Principles of Complex Systems, Vols. 1, 2, & 3D
CSYS/MATH 300, 303, & 394
University of Vermont, Fall 2022
Solutions to Assignment 03
Kingons, or possibly Queons

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1. As in assignment 1, consider a random variable X with a probability distribution given by

$$P(x) = cx^{-\gamma},$$

where c is a normalization constant you determined in the first assignment, and $0 < a \leq x \leq b$. (a and b are the lower and upper cutoffs respectively.) Assume that $\gamma > 1$.

Note: For all answers you obtain for the questions below, replace c by the expression you obtained in the first assignment, and simplify expressions as much as possible.

Compute the nth moment of X which is in general defined as:

$$\langle x^n \rangle = \int_a^b x^n P(x)\, \mathrm{d}x$$

Solution:

From the first assignment, we derived that for $P(x) = cx^{-\gamma}$,

$$c = \frac{\gamma - 1}{a^{-(\gamma-1)} - b^{-(\gamma-1)}} \tag{1}$$

Now,

$$\begin{aligned}
\langle x^n \rangle &= \int_a^b x^n P(x)\,\mathsf{d}x \\
&= \int_a^b x^n c x^{-\gamma}\,\mathsf{d}x \\
&= \int_a^b c x^{n-\gamma}\,\mathsf{d}x \\
&= c\int_a^b x^{n-\gamma}\,\mathsf{d}x \\
&= c\left[\frac{x^{n-\gamma+1}}{n-\gamma+1}\right]_a^b \\
&= c\left(\frac{b^{n-(\gamma-1)}-a^{n-(\gamma-1)}}{n-(\gamma-1)}\right) \\
\langle x^n \rangle &= \frac{-(\gamma-1)}{(n-(\gamma-1))}\left(\frac{b^{n-(\gamma-1)}-a^{n-(\gamma-1)}}{b^{-(\gamma-1)}-a^{-(\gamma-1)}}\right)
\end{aligned} \tag{2}$$

□

2. In the limit $b \to \infty$, how does the nth moment behave as a function of γ?

Solution:

We know that, for $\gamma > 1$

$$\lim_{b\to\infty} c = (\gamma-1)a^{\gamma-1} \tag{3}$$

Now, using equation 3,

$$\begin{aligned}
\lim_{b\to\infty}\langle x^n \rangle &= \lim_{b\to\infty} c * \lim_{b\to\infty}\left(\frac{b^{n-(\gamma-1)}-a^{n-(\gamma-1)}}{n-(\gamma-1)}\right) \\
\lim_{b\to\infty}\langle x^n \rangle &= \frac{(\gamma-1)a^{\gamma-1}}{n-(\gamma-1)}\lim_{b\to\infty}\left(b^{n-(\gamma-1)}-a^{n-(\gamma-1)}\right)
\end{aligned} \tag{4}$$

(a) If $n > (\gamma-1)$, $\lim_{b\to\infty}\langle x^n \rangle \to \infty$ Moment blows up.

(b) If $n < (\gamma-1)$,

$$\lim_{b\to\infty}\langle x^n \rangle = \frac{(-\gamma-1)a^{\gamma-1}}{n-(\gamma-1)}\left(-a^{n-(\gamma-1)}\right)$$

$$\lim_{b\to\infty}\langle x^n \rangle = \frac{(\gamma-1)a^n}{(\gamma-1)-n}$$

This means the moment is finite for the limit $b \to \infty$.

□

3. (a) Find σ, the standard deviation of X for finite a and b, then obtain the limiting form of σ as $b \to \infty$, noting any constraints we must place on γ for the mean and the standard deviation to remain finite as $b \to \infty$.

   Some help: the form of σ^2 as $b \to \infty$ should reduce to

$$= \frac{(\gamma - c_1)}{(\gamma - c_2)(\gamma - c_3)^2} a^2$$

   where c_1, c_2, and c_3 are simple, meaningful constants to be determined (by you).

