Transcritical Bifurcation

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Mathematical Modeling of Complex System

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Chapter 1

Normal Form of Transcritical Bifurcation

1.1 Normal Form of Transcritical Bifurcation

In this section, how Transcritical Bifurcation(TB) occurs depending on varying parameter r will be explored. In addition, the relationship between the stability of equilibrium points of a system and how the solution can be drawn will also be examined.

The **Normal Form** of TB is defined as:

$$x' = f(x,r) = rx - x^2 \quad r, x \in \mathbb{R} \tag{1.1}$$

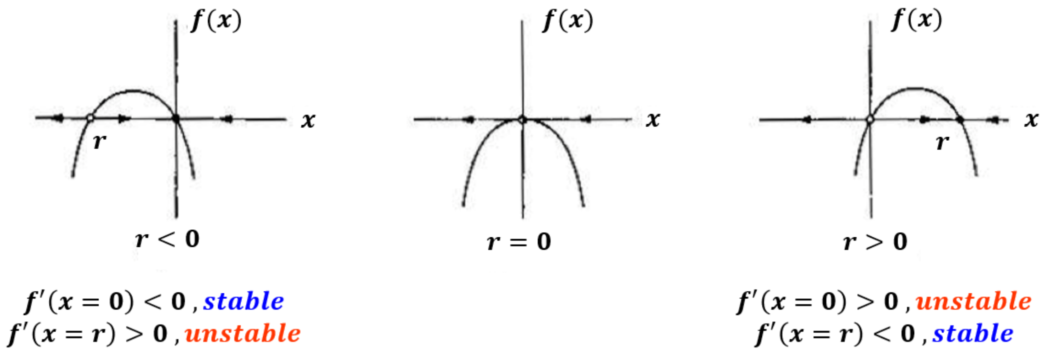
From (1.1), we can find the two equilibrium points of the normal form by setting the equation equal to zero

$$f(x,r) = x(r-x) = 0 \tag{1.2}$$

Then, it will give us

$$x = 0 \quad and \quad x = r \tag{1.3}$$

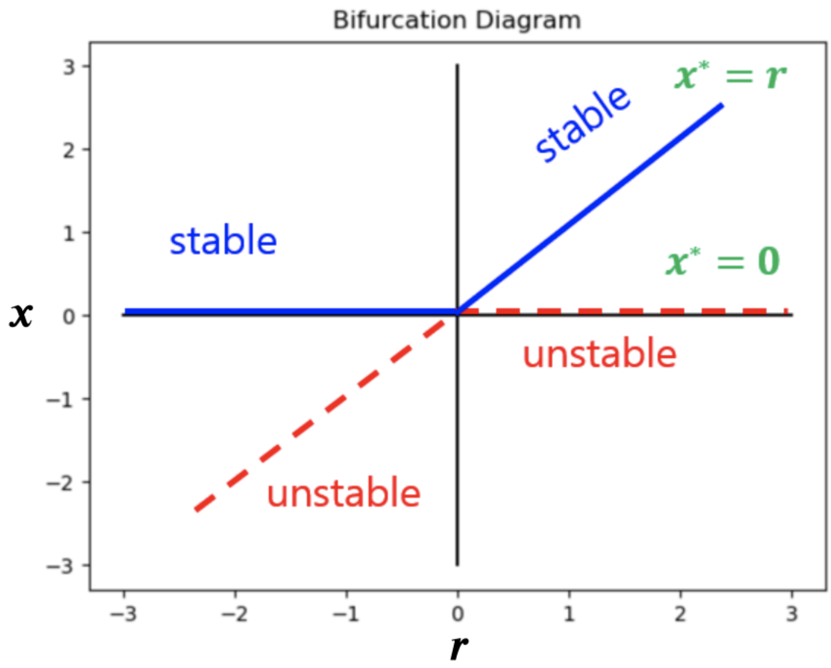
The next logical step would be to examine the stabilities of the two equilibrium points. We will do so by checking the sign of the slopes at the equilibrium points on the vector field of the normal form as r is varied in Fig.1. Here, we set $x' = f(x,r) = f(x)$.



