Principles of Complex Systems, Vols. 1, 2, & 3D
CSYS/MATH 300, 303, & 394
University of Vermont, Fall 2022
Solutions to Assignment 02
Curry with Named Meat 15p

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1. Use a back-of-an-envelope scaling argument to show that maximal rowing speed V increases as the number of oarspeople N as $V \propto N^{1/9}$.

   Assume the following:

   (a) Rowing shells are geometrically similar (isometric). The table below taken from McMahon and Bonner shows that shell width is roughly proportional to shell length ℓ.

   Shell dimensions and performances.

No. of oarsmen	Modifying description	Length, l (m)	Beam, b (m)	l/b	Boat mass per oarsman (kg)	Time for 2000 m (min) I	II	III	IV
8	Heavyweight	18.28	0.610	30.0	14.7	5.87	5.92	5.82	5.73
8	Lightweight	18.28	0.598	30.6	14.7				
4	With coxswain	12.80	0.574	22.3	18.1				
4	Without coxswain	11.75	0.574	21.0	18.1	6.33	6.42	6.48	6.13
2	Double scull	9.76	0.381	25.6	13.6				
2	Pair-oared shell	9.76	0.356	27.4	13.6	6.87	6.92	6.95	6.77
1	Single scull	7.93	0.293	27.0	16.3	7.16	7.25	7.28	7.17

   (b) The resistance encountered by a shell is due largely to drag on its wetted surface.

   (c) Drag force is proportional to the product of the square of the shell's speed (V^2) and the area of the wetted surface ($\propto \ell^2$ due to shell isometry).

   (d) Power $\propto$ drag force $\times$ speed (in symbols: $P \propto D_f \times V$).

   (e) Volume displacement of water by a shell is proportional to the number of oarspeople N (i.e., the team's combined weight).

   (f) Assume the depth of water displacement by the shell grows isometrically with boat length ℓ.

   (g) Power is proportional to the number of oarspeople N.

   Solution: To show Rowing speed

$$V \propto N^{1/9}$$

where V is Rowing speed,
N is the number of oarspeople.

Given,

(a) $D_f \propto V^2\ell^2$ from (c)

(b) $P \propto D_f * V$ from (d)

(c) $\ell^3 \propto N$ from (e)

(d) $P \propto N$ from (g)

Combine (a) in (b), we get

$$P \propto V^2\ell^2 * V$$

$$P \propto V^3\ell^2 \tag{1}$$

From (c)