   (b) For the case of $b \to \infty$, how does σ behave as a function of γ, given the constraints you have already placed on γ? More specifically, how does σ behave as γ reaches the ends of its allowable range?

Solution:

(a) For finite a and b,

$$\langle x^2 \rangle = \frac{-(\gamma - 1)}{(2 - (\gamma - 1))} \left(\frac{b^{2-(\gamma-1)} - a^{2-(\gamma-1)}}{b^{-(\gamma-1)} - a^{-(\gamma-1)}} \right)$$

$$\langle x \rangle^2 = \left(\frac{-(\gamma - 1)}{(1 - (\gamma - 1))} \left(\frac{b^{1-(\gamma-1)} - a^{1-(\gamma-1)}}{b^{-(\gamma-1)} - a^{-(\gamma-1)}} \right) \right)^2$$

Then the variance is defined by,

$$\sigma^2 = \langle x^2 \rangle - \langle x^1 \rangle^2 \tag{5}$$

and the standard deviation is defined by,

$$\sigma = \sqrt{\langle x^2 \rangle - \langle x^1 \rangle^2} \tag{6}$$

Limiting form would be $b \to \infty$,

$$\langle x^2 \rangle = \frac{(\gamma - 1)a^2}{(\gamma - 1) - 2}$$

$$\langle x \rangle^2 = \frac{(\gamma - 1)a}{(\gamma - 1) - 1}$$

Then the variance is defined by,

$$\begin{aligned} \sigma^2 &= \langle x^2 \rangle - \langle x^1 \rangle^2 \\ &= \frac{(\gamma - 1)a^2}{(\gamma - 1) - 2} - \frac{(\gamma - 1)a}{(\gamma - 1) - 1} \qquad (7) \\ \sigma^2 &= \frac{(\gamma - 1)a^2}{(\gamma - 3)(\gamma - 2)^2} \end{aligned}$$

Then the standard deviation σ is,

$$\sigma = \frac{a}{\gamma - 2}\sqrt{\frac{\gamma - 1}{\gamma - 3}} \tag{8}$$

(b) Behavior when $b \to \infty$,

(a) When $\gamma > 3$ the standard deviation is finite.

(b) But when $\gamma \to 3+$, $\sigma \to \infty$

(c) When $\gamma \to \infty$,

$$\begin{aligned} \lim_{\gamma \to \infty} \sigma &= \lim_{\gamma \to \infty} \frac{a}{\gamma - 2}\sqrt{\frac{\gamma - 1}{\gamma - 3}} \\ &= \lim_{\gamma \to \infty} a \frac{1 - \frac{1}{\gamma}}{1 - \frac{2}{\gamma}} \frac{1}{\sqrt{(\gamma - 3)(\gamma - 1)}} \\ &= 0 \end{aligned} \tag{9}$$

