Chapitre 01: Introduction, simple models

Requirements of an LP problem:

- LP problems seek to maximize or minimize a quantity expressed as an objective function
- Real-world restrictions or constraints limit how much the objective can be achieved
- Both the objective function and constraints must be written as linear equations or inequations

Definition:

LP is a mathematical method to efficiently allocate limited resources and determine the optimal solution by maximizing or minimizing the objective function.

Components of LP:

- A set of decision variables
- An objective function
- A set of constraints

Essentials of LP model:

- Limited resources: limited number of labor, material equipment and finance
- Objective: refers to the optimization aim (maximize profits or minimize cost)
- Linearity: the increase of labor input will cause a proportionate increase of output
- Homogeneity: products, workers, efficiency and machines are assumed to be identical
- Divisibility: it is assumed that resources can be devided into fractions

LP Models:

- Blending models (diet problem)
- Production model
- Transportation model

Canonic LP model:

- Maximization canonic model:
  - Objective function is maximization
  - Constraints are "less or equal" inequalities
  - Variables are non-negative ($x \geq 0$)
- Minimization canonic model:
  - Objective function is minimization
  - Constraints are "greater or equal" inequalities
  - Variables are non-negative ($x \geq 0$)
- Mixed canonic model:
  - Objective function is min or max
  - Constraints are mixed ($<=$ or $>=$ or $=$)
  - Variables can include unrestricted or non-negative variables.

LP standard form:

It is a specific, constrained format of an LP and is commonly used for computational purposes

- Objective Function: is min or max
- Constraints: all constraints must be equalities
- Variable restrictions: All variables must be non-negative ($x \geq 0$)

Transformation rules:

- Objective function:

$$Min(Z) = Max(-Z)$$

- Constraints:

$$ax \leq b \iff \begin{cases} ax + s = b \\ s \geq 0 \end{cases}$$

$$ax \geq b \iff \begin{cases} ax - s = b \\ s \geq 0 \end{cases}$$

- Variables:

$$x \leq 0 \iff x' = -x, \; x' \geq 0$$

$$x \in \mathbb{R} \iff x = x' - x''$$

$$x', x'' \geq 0$$

Remarque:

If we want to transform into a canonic form and we have an equality:

$$ax = b \begin{cases} ax \geq b \\ \text{and} \\ ax \leq b \end{cases}$$

Chapitre 02: Geometry, Graphic method

Convex

A convex set is a region in which, for any two points within the set, the straight line segment connecting them lies entirely within the set

Feasible region:

Feasible region or feasible set or solution space is the set of all possible points of an optimization problem that satisfy the problem's constraints

Theorems:

- The feasible region of an LP is a convex
- if the feasible region is non-empty an bounded, then there in an optimal solution

Corner point:

also called an extreme point of the feasible region is a point that is not the middle point of two other points of the feasible region

Theorems:

- If every variable is non-negative, and if the feasible region is non-empty then there is a corner point
- In 2D, a corner point is at the intersection of two equality constraints
- Corner points illustrate optimality
- if a feasible region has a corner point and it has an optimal solution then ther is an optimal solution that is a corner point

Graphical solusion:

- Solution: A set of real values of $X = (x_1, x_2 \dots)$ which satisfies the constraints.
- Feasible solution: a set of real values of $X = (x_1, x_2 \dots x_n)$ which satisfies the constraints and satisfies the non negativity restriction $X \geq 0$
- Feasible region: The collection of all the feasible solution, is called feasible region

Solution case:

- Case 01: "Unique Solution"
One optimal solution, objective line touches the feasible region at one point
- Case 02: "Multiple solutions"
Infinite solutions along a line segment. Objective line is parallel to one edge of the feasible regions

CamScanner

• Case 03: "No solution (infeasible)"

No feasible solution, the constraint define a region with no overlaping points (no feasible region)

• Case 04: "Unbounded"

Feasible region extends infinitely in the direction of optimization.

Chapitre 03: solving; simplex method

Simplex for Maximization:

1) All inequalities must be $\leq$

2) Convert to standard form: transform inequalities into equations using slack variables (RHS must be positive)

3) Standard objective function:

• $Z = c_1x_1 + c_2x_2 \dots + c_nx_n$

becomes:

$Z - c_1x_1 - c_2x_2 \dots + c_nx_n = 0$

4) Create the initial table T_0:

• Each constraint forms a row. Coefficients of variables in each constraint go into the corresponding columns.

• The RHS of each constraint goes in the solution column.

• Example:

Basic	Z	x_1	S_1	S_2	Solution
S_1	0	1	1	0	40
S_2	0	4	0	1	120
Z	1	-40	0	0	0

5) Pivot Column:

Identify the most negative coefficient in the (Z-row)

6) Pivot Row:

Divide the RHS (solution column) by the corresponding positive values in the in the pivot column and we choose the smallest positive value.

