Complexity Analysis: Willans' Formula Algorithms

1 Original Algorithm: Wilson's Theorem

The original formulation by C. P. Willans utilizes Wilson's Theorem to detect prime numbers. The prime detector $D(j)$ is defined as:

$$D_{wilson}(j) = \left\lfloor \cos^2 \pi \frac{(j-1)!+1}{j} \right\rfloor \tag{1}$$

Complexity Derivation

The computational cost is dominated entirely by the factorial term $(j-1)!$.

1. **Detector Cost:** Calculating $(j-1)!$ involves $j-2$ multiplications. However, the magnitude of the number grows super-exponentially. The bit-complexity of computing $j!$ is roughly $O((j \log j)^2)$. For simplicity, we denote the complexity of the detector as $O(j!)$.

2. **Total Complexity:** Willans' formula sums from $i = 1$ to 2^n. The nested structure implies that finding the n-th prime requires evaluating the detector for integers up to 2^n.

$$T_{orig}(n) \approx \sum_{i=1}^{2^n} (i!) \approx O((2^n)!)$$

Using Stirling's approximation, this is **Super-Exponential**.

2 Modified Algorithm: Trial Division

We replace the factorial-based detector with a standard Trial Division algorithm. The detector $D(j)$ returns 1 if j is prime and 0 otherwise.

Complexity Derivation

1. **Detector Cost:** To determine if j is prime, we check divisibility by integers up to $\sqrt{j}$.

$$\text{Cost}(D_{trial}(j)) = O(\sqrt{j})$$

2. **Prime Counter Cost ($\pi(i)$):** To find the count of primes up to i, we sum the detector cost for all $k \leq i$:

$$\text{Cost}(\pi(i)) = \sum_{k=1}^{i} \sqrt{k} \approx \int_1^i x^{0.5} dx \approx \frac{2}{3} i^{1.5} = O(i^{1.5})$$

3. **Total Complexity:** The master formula iterates i from 1 to 2^n. We sum the cost of the prime counter for each step:

$$T_{mod}(n) \approx \sum_{i=1}^{2^n} i^{1.5} \approx \int_1^{2^n} x^{1.5} dx$$

$$T_{mod}(n) \approx \left[\frac{x^{2.5}}{2.5} \right]_1^{2^n} \approx (2^n)^{2.5} = 2^{2.5n}$$

This results in **Exponential Time** complexity $O(2^{2.5n})$, which is significantly more efficient than factorial time.

3 Side-by-Side Comparison

The table below compares the theoretical time complexity and practical feasibility of both algorithms.

Parameter	**Original (Wilson's)**	**Modified (Trial Division)**
Detector Logic	$(j-1)! \equiv -1 \pmod{j}$	$j \pmod{k} \neq 0, \forall k \leq \sqrt{j}$
Cost per Integer	$O(j!)$	$O(\sqrt{j})$
Total Complexity	$\boldsymbol{O((2^n)!)}$	$\boldsymbol{O(2^{2.5n})}$
Complexity Class	Super-Exponential	Exponential
Feasibility ($n = 10$)	**Impossible** (Requires 1023!)	**Instant** (< 1 ms)
Feasibility ($n = 50$)	Impossible	Impossible (Too slow)

Table 1: Algorithm Performance Comparison