{cases}.$$

Moreover,

$$\begin{cases} x \mapsto \int f_x d\nu \\ y \mapsto \int f^y d\mu \end{cases} \quad \text{are measurable}$$

and if $f \in L^1$ both are also L^1. Also

$$\int f d(\overline{\mu \times \nu}) = \int \int f(x,y) d\nu d\mu = \int \int f(x,y) d\mu d\nu.$$

Lemma 2.7.2. $(X, \mathcal{A}, \mu)$ and $\overline{\mathcal{A}}$ is the completion with respect to μ and f is $\overline{\mathcal{A}}$-measurable. Then $\exists g$ such that g is $\mathcal{A}$-measurable with $g = f$ μ-a.e.

Lemma 2.7.3. If $h = 0$ almost everywhere on $X \times Y$ then we have

$$\int h = \int \int h d\mu d\nu = 0.$$

Lecture 19: Ninteenth Lecture

2.8 Lebesgue in $\mathbb{R}^N$

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Definition 2.8.1. $(\mathbb{R}^n, \mathcal{L}^n, m^n)$ Lebesgue measure with $m^n =$ completion of $m \times \cdots \times m$.

$$\mathcal{L}^n = \{\text{Lebesgue measurable sets}\} \supset \mathcal{B}_{\mathbb{R}^n}.$$

Notation. We typically write $\int f dm$ on $\int f(x) dx$ instead of $\int f dm^n$.

Theorem 2.8.1 (Fubini-Tonelli:). If $f \in L^+(\mathbb{R}^n)$ or $f \in L^1(m)$ then

$$\int f(x) dx = \int \cdots \int f(x_1, \dots, x_n) dx_1\, dx_2 \cdots dx_n.$$

We can also change order of integration.

Example. Show $\int_0^\infty e^{-sx}\frac{\sin^2(x)}{x} = \frac{1}{4}\log\left(1+4s^{-2}\right)$ for $s > 0$. Hint: Integrate $e^{-sx}\sin(2xy)$ over $0 < x < \infty$ and $0 < y < 1$.

2.8.1 Properties of Lebesgue

We will discuss the analagous properties of the 1-dimensional case. Namely,

(1) Regularity

(2) Approximation

(3) Behavior under affine transformation

This corresponds to (pg. 70-76) in Folland.

Theorem 2.8.2 (Regularity)**.** If $E \in \mathcal{L}^n$, then

(a) $m(E) = \inf\{m(U) \mid U \text{ open}, U \supset E\} = \sup\{m(K) \mid K \text{ compact}, K \subset E\}$

(b) If $m(E) < \infty$, then for any $\epsilon > 0$, there exists disjoint rectangles $R_1, \ldots, R_N$ with interval sides such that

$$m(E \triangle \bigcup_{i=1}^{N} R_i) < \epsilon.$$

Sketch of (a). Reduce to 1-dimensional case: Fix $\epsilon > 0$ then $\exists\{T_i\}_{i=1}^\infty$ of rectangles, with

$$\begin{cases} E \subset \bigcup_{i=1}^\infty T_i, \\ \sum_{i=1}^\infty m(T_i) \le m(E) + \epsilon \end{cases}.$$

Therefore, $T_i = A_{i_1} \times \cdots \times A_{i_n}$ with $A_{i_j} \in \mathcal{L}(\mathbb{R})$. Approximate by open U_i in $\mathbb{R}$. Then the product $U_i \subset \mathbb{R}^n$ is open,

$$\begin{cases} T_i \subset U_i, \\ m(U_i) \le m(T_i) + \frac{\epsilon}{2^i} \end{cases}.$$

Now:

$$E \subset \bigcup_{i=1}^\infty T_i \subset \bigcup_{i=1}^\infty U_i = \text{open}, .$$

Thus,

$$m(\bigcup_{i=1}^\infty U_i) \le \sum_{i=1}^\infty m(U_i) \le m(E) + \epsilon.$$

The compact case is more or less the same. ■

Sketch of (b). Instead of approximating T_j by U_j open product, approximate T_j by a countable collection of coordinate rectangles $\{R_{j,i}\}_{i=1}^\infty$. Then we can throw away ∞-many such that we are left with $R_{j,1}, \ldots, R_{j,m_j}$ with

$$\sum_{i=1}^{m_j} m(R_{j,i}) \le m(T_j) + \frac{\epsilon}{2^i}.$$

Now throw away ∞-many T_j such that

$$\sum_{i,j}^{\infty} m(R_{j,i}) \leq \sum_{j=1}^{N} m(T_j) + \frac{\epsilon}{2} \leq m(E) + \epsilon.$$

■

Theorem 2.8.3. If $f \in L^1(m)$, $\epsilon > 0$, then

(1) $\exists \phi = \sum_{j=1}^{N} c_j \mathbf{1}_{R_j}$ with R_j a rectangle with interval sides and $\int_{\mathbb{R}^n} |\phi - f| dm < \epsilon$

(2) $\exists \phi \in C_C^0(\mathbb{R}^n)$ with $\int_{\mathbb{R}^n} |\phi - f| dm < \epsilon$

Sketch of (1). Consider $f \geq \tilde{\phi} = \sum_{j=1}^{N} c_j \mathbf{1}_{E_j}$ with E_j $\mathcal{L}$-measurable. From the last theorem, $E_j \sim \bigcup_{j=1}^{M} R_j$ disjoint. Then

$$\int \left| \sum_k c_k \mathbf{1}_{\mathbb{R}_k} - \sum_j \mathbf{1}_{E_j} \right| dm < \frac{\epsilon}{2}.$$

Then just use triangle inequality. ■

Sketch of (2). $\mathbf{1}_{R_j} = \mathbf{1}_{R_{j,1}}(x_1) \cdot \mathbf{1}_{R_{j,2}}(x_2) \cdots$ with $R_j = R_{j,1} \times \cdots \times R_{j,n}$. Then for $\mathbf{1}_{R_{1,i}}(x_i)$ can be approximated by $g_i(x_i) \in C_c^0(\mathbb{R})$. Product approximates $\mathbf{1}_{R_j}$. ■

Notation (Translation). Given $a \in \mathbb{R}^n$, a translation is the map $\tau_a : \mathbb{R}^n \to \mathbb{R}^n$ defined by $\tau_a(x) = x + a$.

Theorem 2.8.4 (Folland 2.4.2). If $E \in \mathcal{L}^n$, then

(1) $\tau_a(E) \in \mathcal{L}^n$ with $m(E) = m(\tau_a(E))$

(2) If $f : \mathbb{R}^n \to \mathbb{C}$ is Lebesgue measurable, so is $f \circ \tau_a$ and

$$\int (f \circ \tau_a) dm = \int f dm.$$

Proof of (1). Borel $E \mapsto \tau_a$ is a homeomorphism. Therefore, $\tau_a(E)$ is also Borel. So if f is Borel, then so is $f \circ \tau_a$. Additionally, $\tau_a(N)$ is a null set for any null set N. From the 1-d case, we have that $m(E) = m(\tau_a(E))$ for all rectangles E. ■

Proof of (2). Let f be Lebesgue measruable and $B \subset \mathbb{C}$ Borel, where $f^{-1}(B) = E \cup N$ with E Borel and $m(N) = 0$. Then $\tau_a^{-1}(E)$ is also Borel and $m(\tau_a^{-1}(N)) = 0$. Thus, $f \circ \tau_a$ is Lebesgue measurable. Conclude by approximating with simple functions. ■

Lecture 20: Twentieth Lecture

2.8.2 Linear Transformations

20 Feb. 11:00

Let $T : \mathbb{R}^n \to \mathbb{R}^n$ with matrix T_{ij}. Recall that T is invertible iff $\det T \neq 0$, with $T \in GL(n, \mathbb{R})$.

Theorem 2.8.5. Fix $T \in GL(n, \mathbb{R})$

(a) $E \in \mathcal{L}^n \Rightarrow T(E) \in \mathcal{L}^n$ and $m(T(E)) = |\det T| m(E)$.

(b) If $f : \mathbb{R}^n \to \mathbb{C}$ is Lebesgue measurable, so is $f \circ T$. If $f \in L^+$ or $f \in L^1(m)$, then $f \circ T \in$

$L^+, L^1(m)$ and

$$\int f \circ T dm = |\det T| \int f dm.$$

Lemma 2.8.1. Any $T \in GL(n, \mathbb{R})$ can be written as a finite composition of invertible linear maps of the following types:

(1) $(x_1, \dots, x_n) \mapsto (x_1, \dots, cx_j, \dots, x_n)$

(2) $(x_1, \dots, x_n) \mapsto (x_1, \dots, x_j + cx_k, \dots, x_n)$

(3) $(x_1, \dots, x_k, \dots, x_j, \dots, x_n) \mapsto (x_1, \dots, x_j, \dots, x_k, \dots x_n)$

Proof. Gaussian Elimination. ∎

Sketch of theorem. We will proceed in steps:

Step 1: First assume that E is a Borel set, f is Borel measurable.

Then $T, T^{-1} : \mathbb{R}^n \to \mathbb{R}^n$ continuous. Therefore, $T(E)$ is a Borel set, and $f \circ T$ is Borel measurable. If (a)-(b) is true for S, T, also true for $S \circ T$ because

$$|\det(S \circ T)| = |\det(S)| \cdot |\det(T)|.$$

Therefore, it suffices to show that (a)-(b) for T as in (1)-(3) from the lemma. Use Fubini-Tonelli to reduce to 1-dimensional case.

> **Note.** Clearly, (b) implies (a) because we can just set $f = \mathbf{1}_E$.

For (1), $\det T = c$. Note

$$\begin{aligned}\int f \circ T dm &= \int \cdots \underbrace{\int f(x_1, \dots, cx_j, \dots x_n) dx_j}_{=|c| \int f(x_1, \dots, x_n) dx_1} dx_1 \cdots \hat{dx_j} \cdots dx_n \\ &= |c| \int f dm.\end{aligned}$$

Step 2: The collection of Borel (sub)null sets is preserved by T, T^{-1}.

This is clear from step 1.

Step 3: Suppose $E \in \mathcal{L}^n$.

Then $E = F \cup N$, where F is Borel and N is Borel (sub)null. Note that

$$T(E) = \underbrace{T(F)}_{\text{Borel}} \cup \underbrace{T(N)}_{\text{Subnull}}.$$

Therefore,

$$m(T(E)) = m(T(F)) = |\det T| m(F) = |\det T| m(E).$$

∎

Proof of (a) implies (b). DYI. Approximate by simple functions. ∎

Definition 2.8.2. T is a "rotation" if $T \cdot T^* = id$ i.e. $|\det T| = 1$.

Corollary 2.8.1. Lebesgue measure is rotation invariant. If T is a rotation, then

$$m(T(E)) = m(E).$$

Corollary 2.8.2. Any real vector space V has a Lebesgue measure, uniquely determined up to a constant.

Proof. After a choice of basis, $V \cong \mathbb{R}^n$. We can go between different choices of bases via some T. ■

Ending chapter remarks:

- We are skipping Jordan Content.
- Read staement of Theorem 2.47; change of variables.
- Read polar coordinates (2.7) "$dm = r^{n-1} d\sigma dr$"
  - 2d - $dA = r d\theta dr$
  - 3d - $dV = \rho^2 \sin\phi d\phi d\theta d\rho$

  $dr =$ Lebesgue on $(0, \infty)$, $d\sigma =$ spherical measure on S^{n-1}

Corollary 2.8.3. If $f \in L^1(m)$ or $L^+(\mathbb{R}^n)$ is a rotation invariant[a] function, then

$$\int_{\mathbb{R}^n} f dm = c_n \int_0^\infty r^{n-1} g(r) dr.$$

where $f(x) = g(|x|)$

[a] $f(x)$ only depends on $|x| = \left(\sum_{i=1}^n x_i^2\right)^{\frac{1}{2}}$

Chapter 3

Differentiation on $\mathbb{R}^n$

We want to generalize the following calculus facts: If $f : [a, b] \to \mathbb{C}$ is "nice", then

- $\int_a^b f'(x)dx = f(b) - f(a)$
- $\frac{d}{dx} \int_a^x f(t)dt = f(x)$
- $\lim_{r \to 0^+} \frac{1}{r} \int_x^{x+r} f(t)dt = f(x) = \lim_{r \to 0^+} \frac{1}{r} \int_{x-r}^{x} f(t)dt$

Remark. We are starting with (3.4). We will do rest of (3) later

Definition 3.0.1. Say $f : \mathbb{R}^n \to \mathbb{C}$ is locally integrable if it is Lebesgue measurable and

$$\int_K |f| dm < \infty$$

for any compact $K \subset \mathbb{R}^n$. Equivalently,

$$\int_B |f| dm < \infty$$

for every ball $B \subset \mathbb{R}^n$.

Definition 3.0.2. $L^1_{\text{loc}}(\mathbb{R}^n) = \{f \mid f : \mathbb{R}^n \to \mathbb{C}, f \text{ is locally integrable}\}$

Example. $f \in C^0(\mathbb{R}^n)$, f simple function are locally integrable.

Example. $f(x) = |x|^p$ on $\mathbb{R}^n$. $f \in L^1_{\text{loc}}(\mathbb{R}^n) \Leftrightarrow p > -n$.

$$\int_B |f| = c_n \int_0^R r^{n-1+p} dr < \infty \Leftrightarrow n - 1 + p > -1.$$

Lecture 21: Twenty-first Lecture

21 Feb. 11:00

Definition 3.0.3. For $x \in \mathbb{R}$, $r > 0$, $f \in L^1$ locally.

$$A_r f(x) = \frac{1}{m(B(x,r))} \int_{B(x,r)} f dm =: \underbrace{\int_{B(x,r)} f}_{\text{average of } f}.$$

Problem 3.0.1. Does $\lim_{r\to 0^+} A_r f(x) = f(x)$?

Lemma 3.0.1. For $f \in L^1$, $A_r f(x)$ is jointly continuous in (x, r).

Proof. Suppose $(x_j, r_j) \to (x, r)$ in $\mathbb{R}^n \times \mathbb{R}_{\geq 0}$. Then $m(B(x_j, r_j)) \to m(B(x, r))$. By DCT,

$$\int \mathbf{1}_{B(x_j,r_j)} f \to \int \mathbf{1}_{B(x,r)} f.$$

Thus, $A_{r_j} f(x_j) \to A_r f(x)$. ■

Definition 3.0.4. The Hardy-Littlewood maximal function of $f \in L^1_{\text{loc}}$ is $Hf : \mathbb{R}^n \to [0, \infty]$ defined by

$$Hf(x) \coloneqq \sup_{r>0} A_r |f|(x).$$

Lemma 3.0.2. $Hf : \mathbb{R}^n \to [0, \infty]$ is measurable.

Proof. $a \in \mathbb{R} \Rightarrow (Hf)^{-1}((a, \infty]) = \bigcup_{r>0} \underbrace{(A_r|f|)^{-1}((a, \infty])}_{\text{open by } L^1}$. ■

Lemma 3.0.3 (Vitali). Let $B_1, \ldots, B_k$ be open balls in $\mathbb{R}^n$. Then there is a subcollection $B_1', \ldots, B_m'$ of disjoint balls such that $\bigcup_{i=1}^k B_i \subseteq \bigcup_{i=1}^m B_i'$.

