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Math 2605,

2015 Nov. 23rd

The relationship between determinant, trace, and rate of convergence

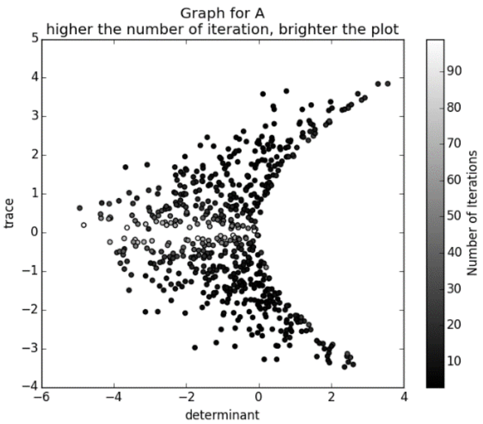

Figure 1

After generating the scatterplots for matrices in the determinant-trace plane, I realized a pattern in the scatterplots. That is, the scatter plots form a parabolic-shape, and the ones that are near the $\mathrm{tr(A)} = 0$ line need more power method iterations than others.

The case with plots near $\mathrm{tr(A)} = 0$ needing more iterations can be explained by the fact that the trace represents the sum of all the eigenvalues. Since $\mathrm{tr(A)} = \sum_{k=1}^{n} \lambda_k$, it could be assumed that if we assume one eigenvalue is close to the negative of another, both eigenvalues will have same absolute value, and thus, making the ratio $\frac{\lambda_1}{\lambda_2}$ (with λ_1 being the biggest eigenvalue, while λ_2 is second biggest) to be close to 1, which means the power method is converging slowly and will need a lot of iterations to meet the tolerance requirements (if the ratio was at 0, it would mean the power method is oscillating between the two eigenvalues).

The case of plots near parabola requiring high numbers of iterations has to do with the fact that some matrices come with (a) pair(s) of complex eigenvalues. This is can be explained by fact that eigenvalues of A are determined by: $\frac{\mathrm{tr(A)} \pm \sqrt{\left(\left(tr(A)\right)^2 - 4det(A)\right)}}{2}$.

From this equation, we can see that if $(tr(A))^2 - 4det(A) < 0$ or $\frac{(tr(A))^2}{4} < D$, the matrix is going to have one or more complex pairs of eigenvalues. Since complex eigenvalues tend to come in (a) complex conjugate pair(s), $(a + bi, a - bi)$, their magnitude would be same, and therefore, it will make convergence ratio $\frac{\lambda_1}{\lambda_2}$ to become 1. This causes plot to not form in the right side of parabola, where determinant is greater than $\frac{(tr(A))^2}{4}$.

However, the graph generated by inverse matrix (Figure 2) looked more haphazard than one generated from the original random matrices. It seems like there are quite a lot of outliers due to fact that original matrices had some very small trace and determinants, causing the Figure 2 to be much more spread out. Also, it seems that the precision loss due to float's limited digit storing had distorted the pattern that was visible in the Figure 1.

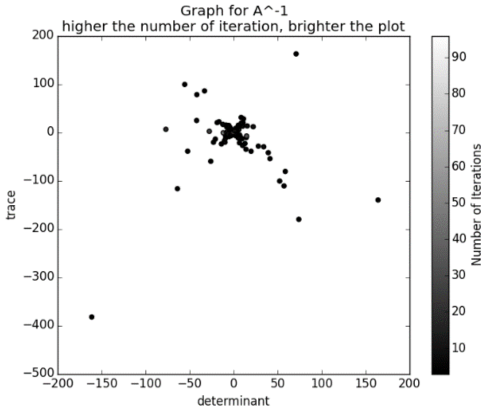

Figure 2

Works Cited

http://math.unice.fr/~frapetti/CorsoF/cours4part2.pdf (Webnotes)

http://isites.harvard.edu/fs/docs/icb.topic251677.files/notes16.pdf (Harvard)