(Figure 1)

From the figure above, we can see that the sign of the slopes at $f'(x=0)$ and $f'(x=r)$ are exchanged based on $r=0$. When $r<0$, the stable equilibrium point is $x=0$ and $x=r$

is unstable. For $r > 0$, the situation is reversed. $x = 0$ is now unstable and $x = r$ becomes stable. That's how a transcritical bifurcation takes place at $r = 0$ (Two equilibrium points are exchanging their stability). (Note: When $r = 0$ there would only be one equilibrium point). It will be more clear if we examine the Bifurcation Diagram of the system below: As shown on the diagram, when $r < 0$, the first curve of equilibrium points being $x = 0$ is stable. When $r > 0$, the second curve, $x = r$ becomes stable.



(Figure 2)

1.2 Stability and Solution of the system

From the equation (1.1), let's partial derivative with respect to x,

$$f_x(x, r) = f'(x) = r - 2x \tag{1.4}$$

If we put the two equilibrium points $x = 0$ and $x = r$ into equation (1.4),

$$f_x(x = 0, r) = f'(x = 0) = r \quad \textit{and} \quad f_x(x = r, r) = f'(x = r) = -r \tag{1.5}$$

we get $f'(x = 0) = r$ and $f'(x = r) = -r$. This implies that the stability of the two equilibrium will be exchanged at $r = 0$.(Fig.2)

Now we can explore the relationship between the stability of equilibrium points and the solution of the system. But first, what does an exchange of stability mean and how does it relate to the solution of the system? To help answer this question, we propose the following:

Proposition 1.1 The solution of the system tends to converge to the stable equilibrium point when t is increasing.

Proof: First, we need to find the exact solution of the normal form (1.1). We begin by using solutions from Bernoulli's Equation. Now set $z = x^{(1-2)} = x^{-1}$. Taking the derivative of this with respect to t, then we get:

$$\frac{dz}{dt} = -x^{-2}x' = -x^{-2}(rx - x^2) = -rx^{-1} + 1 = 1 - rz$$

So the new equation that needs to be solved is:

$$z' = 1 - rz \tag{1.6}$$

Step 1: Consider the Homogeneous Term, $\frac{dz}{dt} = z' = -rz$, using the technique of separation of variable we get $\frac{dz}{z} = -r\frac{dt}{1}$. Solving this further gives $ln|z| = -rt + C$, for C is a constant. Finally the solution is:

$$z = Ce^{-rt} \tag{1.7}$$

Step 2: Using variation of constants. Taking the derivative of (1.7) with respect to t gives:

$$z' = C'e^{-rt} - Cre^{-rt} \tag{1.8}$$

Step 3: Substitute z' into (1.6). This gives:

$$C'e^{-rt} - Cre^{-rt} = -Cre^{-rt} + 1$$

$$C'(t) = e^{rt}$$

$$C(t) = \frac{1}{r}e^{rt} + C_1 \quad C_1 \quad a \quad constant$$

Step 4: Putting $C(t)$ into (1.7), then this will give:

$$z = e^{-rt}(\frac{1}{r}e^{rt} + C_1) = \frac{1}{r} + C_1e^{-rt} = \frac{1}{x}$$