Proof. Let B_1' be the ball among $B_1, \ldots, B_k$ of maximal radius. Let B_2' be the ball among $B_1, \ldots, B_k$ of maximal radius that is disjoint from B_1'. Continue this procedure until no balls are left. Pick any B_i. Then it is sufficient to show that $\exists j$ such that $B_i \subset 3B_j'$. Note $\exists! j$ such that $B_i \cap B_j' \neq \emptyset$ and j minimal. Then for every $1 \leq \ell < j$, we have that $B_i \cap B_\ell' = \emptyset$. By construction, it follows that $\text{rad}(B_j') \geq \text{rad}(B_i)$. Thus, we have that $3B_j' \supseteq B_i$. ■

Theorem 3.0.1 (Hardy-Littlewood Maximal Theorem). $\exists C > 0$ such that for all $f \in L^1$ and all $\alpha > 0$

$$m(\{x : Hf(x) > \alpha\}) \leq \frac{C}{\alpha} \int |f(x)| dx.$$

Proof. Set $E_\alpha = \{Hf > \alpha\} \subseteq \mathbb{R}^n$ this is measurable by above lemma. Then $x \in E_\alpha \Rightarrow \exists r_x > 0$ such that $A_r|f|(x) > \alpha$ because $Hf = \sup A_r|f|$. Then

$$m(B(x, r_x)) < \frac{1}{\alpha} \int_{B(x,r_x)} |f|.$$

By inner regularity of m, for any $K \subseteq E_\alpha$ compact, $\exists x_1, \ldots, x_m \in K$ such that $K \subseteq \bigcup_{i=1}^m B(x_i, r_{x_i})$. From Vitali's lemma above, we have that $\exists \ell \leq m$ such that for some $B_1, \ldots, B_\ell$ disjoint, we have that $K \subseteq \bigcup_{i=1}^\ell 3B_i$. Then

$$\begin{aligned} m(K) \leq \sum_{i=1}^\ell m(3B_i) &= 3^n \sum_{i=1}^\ell m(B_i) \\ &< 3^n \sum_{i=1}^\ell \frac{1}{\alpha} \int_{B_i} |f| \\ &\leq \frac{3^n}{\alpha} \int |f|. \end{aligned}$$

By taking the supremum over $K \subset E_\alpha$, we get that

$$m(E_\alpha) \leq \frac{3^n}{\alpha} \int |f|.$$

■

Lecture 22: Twenty-Second Lecture

6 Mar. 11:00

Definition 3.0.5. The Lebesgue set of f is

$$L_f := \{x \in \mathbb{R}^n \mid \lim_{r \to 0^+} \int_{B(x,r)} |f(y) - f(x)| dy = 0\}.$$

Theorem 3.0.2 (Lebesgue Differentiation Theorem). (L_f is measurable and) $m(L_f^c) = 0$.

$$x \in L_f \Rightarrow \lim_{r \to 0^+} \frac{1}{m(B(x,r))} \int_{B(x,r)} f(y) dy = f(x).$$

Corollary 3.0.1. If $E \subset \mathbb{R}^n$ is Lebesgue measurable, then

$$\underbrace{D_E(x)}_{\text{density of } E \text{ at } x} := \lim_{r \to 0^+} \frac{m(E \cap B(x,r))}{m(B(x,r))} = \begin{cases} 1, & \text{for a.e. } x \in E = L_{\mathbf{1}_E} \\ 0, & \text{for a.e. } x \in E^c \end{cases}.$$

Proof. $f = \mathbf{1}_E$ ■

Example. $E = \{0\} \cup \bigcup_{j=0}^{\infty} [\frac{2}{3} \cdot 2^{-j}, 2^{-j}] \subset [0,1] \subset \mathbb{R}$.

- $x \in \bigcup_{j=0}^{\infty} I_j^0 \Rightarrow D_E(x) = 1$
- $x \in \mathbb{R} \setminus E \Rightarrow D_E(x) = 0$
- $x \in \bigcup_{j=0}^{\infty} \partial I_j \Rightarrow D_E(x) = \frac{1}{2}$
- $x = 0$
  - Need to find $m(E \cap (-r, r))$
  - In fact; $\lim_{r \to 0^+} \frac{m(E \cap [0,r))}{2r}$ does not exist.
  - Take two subsequences: left and right endpoints.
    * $r = 2^{-j} \Rightarrow m(E \cap [0,r)) = \sum_{i=j}^{\infty} m(I_i) = \sum_{i=j}^{\infty} \frac{1}{3} 2^{-i} = \frac{1}{3} 2^{1-j}$

$$m(E \cap [0,r)) \setminus 2^{-j} = \frac{2}{3}.$$

    * $r = \frac{2}{3} \cdot 2^{-j} = \sum_{i=j+1}^{\infty} m(I_i) = \frac{1}{3} 2^{-j}$

$$m(E \cap [0,r)) \setminus 2^{-j} = \frac{1}{3} 2^{-j} \Big/ \frac{2}{3} 2^{-j} = \frac{1}{2}.$$

Problem 3.0.2 (HW7 #4). Example where $D_E(0) = \frac{\alpha}{2}$ for $0 < \alpha < 1$ and $E \subset [0,1]$.

Theorem. $x \in \mathbb{R}^n$, $E_r, r > 0$

$$\alpha B(x,r) \subset E_r \subset B(x,r).$$

Say E_r "shrink nicely" to x as $r \to 0^+$.

Theorem 3.0.3. $f \in L^1_{\text{loc}}$ and $x \in L_f$ then

$$\lim_{r \to 0^+} \frac{1}{m(E_r)} \int_{E_r} |f(y) - f(x)| dy = 0.$$

Corollary 3.0.2. Given $f \in L^1_{\text{loc}}$ set

$$F(x) \coloneqq \int_0^x f(t)dt = \int_{[0,x)} f dm.$$

Then $F'(x)$ exists for a.e. $x \in \mathbb{R}$ and $F'(x) = f(x)$.

Chapter 4

L^p spaces

4.1 Normed Vector Spaces

Formally, we have {topological spaces} $\supset$ {metric spaces} $\supset$ {normed vector spaces}.

Definition 4.1.1. A metric space is (X, ρ) such that $\rho : X \times X \to [0, \infty)$, where ρ has some useful properties.

Example. A few examples:

- $\mathbb{Q}, \mathbb{R}$ then $\rho(x, y) = |x - y|$
- For $\mathbb{R}_{>0}$, $\rho(x, y) = |\log\left(\frac{x}{y}\right)|$
- $\mathbb{R}^n$, $\rho(x, y) = \sum_1^n |x_i - y_i|$
- $\mathbb{R}^n$, $\rho(x, y) = \sum_1^n |x_i - y_i|^2$
- $\mathbb{R}^n$, $\rho(x, y) = \max_i |x_i - y_i|^2$
- $C^0([0, 1])$, $\rho(f, g) = \sup |f - g|$
- $L^1(\mu)$, $\rho(f, g) = \int |f - g| d\mu$

Definition. Let $K = \mathbb{R}$ or $\mathbb{C}$.

Definition 4.1.2. $\|\cdot\| : V \to [0, \infty)$ is a seminorm if

- $\|v + w\| \leq \|v\| + \|w\|$.
- $\|cv\| = |c| \cdot \|v\|$.

It is a norm if $\|v\| \Rightarrow v = 0$.

Definition 4.1.3. A normed vector space is the pair $(V, \|\cdot\|)$ where V is a K-vector space and $\|\cdot\| : V \to [0, \infty)$ is a norm.

Remark. The norm $\|\cdot\|$ induces a metric $\rho(v, w) := \|v - w\|$ such that

$$v_n \to v \Leftrightarrow \|v_n - v\| \to 0.$$

Example. $V = \mathbb{R}^n$ with $\|(x_1, \ldots, x_n)\| = \sum_{i=1}^n |x_i|^2$

Example. $V = C^0([0,1])$ with $\|f\| = \sup_{0\le x\le 1}|f(x)|$

Example. $V = L^1(X, \mathcal{A}, \mu)$ with $\|f\| = \int |f| d\mu$. Norm if we identify functions as equal a.e.

Definition 4.1.4. A Banach space is a complete normed vector space i.e. every Cauchy sequence converges.

Definition 4.1.5. A sequence $(v_n)_{n=1}^{\infty}$ is Cauchy if $\forall \epsilon > 0$, $\exists N \ge 1$ such that

$$m, n \ge N \Rightarrow \|v_n - v_m\| < \epsilon.$$

.

Definition 4.1.6. A sequence $(v_n)_{n=1}^{\infty}$ converges if $\forall \epsilon > 0$, $\exists N \ge 1$ such that

$$n \ge N \Rightarrow \|v_n - v\| < \epsilon.$$

.

Exercise. $V = \mathbb{R}^n$ with either of the 3 norms is a Banach space.

Definition 4.1.7. If $(v_n)_{n=1}^{\infty}$ is a sequence in a normed vector space, then

- $\sum_{n=1}^{\infty} v_n$ converges is $\exists v \in V$ such that $\lim_{N\to\infty} \sum_{n=1}^{N} v_n = v$
- $\sum_{n=1}^{\infty} v_n$ absolutely convergent if $\sum_{n=1}^{\infty} \|v_n\| < \infty$.

Theorem 4.1.1. A Normed Vector Space $(V, \|\cdot\|)$ is a Banach space if and only if every absolutely convergent series converges.

Proof. For the $\Rightarrow$ direction, DIY: show that the partial sums form a Cauchy sequence. Now consider the $\Leftarrow$ direction. Let $(v_j)_1^{\infty}$ be a Cauchy sequence. Idea: telescoping series i.e.

$$v_n = \sum_{i=1}^{n} (v_i - v_{i-1}), \quad v_0 := 0.$$

Then we only need to show that v_n converges. It is sufficient to show that

$$\sum_{n=1}^{\infty} \|v_n - v_{n-1}\| < \infty.$$

In order to do this, we need to make a slight modification. For $j = 1, 2, \ldots$ pick n_j such that $m, n \ge n_j \Rightarrow \|v_m - v_n\| \le 2^{-j}$. Without loss of generality, $n_1 < n_2 < \cdots$ such that

$$v_{n_j} = \sum_{i=1}^{j} (v_{n_i} - v_{n_{i-1}}), \quad v_{n_0} = 0$$

and

$$\sum_{j=1}^{\infty} \|v_{n_j} - v_{n_{j-1}}\| < \infty.$$

Thus, $v_{n_j} \to v$ fro some $v \in V$. Now use (v_n) is Cauchy to conclude original sequence must converge. ∎

Corollary 4.1.1. $L^1(\mu)$ is complete i.e. a Banach space.

Proof. Suppose $f_n \in L^1$ such that $\sum_{n=1}^{\infty} \int |f_n| < \infty$. Set $g_n = \sum_{i=1}^{n} f_i$. Must show $\exists g \in L^1$ such that

$$\int |g_n - g| \to 0.$$

Set $h := \sum_{n=1}^{\infty} |f_n| \in L^1$. Therefore, $h < \infty$ almost everywhere. It follows that $g(x) := \lim_{n\to\infty} g_n(x)$ exists almost everywhere. Note

$$|g_n - g| \leq 2h.$$

By DCT, we have that

$$\int |g_n - g| \to 0.$$

■

Lecture 23: Twenty-Third Lecture

4.2 L^p spaces

8 Mar. 11:00

We have $(X, \mathcal{A}, \mu)$ is a measure space and $0 < p < \infty$. If $f : X \to \mathbb{C}$ is measurable, then set

$$\|f\|_p := \left(\int |f|^p\right)^{1/p} \in [0, \infty]$$

and

$$L^p(\mu) := \{f \mid \|f\|_p < \infty\}/\sim,$$

where $f \sim g$ if $f = g$ a.e.

Example. $(\mathbb{R}, \mathcal{L}, m)$

(1) $f(x) = \frac{1}{x^\alpha}\mathbf{1}_{(0,1)}$, $f \in L^p \Leftrightarrow p\alpha < 1$

- This is equivalent to $\int_0^1 \frac{1}{x^{p\alpha}}dx < \infty$

(2) $f(x) = \frac{1}{x^\alpha}\mathbf{1}_{(1,\infty)}$, $f \in L^p \Leftrightarrow p\alpha > 1$

- This is equivalent to $\int_1^\infty \frac{1}{x^{p\alpha}}dx < \infty$

Example. $\ell^p := \ell^p(\mathbb{N}) := L^p(\mathbb{N}, \mathcal{P}(\mathbb{N}), \text{counting})$ Then this is equivalent ot

$$\{(a_n)_1^\infty \mid \sum_{n=1}^{\infty} |a_n|^p < \infty\}.$$

Problem 4.2.1. Is L^p a Banach space?

Answer. Yes! ⊛

Lemma 4.2.1. L^p is a vector space.

Proof. We must check the two axioms:

1. If $f \in L^p$, $c \in K$, then $cf \in L^p$ because
$$\int |cf|^p = |c|^p \int |f|^p < \infty.$$
2. If $f, g \in L^p$ then $f + g \in L^p$.
$$\begin{aligned} |f+g|^p &\leq (|f| + |g|)^p \\ &\leq (2\max\{|f|, |g|\})^p \\ &= 2^p \max\{|f|^p, |g|^p\} \\ &\leq 2^p(|f|^p + |g|^p). \end{aligned}$$
Thus,
$$\int |f+g|^p \leq 2^p \left(\int |f|^p + \int |g|^p \right) < \infty.$$

■

Problem 4.2.2. Is L^p a normed vector space?

We need to check the following things

- $\|cf\|_p = |c| \cdot \|f\|_p$
- $\|f + g\|_p \leq \|f\|_p + \|g\|_p$
  - This is ok when $p = 1$.
  - False if $0 < p < 1$. (See book)
  - We only consider $1 < p < \infty$

We are going to make use of the conjugate exponent q such that
$$\frac{1}{p} + \frac{1}{q} = 1, \quad (q = \frac{p}{p-1}).$$

Theorem 4.2.1 (Holder's Inequality). If $f, g : X \to \mathbb{C}$ are measurable then

(a) $\|fg\|_1 \leq \|f\|_p \cdot \|g\|_q$

(b) If $f \in L^p$, $g \in L^q$, equality holds if and only if $|f|^p$ and $|g|^q$ are proportional i.e.
$$\alpha |f|^p \overset{\text{a.e.}}{=} \beta |g|^q, \quad {\scriptstyle \substack{\alpha,\beta \geq 0 \\ (\alpha,\beta) \neq (0,0)}}.$$

Proof. We will proceed by steps:

(1) Reductions :

- Without loss of generality, $\|f\|_p > 0$, $\|g\|_q > 0$. Otherwise, $f = 0$ almost everywhere or $g = 0$ almost everywhere. Then $fg = 0$ almost everywhere and $\|fg\|_1 = 0$.
- Without loss of generality, $\|f\|_p < \infty$, $\|g\|_q < \infty$.
- Without loss of generality, $\|f\|_p = \|g\|_q = 1$. (Replace f by $f/\|f\|_p$ and g by $g/\|g\|_q$)

(2) Pointwise estimate :

$a, b \geq 0 \Rightarrow ab \leq \frac{a^p}{p} + \frac{b^q}{q}$ with equality if and only if $a^p = b^q$.

Proof. This is clear if $a = 0$ or $b = 0$. So without loss of generality, $a, b > 0$. Then it suffices to show that

$$\frac{a}{b^{q-1}} \leq \frac{a^p}{p \cdot b^q} + \frac{1}{q}.$$

Since $\frac{1}{p} + \frac{1}{q} = 1$, we have that $p = \frac{q}{q-1}$ such that

$$\frac{a^p}{b^q} = \left(\frac{a}{b^{q-1}}\right)^p.$$

If we set $t = \frac{a}{b^{q-1}} \in (0, \infty)$ we must show

$$t \leq \frac{t^p}{p} + \frac{1}{q}.$$

DYI (Calculus). Equality iff $t = 1$. ■

(3) Conclusion :

By (2), we have

$$\begin{aligned} \|fg\|_1 &= \int |f(x)g(x)|d\mu(x) \\ &\overset{(2)}{\leq} \int \frac{|f(x)|^p}{p} d\mu(x) + \int \frac{|g(x)|^q}{q} d\mu(x) \\ &= \frac{1}{p} + \frac{1}{q} = 1. \end{aligned}$$

Furthermore, equality holds iff $|f(x)|^p = |g(x)|^q$ for almost every x.

■

Remark. If $p = q = 2$, then this is Cauchy-Schwartz inequality.

Corollary 4.2.1 (Minkowski's Inequality)**.** If $1 \leq p < \infty$, then

$$\|f + g\|_p \leq \|f\|_p + \|g\|_p.$$

It follows that L^p is a normed vector space.

Proof. Without loss of generality, $p > 1$, $|f| + |g| > 0$ almost everywhere. Then

$$|f + g|^p \leq (|f| + |g|) \cdot |f + g|^{p-1}, \quad \text{pointwise } \forall x.$$

By Holder's inequality,

$$\begin{aligned} \int |f + g|^p &\leq \int |f| \cdot |f + g|^{p-1} + \int |g||f + g|^{p-1} \\ &\overset{\text{Holder}}{\leq} \|f\|_p \|f + g\|_q^{p-1} + \|g\|_p \||f + g|^{p-1}\|_q \\ &= \left(\int |f + g|^{p-1}q\right)^{\frac{1}{q}} = \|f + g\|_p^{p/q}. \end{aligned}$$

Therefore,

$$\|f + g\|_p^p \leq \|f + g\|_p^{p/q} \cdot (\|f\|_p + \|g\|_p).$$

Now $p - \frac{p}{q} = 1$. Thus,

$$\|f+g\|_p = \frac{\|f+g\|_p^p}{\|f+g\|_p^{p/q}} \leq (\|f\|_p + \|g\|_p).$$

■

Lecture 24: Twenty-Fourth Lecture

10 Mar. 11:00

Definition 4.2.1. We define L^p as follows:

$$L^p := \{f : X \to \mathbb{C} \mid \|f\|_p^p := \int |f|^p < \infty\} / \sim .$$

Problem 4.2.3. $f \in L^p \Leftrightarrow f^p \in L^1$?

Answer. This is not true because you cannot take arbitrary powers of arbitrary complex numbers. However, $|f| \in L^p \Leftrightarrow |f|^p \in L^1$. ⊛

Theorem 4.2.2. For $p \geq 1$, L^p is a Banach space (complete).