$$l \propto N^{1/3}$$

,

Then Equation1 becomes

$$P \propto V^3 N^{2/3}$$

using (d),

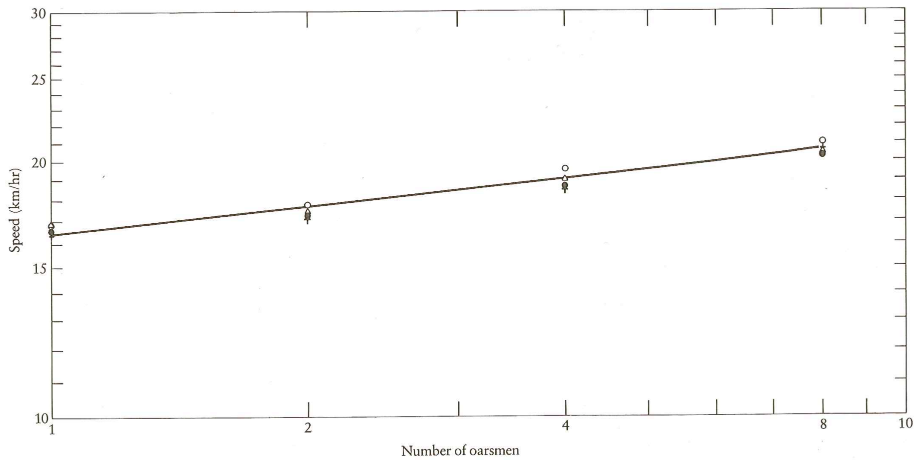
$$N \propto V^3 N^{2/3}$$

taking cube root on both sides, and rearranging N,

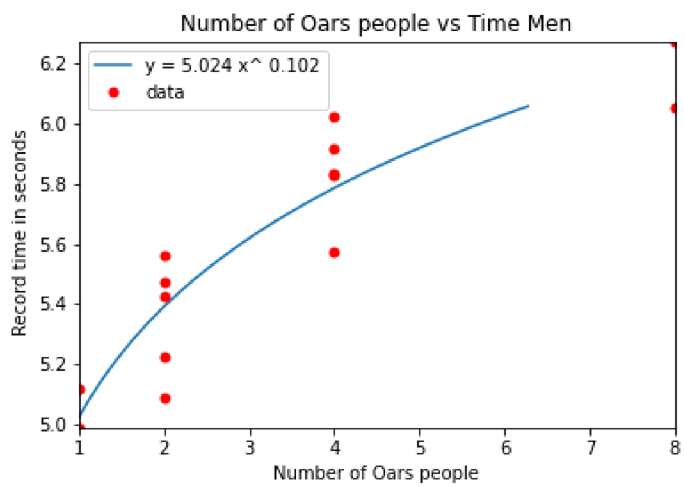
$$V \propto N^{1/9}$$

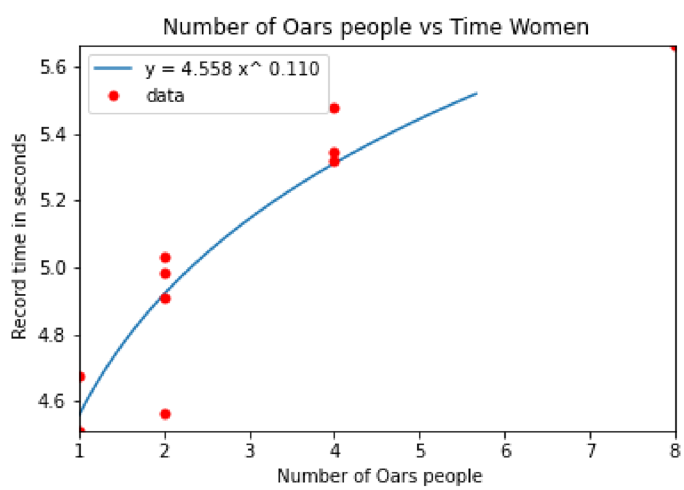
Hence proved. □

2. Find the modern day world record times for 2000 metre races and see if this scaling still holds up. Of course, our relationship is approximate as we have neglected numerous factors, the range is extremely small (1–8 oarspeople), and the scaling is very weak (1/9). But see what you can find. The figure below shows data from McMahon and Bonner.



Solution:





We can see that the exponent found from line fit on log10 transformed data approximately matches up with $1/9(0.11)$ scaling for women(0.110) slightly less for men(0.10).

$\square$

3. Finish the calculation for the platypus on a pendulum problem so show that a simple pendulum's period τ is indeed proportional to $\sqrt{\ell/g}$.

   Basic plan from lectures: Create a matrix A where ijth entry is the power of dimension i in the jth variable, and solve by row reduction to find basis null vectors.

   In lectures, we arrived at:

$$A\vec{x} = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{2}$$

   You only have to take a few steps from here.

   Solution:

   From the video, we are left with one equation

$$F(\pi_1) = 0$$

   .

$$[\pi_1] = (L)^{x_1}(M)^{x_2}(LT^{-2})^{x_3}(T)^{x_4}$$

   Lets perform the row reduction of the augmented form of matrix A.

$$\left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -2 & 1 & 0 \end{array}\right] = \left[\begin{array}{cccc|c} 1 & 0 & 0 & \frac{1}{2} & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -\frac{1}{2} & 0 \end{array}\right]$$

   From the row reduced echelon form, we can get the nullspace basis vectors.

   Nullspace basis vector rewritten in terms of x_4,

   $x_1 = -\frac{1}{2}x_4$
   $x_2 = 0$
   $x_3 = \frac{1}{2}x_4$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = x_4 \begin{bmatrix} -\frac{1}{2} \\ 0 \\ \frac{1}{2} \\ 1 \end{bmatrix}$$

We have

$$\pi_1 = l^{x_1} m^{x_2} g^{x_3} \tau^{x_4}$$

$$[\pi_1] = 1$$

$$[\pi_1] = (L)^{x_1} (M)^{x_2} (LT^{-2})^{x_3} (T)^{x_4}$$

with

- $[\tau] = T$
- $[l] = L$
- $[g] = LT^{-2}$
- $[m] = M$

Plugging in exponents from the column vector,

$$\pi_1 = l^{-\frac{1}{2}} g^{\frac{1}{2}} \tau \propto 1$$

Rearranging, we get

$$\tau \propto \sqrt{\frac{l}{g}}$$

$\square$

4. Show that the maximum speed of animals $V_{\max}$ is proportional to their length L Here are five dimensionful parameters:

   - $V_{\max}$, maximum speed.
   - ℓ, animal length.
   - ρ, organismal density.
   - σ, maximum applied force per unit area of tissue.
   - b, maximum metabolic rate per unit mass (b has the dimensions of power per unit mass).