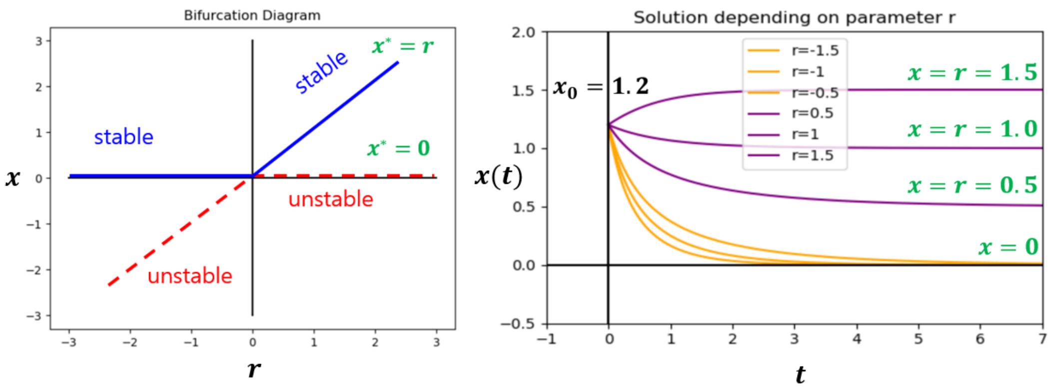
and finally:

$$x(t) = \frac{r}{1 + C_1re^{-rt}}$$

If we consider the initial value $x(0) = x_0$, then $x_0 = \frac{r}{1+C_1r}$ and it implies that $C_1 = \frac{1}{x_0} - \frac{1}{r}$. And the final solution is the following expression:

$$x(t) = \frac{rx_0}{x_0 + (r - x_0)e^{-rt}} \tag{1.9}$$

And here is a diagram for the solution of (1.1) depending on various r (when $x_0 = 1.2$).



(Figure 3)

If

$$r < 0 \quad \rightarrow \quad x = 0 \quad is \quad stable \quad \rightarrow \quad x(t) \quad coverges \quad to \quad x = 0 \quad (orange \quad color)$$

$$r > 0 \quad \rightarrow \quad x = r \quad is \quad stable \quad \rightarrow \quad x(t) \quad coverges \quad to \quad x = r \quad (purple \quad color)$$

Conclusion : We confirm that the solution of the system tends to converge to the stable equilibrium point.■

Chapter 2

Conditions for Transcritical Bifurcation

In this chapter, we will show how any general one-dimensional system undergoes a transcritical bifurcation if it satisfies the conditions of **(Theorem2.1)** and apply these conditions to the normal form. We will also derive how any general system can be transformed so their point of occurrence for a transcritical bifurcation can be translated to the origin at $(0,0)$ in **(Theorem2.2)**.

First, let's define the general system as

$$x' = f(x,r) \quad for \quad x, r \in \mathbb{R} \tag{2.1}$$

And we introduce the assumption that

$$f(0,r) = 0 \quad for \quad r \in \mathbb{R} \tag{2.2}$$

What we know from assumption (2.2) is that $x = 0$ is always one of the equilibrium points. So we can express that the general form (2.1) as

$$x' = f(x,r) = xF(x,r) \quad x, r \in \mathbb{R} \tag{2.3}$$

If we use Taylor expansion for the system at $x = 0$,

$$x' = f(0+x,r) = f(x,r) = f(0,r) + xf_x(0,r) + \frac{x^2}{2}f_{xx}(0,r) + \frac{x^3}{6}f_{xxx}(0,r) + \mathcal{O}(x^4) \tag{2.4}$$

we know $f(0,r) = 0$, therefore,

$$x' = f(x,r) = x[f_x(0,r) + \frac{x}{2}f_{xx}(0,r) + \frac{x^2}{6}f_{xxx}(0,r) + \mathcal{O}(x^3)] = xF(x,r) \tag{2.5}$$

where

$$F(x,r) = f_x(0,r) + \frac{x}{2}f_{xx}(0,r) + \frac{x^2}{6}f_{xxx}(0,r) + \mathcal{O}(x^3) \tag{2.6}$$

Using equations (2.3) and (2.6), we will now derive how a general form undergoes a transcritical bifurcation if it satisfies the given conditions in Theorem 2.1.