Proof. (Did this for $p = 1$). Recall that it suffices to show that every absolutely convergent series converges. Suppose $f_k \in L^p$ such that $B := \sum_1^\infty \|f_k\|_p < \infty$. Must show that $g_n := \sum_1^n f_k$ converges in L^p to some $g \in L^p$. Set

$$G_n := \sum_1^n |f_k|, \quad G := \sum_1^\infty |f_k|$$

such that $G_n, G \in L^+$. By Minkowski,

$$\|G_n\|_p \leq \sum_1^n \||f_k|\|_p = \sum_1^n \|f_k\|_p \leq B.$$

Note that $G_n^p \nearrow G^p$ so $\|G\|_p = \lim_{n\to\infty} \|G_n\|_p \leq B$ by MCT. Note

$$\|G\|_p^p = \int G^p < \infty \Rightarrow G^p < \infty \text{ a.e.} \Leftrightarrow G < \infty \text{ a.e.}.$$

Then

$$G(x) = \sum_1^\infty |f_k(x)| < \infty \text{ for a.e. } x.$$

For such x, the series $g(x) := \sum_1^\infty f_k(x)$ converges. Now it remains to show that $g \in L^p$ and $g_n \to g$ in L^p i.e. $\|g_n - g\|_p \to 0$. We want to use DCT to show $g_n \to g$ almost everywhere i.e. we want

$$\int |g_n - g|^p \to 0.$$

We know that $|g_n|, |g| \leq G$ almost everywhere. Therefore,

$$|g_n - g|^p \leq 2^p G^p \quad \text{and} \quad \int 2^p G^p < \infty.$$

Thus, by DCT we have

$$\int |g_n - g|^p \to 0$$

as desired. ■

Proposition 4.2.1. If $1 \leq p < \infty$, then simple functions of the form

$$f = \sum_{j=1}^{n} a_j \mathbf{1}_{E_j}$$

where $\mu(E_i) < \infty$, are dense in L^p.

Proof. Given $f \in L^p$ pick f_n simple such that $f_n \to f$ almost everywhere and $|f_n| \nearrow |f|$. Then

$$f_n \in L^p \quad \text{and} \quad \|f_n - f\|_p \to 0 \text{ by DCT.}$$

Write f_n in standard form:

$$f_n = \sum_{j=1}^{m} a_{n,j} \mathbf{1}_{E_{n,j}}, \qquad \substack{a_{n,j} \neq 0 \\ E_{n,j} \text{ disjoint}}.$$

Then

$$|f_n|^p = \sum_{j}^{m} |a_{n,j}|^p \mathbf{1}_{E_{n,j}}$$

and

$$\sum_{j=1}^{m} \underbrace{|a_{n,j}|^p}_{>0} \mu(E_{n,j}) = \int |f_n|^p < \infty.$$

■

Corollary 4.2.2. $C_c^0(\mathbb{R}^n) \subset L^p(\mathbb{R}^n, m)$ dense for $1 \leq p < \infty$.

Proof. Proof as for $p = 1$: DIY. ■

Note. This corollary is for $\mathbb{R}^n$ not for abstract measure spaces like simple functions.

Problem 4.2.4. $L^\infty = ?$

Example. $X = \{1, 2, \ldots, n\}$, $\mathcal{A} = \mathcal{P}(X)$, $\mu = \text{count}$. $L^p = \mathbb{C}^n$ with norm

$$\|(a_1, \ldots, a_n)\|_p = \left(\sum_{j=1}^{n} |a_j|^p \right)^{1/p} \underbrace{\to}_{\text{DIY}} \max_j |a_j| \text{ as } p \to \infty.$$

In this case, $L^\infty = \mathbb{C}^n$ with $\|(a_1, \ldots, a_n)\|_\infty = \max_j |a_j|$.

Problem 4.2.5. In general, set $\|f\|_\infty = \sup_{x \in X} |f(x)|$?

Answer. Not quite. Do a.e. version because for $f = \begin{cases} 1, & \text{if } x = 1 \\ 0, & \text{otherwise} \end{cases}$. Then $\|f\|_\infty = 1$, but f should be identified with 0. ⊛

Definition 4.2.2. If $f : X \to \mathbb{C}$ is measurable, set

$$\begin{aligned} \|f\|_\infty &:= \operatorname{ess\,sup}(f) \\ &:= \inf\{a \in [0,\infty) \mid \mu\{|f| > a\} = 0\} \\ &= \inf\{\sup |g| \mid g = f \text{ a.e.}\}. \end{aligned}$$

Definition 4.2.3. Now we define L^∞ as

$$L^\infty := \{f : X \to \mathbb{C} \text{ measurable} \mid \|f\|_\infty < \infty\}.$$

Definition 4.2.4. Additionally, we have

$$\ell^\infty = \{(a_j)_{j=1}^\infty \mid a_j \in \mathbb{C},\ \|(a_j)\| := \sup_j |a_j| < \infty\}.$$

Example. $f(x) = x \cdot \mathbf{1}_{\mathbb{Q}}$. Then $f \in L^\infty$ because $\|f\|_\infty = 0$.

Lemma 4.2.2. If $f \in L^\infty$ then

(a) $|f(x)| \le \|f\|_\infty$ for a.e. x

(b) $\exists g$ bounded such that $f = g$ a.e.

Theorem 4.2.3. We have the following similar results for L^∞:

(1) $\|fg\|_1 \le \|f\|_1 \cdot \|g\|_\infty$, "$\frac{1}{1} + \frac{1}{\infty} = 1$"

(2) L^∞ is a normed vector space.

(3) $f_n \to f \in L^\infty \Leftrightarrow \exists$ null set at E such that $f_n \to f$ uniformly on E^c

(4) Simple functions are dense in L^∞.

Proof. DIY. ■

Remark. $C_c^0(\mathbb{R}^n) \subset L^\infty(\mathbb{R}^n, m)$ is not dense. The function $f = \mathbf{1}_{\mathbb{R}^n} \in L^\infty$ cannot be approximated by elements of $C_c^0(\mathbb{R}^n)$.

Theorem 4.2.4. L^∞ is a Banach space.

Proof. Suppose $f_n \in L^\infty$ and $B := \sum_1^\infty \|f_k\|_\infty < \infty$. Set

$$E_k = \{|f_k| > \|f_k\|_\infty\}, \quad E = \bigcup_k E_k \Rightarrow \mu(E) = 0.$$

For $x \in E^c$, we have that

$$\sum_1^\infty |f_k(x)| \le \sum_1^\infty \|f_k\|_\infty \le B < \infty.$$

Therefore, the series $\sum_1^\infty f_k(x) =: g(x)$ converges for $x \in E^c$. Can extend g to X by $g = e^\pi$ on E.

Set

$$g_n(x) = \begin{cases} \sum_1^n f_k(x), & \text{if } x \in E^c \\ e^\pi, & \text{if } x \in E \end{cases}.$$

Then $\|g_n - g\|_\infty \to 0$. ∎

4.3 Relationships between L^p spaces

Example. $(\mathbb{R}, \mathcal{L}, m)$, $f(x) = \frac{1}{x^\alpha}\mathbf{1}_{(0,1)}$, $g(x) = \frac{1}{x^\alpha}\mathbf{1}_{1,\infty}$. Then

$$\begin{cases} f \in L^p \Leftrightarrow p\alpha < 1 \\ g \in L^p \Leftrightarrow p\alpha > 1 \end{cases}.$$

Proposition 4.3.1. If $\mu(X) < \infty$, then $L^p \subset L^q$ for $0 < p < q \leq \infty$.

Lecture 25: Twenty-Fifth Lecture

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Proposition 4.3.2. If $\mu(X) < \infty$ and $0 < p < q \leq \infty$, then $L^p \supset L^q$

Proof. Suppose $f \in L^q$. There are two cases

Case 1: $q = \infty$

We have that $|f(x)| \leq \|f\|_\infty$ for a.e. x. Therefore,

$$\int |f|^p \leq \int \|f\|_\infty^p = \|f\|_\infty^p \cdot \underbrace{\mu(X)}_{<\infty}.$$

Case 2: $q < \infty$

Use Holder with conjugate exponents $\frac{q}{p}$, $\frac{q}{q-p}$.

$$\begin{aligned} \int |f|^p &= \int |f|^p \cdot 1 \\ &\leq \left(\int |f|^{p \cdot \frac{1}{p}}\right)^{\frac{p}{q}} \cdot \left(\int 1^{\frac{q}{q-p}}\right)^{\frac{q-p}{q}} \\ &= \|f\|_q^p \cdot \mu(X)^{\frac{q-p}{q}} < \infty. \end{aligned}$$

Alternatively,

$$\|f\|_p \leq \|f\|_q \cdot \mu(x)^{\frac{1}{p} - \frac{1}{q}}.$$

∎

Definition 4.3.1. f is log concave if $0 < p < q < r \leq \infty$ implies that

$$\|f\|_q \leq \|f\|_p^\lambda + \|f\|_r^\lambda$$

where $\frac{1}{q} = \lambda \cdot \frac{1}{p} + (1-\lambda) \cdot \frac{1}{r}$. In particular, $L^p \cap L^r \subset L^q$.

Proposition 4.3.3. If $f : X \to \mathbb{C}$ is measurable, then the function

$$[0, \infty) \ni t \mapsto \|f\|_{1/t} \in [0, \infty)$$

is log concave.

Proof. We have 2 cases:

Case 1: $r = \infty$

$\lambda = p/q$ so we must prove that $\|f\|_q \leq \|f\|_p^{p/q} \cdot \|f\|_\infty^{1-p/q}$. Since $f|^q \leq |f|^p \cdot \|f\|_\infty^{q-p}$ a.e. , we have that

$$\begin{aligned} \|f\|_q &= \left(\int |f|^q \right)^{1/q} \\ &\leq \left(\int |f|^p \cdot \|f\|_\infty^{q-p} \right)^{\frac{1}{q}} \\ &= \|f\|_p^{p/q} \cdot \|f\|_\infty^{1-p/q}. \end{aligned}$$

Case 2: $r < \infty$

Use Holder with conjugate exponents $\frac{p}{\lambda q}$, $\frac{r}{(1-\lambda)q}$.

$$\begin{aligned} \int |f|^q &= \int |f|^{\lambda q} \cdot |f|^{(1-\lambda)q} \\ &\leq \left(\int |f|^p \right)^{\frac{\lambda q}{p}} \cdot \left(\int |f|^r \right)^{\frac{(1-\lambda)q}{r}}. \end{aligned}$$

Take qth roots.

■

Remark. We have the following fact: any log concave function $h : (a, b) \to \mathbb{R}$ is continuous.

Corollary 4.3.1. If $f \in L^P \cap L^r$, then

$$(p, r) \ni q \mapsto \|f\|_q \in [0, \infty)$$

is continuous.

Proposition 4.3.4. If $0 < p < q \leq \infty$, then $\begin{cases} \ell^p \subset \ell^q \\ \|f\|_q \leq \|f\|_p, \forall f \in \ell^p \end{cases}$

Proof. DIY. Show $\|f\|_\infty \leq \|f\|_p$, then use proposition. ■

Proposition 4.3.5. If $0 < p < q < r \leq \infty$, then $L^q \subset L^p + L^r$.

Sketch. Given $f \in L^q$, set $E = \{|f| > 1\}$. Then we can write

$$f = \underbrace{f \cdot \mathbf{1}_E}_{\in L^p} + \underbrace{f \cdot \mathbf{1}_{E^c}}_{\in L^r}.$$

■

Exercise. Find $f \in L^1(\mathbb{R})$ such that $f \notin L^p(u)$ for any $p > 1$ and any nonempty open $u \subset \mathbb{R}$.

Answer. Use if $\alpha > 0$ then $\int_0^1 \frac{dx}{x^\alpha} = \begin{cases} \frac{1}{1-\alpha}, & \text{if } \alpha < 1 \\ \infty, & \text{if } \alpha \geq 1 \end{cases}$. Pick an enumeration $(x_n)_1^\infty$ of $\mathbb{Q}$. Consider

$$\sum_{n=1}^{\infty} 2^{-n} \frac{1-\alpha_n}{(x-x_n)^{\alpha_n}} \cdot \mathbf{1}_{(x_n, x_{n+1})}.$$

For some $\alpha_n \nearrow 1$, $\alpha_n < 1$ i.e. $\alpha_n = 1 - \frac{1}{n}$. ⊛

Exercise. Prove $f \in L^3([-1,1])$, then

$$\int_{-1}^{1} \frac{|f(x)|}{\sqrt{|x|}} dx < \infty.$$

Answer. We have that

$$\int_{-1}^{1} |f(x)| \cdot |x|^{-1/2} dx \leq \left(\int_{-1}^{1} |f(x)|^3 dx \right)^{\frac{1}{3}} \cdot \left(\int_{-1}^{1} |x|^{-\frac{1}{2} \cdot \frac{3}{2}} \right)^{\frac{2}{3}} < \infty.$$

⊛

4.4 Some Functional Analysis

Definition. We will mainly work over $K = \mathbb{R}$ or $\mathbb{C}$. Let $(V, \|\cdot\|)$ be a normed vector space

Definition 4.4.1. $f : V \to K$ is a linear function if

- $f(\lambda v) = \lambda f(v)$, for $\lambda \in K$, $v \in V$
- $f(v + w) = f(v) + f(w)$, for $v, w \in V$

Definition 4.4.2. Let $f : V \to K$ be a linear functional. We say f is bounded if $\exists C > 0$ such that

$$|f(v)| \leq C \cdot \|v\|.$$

Proposition 4.4.1. The following are equivalent:

(1) f is bounded

(2) f is continuous

(3) f is continuous at $0 \in V$

(1) $\Rightarrow$ (2). Assume f is bounded. Then for $v, w \in V$ we have that

$$|f(v) - f(w)| = |f(v - w)| \leq C \cdot \|v - w\|.$$

Thus, as $w \to v$ we have that $f(w) \to f(v)$. ■

(2) $\Rightarrow$ (3). This is trivial. ■

(3) $\Rightarrow$ (1). We know that $\exists \delta > 0$ such that if $\|v\| < \delta$ then $|f(v)| \leq 1$ $(\epsilon = 1)$.

Therefore, for $v \neq 0$ we have that

$$|f(v)| = \left|f\left(\frac{\delta}{\|v\|} \cdot v\right)\right| \cdot \frac{\|v\|}{\delta} \leq \frac{\|v\|}{\delta}.$$

■

Lecture 26: Dual of L^p

15 Mar. 11:00

Definition 4.4.3. If V is a normed vector space, then the dual V^* of V is the space of bounded linear functionals on V

- V^* is a Banach space.
- $L^{p*} = L^q$ for $1 < p, q < \infty$.

Definition 4.4.4. Given $\phi \in V^*$, set

$$\begin{aligned}\|\phi\| &:= \inf\{C > 0 | |\phi(v)| \leq C\|v\|, \forall v \in V\} \\ &\underbrace{=}_{\text{check}} \sup_{v\neq 0} \frac{|\phi(v)|}{\|v\|} \\ &\underbrace{=}_{\text{check}} \sup_{v=1} |\phi(v)|.\end{aligned}$$

Theorem 4.4.1. V^* is a Banach space i.e.

(1) V^* is a vector space

(2) V^* is a normed vector space

Proof of (1). DIY. ■

Proof of (2). Let $\phi, \psi \in V^*$. We must show that $\|\phi + \psi\| \leq \|\phi\| + \|\psi\|$. We know that

$$|\phi(v)| \leq \|\phi\| \cdot \|v\| \quad \text{and} \quad |\psi(v)| \leq \|\psi\| \cdot \|v\|, \quad \forall v \in V.$$

Therefore,

$$\begin{aligned}|(\phi + \psi)(v)| &\leq |\phi(v)| + |\psi(v)| \\ &\leq (\|\phi\| + \|\psi\|)\,\|v\|.\end{aligned}$$

It follows that

$$\|\phi + \psi\| \leq \|\phi\| + \|\psi\|.$$

Similarly, $\|\lambda\phi\| = |\lambda|\|\phi\|$ for $\lambda \in K$, $\phi \in V^*$.

Now we must show that V^* is complete. Suppose $(\phi_n)_n$ is a Cauchy sequence in V^*. Then $\forall v \in V$

$$\begin{aligned}|\phi_n(v) - \phi_m(v)| &= |(\phi_n - \phi_m)(v)| \\ &\leq \underbrace{\|\phi_n - \phi_m\|}_{\to 0 \text{ as } n,m\to\infty} \cdot \|v\|.\end{aligned}$$

Therefore, $\forall v$ $(\phi_n(v))_n$ is a Cauchy sequence in $K = \mathbb{R}$ or $\mathbb{C}$ which are complete. It follows that

$\phi(v) := \lim_{n\to\infty} \phi_n(v)$ exists. Check:

$$\begin{cases} \phi : V \to K \text{ is an element of } V^* \\ \|\phi_n - \phi\| \to 0 \text{ as } n \to \infty \end{cases}.$$

■

Problem 4.4.1. Does $(L^p)^* = L^q$ for $p^{-1} + q^{-1} = 1$?