7) Pivot:

The pivot is the intersection of the pivot row and column.

8) Create the tables T_1 :

- Devide the pivot row by the pivot (pivot case becomes 1)
- All the values in the pivote column becomes 0 (except pivot)
- The rest of the values are filled using pivote operations:

$$a = a_{T_0} - \frac{b \times c}{p}$$

a = The new value

a_{T_0} = The value in the previous Table

b : The projection of a_{T_0} on the pivot column

c : The projection of a_{T_0} on the pivot row

p : pivot

9) **Repeat** : repeat steps 5 to 8 until all Z-row coefficients are non-negative

10) **Solution** : The RHS (solution column gives the solution.

Non basic variables are set to 0!

Remark :

- The leaving variable in each table is the one of the pivot row
- The entring variable is the one of the pivot column.

Big M methode !

1) Standard form : (RHS ≥ 0)

- For equal constraint :
  - Add Artificial variable.
- For "$\geq$" constraint :
  - Add surplus variable
  - Add Artificial variable
- For "$\leq$" constraint :
  - Add slack variable

2) Standard objective function:

- Add a large penalty (M) for artificial variables in the objective function.

$$Z = c_1 x_1 + c_2 x_2 + \dots + c_n x_n - M_1 a_1 - M_2 a_2 - \dots - M_n a_n + 0 S_1 + 0 S_2 \dots + 0 S_n$$

3) Create the initial table T_0:

- C_j: The coefficients in the objectif function
- X_B: corresponds to the RHS of the equations
- B: Basic variables
- C_B: Coefficients of basic
- Z_j: Sum of products of CB and x_i coefficient

$$Z_j = \sum CB \times x_i$$

4) Pivote column:

- Identify the most negative value in the $(Z_j - C_j)$ row.
- The variable of the pivote column is the entring variable.

5) Pivote row:

- We devide the XB column (solution column) by the correspondant positive value in the pivot column and we choose the smallest positive value.
- The variable of the pivot row is the leaving variable.

6) Pivot:

The pivot is the intersection of the pivot row and column.

7) Create table T_1:

- The column of the variable leaving the basis can be removed since it's no longer contributes to the solution
- Devide the pivot row by the pivot
- All values in the pivot column becomes 0 (except pivot)
- The rest of the values are filled using pivot operations

8) Repeat: repeat steps 4 to 7 untill all $(Z_j - C_j)$ row values are non-negative

9) Solution: The XB column gives the solution.

Two Phase methode :

Phase 01:

- Standard Form : we convert the inequalities into equations using slack, surplace and artificial variables the same way in the Big M method
- Define the auxiliary objective fuction :

  this function only contains the artificial variables

  $Z' = -M_1 A_1 - M_2 A_2 - \dots - M_n A_n$
- Constructing the table T_0:

  We use the auxiliary objective function insteded of the objective function

  The rest of the table remains the same as in the Big M methode.
- Solving for Z':

  we solve for Z' in the same way in Big M methode
- End of the phase :

  To be able to move to phase two there are 3 conditions :

  - The value of $Z' = 0$
  - All $Z_j - C_j$ are non-negative
  - All the artifitial variables are removed from the table

Phase 2:

- Constructing the initial table :

  We use the origino objective function (without artifitial variables), while keeping the rest values from the last table of phase 1

- Solving for Z:

we solve the same way we did in Big M method, we stop when all the values in $(Z_j - C_j)$ row are non-negative

Special case:

Degeneracy:

- Detection in table:

when determining the pivot row, two or more rows have the same minimum ratio (solution / pivot column).

- Interpretation:

This can result in degenoracy where the program might cycle without any progress

- Resolution:

- Choose any row arbitrarely from the tied rows
- Use techniques like Bland's Rule to prevent cycling.

Multiple Optimal Solutions:

- Detection in table:

when a non-basic variable takes a 0 value in the Z-row (or $Z_j - C_j$ row) in the optimal table.

- Interpretation:

The objective function is parallel to one of the constraints and the PL has multiple optimal solutions (alternative solutions)

Unbounded:

- Detection in table:

When selecting a pivot row, all ratios (solution / pivot column) are non-positive or undefined

• Interpretation:

The objective fuction can increase (in maximization) or decrease (in minimization) indefinitely without violating constraints which make the LP unbounded

Infeasibility:

• Detection in table:

At the end table (or the last table in phase I in the two phase method) if artificial variable remains in the basis (in the two phase method if the artificial variable are not removed) with a non-zero value.

• Interpretation:

The problem is infeasible (has no feasible solution) as the artificial variables represent a violation of the original constraints