2.1 General Form

Theorem 2.1 Consider the system,

$$x' = f(x,r) = xF(x,r) \quad x, r \in \mathbb{R} \tag{2.7}$$

with $f\colon \mathbb{R}\times\mathbb{R}\to\mathbb{R}$ sufficiently smooth. If the system satisfies the following conditions of:

$$(i) f(0,0)=0, \quad (ii) f_x(0,0)=0, \quad (iii) f_r(0,0)=0, \quad (iv) f_{xx}(0,0)\neq 0, \quad (v) f_{xr}(0,0)\neq 0$$

then the system undergoes a transcritical bifurcation at the origin $(0,0)$.

Proof: First, take the partial derivative of $f(x,r)$ with respect to x and get

$$f_x(x,r) = F(x,r) + xF_x(x,r) \tag{2.8}$$

$$\implies f_x(0,0) = F(0,0) = 0 \quad by \quad (ii) \tag{2.9}$$

Differentiate (2.8) again

$$f_{xx}(x,r) = 2F_x(x,r) + xF_{xx}(x,r) \tag{2.10}$$

$$\implies f_{xx}(0,0) = 2F_x(0,0) \neq 0 \implies F_x(0,0) = \frac{f_{xx}(0,0)}{2}, \quad F_x(0,0)\neq 0 \quad by \quad (iv) \tag{2.11}$$

So, we have $F(0,0)=0$ and $F_x(0,0)\neq 0$, and using the Implicit Function Theorem(IFT), $F(x,r)=0$ has an unique solution $x=h(r)$ which satisfies $h(0)=0$ and $F(h(r),r)=0$. So we get,

$$F(h(r),r) = 0, \quad h(0) = 0 \quad by \quad (IFT) \tag{2.12}$$

Replacing x with $h(r)$ in equation (2.6) gives us

$$F(h(r),r) = f_x(0,r) + \frac{h(r)}{2} f_{xx}(0,r) + \frac{(h(r))^2}{6} f_{xxx}(0,r) + \mathcal{O}(h(r)^3) = 0$$

$$\implies \quad \frac{h(r)}{2} f_{xx}(0,r) + \frac{(h(r))^2}{6} f_{xxx}(0,r) + \mathcal{O}(h(r)^3) = -f_x(0,r) \tag{2.13}$$

Differentiate (2.13) with respect to r and simplifying gives

$$\frac{h'(r)}{2} f_{xx}(0,r) + h(r)[\frac{1}{2} f_{xxr}(0,r) + \frac{1}{3} h'(r) f_{xxx}(0,r) + \mathcal{O}(h(r))] = -f_{xr}(0,r) \tag{2.14}$$

Setting $r=0$ gives,

$$\frac{h'(0)}{2} f_{xx}(0,0) + h(0)[\frac{1}{2} f_{xxr}(0,0) + \frac{1}{3} h'(0) f_{xxx}(0,0) + \mathcal{O}(h(0))] = -f_{xr}(0,0) \tag{2.15}$$

By (2.12), $h(0)=0$, therefore,

$$\frac{h'(0)}{2} f_{xx}(0,0) = -f_{xr}(0,0) \implies h'(0) = \frac{-2f_{xr}(0,0)}{f_{xx}(0,0)} \neq 0 \quad by \quad (iv),(v) \tag{2.16}$$

So, we know that $x=0$ and $x=h(r)$ are different equilibrium points. Now, we will check the dynamics of their stabilities. First, insert $x=0$ into equation (2.8) and get

$$f_x(x,r) = F(x,r) + xF_x(x,r)$$

$$\implies f_x(0,r) = F(0,r) \tag{2.17}$$

The above equation can be expressed in terms of r by using Taylor expansion.

$$F(0,r) = F(0,0+r) = F(0,0) + rF_r(0,0) + \frac{r^2}{2} F_{rr}(0,0) + \mathcal{O}(r^3)$$

$$= r[F_r(0,0) + \frac{r}{2} F_{rr}(0,0) + \mathcal{O}(r^2)] \quad (F(0,0)=0) \tag{2.18}$$