Answer. Define

$$\begin{aligned} F : L^q &\longrightarrow (L^p)^* \\ g &\longmapsto \phi_g \end{aligned}$$

where $\phi_g(f) = \int fg$ for $f \in L^p$. This is well-defined by Holder's inequality i.e.

$$\|fg\|_1 \le \|f\|_p \cdot \|g\|_q.$$

It is easy to show that $\phi_g : L^p \to \mathbb{C}$ satisfies:

$$\begin{cases} \phi_g(f_1 + f_2) = \phi_g(f_1) + \phi_g(f_2) \\ \phi_g(\lambda f) = \lambda \phi_g(f) \end{cases}.$$

Also $|\phi_g(f)| = |\int fg| \le \int |fg| \le \|f\|_p \cdot \|g\|_q$. Therefore, ϕ_g is a bounded linear functional, namely $\|\phi_g\| \le \|g\|_q$. We get a map

$$\begin{aligned} L^q &\longrightarrow (L^p)^* \\ g &\longmapsto \phi_g \end{aligned}$$

that satisfies the boundedness above. This map is linear:

$$\begin{cases} \phi_{g_1+g_2} = \phi_{g_1} + \phi_{g_2} \\ \phi_{\lambda g} = \lambda \phi_g \end{cases}.$$

This is a linear map of vector spaces from $L^q \to (L^p)^*$, where $\|\phi_g\| \le \|g\|_q$. ⊛

Definition 4.4.5. An isometry is a map f such that $\|f(g)\| = \|g\|$.

Theorem 4.4.2. For $1 < p, q < \infty$ where $p^{-1} + q^{-1} = 1$, the linear map

$$\begin{aligned} L^q &\longrightarrow (L^p)^* \\ g &\longmapsto \phi_g \end{aligned}$$

is a bijection and an isometry (isometric isomorphism).

Proof. We will prove this through a sequence of claims.

Claim. $\|\phi_g\| = \|g\|_p$, $\forall g \in L^q$ (isometry).

Proof. Follows from below Lemma. ■

Claim. $g \mapsto \phi_g$ is injective.

Proof. Follows from previous claim. ■

Claim. $g \mapsto \phi_g$ is surjective.

Thus, the result follows directly from the claim. ∎

Lemma 4.4.1. If $g \in L^q$ for $1 < p, q < \infty$, then

$$\|g\|_q = \|\phi_g\| = \sup\left\{\left|\int fg\right| \;\middle|\; \|f\|_p \le 1\right\}.$$

Proof. We know from Holder that $\|\phi_g\| \le \|g\|_q$. To prove $\ge$ we may assume that $g \ne 0$ almost everywhere. From Holder, we know that we have equality when $|f|^p$ and $|g|^q$ are are proportional. Therefore, we must pick an f such that this is the case. Set

$$f := \frac{|g|^{q-1} \cdot \overline{\operatorname{sgn}(g)}}{\|g\|_q^{q-1}}.$$

where

$$\operatorname{sgn}(a) = \begin{cases} a/|a|, & \text{if } a \ne 0 \\ 0, & \text{if } a = 0 \end{cases}.$$

Then

$$fg = \frac{|g|^{q-1} \cdot |g|}{\|g\|_q^{q-1}} \ge 0.$$

Therefore,

$$\int fg = \frac{1}{\|g\|_q^{q-1}} \int |g|^q = \frac{\|g\|_q^q}{\|g\|_q^{q-1}} = \|g\|_q$$

and

$$\|f\|_p^p = \int |f|^p = \frac{1}{\|g\|_q^{p(q-1)}} \int |g|^{(q-1)p} = 1.$$

Thus, $(q-1)p = q$. ∎

Remark. The Lemma is also true for

- $(p, q) = (\infty, 1)$
- $(p, q) = (1, \infty)$ if μ is σ-finite

Remark. We also have the following:

- If μ is σ-finite, $L^\infty \to (L^1)^*$ is also an isometric isomorphism.
- In general, $L^1 \to (L^\infty)^*$ is an injective isometry but not surjective.

Corollary 4.4.1. If $1 < p, q < \infty$, then L^p is a reflexive Banach space i.e. $(L^p)^{**} = L^p$

Note. In the special case $p = q = 2$, we have that $L^2 \to (L^2)^*$ is an isometric isomorphism. This is a Hilbert space (will discuss later).

Proposition 4.4.2 (Chebyshev for L^p)**.** If $f \in L^p$ for $0 < p < \infty$ and $\alpha > 0$, then

$$\mu(\{|f| > \alpha\} \leq \left(\frac{\|f\|_p}{\alpha}\right)^p.$$

Proof. The proof is the same as for $p = 1$. Set $E_\alpha = \{|f| > \alpha\}$. Then

$$\int |f|^p \geq \int_{E_\alpha} |f|^p \geq \int_{E_\alpha} \alpha^p = \alpha^p \cdot \mu(E_\alpha).$$

■

Chapter 5

Complex and Signed Measures

Lecture 26: Signed Measures

Suppose we have some $f \in L^+(X, \mathcal{A}, \mu)$, then: 17 Mar. 11:00

- $\nu(E) = \int_E f d\mu$ defines a new measure on $(X, \mathcal{A})$

What if $f \in L^1$? i.e $f : X \to \mathbb{R}$ or $f : X \to \mathbb{C}$.

5.1 Signed Measures

Definition 5.1.1. Fix $(X, \mathcal{A})$ measurable space. A signed measure is a function

$$\nu : \mathcal{A} \to (-\infty, \infty] \text{ or } [-\infty, \infty)$$

such that

- $\nu(\emptyset) = 0$,
- If E_j, $j \in \mathbb{N}$ are disjoint sets in $\mathcal{A}$, then
  - $\nu(\bigcup_{j=1}^{\infty} E_j) = \sum_{j=1}^{\infty} \nu(E_j)$
  - If $\left|\nu(\bigcup_{j=1}^{\infty} E_j)\right| < \infty$, then $\sum_{j=1}^{\infty} |\nu(E_j)| < \infty$

Example. The following are examples of signed measures:

(a) If ν is a positive measure i.e. $\nu(A) \in [0, \infty]$, then ν is a signed measure.

(b) If μ_1, μ_2 positive measure, μ_1 or μ_2 is finite ($\mu_i(X) < \infty$) then

$$\nu := \mu_1 - \mu_2 \quad \text{is a signed measure.}$$

(c) If μ is a positive measure. $f : X \to \overline{\mathbb{R}}$ is measurable such that

$$\int f^+ d\mu < \infty \quad \text{or} \quad \int f^- d\mu < \infty$$

then

$$\nu(E) := \int_E f d\mu \quad \text{is a signed measure.}$$

Lemma 5.1.1. If $A \subset B$ then it is not necessarily the case that $\nu(A) \leq \nu(B)$ but

$$\begin{cases} \nu(A) = \infty \Rightarrow \nu(B) = \infty \\ \nu(B) = -\infty \Rightarrow \nu(A) = -\infty \end{cases}.$$

Proof. DIY: $\nu(B) = \nu(A) + \nu(B \setminus A)$. ■

Lemma 5.1.2. Suppose $E_j \in \mathcal{A}$, $j \in \mathbb{N}$ then

(a) If $E_1 \subset E_2 \subset \cdots$ then $\nu(\bigcup_{j=1}^{\infty} E_j) = \lim_{j\to\infty} \nu(E_j)$

(b) If $E_1 \supset E_2 \supset \cdots$ and $-\infty < \nu(E_1) < \infty$ then

$$\nu(\bigcap_{j=1}^{\infty} E_j) = \lim_{j\to\infty} \nu(E_j).$$

Proof. DIY. Use above lemma for (b). ■

Definition 5.1.2. Given $(X, \mathcal{A})$, signed measure ν, and $E \in \mathcal{A}$ say:

- E is positive for ν if $E \supset F \in \mathcal{A} \Rightarrow \nu(F) \geq 0$.
- E is negative for ν if $E \supset F \in \mathcal{A} \Rightarrow \nu(F) \leq 0$.
- E is null for ν if $E \supset F \in \mathcal{A} \Rightarrow \nu(F) = 0$.

Lemma 5.1.3. If E is positive for ν, $E \supset F \in \mathcal{A}$ then

$$0 \leq \nu(F) \leq \nu(E) \quad \text{and} \quad \nu(E \setminus F) \geq 0.$$

Proof. $\nu(E) = \nu(E \setminus F) + \nu(F)$. ■

Lemma 5.1.4. Suppose $(X, \mathcal{A})$ and ν is a signed measure.

(a) If E is positive/negative/null for ν, $E \supset F \in \mathcal{A}$ then F is positive/negative/null for ν.

(b) If E_j is positive for ν, $j \in \mathbb{N}$ then $\bigcup_{j=1}^{\infty} E_j$ postive for ν.

> **Remark.** Same is true for "negative" and "null".

Proof. (a) is clear (DIY). For (b), set $G_j := E_j \setminus E_{j-1}$ for $j \in \mathbb{N}$ with $E_0 = \emptyset$. Then G_j is positive for ν by (a). We then have that $\bigcup_{j=1}^{\infty} G_j = \bigcup_{j=1}^{\infty} E_j$, where the G_j's are disjoint. Suppose that $F \in \bigcup_{j=1}^{\infty} E_j$. Then

$$F = \bigcup_{j=1}^{\infty} F \cap G_j \Rightarrow \nu(F) = \sum_{j=1}^{\infty} \nu(F \cap G_j) \geq 0$$

because $F \cap G_j \subset E_j$ and each E_j is positive for ν. ■

Theorem 5.1.1. Suppose $(X, \mathcal{A})$ and ν is a signed measure. Then $\exists P, N \in \mathcal{A}$ such that

- $X = P \cup N$ and $P \cap N = \emptyset$.
- P positive for ν.
- N negative for ν.

Furthermore, if (P', N') is another such pair, then $P \triangle P' = N \triangle N'$ is null for ν.

Proof of uniqueness. $P \triangle P' = N \triangle N'$ is both positive and negative for ν. Thus, it must be null. To see this

$$P \triangle P' = (P \setminus P') \sqcup (P' \setminus P),$$

which must be positive. Similarly, we must have $N \triangle N'$ must be negative. ■

Proof of existence. Without loss of generality, $\nu(A) \neq \infty$ for every $A \in \mathcal{A}$. Otherwise, look at $-\nu$. Set

$$m \coloneqq \sup\{\nu(E) \mid X \supset E \in \mathcal{A} \text{ positive for } \nu\}.$$

Pick P_j positive such that $\nu(P_j) \to m$. From previous lemmas,

$$\begin{cases} P :- \bigcup_{j=1}^{\infty} P_j \text{ is positive for } \nu \\ \nu(P) \geq \nu(P_j), \text{ for every } j \end{cases}.$$

Therefore, we must have that $\nu(P) = m$ for $m \in [0, \infty)$.

Set $N \coloneqq P^c$. We must show that N is negative for ν. Suppose that N is not negative. Then $\exists F \subset N$ such that $\nu(F) > 0$. Therefore, we can pick $n_1 \in \mathbb{N}$ minimal such that $\exists A_1 \subset n$ with $\nu(A_1) \geq \frac{1}{n_1}$. We cannot have that A_1 is positive for ν because then

$$\begin{gathered} P \sqcup A_1 \text{ is positive for } \nu \\ \nu(P \sqcup A_1) > m. \end{gathered}$$

Therefore, we must have some $F_1 \subset A_1$ such that $\nu(F_1) < 0$. So we can pick $n_2 \in \mathbb{N}$ minimal such that $\exists B_2 \subset A_1$ with $\nu(B_2) \leq \frac{-1}{n_2}$. Set $A_2 \coloneqq A_1 \setminus B_2$ such that $\nu(A_2) \geq \nu(A_1) + \frac{1}{n_2}$. If we continue like this, then we will get $N \coloneqq A_0 \supset A_1 \supset A_2 \supset \cdots$ such that for every j

$$\begin{cases} \nu(A_j) \geq \nu(A_{j-1}) + \frac{1}{n_j} \\ \text{If } A \subset A_{j-1}, \text{ then } \nu(A) < \nu(A_{j-1}) + \frac{1}{n_j - 1} \end{cases}.$$

Set $A \coloneqq \bigcap_{j=1}^{\infty} A_j$. Then

$$\infty > \nu(A) = \lim_{j \to \infty} \nu(A_j) \geq \sum_{j=1}^{\infty} \frac{1}{n_j}.$$

Therefore, $n_j \to \infty$ as $j \to \infty$. We cannot have A positive for ν. Therefore, $\exists B \subset A$ such that $\nu(B) < 0$. Set $A' = A \setminus B$ such that

$$\nu(A') = \nu(A) - \nu(B) > \nu(A) + \frac{1}{n}.$$

For $j \gg 1$ we have $n_j > n$ such that $A' \subset A_{j-1}$ which contradicts choice of n_j. ■

Lecture 27: The Jordan Decomposition

20 Mar. 11:00

Definition 5.1.3. Given a signed measure ν on $(X, \mathcal{A})$ define

$$\begin{cases} \nu^+(E) \coloneqq \nu(E \cap P) \geq 0 \\ \nu^-(E) \coloneqq -\nu(E \cap N) \geq 0 \end{cases} \quad E \in \mathcal{A}.$$

Lemma 5.1.5. We have the following:

- ν^+, ν^- are positive measures on $(X, \mathcal{A})$
- At least one of ν^+, ν^- is finite i.e. $\nu^+(X) < \infty$ or $\nu^-(X) < \infty$.
- $\nu = \nu^+ - \nu^-$

Proof. DIY. ■

Definition 5.1.4. If ν_1, ν_2 are signed measures on $(X, \mathcal{A})$, then we say ν_1, ν_2 are mutually singular, written $\nu_1 \perp \nu_2$, if

$$X = E_1 \sqcup E_2, \text{ where } \begin{cases} E_1 \text{ null for } \nu_2 \\ E_2 \text{ null for } \nu_1 \end{cases}.$$

Example. $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$. We have the following measures:

- $\mu_1 = m$ Lebesgue.
- $\mu_2 = \sum_{j=1}^{\infty} c_j \delta_{x_j}$ discrete measure.
- $\mu_3 = \mu_C$ Cantor measure.

These are all mutually singular. For $\mu_1 \perp \mu_2$, we have $E_1 = \mathbb{R} \setminus \{x_j\}_1^\infty$, $E_2 = \{x_j\}_1^\infty$.

Example. ν^+ and ν^- are mutually singular.

$$\begin{cases} P \text{ null for } \nu^- \\ N \text{ null for } \nu^+ \end{cases}.$$

Theorem 5.1.2. If ν is a signed measure on $(X, \mathcal{A})$, then there is a unique positive measure ν^+ on $(X, \mathcal{A})$ such that

$$\begin{cases} \nu^+ \perp \nu^- \\ \text{at least one of } \nu^+, \nu^- \text{ is finite} \\ \nu = \nu^+ - \nu^- \end{cases}.$$

Proof. To prove existence, we can just define $\nu^\pm$ as above. Now we must prove uniqueness. Suppose $\nu = \nu^+ - \nu^- = \tilde{\nu}^+ - \tilde{\nu}^-$, where $\tilde{\nu}^\pm$ such that

$$\begin{cases} \tilde{\nu}^+ \perp \tilde{\nu}^- \\ \text{at least one of } \tilde{\nu}^+, \tilde{\nu}^- \text{ is finite} \end{cases}.$$

Write $X = \tilde{P} \sqcup \tilde{N}$, where $\begin{cases} \tilde{P} \text{ null for } \tilde{\nu}^- \\ \tilde{N} \text{ null for } \tilde{\nu}^+ \end{cases}$. Therefore,

$$\begin{cases} \tilde{P} \text{ positive for } \nu \\ \tilde{N} \text{ negative for } \nu \end{cases}.$$

By Hahn decomposition, $P \triangle \tilde{P} = N \triangle \tilde{N}$ are null for ν. Now we want to show that $\tilde{\nu}^+ = \nu^+$ which implies the same equality for the $-$ counterpart. Pick any $A \in \mathcal{A}$. Then

$$\tilde{\nu}^+(A) = \tilde{\nu}^+(A \cap \tilde{P}) + \underbrace{\tilde{\nu}^+(A \cap \tilde{N})}_{=0} = \tilde{\nu}^+(A \cap \tilde{P})$$

However,

$$\tilde{\nu}^+(A \cap \tilde{P}) = \nu(A \cap \tilde{P}) - \underbrace{\tilde{\nu}^-(A \cap \tilde{P})}_{=0} = \nu(A \cap \tilde{P}).$$

We also have that

$$\nu^+(A) = \nu(A \cap P).$$

But $P \triangle \tilde{P}$ is null for ν. Thus, $\nu(A \cap (P \triangle \tilde{P})) = 0$ and $\nu(A \cap P) = \nu(A \cap \tilde{P})$. ■

Definition 5.1.5. The total variation of a signed measure ν is

$$|\nu| := \nu^+ - \nu^-$$
$$\underbrace{|\nu|(A)}_{\neq |\nu(A)| \text{ in general}} := \nu^+(A) + \nu^-(A).$$

Lemma 5.1.6. $|\nu|$ is a positive measure on $(X, \mathcal{A})$.

