MTH 308A

MINI PROJECT 2

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OBJECTIVE:

For given a, b ∈ R (a < b) and a positive integer n, we had to choose a discrete computational grid Xgrid = {xj} with xj ∈ [a, b], j = 1, . . . , n. Let Fgrid = {f(xj)} be the function data obtained by evaluating a given infinitely differentiable functions f : [a, b] → R whose first 10 derivatives at points a and b were 0.
We had to design and implement an approximation function to f based on the discrete data using interpolation techniques.

THEORY:

Using the given information that the derivatives at the end points a and b are zero for the first 10 derivatives, we can assume that the best approximation for the given function could be periodic function and therefore we try to use trigonometric interpolation to obtain the final approximation.

The approximate trigonometric polynomial is chosen as:

$$p(x) = a_0 + \sum_{k=1}^{n} a_k cos(kx) + \sum_{k=1}^{n} b_k sin(kx)$$

The next important part to be considered is the way in which the points for the discrete data are to be chosen. For this, we consider an affine translation of the *Chebyshev nodes* as our choice. These are the roots of the *Chebyshev polynomial* of the first kind of degree *n*.

The *Chebyshev polynomial* is defined on the set x ∈ [−1, 1] :

$$T(x) = \sum_{k=0}^{n} a_k cos(k * cos^{-1}(x))$$

It can be shown that upon choosing the *Chebyshev nodes* as the interpolation points and using the method of *Chebyshev Interpolation* we can obtain the following error bound on our approximate function:

$$|f(x) - p(x)| \leq \left(\frac{(b-a)}{2}\right)^2 \frac{max_{x\in[a,b]} f^{(n+1)}(x)}{2^n(n+1)!}$$

ALGORITHM

The following algorithm is due to the work of John P. Boyd [1].

1. First we select the *Chebyshev nodes*, xGrid as following:

   $X_k = ((b - a)/2) *cos(\pi*(k-1)/ (n-1)) + (b + a)/2$ for k=1,2,…,n.

2. Compute the fGrid at the interpolation points as
   fGrid (k) = f(xGrid(k)) , k = 1,2,...,n.

3. Compute the interpolation matrix I defined as follows:
   $I_{jk} = (2 /p_j*p_k*(n-1))* cos((j-1)*\pi* (k-1)/ (n-1))$
   where, $p_j = 2$, if j = 1 or n, else pj = 1.

4. Compute the coefficient vector A as follows:
   $A(k) = \sum I_{jk}*fGrid(k)$ where the summation is from j=1 to n.

5. The function f is now approximated with p(x) as follows:

   $p(x) = \sum A(k)*cos ((k - 1)*cos^{-1} ((2x - a - b)/ (b - a))$
   where the summation is from k=1 to n.

REFERENCES

[1] *Finding the Zeros of a Univariate Equation: Proxy Rootfinders, Chebyshev Interpolation, and the Companion Matrix , John P. Boyd*

https://pdfs.semanticscholar.org/09c7/d626cd09de525b62eedf70cc91e43f26273e.pdf