Regarding the value of $[F_r(0,0) + \frac{r}{2}F_{rr}(0,0) + \mathcal{O}(r^2)]$, when r is very close to zero, $[F_r(0,0) + \frac{r}{2}F_{rr}(0,0) + \mathcal{O}(r^2)] \approx F_r(0,0)$ because r is insignificantly small and the term

$\frac{r}{2}F_{rr}(0,0) + \mathcal{O}(r^2)$ will not affect the overall value. We can also get the same result by using,

$$\lim_{r\to 0}[F_r(0,0) + \frac{r}{2}F_{rr}(0,0) + \mathcal{O}(r^2)] = F_r(0,0) \tag{2.19}$$

To find $F_r(0,0)$, take the derivative of (2.8) with respect to r,

$$f_{xr}(x,r) = F_r(x,r) + xF_{xr}(x,r) \implies F_r(0,0) = f_{xr}(0,0) \tag{2.20}$$

Therefore, equation (2.18) can be approximated as

$$f_x(0,r) = F(0,r) \approx rf_{xr}(0,0) \tag{2.21}$$

near $r = 0$.

Next, insert another equilibrium point $x = h(r)$ into equation (2.8) and get

$$f_x(h(r),r) = F(h(r),r) + h(r)F_x(h(r),r) = h(r)F_x(h(r),r) \quad (F(h(r),r) = 0) \tag{2.22}$$

The equation above can also be expressed in terms of r by using Taylor expansion.

$$\begin{aligned} h(r)F_x(h(r),r) &= h(0+r)F_x(h(r),r) \\ &= (h(0) + rh'(0) + \frac{r^2}{2}h''(0) + \mathcal{O}(r^3))F_x(h(r),r) \\ &= r[(h'(0) + \frac{r}{2}h''(0) + \mathcal{O}(r^2))F_x(h(r),r)] \quad (h(0) = 0) \end{aligned} \tag{2.23}$$

Regarding the value of $[(h'(0)+\frac{r}{2}h''(0)+\mathcal{O}(r^2))F_x(h(r),r)]$, when r is very close to zero, the function inside the bigger bracket can be approximated as $h'(0)F_x(h(0),0)$ with the same reasoning as the previous case. Using limit,

$$\lim_{r\to 0}[h'(0) + \frac{r}{2}h''(0) + \mathcal{O}(r^2))F_x(h(r),r)] = h'(0)F_x(h(0),0) = h'(0)F_x(0,0) \tag{2.24}$$

So, equation (2.23) can be approximated as

$$f_x(h(r),r) = h(r)F_x(h(r),r) \approx rh'(0)F_x(0,0) \tag{2.25}$$

near $r = 0$. Equation (2.25) can be simplfied. We know that

$$h'(0) = \frac{-2f_{xr}(0,0)}{f_{xx}(0,0)} \quad and \quad F_x(0,0) = \frac{1}{2}f_{xx}(0,0)$$

therefore,

$$f_x(h(r),r) \approx r\frac{-2f_{xr}(0,0)}{f_{xx}(0,0)}\frac{1}{2}f_{xx}(0,0) = r(-f_{xr}(0,0)) \tag{2.26}$$

Now let's compare the results of equations $f_x(0,r) \approx r(f_{xr}(0,0))$ and $f_x(h(r),r) \approx r(-f_{xr}(0,0))$. No matter what the starting sign of $f_{xr}(0,0)$ is, they will be switched when $r = 0$. The dynamics of the cases are summarized as below:

I) $f_{xr}(0,0) > 0$

$$r > 0 \implies \begin{cases} f_x(0,r) \approx rf_{xr}(0,0) > 0, & x = 0 \text{ is unstable.} \\ f_x(h(r),r) \approx r(-f_{xr}(0,0)) < 0, & x = h(r) \text{ is stable.} \end{cases} \tag{2.27}$$

$$r < 0 \implies \begin{cases} f_x(0,r) \approx rf_{xr}(0,0) < 0, & x = 0 \text{ is stable.} \\ f_x(h(r),r) \approx r(-f_{xr}(0,0)) > 0, & x = h(r) \text{ is unstable.} \end{cases} \tag{2.28}$$