Proof. DIY. ■

Definition 5.1.6. We set

$$L^1(\nu) := L^1(|\nu|) = \{f : X \to \mathbb{C} \text{ measurable} \mid \int |f| d|\nu| < \infty\}$$
$$\underbrace{=}_{\text{check}} L^1(\nu^+) \cap L^1(\nu^-).$$

If $f \in L^1(\nu)$ set

$$\int f d\nu = \int f d\nu^+ - \int f d\nu^-.$$

Example. If μ is a positive measure on $(X, \mathcal{A})$ and $f : X \to \mathbb{R}$ is integrable, and

$$\nu(E) := \int_E f d\mu$$

then ν is a signed measure and

$$\nu^\pm(E) = \int_E f^\pm d\mu, \text{ where } f = f^+ - f^-.$$

Then $P = \{f \geq 0\}$ and $N = \{f \leq 0\}$, and $|\nu|(E) = \int_E |f| d\mu$.

Lemma 5.1.7. If ν is a signed measure, then

(a) $|\nu(E)| \leq |\nu|(E)$ for every $E \in \mathcal{A}$.

(b) E is a $\underbrace{\text{null set}}_{\substack{|\nu|(E)=0 \\ \nu^+(E)=\nu^-(E)=0}}$ for ν.

(c) If κ is a signed measure then

$$\kappa \perp \nu \Leftrightarrow \kappa \perp |\nu| \Leftrightarrow \kappa \perp \nu^+ \text{ and } \kappa \perp \nu^-.$$

Definition 5.1.7. A signed measure ν is

$$\begin{cases} \text{finite} \\ \sigma\text{-finite} \end{cases} \Leftrightarrow |\nu| \text{ is } \begin{cases} \text{finite} \\ \sigma\text{-finite} \end{cases}.$$

5.2 The Lebesgue-Radon-Nikodyn Theorem

Problem 5.2.1. $\mu := m + \mu_C + \sum_1^\infty c_j \delta_{x_j}$ on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$. Given μ, can we reconstruct this decomposition?

Problem 5.2.2. Given μ positive measure and ν signed measure, how to see whether

$$\nu(E) = \int_E f \, d\mu \tag{$\star$}$$

for some $f \in L^1(\mu)$ ($\mathbb{R}$-valued), and if so how to recover f?

Definition 5.2.1. If μ is a positive measure and ν is a finite signed measure, say

$$\nu \ll \mu$$

ν is absolutely continuous with respect to μ if

$$\mu(E) = 0 \Rightarrow \nu(E) = 0.$$

Lecture 28: The Radon-Nykodyn Theorem

22 Mar. 11:00

Proposition 5.2.1. If ν is a finite signed measure and μ is a positive measure on $(X, \mathcal{A})$ then the following are equivalent

(i) $\nu \ll \mu$

(ii) $\forall \epsilon > 0 \; \exists \delta > 0$ such that $\mu(E) < \delta \Rightarrow |\nu(E)| < \epsilon$.

Proof. For (ii) $\Rightarrow$ (i), it is clear. Now for (i) $\Rightarrow$ (ii), assume $\exists \epsilon > 0$ such that $\forall n \in \mathbb{N} \; \exists E_n \in \mathcal{A}$ such that $\mu(E_n) < 2^{-n}$ but $|\nu(E_n)| \geq \epsilon$. For $k \in \mathbb{N}$ set $F_k := \bigcup_{n \geq k} E_n$, $F := \bigcap_k F_k = \limsup_n E_n$. Then

- $\mu(F_k) < \sum_{n=k}^\infty \mu(E_n) = 2^{1-k}$. Therefore, $\mu(F) = 0$.
- Without loss of generality, $\nu(E_n) \geq \epsilon$ because one of ν^+ or ν^- is positive by Jordan decomposition. Then $\nu(F_k) \geq \epsilon \Rightarrow \nu(F) = \lim_{k \to \infty} \nu(F_k) \geq \epsilon$ (need ν finite here).

Thus, $\nu \not\ll \mu$ and we have a contradiction. ■

Corollary 5.2.1. If μ is a positive measure and $f \in L^1(\mu)$ then $\forall \epsilon > 0 \; \exists \delta > 0$ such that

$$\mu(E) < \delta \Rightarrow \int_E |f| \, d\mu < \epsilon.$$

Theorem 5.2.1 (Radon-Nikodyn Theorem)**.** If μ is a σ-finite positive measure, ν is a σ-finite signed measure on $(X, \mathcal{A})$ such that $\nu \ll \mu$ then $\exists f : X \to \mathbb{R}$ extended integrable such that $\nu(E) = \int_E f \, d\mu$ for every $E \in \mathcal{A}$.

Notation. We write $d\nu = f d\mu$, or $f = \frac{d\nu}{d\mu}$.

Example. μ_F Lebesgue-Stieltjes measure on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ associated to $F(x) = e^{2x}$ such that

$$\mu_F((a, b]) = e^{2b} - e^{2a}.$$

Compare to m = Lebesgue measure. Set $\kappa(E) := \int_E 2e^{2x} dx = \int_E f dm$ for $f(x) = 2e^{2x}$. Then $\kappa((a, b]) = \int_a^b 2e^{2x} dx = e^{2b} - e^{2a} = \mu_F((a, b])$. By uniqueness, $\kappa = \mu_F$. Therefore, $\mu_F \ll m$ and

$$\frac{d\mu_F}{dm} = 2e^{2x} = F'(x).$$

Example (Non-example)**.** $F : \mathbb{R} \to \mathbb{R}$ Cantor function such that $F'(x) = 0$ m almost everywhere. Then μ_F is the Cantor measure. However, $\frac{d\mu_F}{dm} \neq F'(x) = 0$. $\mu_F \not\ll m$. In fact, they are mutually singular.

Lemma 5.2.1. If μ, ν are finite positive measures on $(X, \mathcal{A})$ such that $\mu \not\perp \nu$, then $\exists E \in \mathcal{A}$ such that $\mu(E) > 0$ and $\nu \geq \epsilon\mu$ on E (i.e. on $F \subset E$).

Proof. For $n \in \mathbb{N}$, $X = P_n \sqcup N_n$ by Hahn-decomposition with respect to $\nu - \frac{1}{n}\mu$. Set $P = \bigcup_{n=1}^{\infty} P_n$, $N = P^c = \bigcap_{n=1}^{\infty} N_n$. Then

$$\nu \leq \frac{1}{n}\mu \text{ on } N_n \Rightarrow \nu = 0 \text{ on } N.$$

If $\mu(P) = 0$, then $\mu \perp \nu$. Otherwise, $\mu(P_n) > 0$ for some n. Pick $\epsilon = \frac{1}{n}$, $E = P_n$. ∎

Sketch of RN Theorem Proof. We will proceed by steps.

Step 1: First suppose μ, ν are finite postive

For uniqueness, suppose $d\nu = f_1 d\mu = f_2 d\mu$. For $f_i \in L^1(\mu)$, $f_i \geq 0$. Set $g = f_1 - f_2 \in L^1(\mu)$. Then

$$\int g_E d\mu = 0 \ \forall E \in \mathcal{A} \Rightarrow g = 0 \ \mu \text{ a.e.}.$$

For existence, define

$$\mathcal{F} := \left\{ f \in L^+ \middle| \int_E f d\mu \leq \nu(E), \ \forall E \in \mathcal{A} \right\} \Big/ \underbrace{\sim}_{\mu \text{ a.e.}}.$$

Now we can define the partial order on $\mathcal{F}$: $f_1 \leq f_2$ iff $f_1(x) \leq f_2(x)$ for μ a.e. x. Then we have the minimal element $0 \in \mathcal{F}$. Then the idea is to prove that $\mathcal{F}$ has a unique maximal element f, such that $d\nu = f d\mu$.

Claim. If $f, g \in \mathcal{F}$, then $\max\{f, g\} \in \mathcal{F}$.

Proof. Set $A := \{f > g\}$. Then

$$\begin{aligned}\int_E \max\{f, g\} &= \int_{E\cap A} f d\mu + \int_{E \cap A^c} g d\mu \\ &\leq \nu(E \cap A) + \nu(E \cap A^c) = \nu(E).\end{aligned}$$

∎

Set $a := \sup\{\int_X f d\mu \mid f \in \mathcal{F}\}$. Then $a \leq \nu(X) < \infty$. Pick $g_n \in \mathcal{F}$ such that $\int g_n d\mu \geq a - \frac{1}{n}$. Set $f_n = \max\{g_1, \ldots, b_n\}$. From the claim $f_n \in \mathcal{F}$ and $\int f_n d\mu \geq a - \frac{1}{n}$. Therefore, $f_n \nearrow f \in L^+$. By MCT, for every E, $\int_E f d\mu = \lim_{n\to\infty} \int_E f_n d\mu \leq \nu(X)$. Now we have that $f \in \mathcal{F}$ and $\int f d\mu = a$.

Define ν' positive measure defined by $\nu'(E) = \int_E f d\mu$. We know that $\nu' \leq \nu$, so we can define the finite positive measure $\tilde{\nu} = \nu - \nu'$. Then $\tilde{\nu} \ll \mu$. We want to show that $\tilde{\nu} = 0$. If $\tilde{\nu} \neq 0$, then $\tilde{\nu} \not\perp \mu$. From the lemma, $\exists E \in \mathcal{A}$ such that $\mu(E) > 0$ and $\epsilon > 0$ such that $\tilde{\nu} \geq \epsilon\mu$ on E. Set

$$g := f + \epsilon \mathbf{1}_E.$$

Then $g \in \mathcal{F}$ and $\int g d\mu > a$. This is a contradiction. Thus, $\tilde{\nu} = 0$.

■

Lecture 29: The Radon-Nykodyn Theorem cont.

24 Mar. 11:00

Theorem 5.2.2 (Lebesgue-Radon Nikodyn Theorem)**.** If μ is a σ-finite positve measure and ν is a σ-finite signed measure on $(X, \mathcal{A})$ then $\exists!$ decomposition

$$\nu = \lambda + \rho,$$

where λ, ρ are σ-finite signed measures such that λ is mutually singular with respect to μ and $\rho \ll \mu$. Moreover, $\exists!$ extended μ-measurable function $f : X \to \mathbb{R}$ such that $d\rho = f d\mu$.

> **Remark.** Even if ν is a positive measure, λ may not be positive.

Sketch (see last class + Folland). We will once again proceed by steps:

Step 1: Assume μ, ν are finite positive measures.

Set $\mathcal{F} = \left\{f \in L^+ \mid \int_E f d\mu \leq \nu(E), \forall E \in \mathcal{A}\right\} \Big/ \sim$. Prove as last time that $\mathcal{F}$ is a poset and $\mathcal{F}$ has a unique maximal element $f \in L^+ \cap L^1$. Define ρ finite positive measure by $d\rho = f d\mu$. Define $\lambda = \nu - \rho$ finite signed measure. Then from the lemma, we get that $\lambda \perp \mu$.

Step 2: Now assume μ, ν are σ-finite positive measures.

We can write $X = \bigsqcup_{j=1}^\infty A_j$ such that $\mu(A_j), \nu(A_j) < \infty$. Now apply Step 1 to $\mu_j := \mu|_{A_j}$, $\nu_j := v|_{A_j}$. Then

$$\mu_j = \lambda_j + \rho_j, \text{ where } \begin{cases} \lambda_j \perp \mu_j \\ d\rho_j = f_j d\mu_j \end{cases}.$$

For $f_j \in L^+ \cap L^1(\mu_j)$. Then we can "glue together" λ_j, ρ_j, f_j to get λ, ρ, f i.e.

$$f(x) = f_j(x), \quad x \in A_j$$
$$\lambda(E) = \sum_{j=1}^\infty \lambda_j(E \cap A_j).$$

Step 3: μ σ-finite positive measure, ν σ-finite signed measure.

Apply Step 2 to Jordan Decomposition $\nu^\pm$.

Step 4: Uniqueness.

Suppose we have that

$$\nu = \lambda_1 + \rho_1 = \lambda_2 + \rho_2, \quad \lambda_i \perp \mu,\ \rho_i \ll \mu.$$

If μ, ν finite then

$$\lambda_1 - \lambda_2 = \rho_2 - \rho_1 =: \kappa.$$

Then $\kappa \perp \mu$ and $\kappa \ll \mu$ implies that $\kappa = 0$. If μ, ν are σ-finite look at $X = \bigsqcup_{j=1}^{\infty} A_j$ as before. If $\lambda \ll \mu$ then $d\lambda = f d\mu$ with f unique almost everywhere. Did this last time when λ finite. In general look at $X = \sqcup_{j=1}^{\infty} A_j$.

■

Remark. The following are a few applications:

- Conditional expectation
- Measures on $\mathbb{R}$.

Example. $F(x) = \begin{cases} e^{3x}, & \text{if } x < 0 \\ 1, & \text{if } 0 \leq x < 2 \\ 3, & \text{if } x \geq 2 \end{cases}$ Then let μ_F be the LS-measure for F with $\mu = m$. Then

$$\mu_F(E) = \underbrace{\int_{E \cap \mathbb{R}_-} 3e^{3x} \mathrm{d}x}_{\rho(E)} + \underbrace{2\delta_2(E)}_{\lambda(E)},$$

such that $\rho \ll m$, $\underbrace{\lambda}_{\{2\}} \perp \underbrace{m}_{\mathbb{R}-\{2\}}$. Therefore,

$$\frac{\mathrm{d}\rho}{\mathrm{d}m} = 3e^{3x} \cdot \mathbf{1}_{\mathbb{R}_-}$$

Proposition 5.2.2 (Properties of RN derivatives)**.** We have the following properties:

(1) $\frac{\mathrm{d}(\nu_1+\nu_2)}{\mathrm{d}\mu} = \frac{\mathrm{d}\nu_1}{\mathrm{d}\mu} + \frac{\mathrm{d}\nu_2}{\mathrm{d}\mu}$ and $\frac{\mathrm{d}(c\nu)}{\mathrm{d}\mu} = c \cdot \frac{\mathrm{d}\nu}{\mathrm{d}\mu}$ μ a.e. $[\nu_i \ll \mu]$.

(2) $\int g \mathrm{d}\nu = \int g \frac{\mathrm{d}\nu}{\mathrm{d}\mu} \mathrm{d}\mu$. $g \in L^1(\nu)$, $\nu \ll \mu$.

(3) If $\nu \ll \mu$, $\mu \ll \lambda$ then $\left\{ \nu \ll \lambda, \frac{\mathrm{d}\nu}{\mathrm{d}\lambda} = \frac{\mathrm{d}\nu}{\mathrm{d}\mu} \cdot \frac{\mathrm{d}\mu}{\mathrm{d}\lambda} \right.$ $(\lambda$ a.e. $)$.

(4) If $\nu \ll \mu$, $\mu \ll \nu$ then $\frac{\mathrm{d}\nu}{\mathrm{d}\mu} \cdot \frac{\mathrm{d}\mu}{\mathrm{d}\nu} = 1$ a.e. with respect to μ and/or ν.

Proof. DIY (Folland pg. 91). ■

5.3 Complex Measures

Definition 5.3.1. A complex measure on a measurable space $(X, \mathcal{A})$ is a function $\mu : A \to \mathbb{C}$ such that

(1) $\mu(\emptyset) = 0$,

(2) If E_j for $j \in \mathbb{N}$ are disjoint sets in $\mathcal{A}$ then

$$\mu(\bigcup_{j=1}^{\infty} E_j) = \sum_{j=1}^{\infty} \mu(E_j)$$

and $\sum_{j=1}^{\infty} |\mu(E_j| < \infty$.

Note. $\operatorname{Re}\mu$, $\operatorname{Im}\mu$ are finite signed measures.

Theorem 5.3.1 (Lebesgue-Radon Nikodyn for Complex Measures)**.** If μ is a σ-finite positve measure and ν is a complex measure on $(X, \mathcal{A})$ then $\exists!$ decomposition

$$\nu = \lambda + \rho,$$

where λ, ρ are complex measures such that λ is mutually singular with respect to μ and $\rho \ll \mu$. Moreover, $\exists! f \in L^1(\mu)$ such that $f : X \to \mathbb{C} \in L^1(\mu)$ and $d\rho = f d\mu$.

Proof. Follows from LRNT applied to $\operatorname{Re}\nu$, $\operatorname{Im}\nu$. ■

Remark. Next time we will see total variation measure $|\nu|$

$$\|\nu\| := |\nu|(x).$$

Then {complex measures} is a Banach space.

