

The Inf-Sup Condition (Ladyzhenskaya–Babuška–Brezzi Condition): Theory, Discretization, and Applications in Interface and CZM Modeling

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Abstract

The inf-sup condition, also known as the Ladyzhenskaya–Babuška–Brezzi (LBB) condition, is a fundamental criterion for the stability of mixed finite element methods in constrained mechanical systems, such as incompressible materials, fluid flow, and interface problems. This report provides an overview of the LBB condition in continuous and discrete settings, establishes its connection with the singular value decomposition (SVD), and explores its critical role in interface mechanics and cohesive zone modeling (CZM). Theoretical foundations are supported by references to key contributors including Fortin, Lorentz, Brezzi, and Bathe.

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1 Introduction

The Ladyzhenskaya–Babuška–Brezzi (LBB) or inf-sup condition is a foundational stability requirement in the numerical solution of constrained systems, particularly mixed finite element methods (FEM). Originally developed for incompressible fluid mechanics and elasticity, the LBB condition ensures the well-posedness of saddle-point problems arising from the use of Lagrange multipliers. Its importance extends to contact, fracture, and interface problems, especially those using cohesive zone models (CZM).

2 Examples of the LBB Condition in Mechanical Modeling

2.1 Incompressible Elasticity and Fluid Flow

In incompressible elasticity and Stokes flow, the displacement (or velocity) field u is constrained to be divergence-free. This constraint is imposed using a pressure-like Lagrange multiplier p , leading to the mixed formulation:

$$\begin{aligned} Au + B^T p &= f \\ Bu &= 0 \end{aligned}$$

The continuous inf-sup condition requires:

$$\inf_{p \neq 0} \sup_{u \neq 0} \frac{(Bu, p)}{\|u\| \|p\|} \geq \beta > 0$$

2.2 Other PDEs where LBB is Relevant

- **Darcy Flow (Porous Media):** The velocity-pressure formulation of Darcy's law leads to the need for inf-sup stability between velocity and pressure spaces.
- **Mixed Formulation of Linear Elasticity:** In stress-displacement mixed formulations, LBB ensures stability between stress and displacement fields.
- **Time-Harmonic Maxwell's Equations:** Mixed methods for solving Maxwell equations in frequency domain rely on LBB-type stability between electric and magnetic fields.
- **Nearly Incompressible Materials:** Volumetric locking in elasticity is avoided by satisfying the inf-sup condition between pressure and displacement spaces.
- **Hydro-mechanical Coupling in Poroelasticity:** Mixed formulations coupling pressure and displacement require LBB for the pressure-displacement stability.

2.3 Finite Element Discretization Examples

- **Stable:** Taylor–Hood elements (e.g., P_2/P_1 for Stokes).
- **Unstable:** Equal-order interpolation (P_1/P_1) leads to pressure oscillations.
- **MINI element:** Enriched P_1 space with bubble functions and constant pressure.

3 The Discrete Inf-Sup Condition and Singular Value Decomposition

After discretization, the mixed problem results in a block matrix system. The discrete inf-sup condition is equivalent to requiring the matrix B to have full rank. This is quantitatively captured by its smallest nonzero singular value:

$$\inf_{q_h \neq 0} \sup_{v_h \neq 0} \frac{q_h^T B v_h}{\|q_h\| \|v_h\|} = \sigma_{\min}(B)$$

The SVD of B provides a complete picture of the stability: zero singular values imply a violation of the inf-sup condition, resulting in unstable modes. This link justifies inf-sup testing through eigenvalue analysis of $BA^{-1}B^T$.

4 Applications to Interface and CZM Modeling

4.1 Mixed Formulations with Lagrange Multipliers

Cohesive Zone Models (CZMs) often involve mixed finite element formulations where tractions across an interface are treated as Lagrange multipliers. Ensuring numerical stability requires that the displacement jump and traction spaces form an inf-sup stable pair.

4.2 Work of Eric Lorentz and Michel Fortin

Lorentz introduced a stable mixed interface element for cohesive laws with constant traction interpolation. This avoided overconstraining the system and spurious traction oscillations. Fortin's foundational work on mixed methods and Fortin interpolation operators provides the theoretical underpinning for constructing inf-sup stable spaces.

4.3 Other References and Practices

Methods like mortar methods, stabilized Lagrange multipliers, and reduced multiplier spaces have been explored in CZM and XFEM to satisfy or relax inf-sup constraints. Notable works include Béchet et al. on stable interface multipliers.

5 Historical Summary and Key Contributors

- **Ladyzhenskaya (1960s):** Incompressible flow and uniqueness conditions.
- **Babuška (1971–73):** General inf-sup in FEM.
- **Brezzi (1974):** Saddle-point theory and discrete stability.
- **Fortin (1980s–1990s):** Fortin operators, MINI elements.
- **Bathe (1990s):** Practical inf-sup tests for engineers.
- **Di Pietro (2000s–):** Extension to DG, HHO, modern methods.

References

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