   And here are the three dimensions: L, M, and T.

   Use a back-of-the-envelope calculation to express $V_{\max}/\ell$ in terms of ρ, σ, and b.

   Note: It's argued in that these latter three parameters vary little across all organisms (we're mostly thinking about running organisms here), and so finding $V_{\max}/\ell$ as a function of them indicates that $V_{\max}/\ell$ is also roughly constant.

   Solution:

   From the information given and dimensions, L, M and T, we have,

(a) $[V_{max}] = LT^{-1}$

(b) $[\ell] = L$

(c) $[\rho] = ML^{-3}$

(d) $[\sigma] = [F]/[Area] = ML^{-1}T^{-2}$

(e) $[b] = [P]/[m] = L^2T^{-3}$

$$\frac{V_{max}}{\ell} \propto \rho^{x_1}\ell^{x_2}b^{x_3}$$

Then,

$$[\frac{V_{max}}{\ell}] = [\rho]^{x_1}[\sigma]^{x_2}[b]^{x_3}$$

$$T^{-1} = (ML^{-3})^{x_1}(ML^{-1}T^{-2})^{x_2}(L^2T^{-3})^{x_3}$$

$$T^{-1} = M^{x_1+x_2}L^{-3x_1-x_2+2x_3}T^{-2x_2-3x_3}$$

Writing into a system of equations,

$$x_1 + x_2 = 0$$

$$-3x_1 - x_2 + 2x_3 = 0$$

$$-2x_2 - 3x_3 = -1$$

Matrix equation and row reduction to get null space:

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ -3 & -1 & 2 & 0 \\ 0 & -2 & -3 & -1 \end{array}\right] = \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

gives us

(a) $x_1 = 1$

(b) $x_2 = -1$

(c) $x_3 = 1$

Then the original equation becomes,

$$\frac{V_{max}}{\ell} \propto \frac{\rho b}{\sigma}$$

$\square$

5. Use the Buckingham π theorem to reproduce G. I. Taylor's finding the energy of an atom bomb E is related to the density of air ρ and the radius of the blast wave R at time t:

$$E = \text{constant} \times \rho R^5/t^2. \tag{3}$$

In constructing the matrix, order parameters as E, ρ, R, and t and dimensions as L, T, and M.

Solution:

Defining dimensions of quantities,

(a) The energy of atom bomb $[E] = ML^2T^{-2}$
(b) The density of air$[\rho] = ML^{-3}$
(c) The radius of blast wave $[R] = L$
(d) time $[t] = T$

Constructing the Buckingham π theorem,

$$[\pi_i] = [E]^{x_1}[\rho]^{x_2}[R]^{x_3}[t]^{x_4}$$

$$[\pi_i] = (ML^2T^{-2})^{x_1}(ML^{-3})^{x_2}(L)^{x_3}(T)^{x_4}$$
$$[\pi_i] = M^{x_1+x_2}L^{2x_1-3x_2+x_3}T^{-2x_1+x_4}$$

Expressing in a system of equations for solving the exponents,

$$x_1 + x_2 = 0$$

$$2x_1 - 3x_2 + x_3 = 0$$
$$-2x_1 + x_4 = 0$$

Row reduction of augmented form of matrix A,

$$\left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 0 \\ 2 & -3 & 1 & 0 & 0 \\ -2 & 0 & 0 & 1 & 0 \end{array}\right] = \left[\begin{array}{cccc|c} 1 & 0 & 0 & -\frac{1}{2} & 0 \\ 0 & 1 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 & \frac{5}{2} & 0 \end{array}\right]$$

The nullspace vector,

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = x_4 \begin{bmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ -\frac{5}{2} \\ 1 \end{bmatrix}$$

Solved exponents in π equation, we get,

$$[\pi_i] = [E]^{\frac{1}{2}}[\rho]^{-\frac{1}{2}}[R]^{-\frac{5}{2}}[t]^1$$
$$\pi_i = E^{\frac{1}{2}}\rho^{-\frac{1}{2}}R^{-\frac{5}{2}}t^1$$
$$E = \pi_i \frac{R^5\rho}{t^2}$$

and

$$\pi_i = constant$$

,

$$E = constant * \frac{R^5\rho}{t^2}$$

Hence shown.