Lecture 30: Complex measures

27 Mar. 11:00

Definition 5.3.2. We define:

$$\begin{aligned} L^1(\nu) &:= L^1(\operatorname{Re}\nu) \cap L^1(\operatorname{Im}\nu) \\ &= L^1(|\operatorname{Re}\nu|) \cap L^1(|\operatorname{Im}\nu|). \end{aligned}$$

Definition 5.3.3. Set $\mu := |\operatorname{Re}\nu| + |\operatorname{Im}\nu|$ finite positive measure such that $\nu \ll \mu$. By LRNT, $\mathrm{d}\nu = f\mathrm{d}\mu$ for $f \in L^1(\mu)$. Then define $|\nu|$ by $\mathrm{d}\nu = |f|\mathrm{d}\mu$. Equivalently,

$$|\nu|(E) = \int_E |f|\mathrm{d}\mu.$$

Theorem 5.3.2. Set $\|\nu\| := |\nu|(X) \in [0, \infty)$ and $\mathcal{M} := \{\text{complex measures on } (X, \mathcal{A})\}$. Then $(\mathcal{M}, \|\cdot\|)$ is a complex Banach space.

Proof. HW10. ■

Lemma 5.3.1. We have the following:

- $\nu \ll |\nu|$. In fact: $|\nu(E)| \leq |\nu|(E)$ for every $E \in \mathcal{A}$.
- $\frac{\mathrm{d}\nu}{\mathrm{d}|\nu|}$ has absolute value 1 $|\nu|$-a.e.
- If $f \in L^1(\nu)$ then $\int f\mathrm{d}\nu = \int f \frac{\mathrm{d}\nu}{\mathrm{d}|\nu|}\mathrm{d}|\nu|$ such that

$$\left|\int f\mathrm{d}\nu\right| \leq \int |f|\mathrm{d}|\nu|.$$

Proof. Folland pg. 94 ■

Chapter 6

Old Concepts Revisited

6.1 Important Dualities

$(V, \|\cdot\|)$ normed vector space over $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$. Then

$$\text{Dual } V^* := \{\text{bounded linear functionals } \phi : V \to \mathbb{K}\}$$

where bounded means

$$\|\phi\| = \sup_{v \neq 0} \frac{|\phi(v)|}{\|v\|} < \infty.$$

We also have V^* is a Banach space.

Example. We have that

$$\begin{aligned} V &= C_0^0(\mathbb{R}^n; \mathbb{K}) \\ &= \{f : \mathbb{R}^n \to \mathbb{K} \text{ continuous} \mid \lim_{R \to \infty} \sup_{B(0,R)^c} |f| = 0\} \end{aligned}$$

is a Banach space with norm $\|f\| = \sup_{R^n} |f|$.

Theorem 6.1.1. We have the following:

(1) $\mathbb{K} = \mathbb{R}$, $V^* = \mathcal{M}(\mathbb{R}^n; \mathbb{R}) := \{\text{finite signed Borel measures on } \mathbb{R}^n\}$.

(2) $\mathbb{K} = \mathbb{C}$, $V^* = \mathcal{M}(\mathbb{C}^n; \mathbb{C}) := \{\text{finite complex Borel measures on } \mathbb{R}^n\}$.

Sketch. Define map

$$\begin{aligned} T : \mathcal{M}(\mathbb{R}^n; \mathbb{K}) &\longrightarrow V^* \\ \nu &\longmapsto T(\nu) = \phi_\nu \end{aligned}$$

by $\phi_\nu(f) = \int_{\mathbb{R}^n} f \mathrm{d}\nu \in \mathbb{K}$. This is well defined because

$$\int f \mathrm{d}\nu = \int f \,\mathrm{d}(\operatorname{Re}\nu)^+ - \int f \,\mathrm{d}(\operatorname{Re}\nu)^- + \int f \,\mathrm{d}(\operatorname{Im}\nu)^+ - \int f \,\mathrm{d}(\operatorname{Im}\nu)^-$$

and f is continuous and bounded such that $f \in L^1$. It is easy to see that ϕ_ν is a linear functional on V. From the lemma, we have that

$$|\phi_\nu(f)| = \left| \int f \mathrm{d}\nu \right| \leq \int |f| \mathrm{d}|\nu| \leq \|f\| \cdot |\nu|(\mathbb{R}^n) = \|f\| \|\nu\|.$$

Therefore, ϕ_ν is a bounded linear functional and $\|\phi_\nu\| \leq \|\nu\|$. At this point, we have a linear map

$$T : \mathcal{M} \to V^*$$

and $\|T\nu\| \leq \|\nu\|$. It remains to show that

(1) $\|T\nu\| = \|\nu\|$ i.e. T is an isometry. This would imply that T is injective.

(2) T is surjective. (harder)

■

6.2 Dual of L^p revisited

$(X, \mathcal{A}, \mu)$ measure space with μ positive measure. Additionally, we have $1 \leq p, q < \infty$ such that

$$\frac{1}{p} + \frac{1}{q} = 1.$$

Recall

$$L^p(\mu) := \{f : X \to \mathbb{K} \text{ measurable} \mid \left(\int |f|^p \mathrm{d}\mu\right)^{1/p} < \infty\}.$$

Slogan (with caveats): The dual of L^p is L^q.

Sketch. We have a map

$$\begin{aligned} T : L^q &\longrightarrow (L^p)^* \\ g &\longmapsto T(g) = \phi_g, \end{aligned}$$

where $\phi_g(f) := \int fg\mathrm{d}\mu$. By Holder's inequality, $fg \in L^1$ and $\|fg\|_1 \leq \|f\|_p \cdot \|g\|_q$. Therefore, $\phi_g \in (L^p)^*$ and $\|\phi_g\| \leq \|g\|_q$. Clearly, T is linear and $\|Tg\| \leq \|g\|_q$ for every $g \in L^q$. ■

Theorem 6.2.1. If $1 < p, q < \infty$ then T is an isometric isomorphism such that

$$\begin{cases} T : L^q \to (L^p)^* \text{ bijective} \\ \|Tg\| = \|g\|_q \forall g \in L^q \end{cases}.$$

Proof. Alread proved that $\|Tg\| = \|g\|_q$ for every $g \in L^q$. (Need to find $f \in L^p$ such that $\begin{cases} \|fg\|_1 = \|g\|_q \\ \|f\|_p = 1 \end{cases}$). Therefore, T is injective. Now we need to show that T is surjective. For simplicity, assume μ is finite ($\mu(X) < \infty$). Therefore, any simple function lies in L^p, L^q. Pick any $\phi \in (L^p)^*$. Must find $g \in L^q$ such that $\phi(f) = \int fg\mathrm{d}\mu$ for every $f \in L^p$. Define $\nu : \mathcal{A} \to \mathbb{K}$ by

$$\nu(E) := \phi(\mathbf{1}_E).$$

Claim. ν is a finite signed / complex measure on $(X, \mathcal{A})$.

Proof. Skipped: use $1 < p, q < \infty$. ■

Claim. $\nu \ll \mu$

Proof. Clear: $\mu(E) = 0 \Rightarrow \mathbf{1}_E = 0$ a.e., so $\phi(\mathbf{1}_E) = \phi(0) = 0$. ■

From the claims, we can apply RNT such that $\mathrm{d}\nu = g\mathrm{d}\mu$ for some $g \in L^1(\mu)$. Then it remains to show that $g \in L^q(\mu)$ and $\phi = \phi_g$ i.e.

$$\phi(f) = \int fg\mathrm{d}\mu, \quad \forall f \in L^p.$$

See Folland 6.2. ■

Lecture 30: Differentiation in $\mathbb{R}^n$ Revisited

6.3 Differentiation in $\mathbb{R}^n$ Revisited

27 Mar. 11:00

Definition 6.3.1. A Borel measure ν on $\mathbb{R}^n$ is regular if

(i) ν is locally finite: $\nu(K) < \infty$ for every compact $K \subset \mathbb{R}^n$.

(ii) $\nu(E) = \inf\{\nu(E) \mid U \supset E \text{ open}\}$ for every $E \in \mathcal{B}_{\mathbb{R}^n}$

Remark. We have that (i) $\Rightarrow$ (ii). For proof, see Folland 7.2.

Example. We saw that this is true for $\nu = m$.

Example. We saw that this is true for $\nu =$ LS measure on $\mathbb{R}$

Definition 6.3.2. A signed or complex measure ν is regular if and only if $|\nu|$ is regular.

Remark. An easy result from this is the following: if $f \in L^+(\mathbb{R}^n)$ then $\mathrm{d}\nu = f\mathrm{d}m$ is locally finite if and only if $f \in L^1_{\text{loc}}(\mathbb{R}^n)$

Lemma 6.3.1. If $f \in L^1_{\text{loc}}(\mathbb{R}^n)$, then $f\mathrm{d}m$ is regular.

Proof. See Folland pg. 99 (without using the remark in the definition). ■

Given a signed or complex ν regular. We want to study density of ν with respect to m i.e. given x we want to study

$$\lim_{r\to 0^+} \frac{\nu(B(x,r)}{m(B(x,r))}.$$

We have 3 cases:

Case 1: $\nu \ll m$

Then by RNT, we have that $\mathrm{d}\nu = f\mathrm{d}m$ for $f \in L^1_{\text{loc}}(\mathbb{R}^n)$. By LDT, we have that

$$\frac{\nu(B(x,r))}{m(B(x,r))} = ⨍_{B(x,r)} f(y)\mathrm{d}y \to f(x) \text{ for } m \text{ a.e. } x.$$

Case 2: $\nu \perp m$

Example. $\nu = \delta_{x_0}$. Then

$$\frac{\nu(B(x,r))}{m(B(x,r))} \to \begin{cases} +\infty, & \text{if } x = x_0 \\ 0, & \text{if } x \neq x_0 \end{cases} = 0 \ m \text{ a.e. } .$$

Lemma 6.3.2. $\nu \perp m \Rightarrow \lim_{r\to 0^+} \frac{\nu(B(x,r))}{m(B(x,r))} = 0$ for m a.e. x.

Proof. Since $|\nu| \perp m$ and $|\nu(B(x,r))| \leq |\nu|(B(x,r))$, we may assume that ν is positive ($\nu \mapsto |\nu|$). Pick A Borel set such that $\nu(A) = m(A^c) = 0$. Set

$$F_k := \left\{ x \in A \mid \limsup_{r \to 0^+} \frac{\nu(B(x,r))}{m(B(x,r))} > \frac{1}{k} \right\} \text{ for } k \in \mathbb{N}.$$

HW: F_k is a Borel set. Now it suffices to show that $m(F_k) = 0$ for every k because

$$m(\{x \in A \mid \limsup_{r \to 0^+} \frac{\nu(B(x,r))}{m(B(x,r))} > 0\} = m\left(\bigcup_{k=1}^{\infty} F_k\right) = 0.$$

By regularity of ν, $\exists$ open set $U_\epsilon \supset A$ such that $\nu(U_\epsilon) < \epsilon$. Given $x \in F_k$, $\exists r_x > 0$ such that

$$B_x := B(x, r_x) \subset U_\epsilon \quad \text{ s.t. } \quad \nu(B_x) > \frac{1}{k} m(B_x).$$

By inner regularity of m, pick any $K \subset F_k$ compact. Then

$$K \subset \bigcup_{x \in K} B_x \Rightarrow \exists K' \subset K \text{ finite such that } K \subset \bigcup_{x \in K'} B_x.$$

To avoid overcounting, we can use Vitali covering lemma such that

$$\exists K'' \subset K' \text{ s.t. } B_x, x \in K'' \text{ are disjoint and } K \subset \bigcup_{x \in K''} 3B_x.$$

Then

$$\begin{aligned}
m(K) &\leq \sum_{x \in K''} m(3B_x) \\
&= \sum_{x \in K''} 3^n m(B_x) \\
&\leq 3^n \cdot k \sum_{x \in K''} \nu(B_x) \\
&= 3^n k \nu(\bigcup_{x \in K''} B_x) \leq 3^n \cdot k\nu(U_\epsilon) = 3^n \cdot k \cdot \epsilon.
\end{aligned}$$

By inner regularity, take sup over $K \subset F_k$. Then

$$m(F_k) \leq 3^n \cdot k \cdot \epsilon, \quad \forall \epsilon > 0.$$

Thus, $m(F_k) = 0$. ■

Case 3: General case: ν regular, signed or complex measure on $\mathbb{R}^n$.

Theorem 6.3.1. ν regular, signed/complex measure on $\mathbb{R}^n$ such that

$$\mathrm{d}\nu = \mathrm{d}\lambda + f \mathrm{d}m.$$

Then for m a.e. $x \in \mathbb{R}^n$, we have

$$\lim_{r \to 0^+} \frac{\nu(B(x,r))}{m(B(x,r))} = f(x).$$

> **Note.** In particular, the limit exists!

> **Remark.** Also true when $B(x,r)$ replcaed by E_r shrinking nicely to x i.e. $E_r = [x, x+r]$.

Proof. Follows from previous results assuming we can prove that for

$$\nu = \lambda + \rho \quad \text{(Lebesgue decomposition)}$$

the measures λ, ρ are regular. For this, there are two steps:

Step 1: $|\nu| = |\lambda| + |\rho|$. (Lebesgue decomposition of $|\nu|$ wrt m)

Step 2: If μ_1, μ_2 are positive measures on $\mathbb{R}^n$ such that $\mu_1 \perp \mu_2$ then $\mu_1 + \mu_2$ is regular if and only if μ_1, μ_2 are regular.

HW! ■

Lecture 31: Differentiation in $\mathbb{R}$

30 Mar. 11:00

As previously seen. Any locally finite Borel measure on $\mathbb{R}$ is a LS measure i.e.

- $\mu = \mu_F$, $F : \mathbb{R} \to \mathbb{R}$ increasing, right continuous.
- $\mu_F((a, b]) = F(b) - F(a)$. ($\mu_F$ always regular)

Theorem 6.3.2 (Monotone Differentiation Theorem)**.** Let $F : \mathbb{R} \to \mathbb{R}$ be an increasing function. Set $G(x) := F(x+)$. Then

(a) $\exists$ an at most countable set $Z \subset \mathbb{R}$ such that $\forall x \in Z^c$, F and G are continuous at x and $F(x) = G(x)$.

(b) F and G are differentiable almost everywhere and $F'(x) = G'(x)$ almost everywhere.

Remark. F differentiable at x means that $\lim_{h\to 0} \frac{F(x+h)-F(x)}{h}$ exists.

Remark. From now on almost everywhere will mean Lebesgue almost everywhere.

Example. Examples of these are

- F smooth,
- $F = H$ heaviside function,
- $F =$ Cantor function.

Proof of (a). Sufficient to show that $\forall m, n \in \mathbb{N}$ the set

$$Z_{m,n} := \{x \in [-m, m] \mid F(x+) - F(x-) \geq \frac{1}{n}\}$$

is finite, since we can take $Z = \bigcup_{m,n} Z_{m,n}$. But

$$F(m+) - F((-m)-) \geq \sum_{x \in Z_{m,n}} F(x+) - F(x-) \geq \# Z_{m,n} \cdot \frac{1}{n}.$$

■

Proof of (b). Note G is right continuous, increasing. Therefore, we have the LS-measure μ_G. Apply LDT to $\nu = \mu_G$ and $E_r = (x - r, x]$ or $E_r = (x, x + r]$. These shrink nicely to x such that

$$\frac{\mu_G(x - r, X])}{m((x - r, x])} = \frac{G(x) - G(x - r)}{r}, \quad \frac{\mu_G(x, x + r])}{m((x, x + r])} = \frac{G(x + r) - G(x)}{r}.$$

By LDT, $G'(x)$ exists almost everywhere ($G' \in L^1_{\text{loc}}, G' \geq 0$).

Now we will look at F. Set $H := G - F \geq 0$. Then it suffices to show that $H'(x) = 0$ for $x \notin Z = \bigcup_{m,n} Z_{m,n} \supset \{F \neq G\}$. Set $\mu := \sum_{x \in Z} H(x) \cdot \delta_x$, locally finite Borel measure. Note that $\nu(Z^c) = 0 = m(Z)$. Therefore, $\mu \perp m$. By LDT,

$$\frac{\mu((x-r, x+r))}{2r} \to 0, \qquad \text{as } r \to 0 \text{ for a.e. } x.$$

Therefore,

$$\left|\frac{H(x+h) - H(x)}{h}\right| \leq \frac{H(x+h) + H(x)}{|h|} \leq 4 \cdot \frac{\mu((x - 2|h|, x + 2|h|))}{4|h|} \to 0, \qquad \text{as } h \to 0 \text{ for a.e. } x.$$

Thus, $H'(x) = 0$ for a.e. x. ■

Problem 6.3.1. $F(b) - F(a) = \int_a^b F\text{"}(x)\mathrm{d}x$?

Answer. Not always, but we will see later when this is true. Consider $F = \text{Heaviside}, \text{Cantor}$ ⊛

Problem 6.3.2. Given finite regular signed/complex Borel measure μ on $\mathbb{R}$, what can we say about $F(x) := \mu((-\infty, x])$?

Definition 6.3.3. If $F' : \mathbb{R} \to \mathbb{C}$ is any function, define the total variation of F as the function $T_F : \mathbb{R}[0, +\infty]$ by

$$T_F(x) = \sup\left\{\sum_{j=1}^n |F(x_j) - F(x_{j-1})\} \mid -\infty < x_0 < x_1 < \cdots < x_n = x\right\}.$$

Lemma 6.3.3. We have the following:

(a) T_F is increasing.