II) $f_{xr}(0,0) < 0$

$$r > 0 \implies \begin{cases} f_x(0,r) \approx r f_{xr}(0,0) < 0, & x = 0 \text{ is stable.} \\ f_x(h(r),r) \approx r(-f_{xr}(0,0)) > 0, & x = h(r) \text{ is unstable.} \end{cases} \tag{2.29}$$

$$r < 0 \implies \begin{cases} f_x(0,r) \approx r f_{xr}(0,0) > 0, & x = 0 \text{ is unstable.} \\ f_x(h(r),r) \approx r(-f_{xr}(0,0)) < 0, & x = h(r) \text{ is stable.} \end{cases} \tag{2.30}$$

As we can see from the above dynamics, stabilities are always switching at $r = 0$. This shows that the system undergoes a transcritical bifurcation at $(0,0)$. ■

Now we know that if a 1-D system satisfies the conditions proposed in Theorem 2.1, the system will undergo a TB at (0,0). In reality, however, a TB can take place at any point, not only at the origin. Nevertheless, we can transform any system and shift the point of TB to (0,0).

Theorem 2.2 Any 1-D system that undergoes a TB at (x_0,r_0) can be transformed so the occurrence of the transcritical bifurcation becomes at the origin $(0,0)$ by replacing $x = u + x_0$ and $r = v + r_0$.

Proof: Consider a system $x' = f(x,r)$, which undergoes a TB at (x_0, r_0). From Theorem 2.1 we know that

$$f(x_0,r_0) = f_x(x_0,r_0) = f_r(x_0,r_0) = 0, \quad f_{xx}(x_0,r_0) \neq 0, \quad f_{xr}(x_0,r_0) \neq 0$$

Now, we set the new variables u, v as $x = u + x_0$ and $r = v + r_0$. Consider the new system $g(u,v) = u'$. (**Note:** $\frac{dx}{dt} = \frac{du}{dt} \implies x' = u'$)

$$x' = f(x,r) = f(u + x_0, v + r_0) = g(u,v) = u'$$

Using the conditions established in Theorem 2.1, we will now check if $u' = g(u,v)$ undergoes a TB at $(0,0)$.

$$(i) \quad g(0,0) = f(x_0,r_0) = 0$$

$$\begin{aligned} (ii) \quad g_u(u,v) = f_u(u+x_0,v+r_0) &= \frac{\partial}{\partial x}(f(u+x_0,v+r_0))\frac{\partial x}{\partial u} = f_x(u+x_0,v+r_0) \\ &\implies g_u(0,0) = f_x(x_0,r_0) = 0 \quad (\frac{\partial x}{\partial u} = 1) \end{aligned}$$

$$\begin{aligned} (iii) \quad g_v(u,v) = f_v(u+x_0,v+r_0) &= \frac{\partial}{\partial r}(f(u+x_0,v+r_0))\frac{\partial r}{\partial v} = f_r(u+x_0,v+r_0) \\ &\implies g_v(0,0) = f_r(x_0,r_0) = 0 \quad (\frac{\partial r}{\partial v} = 1) \end{aligned}$$

$$\begin{aligned} (iv) \quad g_{uu}(u,v) = f_{uu}(u+x_0,v+r_0) &= \frac{\partial}{\partial u}(f_u(u+x_0,v+r_0)) = \frac{\partial}{\partial u}(f_x(u+x_0,v+r_0)) \\ &= \frac{\partial}{\partial x}(f_x(u+x_0,v+r_0))\frac{\partial x}{\partial u} = f_{xx}(u+x_0,v+r_0) \\ &\implies g_{uu}(0,0) = f_{xx}(x_0,r_0) \neq 0 \end{aligned}$$

$$\begin{aligned} (v) \quad g_{uv} = f_{uv}(u+x_0,v+r_0) &= \frac{\partial}{\partial v}(f_u(u+x_0,v+r_0)) = \frac{\partial}{\partial v}(f_x(u+x_0,v+r_0)) \\ &= \frac{\partial}{\partial r}(f_x(u+x_0,v+r_0))\frac{\partial r}{\partial v} = f_{xr}(u+x_0,v+r_0) \\ &\implies g_{uv}(0,0) = f_{xr}(x_0,r_0) \neq 0 \end{aligned}$$