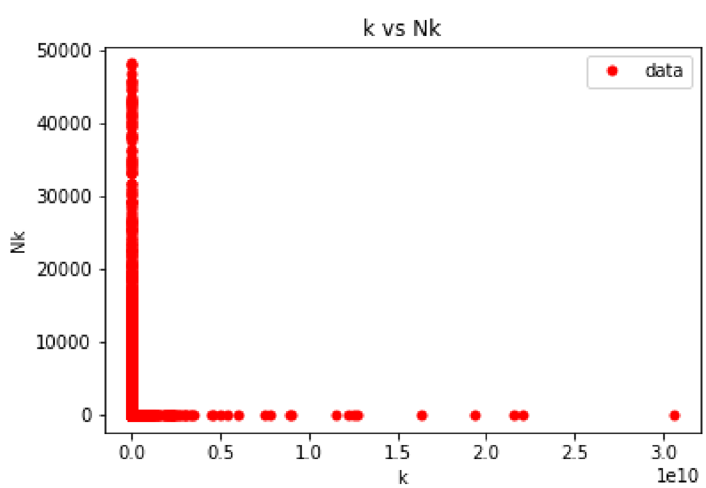
□

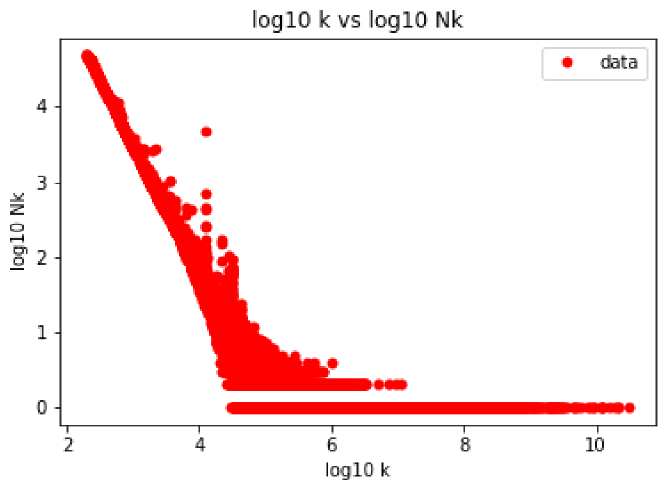
4. Drawing on a Google vocabulary data set:

(a) Plot the frequency distribution N_k representing how many distinct words appear k times in this particular corpus as a function of k.

(b) Repeat the same plot in log-log space (using base 10, i.e., plot $\log_{10} N_k$ as a function of $\log_{10} k$).

Solution:

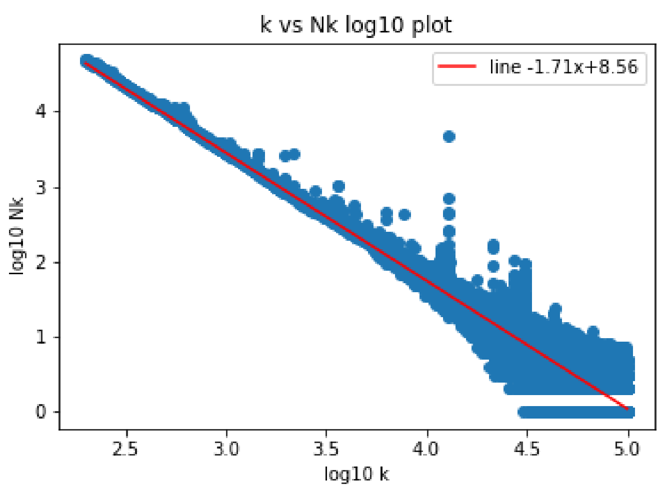
(a) 

(b) 

□

5. Using your eyeballs, indicate over what range power-law scaling appears to hold and, estimate, using least squares regression over this range, the exponent in the fit $N_k \sim k^{-\gamma}$ (we'll return to this estimate later).

Solution:



Looking at the graph, the slope of the line from say $x = (log_{10}k = 2.4$ to $log_{10}k = 4.0)$ seems to be around -2.

We have a line fitted to chosen section where one can fit a line $log_{10}(N_k) = mlog_{10}(k) + b$ with

(a) slope = -1.71

(b) intercept = 8.56

From this the exponent in relation $N_k \propto k^{-\gamma}$,

$$\gamma = 1.71$$

$\square$

6. Compute the mean and standard deviation for the entire sample (not just for the restricted range you used in the preceding question). Based on your answers to the following questions and material from the lectures, do these values for the mean and standard deviation make sense given your estimate of γ?

   Hint: note that we calculate the mean and variance from the distribution N_k; a common mistake is to treat the distribution as the set of samples. Another routine misstep is to average numbers in log space (oops!) and to average only over the range of k values you used to estimate γ.

   Solution:

   Mean,

$$E[k] = \frac{\sum(N_k * k)}{\sum N_k}$$

   Standard Deviation,

$$\sigma_k = \sqrt{\frac{\sum(k - E[k])^2}{\sum N_k}}$$

   (a) Computed Mean = 61651.64

   (b) Computed Standard Deviation = 16783754.85

   For $1 < \gamma = 1.71 < 2$ the standard deviation blows up which make sense and the mean does not describe power law distributions meaningfully. Infact the mean blows up in this range.

$\square$