(b) If $a < b$, then

$$T_F(b) = T_F(a) + \underbrace{\sup\left\{\sum_{j=1}^n |F(x_j) - F(x_{j-1}| \mid a = x_0 < x_1 < \cdots < x_n = b\right\}}_{=T_F([a,b])}.$$

Proof. (a) is clear. For (b), we can demand $a = x_k$ for some k in $T_F(b)$. DIY. ■

Definition 6.3.4. F is of bounded variation, $F \in BV = BV(\mathbb{R})$, if

$$T_F(\infty) = \lim_{x \to \infty} T_F(x) < \infty.$$

Similarly, $BV([a, b])$ etc.

Remark. If F is continuously differentiable ($F \in C^1$) then $F(x_j) - F(x_{j-1}) = \int_{x_{j-1}}^{x_j} F'(t)\mathrm{d}t$. Therefore, $|F(x_j) - F(x_{j-1})| \leq \int_{x_{j-1}}^{x_j} |F'(t)|\mathrm{d}t$. So if $|F'| \leq C$ on $[a, b]$ then $T_F([a, b]) \leq C \cdot |b - a|$. In particualr, $F \in BV([a, b])$.

If Fundamental Theorem of Calculus holds, then we would expect that $T_F([a, b]) = \int_a^b |F'(t)|\mathrm{d}t$.

Example. $F(x) = \sin x$. $F \in BV([a,b])$ for $-\infty < a < b < \infty$. However, $F \notin BV(\mathbb{R})$ because we hvae infinite variation from peak to troph.

Example. $F(x) = x \sin \frac{1}{x}$. $F \notin BV([-1,1])$ (HW). Want to go from peak to troph. End up with harmonic series, which barely diverges.

Proposition 6.3.1. $F \in BV = BV(\mathbb{R}) \Rightarrow T_F \geq 0$, increasing, bounded, $T_F(-\infty) = 0$.

Proof. Sufficient to show that $T_F(-\infty) = 0$. Given $x \in \mathbb{R}$, $\epsilon > 0$. Pick $x_0 < x_1 < \cdots < x_n$ such that $T_F(x) \leq \sum_{j=1}^n |F(x_j) - F(x_{j-1})| + \epsilon$. Then

$$T_F([x_0, x]) \geq T_F(x) - \epsilon$$

Thus,

$$T_F(x_0) = T_F(x) - T_F([x_0, x]) \leq \epsilon.$$

∎

Lecture 32: Bounded Variation Functions + Complex Measures

3 Apr. 11:00

Lemma 6.3.4. We have the following:

(a) If $F : \mathbb{R} \to \mathbb{R}$ bounded increasing then $F \in BV$ and $T_F(x) = F(x) - F(-\infty)$.

(b) If $F, G \in BV$ and $a, b \in \mathbb{C}$ then $aF + bG \in BV$ and $T_{aF+bG} \leq |a|T_F + |b|T_G$.

(c) If F is Lipschitz continuous then $F \in BV$ on any compact interval.

(d) If F is differentiable on $\mathbb{R}$, F' bounded then F of bounded variation on any compact interval.

Proof. For (a), we can get rid of absolute value and are left with telescoping sum. For (b), it is just the triangle inequality. For (c), replace $|F(x_j) - F(x_{j-1})|$ with $M|x_j - x_{j-1}|$. We then get a telescoping series once again. For (d), suppose $|F'| \leq M$. Then $F(x_j) - F(x_{j-1}) = F'(\xi)$ for $\xi \in (x_j, x_{j-1})$. Therefore,

$$|F(x_j) - F(x_{j-1})| \leq M|x_j - x_{j-1}|.$$

Then we are done by (c). ∎

Lemma 6.3.5. If $F \in BV$ then $T_F + F$ and $T_F - F$ are increasing functions on $\mathbb{R}$.

Proof. Pick $x < y$. We must show that $(T_F \pm F)(x) \leq (T_F \pm F)(y)$. Pick $\epsilon > 0$. Can pick $x_0 < x_1 < \cdots < x_n = x$ such that $\sum_{j=1}^n |F(x_j) - F(x_{j-1})| \geq T_F(x) - \epsilon$. Then $x_0 < x_1 < \cdots < x_n = x < y$ such that

$$\sum_{j=1}^n |F(x_j) - F(x_{j-1})| + |F(y) - F(x)| \leq T_F(y).$$

Therefore,

$$\begin{aligned} T_F(y) - T_F(x) &\geq |F(y) - F(x)| - \epsilon \\ &\geq \pm(F(y) - F(x)) - \epsilon \end{aligned}$$

and

$$(T_F \pm F)(y) \geq (T_F \pm F)(x) - \epsilon.$$

■

Theorem 6.3.3. If $F : \mathbb{R} \to \mathbb{R}$ then

$$F \in BV \Leftrightarrow F = F_+ - F_-,$$

where $F_\pm$ bounded, increasing. Can (and will) take $F_\pm = \frac{1}{2}(T_F \pm F)$.

Proof. For the $\Leftarrow$ direction, follows from (a), (b) in one of previous lemmas. Now for the $\Rightarrow$ direction, we have from the previous lemma that $F_\pm := \frac{1}{2}(T_F \pm F)$ is increasing and $F = F_+ - F_-$. We also have that $F_\pm$ bounded because

$$\begin{cases} F \in BV \Rightarrow F \text{ is bounded} \\ T_F \text{ bounded by definition} \end{cases}.$$

■

Problem 6.3.3. In what sense is this decomposition canonical?

Answer. See HW12. ⊛

Lemma 6.3.6. If $F \in BV$ is right-continuous, so if T_F.

Proof. Given $x \in \mathbb{R}$, set $\alpha := T_F(x+) - T_F(x) \geq 0$. Then we want to show that $\alpha = 0$. Pick $\epsilon > 0$. Pick $\delta > 0$ such that if $0 < h < \delta$ then

$$\begin{cases} |F(x+h) - F(x)| < \epsilon \\ 0 \leq T_F(x+h) - T_F(x) < \alpha + \epsilon \end{cases}.$$

Pick $x = x_0 < x_1 < \cdots < x_n = x + h$ such that

$$\sum_{j=1}^{n} |F(x_j) - F(x_{j-1})| \geq \frac{3}{4} \cdot [T_F(x+h) - T_F(x)] \geq \frac{3}{4}\alpha.$$

Then

$$\sum_{j=2}^{n} |F(x_j) - F(x_{j-1})| \geq \frac{3}{4}\alpha - |F(x_1) - F(x)| \geq \frac{3}{4}\alpha - \epsilon.$$

Now pick $x = t_0 < t_1 < \cdots t_m = x_1$ such that

$$\sum_{j=1}^{m} |F(t_j) - F(t_{j-1})| \geq \frac{3}{4} \cdot [T_F(x_1) - T_F(x)] \geq \frac{3}{4}\alpha.$$

Then

$$\begin{aligned} \alpha + \epsilon &> T_F(x+h) - T_F(x) \\ &\geq \sum_{j=1}^{m} |F(t_j) - F(t_{j-1})| + \sum_{j=2}^{n} |F(x_j) - F(x_{j-1}) \geq \frac{3}{4}\alpha + \frac{3}{4}\alpha - \epsilon. \end{aligned}$$

Thus, $\frac{1}{2}\alpha \leq 2\epsilon$ and $\alpha = 0$. ■

Definition 6.3.5. We say that $F \in NBV$, normalized bounded variation, if $F \in BV$, F right-continuous, and $F(-\infty) = 0$.

Note. $NBV \subset BV$ linear subspace.

Lecture 33: Absolutely Continuous Functions

6.4 Absolutely Continuous Functions

5 Apr. 11:00

Theorem 6.4.1. We have the following:

(a) If μ is a complex measure on $\mathbb{R}$, and

$$F(x) \coloneqq \mu((-\infty, x]),$$

then $F \in NBV$.

(b) If $F \in NBV$ then $\exists!$ complex measure $\mu = \mu_F$ on $\mathbb{R}$ such that

$$\mu_F((-\infty, x]) = F(x), \quad \forall x \in \mathbb{R}.$$

(c) μ_F is unique, $|\nu_F| = \mu_{T_F}$. If F real, then $(\mu_F)_\pm = \mu_{F_\pm}$.

Proof of (a). We know that this is true if μ is positive finite Borel measure. In general,

$$\mu = ((\operatorname{Re}\mu)_+ - (\operatorname{Re}\mu)_-) + i((\operatorname{Im}\mu)_+ - (\operatorname{Im}\mu)_-).$$

■

Proof of (b). Since $\operatorname{Re} F, \operatorname{Im} F \in NBV$, we can assume F is $\mathbb{R}$-valued. Then T_F right continuous such that

$$F_\pm = \frac{1}{2}(T_F \pm F) \in NBV, \text{ increasing.}$$

Therefore, $\exists!$ Borel measure $\mu_\pm$ such that

$$\mu_\pm((a, b]) = F_\pm(b) - F_\pm(a).$$

Take $a \to -\infty$ such that $\mu_\pm((-\infty, b]) = F_\pm(b)$. Then we just have $\mu = \mu_+ - \mu_-$. ■

Proof of (c). DIY. ■

Theorem 6.4.2. If $F \in BV$, then

(a) $F(x+)$, $F(x-)$, $F(\pm\infty)$ exist. $\forall x \in \mathbb{R}$.

(b) If $G(x) \coloneqq F(x+)$, then $G \in Bv$ (right continuous).

(c) $\exists$ countable set $Z \subset \mathbb{R}$ such that $\forall x \in \mathbb{R} \setminus \mathbb{Z}$.

(d) $\exists$ null set $W \subset \mathbb{R}$ such that $\forall x \in \mathbb{R} \setminus W$, $F'(x)$, $G'(x)$ exist and $F'(x) = G'(x)$.

Proof. True when F increasing. In general, use $F = F_+ - F_-$. ■

Problem 6.4.1. Given $F \in NBV$, when is $\mu_F \perp m$, $\mu_F \ll m$?

Example. F = Heaviside or F = Cantor function. Then $\mu_F = \delta_0$ or μ_F = Cantor measure and $\mu_F \perp m$.

Proposition 6.4.1. If $F \in NBV$, then $F' \in L^1(m)$. Moreover:

- $\mu_F \perp m \Leftrightarrow F' = 0$ a.e.
- $\mu_F \ll m \Leftrightarrow F(x) = \int_{-\infty}^{x} F'(t)\mathrm{d}t\ \forall x \in \mathbb{R}$.

Proof. $F'(x) = \lim_{r\to 0} \frac{\mu_F(E_r)}{m(E_r)}$, where $E_r = \begin{cases}(x, x+r], & \text{if } r > 0\\ (x+r, x], & \text{if } r < 0\end{cases}$ such that E_r shrinks nicely to x. Now apply LDT. Check! ■

Definition 6.4.1. $F : \mathbb{R} \to \mathbb{C}$ is absolutely continuous if $\forall \epsilon > 0$, $\exists \delta > 0$ such that whenever $(a_1, b_1), \ldots, (a_N, b_N)$ are disjoint intervals:

$$\sum_{j=1}^{N} (b_j - a_j) < \delta \Rightarrow \sum_{j=1}^{N} |F(b_j) - F(a_j)| < \epsilon.$$

Remark. F' exists, bounded $\Rightarrow$ F Lipschitz continuouse $\Rightarrow$ F absolutely continuous $\Rightarrow$ F uniformly continuous.

Example. F = Cantor function. If we truncate

$$F(x) = \begin{cases} 0, & \text{if } x \le 0\\ 1, & \text{if } x \ge 1\\ F(x), & \text{otherwise}\end{cases}$$

then F is uniformly continuous but not absolutely continuous. Since $\mu_F = \mu_C$ = Cantor measure, $\mu_C \perp m$. Therefore, $F' = 0$ almost everywhere as desired.

Proposition 6.4.2. If $F \in NBV$, then $\mu_F \ll m \Leftrightarrow F$ absolutely continuous.

Proof. For the $\Rightarrow$ direction, we saw earlier that if $\mu_F \ll m$ then $\forall \epsilon > 0$, $\exists \delta > 0$, such that

$$m(E) < \delta \Rightarrow |\mu_F|(E) < \epsilon.$$

Now apply to $E = \bigcup_{j=1}^{N} (a_j, b_j)$.

For the $\Leftarrow$ direction, suppose $E \in \mathcal{B}_{\mathbb{R}}$ such that $m(E) = 0$. Then we want to show that $\mu_F(E) = 0$. Since m, μ_F are regular measures, there exists some open subsets $U_1 U_2 \supset \cdots \supset E$ such that

$$\begin{cases} \mu(U_j) \searrow m(E) = 0\\ \mu_F(U_j) \to \mu_F(E)\end{cases}.$$

Pick $\epsilon > 0$, $\delta > 0$ given F absolutely continuous. Without loss of generality, we can assume $m(U_j) < \delta$ for every j. Write U_j as disjoint union of open intervals $(a_j^k, b_j^k)_k$. Then $\forall N$,

$$\sum_{k=1}^{N} (b_j^k - a_j^k) = m(\bigcup_{k=1}^{N} (b_j^k - a_j^k) < \delta.$$

Since F is absolutely continuous,

$$\epsilon > \sum_{k=1}^{N} |F(b_j^k) - F(a_j^k)| \ge \sum_{k=1}^{N} |\mu_F(a_j^k, b_j^k)|.$$

Now let $N \to \infty$ such that $|\mu_F(U_j)| < \epsilon$. Then as $j \to \infty$, $|\mu_F(E)| \le \epsilon$. Thus, $\mu_F(E) = 0$ as $\epsilon \to 0$. ■

Lecture 34: Fundamental Theorem of Calculus + Prob Measures

7 Apr. 11:00

Theorem 6.4.3. If $-\infty < a < b < \infty$ and $F : [a, b] \to \mathbb{C}$ then the following are equivalent

(a) F absolutely continuous on $[a, b]$.

(b) F is differentiable almost everywhere, $F' \in L^1([a, b], m)$ and

$$F(x) - F(a) = \int_a^x F'(t) \mathrm{d}t, \quad x \in [a, b].$$

Proof of (a) $\Rightarrow$ (b). Without loss of generality, $F(a) = 0$. Otherwise, consider $F \mapsto F - F(a)$. Extend F to all of $\mathbb{R}$ by setting $F(x) = \begin{cases} 0, & \text{if } x \le a \\ F(b), & \text{if } x \ge b \end{cases}$

Claim. If $F \in BV([a, b])$ then this extension of $F \in NBV$.

Proof. Pick $\delta > 0$ in definition of absolute continuity corresponding to $\epsilon = 1$. Pick $N > \frac{b-a}{\delta}$. Suppose $a \le x_0 < x_1 < \cdots < x_n \le b$. Then it is sufficient to show that

$$\sum_{j=1}^{n} |F(x_j) - F(x_{j-1})| \le N.$$

Recall that subdividing this partition into smaller intervals can only increase the total variation. After inserting extra points, we can partition

$$\{1, 2, \ldots, n\} = A_1 \sqcup A_2 \sqcup \cdots \sqcup A_N$$

such that

$$\sum_{j \in A_k} (x_j - x_{j-1}) < \delta, \qquad \text{for } k = 1, \ldots, N.$$

Therefore,

$$\sum_{j \in A_k} |F(x_j) - F(x_{j-1})| < 1.$$

and

$$\sum_{j=1}^{n} |F(x_j) - F(x_{j-1})| < N.$$

■

By the claim, we have that $F \in NBV$. By a previous proposition, $\mu_F \ll m$. Then from another proposition, we have that

$$F(x) = \int_{-\infty}^{x} F'(t) \mathrm{d}t, \quad x \in \mathbb{R}.$$

Since $F(x) = 0$ for $x \le a$, we have that

$$F(x) = \int_a^x F'(t) \mathrm{d}t, \quad x \in \mathbb{R}.$$

■

Proof of (b) $\Rightarrow$ (a). From previous proposition, it suffices to show that $\mu_F \ll m$. As we have extended F to $\mathbb{R}$, we know that $F' \in L^1(\mathbb{R}, m)$ and

$$F(x) = \int_{-\infty}^{x} F'(t)\mathrm{d}t, \quad x \in \mathbb{R}.$$

Therefore,

$$\mu_F((c,d]) = F(d) - F(c) = \int_c^d F'(t)\mathrm{d}t = \int_{[c,d]} F'(t)\mathrm{d}t.$$

Thus, $\mathrm{d}\mu_F = F'\mathrm{d}m$ such that $\mu_F \ll m$ as desired. ∎

6.5 Decomposition of Complex Borel Measures on $\mathbb{R}$

Definition. Suppose μ is a complex measure on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$.

Definition 6.5.1. We say that μ is discrete (or atomic) if $\exists$ a countable set $\{x_j\}_{j=1}^{\infty}$ and $c_{j \in \mathbb{C}}$ such that

$$\sum_{j=1}^{\infty} |c_j| < \infty, \quad \mu = \sum_{j=1}^{\infty} c_j \delta_{x_j}.$$

Definition 6.5.2. We say that μ is continuous if $\mu(\{x\}) = 0$ for every $x \in \mathbb{R}$.