□

6. Use the Buckingham π theorem to derive Kepler's third law, which states that the square of the orbital period of a planet is proportional to the cube of its semi-major axis.

   Let's shed some enlightenment and assume circular orbits.

   Parameters:

   - Planet's mass m;
   - Sun's mass M_{sun};
   - Orbital period τ;
   - Orbital radius r;
   - Gravitational constant G.

   (a) What are the dimensions of these five quantities?

   (b) You will find that there are two dimensionless parameters using the Buckingham π theorem, and that you can choose one to be $\pi_2 = m/M_{\text{sun}}$. Find the other dimensionless parameter, π_1.

   (c) Now argue that $\tau^2 \propto r^3$.

   (d) For our solar system's nine (9) planets (yes, Pluto is on the team here), plot τ^2 versus r^3, and using basic linear regression report on how well Kepler's third law holds up.

Solution:

(a) The dimensions of the five quantities:

- Planet's mass $[m] = M$
- Sun's mass $[M_{\text{sun}}] = M$
- Orbital period $[\tau] = T$
- Orbital radius $[r] = L$;
- Gravitational constant $[G] = [\frac{Nm^2}{kg^2}] = L^3T^{-2}M^{-1}$.

(b) π_1 equation

From Buckingham π theorem,

$$\pi_1 = m^{x_1} M_{\text{sun}}^{x_2} r^{x_3} \tau^{x_4} G^{x_5}$$

$$[\pi_i] = (M)^{x_1}(M)^{x_2}(L)^{x_3}(T)^{x_4}(L^3T^{-2}M^-1)^{x_5}$$

$$[\pi_i] = M^{x_1+x_2-x_5} L^{x_3+3x_5} T^{x_4-2x_5} = 1$$

Matrix form we get a row reduced echelon form already,

$$\left[\begin{array}{ccccc|c} 1 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 3 & 0 \\ 0 & 0 & 0 & 1 & -2 & 0 \end{array}\right]$$

Solving for $\vec{x}$ in $A\vec{x} = \vec{0}$,

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = x_5 \begin{bmatrix} 1 \\ 0 \\ -3 \\ 2 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Comparing with,

$$\vec{x} = \lambda_1 \vec{v_1} + \lambda_2 \vec{v_2}$$

The $\vec{v_2}$ vector gives us $\pi_2 = \frac{m}{M}$, The $\vec{v_1}$ vector would gives us π_1 equation.

$$\pi_1 = m^{x_1} M_{\text{sun}}^{x_2} r^{x_3} \tau^{x_4} G^{x_5}$$

$$\pi_1 = mr^{-3}\tau^2 G$$

$$\pi_1 = \frac{m\tau^2 G}{r^3}$$

(c) We have

$$\pi_1 = \frac{m\tau^2 G}{r^3} = c$$

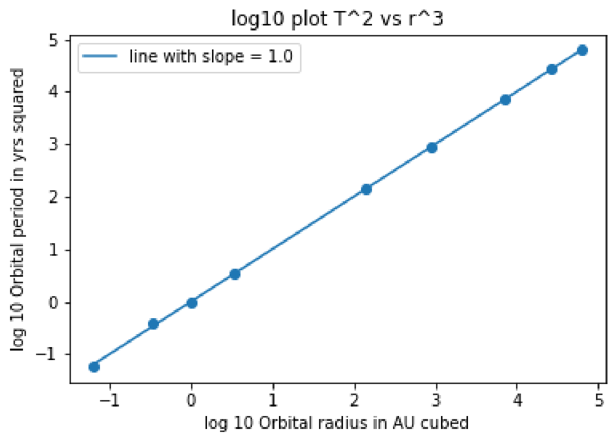
Then rearranging,

$$m\tau^2 G = cr^3$$

From this we can conclude,

$$\tau^2 \propto r^3$$

(d)



Here we can see from the regression, it clearly holds with the relationship predicted by Kepler's law with the fitted line's slope = 1 proving correspondence.

□