Proposition 6.5.1. Any complex measure μ on $\mathbb{R}$ can be uniquely decomposed as

$$\mu = \underbrace{\mu_d}_{\text{discrete}} + \underbrace{\mu_c}_{\text{continuous}}.$$

Proof of uniqueness. Suppose that $\mu = \mu_d + \mu_c = \mu_d' + \mu_c'$. Then

$$\mu_c - \mu_c' = \mu_d' - \mu_d.$$

Then this defines a measure that is both discrete and continuous. The only such measure is 0. ∎

Proof of existence. Define $E_n := \{x \in \mathbb{R} \mid |\mu(\{x\})| \in [\frac{1}{n}, \frac{1}{n-1}),\ n \in \mathbb{N}\}$. Then

$$\infty > |\mu|(E_n) \Rightarrow E_n \text{ finite.}$$

Therefore, $E = \bigcup_{n=1}^{\infty} E_n$ is countable. Set

$$\mu_d(A) := \mu(A \cap E)$$
$$\mu_c(A) := \mu(A \cap E^c).$$

∎

Note. We have that $\mu_d \perp m$. This gives a Lebesgue decomposition

$$\mu_c = \underbrace{\mu_{ac}}_{\mu_{ac} \ll m} + \underbrace{\mu_{sc}}_{\mu_{sc} \perp m},$$

where μ_{ac} and μ_{sc} are both still continuous and

$$\mu = \mu_d + \mu_{ac} + \mu_{sc}.$$

6.6 Some probability theory

Definition. Let $(\Omega, \mathcal{A})$ measurable space.

Definition 6.6.1. A probability measure $\mathbb{P}$ on $(\Omega, \mathcal{A})$ is a positive measure such that $\mathbb{P}(\Omega) = 1$.

Definition. We call $(\Omega, \mathcal{A}, \mathbb{P})$ a sample space, where elements of $\mathcal{A}$ are events.

Definition 6.6.2. A random variable is a measurable mapping $X : \Omega \to \mathbb{R}$.

Definition 6.6.3. The expectation of a random variable X is

$$E[X] := \int X \, \mathrm{d}\mathbb{P}.$$

if $X \in L^1$ (X has finite first moment).

Suppose $X \in L^1$. Then we get a probability measure $\mathbb{P}_X$ on $\mathbb{R}$, where $\mathbb{P}_X$ has distribution function

$$F(t) := \mathbb{P}(\{X \leq t\}), \quad t \in \mathbb{R}.$$

In elementary probability theory, we assume that $\mathbb{P}_X$ is discrete or $\mathbb{P}_X \ll m$.

Lecture 35: Distribution Functions + Jensen's Inequality

7 Apr. 11:00

6.7 Distribution functions

Definition. Let $(X, \mathcal{A}, \mu)$ be a measurable space and $f : X \to \mathbb{C}$ be a measurable function.

Definition 6.7.1. The distribution function of f is the function

$$\begin{aligned} \lambda_f : &\longrightarrow [0, +\infty] \\ \alpha &\longmapsto \mu(\{x \in X \mid |f(x)| > \alpha\}). \end{aligned}$$

Proposition 6.7.1. A distribution function has the following properties

(a) λ_f only depends on $|f|$.

(b) λ_f is decreasing and right-continuous.

(c) $|f| \leq g \Rightarrow \lambda_f \leq \lambda_g$.

(d) $|f_n| \nearrow |f| \Rightarrow \lambda_{f_n} \nearrow \lambda_f$.

Proof. DIY. ■

Proposition 6.7.2 (Chebyshev for L^p). If $f \in L^p$ for $0 < p < \infty$ then

$$\lambda_f(\alpha) \leq \left(\frac{\|f\|_p}{\alpha} \right)^p .$$

Proof. Pretty much the same as regular Chebyshev. ■

Remark. We can actually recover $\|f\|_p$ from λ_f.

Proposition 6.7.3. If $0 < p < \infty$, then

$$\int_X |f|^p \mathrm{d}\mu = p\int_0^\infty \alpha^{p-1}\lambda_f(\alpha)\mathrm{d}\alpha.$$

In particular,

$$\|f\|_1 = \int_0^\infty \lambda_f(\alpha)\mathrm{d}\alpha.$$

Proof. Without loss of generality, $\lambda_f(\alpha) < \infty$ for all $\alpha > 0$. Otherwise, the equality always holds trivially. First suppose that f is a simple function. Then

$$|f| = \sum_{j=1}^N c_j \mathbf{1}_{A_j},$$

where A_j's are disjoint and $c_1 > c_2 > \cdots > c_N > 0$. Therefore,

$$\int |f|^p \mathrm{d}\mu = \sum_{j=1}^N c_j^p \mu(A_j).$$

Note that

$$\lambda_f(\alpha) = \begin{cases} \sum_{j=1}^N \mu(A_j), & \text{if } 0 < \alpha < c_N \\ \sum_{j=1}^{n-1} \mu(A_j), & \text{if } c_n \leq \alpha < c_{n-1} \\ 0, & \text{if } \alpha \geq c_1 \end{cases}.$$

Then

$$\begin{aligned} p\int_0^\infty \alpha^{p-1}\lambda_f(\alpha)\mathrm{d}\alpha &= \left(\sum_{j=1}^N \mu(A_j)\right)\cdot \int_0^{c_N} p\alpha^{p-1}\mathrm{d}\alpha \\ &\quad + \sum_{n=2}^N \left(\sum_{j=1}^{n-1}\mu(A_j)\right)\int_{c_n}^{c_{n-1}} p\alpha^{p-1}\mathrm{d}\alpha. \end{aligned}$$

Therefore,

$$p\int_0^\infty \alpha^{p-1}\lambda_f(\alpha)\mathrm{d}\alpha = \sum_{j=1}^N \mu(A_j)c_N^p + \sum_{n=2}^N\sum_{j=1}^{n-1}\mu(A_j)(c_{n-1}^p - c_n^p).$$

Thus, we have the desired result for simple functions.

In general, pick g_n simple such that $0 \leq g_n \nearrow |f|$. By proposition, $\lambda_{g_n} \nearrow \lambda_f$. By MCT, $\int g_n^p \to \int |f|^p$ such that

$$p\int_0^\infty \alpha^{p-1}\lambda_{g_n}(\alpha)\mathrm{d}\alpha \to p\int_0^\infty \alpha^{p-1}\lambda_f(\alpha)\mathrm{d}\alpha.$$

■

6.8 Jensen's Inequality

Definition 6.8.1. If $I \subset \mathbb{R}$ is an interval, then

$$F : I \to \mathbb{R} \cup \{+\infty\}$$

is convex if for every $t_0, t_1 \in I$ and $\lambda_0, \lambda_1 \geq$ with $\lambda_0 + \lambda_1$ then

$$F(\lambda_0 t_0 + \lambda_1 t_1) \leq \lambda_0 F(t_0) + \lambda_1 F(t_1).$$

Lemma 6.8.1. Extend F to $F : \mathbb{R} \to \mathbb{R} \cup \{+\infty\}$ by $F \equiv +\infty$ on $\mathbb{R} \setminus I$. Then F is convex.

Lemma 6.8.2. If $F : \mathbb{R} \to \mathbb{R} \cup \{+\infty\}$ is convex then $\{t \in \mathbb{R} \mid F(t) < +\infty\}$ is an interval.

Proof. DIY. ∎

Theorem 6.8.1 (Jensen's Inequality)**.** Let $(X, \mathcal{A}, \mu)$ be a measure space with $\mu(X) = 1$. Suppose $g \in L^1(\mu)$ and $F : \mathbb{R} \to \mathbb{R} \cup \{+\infty\}$ is a convex function. Then $F \circ g$ is extended μ-integrable, and

$$F\left(\int g \mathrm{d}\mu\right) \leq \int (F \circ g) \mathrm{d}\mu.$$

Example. $X = \{1, 2, \ldots, n\}$, $\mathcal{A} = \mathcal{P}(X)$. Define $\lambda_i := \mu(\{i\})$ and $t_i := g(i)$ for $i \in X$, where $\sum_{i=1}^n \lambda_i = 1$. Then

$$F(\sum_{i=1}^n \lambda_i t_i) \leq \sum_{i=1}^n \lambda_i F(t_i).$$

Example. $F(x) = e^x$. Then

$$\exp\left(\sum_{i=1}^n \lambda_i t_i\right) \leq \sum_{i=1}^n \lambda_i \exp(t_i).$$

Set $y_i := e^{t_i}$ such that

$$\prod_{i=1}^n y_i^{\lambda_i} \leq \sum_{i=1}^n \lambda_i y_i.$$

For $\lambda_i = \frac{1}{n}$, this gives us that the geometric mean $\leq$ arithmetic mean.

Lemma 6.8.3. If $F : \mathbb{R} \to \mathbb{R} \cup \{+\infty\}$ is convex and $F(t_0) < \infty$ for some $t_0 \in \mathbb{R}$ then $\exists \beta \in \mathbb{R}$ such that

$$F(t) \geq F(t_0) + \beta(t - t_0).$$

Proof. DIY. (HW) ∎

Proof of Jensen's Inequality. For simplicity, assume $F(t) < \infty$ for every t. Set $t_0 := \int g \mathrm{d}\mu \in \mathbb{R}$. From the lemma, $F(t) \geq F(t_0) + \beta(t - t_0)$. Then

$$F \circ g \geq F(t_0) + \beta g - \beta t_0$$

such that

$$\int F \circ g \geq F(t_0)\mu(X) + \beta \underbrace{\int g \mathrm{d}\mu}_{t_0} - \beta t_0 \mu(X).$$

Since μ is a probability measure, $\mu(X) = 1$ and

$$\int (F \circ g)\mathrm{d}\mu \geq F(t_0).$$

■

Lecture 36: Final Lecture :(

17 Apr. 11:00

6.9 Minkowski's inequality for integrals

Definition. Let V be a real vector space.

Definition 6.9.1. . We say that $\phi : V \to \mathbb{R}$ is convex if

$$\phi(\lambda_1 v_1 + \lambda_2 v_2) \leq \lambda_1 \phi(v_1) + \lambda_2 \phi(v_2)$$

for $v_i \in V$, $\lambda_i \geq 0$, and $\lambda_1 + \lambda_2 = 1$.

Definition 6.9.2. We say that ϕ is homogeneous if $\phi(\lambda v) = \lambda \phi(v)$.

Lemma 6.9.1. If ϕ is homogeneous then ϕ is convex if and only if ϕ is subadditive i.e.

$$\phi(v_1 + v_2) \leq \phi(v_1) + \phi(v_2).$$

Proof. DIY. ■

Remark. $\|\cdot\|_p : L^p \to \mathbb{R}$ is homogeneous. By Minkowski, $\|\cdot\|_p$ is convex for $1 \leq p \leq \infty$.

Theorem 6.9.1 (Minkowski for integrals)**.** Let $(X, \mathcal{A}, \mu)$ and $(Y, \mathcal{B}, \nu)$ be σ-finite measure spaces and $1 \leq p \leq \infty$. Then

$$\left\| \int_Y f(\cdot, y)\mathrm{d}\nu(y) \right\|_p \leq \int \|f(\cdot, y)\|_p \,\mathrm{d}\nu(y)$$

provided that $f \in L^+(\mu \otimes \nu)$ or

(a) $f \in L^p(\mu \otimes \nu)$,

(b) $f(\cdot, y) \in L^p(\mu)$ for ν a.e. y,

(c) $y \mapsto \|f(\cdot, y)\|_p$ is μ integrable.

Sketch. Replace f by $|f|$, then use Holder and Fubini-Tonelli. ■

Example. $Y = \{1, 2\}$, $\mathcal{B} = \mathcal{P}(Y)$, $\nu = \delta_1 + \delta_2$. Then this is just usual Minkowski.

Theorem 6.9.2. If $f \in L^p((0,\infty))$, $1 < p < \infty$, set

$$T_f(x) := \frac{1}{x}\int_0^x f(t)\mathrm{d}t$$

then $\|T_f\|_p \leq \frac{p}{p-1}\|f\|_p$.

Proof. Since $|T_f| \leq T_{|f|}$, without loss of generality $f \geq 0$.

$$T_f(x) = \int_0^x \frac{f(t)}{x}\mathrm{d}t \underbrace{=}_{t=xy} \int_0^1 f(xy)\mathrm{d}y.$$

Apply Minkowski to $(X, \mathcal{A}, \mu) = (\mathbb{R}, \mathcal{L}, m)$ and $(Y, \mathcal{B}, \nu) = ([0,1], \mathcal{L}, m)$. Let g be defined by $(x,y) \mapsto f(xy)$. Then

$$\begin{aligned}\|g(\cdot,y)\|_p &= \left(\int_0^\infty f(xy)^p \mathrm{d}x\right)^{\frac{1}{p}} \\ &\underbrace{=}_{t=xy} \left(\int_0^\infty y^{-1} f(t)\mathrm{d}t\right)^{\frac{1}{p}} = y^{-\frac{1}{p}}\|f\|_p.\end{aligned}$$

By Minkowski,

$$\begin{aligned}\|T_f\|_p &\leq \int_0^1 \|g(\cdot,y)\|_p \mathrm{d}y \\ &= \int_0^1 y^{-\frac{1}{p}}\|f\|_p \mathrm{d}y = \frac{p}{p-1}\|f\|_p.\end{aligned}$$

■

6.10 Convolutions

Definition 6.10.1. Give $f : \mathbb{R}^n \to \mathbb{C}$ any $y \in \mathbb{R}^n$ set

$$\tau_y f(x) := f(x-y).$$

Lemma 6.10.1. $\|\tau_y f\|_p = \|f\|_p$ for $1 \leq p \leq \infty$.

Proof. DIY: translation invariance of m. ■

Proposition 6.10.1. If $f \in L^p$, $1 \leq p < \infty$, then

$$\lim_{z\to y}\|\tau_z f - \tau_y f\|_p = 0.$$

Sketch. Easy for $f \in C_c^0(\mathbb{R}^n)$ by uniform continuity. In general case, approximation $C_c^0 \subset L^p$ dense. ■

Definition 6.10.2. If $f, g : \mathbb{R}^n \to \mathbb{C}$ measurable, then

$$(f \star g)(x) := \int_{\mathbb{R}^n} f(x-y)g(y)\mathrm{d}y$$

for all x such that the integral is defined.

> **Note.** $f \star g = \int (\tau_y f)g(y)\mathrm{d}y$

Example. $g = \frac{1}{m(B(0,r))}\mathbf{1}_{B(0,r)}$. Then

$$f \star g(x) = (A_r f)(x) = \frac{1}{B(x,r)} \int_{B(x,r)} f \mathrm{d}m.$$

This is actually continuous in x.

Remark. We like $f \star g$ because it is at least "as nice" as f and g.

Lemma 6.10.2. $f \star g = g \star f$ and $f \star (g \star h) = (f \star g) \star h$.

Sketch. Use translation invariance of m and Fubini.

$$\begin{aligned}(f \star g)(x) &= \int f(x-y)g(y)\mathrm{d}y \\ (g \star f)(x) &= \int g(x-y)f(y)\mathrm{d}y.\end{aligned}$$

By the change of variables $y \mapsto x - y$, it is easy to see that these are the same. ∎

Proposition 6.10.2. If $f \in L^p$, $g \in L^1$ for $1 \leq p < \infty$ then

$$f \star g \in L^p \quad \text{and} \quad \|f \star g\|_p \leq \|f\|_p \cdot \|g\|_1.$$

Sketch. By Minkowski,

$$\begin{aligned}\|f \star g\|_p &= \|\int (\tau_y f) g(y)\mathrm{d}y\|_p \\ &\leq \int \underbrace{\|\tau_y f\|_p}_{\|f\|_p} |g(y)|\mathrm{d}y = \|f\|_p \|g\|_1.\end{aligned}$$

∎

Proposition 6.10.3. If $f \in L^p$, $g \in L^q$ for $1 \leq p, q \leq \infty$ and $p^{-1} + q^{-1} = 1$ then $f \star g$ is bounded and uniformly continuous. If $1 < p, q < \infty$, then $f \star g \in C_0^0$.

Sketch. By Holder's inequality,

$$|f \star g(x)| \leq \|f\|_p \|g\|_q, \quad \forall x.$$

Therefore, $f \star g$ is well-defined everywhere and uniformly bounded. For continuity, suppose $1 \leq p < \infty$. Then

$$\begin{aligned}|(\tau_z(f \star g) - f \star g)(x)| &= \left|\int (f(x-y-z) - f(x-y))g(y)\mathrm{d}y\right| \\ &= |(\tau_z f - f) \star g(x)| \\ &\leq \underbrace{\|\tau_z f - f\|_p}_{\to 0} \|g\|_q \to 0 \text{ uniformly in } x.\end{aligned}$$

∎