So, any system that undergoes TB at (x_0, r_0) can be transformed into the new system $u' = g(u,v)$ that undergoes a TB at $(0,0)$.■

2.2 Normal Form

In this section, we will apply the conditions in Theorem 2.1 to the normal form to confirm its TB occuring at the origin.

Proposition 2.3 The normal form, $x' = f(x,r) = rx - x^2$, $x, r \in \mathbb{R}$ has a TB at (0,0).
Proof: We will show this by checking the conditions proposed earlier.

$$f(x,r) = rx - x^2 \to f(0,0) = 0$$

$$f_x(x,r) = r - 2x \to f_x(0,0) = 0$$
$$f_r(x,r) = x \to f_r(0,0) = 0$$
$$f_{xx}(x,r) = -2 \to f_{xx}(0,0) \neq 0$$
$$f_{xr}(x,r) = 1 \to f_{xr}(0,0) \neq 0$$

So the normal form has a TB at (0,0) and the proposed conditions work well. ■

Chapter 3

Examples of TB

In this chapter, we will present two different examples and show the the two previous theorems can be applied. The first one will have a polynomial form, and the second one has a logarithm term. We will also show how to transform these systems so they can have a TB at $(0, 0)$.

3.1 Polynomial Function

Consider the system from the reference book [2],

$$x' = x^3 + x^2 - (2 + r)x + r \tag{3.1}$$

The system has a TB at $(x, r) = (1, 3)$. Now let's transform the system so that the point of TB can be translated to $(0, 0)$ instead using Theorem2.2.

Replacing $x = u + 1$ and $r = v + 3$ gives

$$x' = f(x, r) = (u + 1)^3 + (u + 1)^2 - (v + 5)(u + 1) + v + 3 = g(u, v) = u(u^2 + 4u - v) = u'$$

$$u' = g(u, v) = u(u^2 + 4u - v) \quad u, v \in \mathbb{R} \tag{3.2}$$

As done previously, the system is transformed as $u' = g(u, v)$. In addition, all the conditions proposed in Theorem 2.1 are satisfied as below. So, it is clear that the transformed system (3.2) has a TB at $(u, v) = (0, 0)$.

$$g(u, v) = u^3 + 4u^2 - vu \to g(0, 0) = 0$$

$$g_u(u, v) = 3u^2 + 8u - v \to g_u(0, 0) = 0$$

$$g_v(u, v) = -u \to g_v(0, 0) = 0$$

$$g_{uu}(u, v) = 6u + 8 \to g_{uu}(0, 0) = 8 \neq 0$$

$$g_{uv}(u, v) = -1 \to g_{uv}(0, 0) = -1 \neq 0$$

To find equilibrium points, we set $u' = u^3 + 4u^2 - vu = 0$, and we get the results, $u_1 = 0$, $u_2 = -2 + \sqrt{4 + v}$, and $u_3 = -2 - \sqrt{4 + v}$. Here, $(v \geq -4)$ because we restrict our analysis for real numbers only. (**Note**: If the value of v is equal to 0, u_1 and u_2 will coalesce. When $v = -4$, u_2 and u_3 will coalesce) u_1 and u_2 pass through (0,0) but u_3 doesn't. So, there might be a TB occurring between u_1 and u_2. To examine this further, we check their stability. Let $g'(u) = 3u^2 + 8u - v$, For u_1

$$g'(u_1 = 0) = -v \begin{cases} < 0 \quad if \qquad\quad v > 0 \rightarrow stable \\ > 0 \quad if \quad -4 < v < 0 \rightarrow unstable \end{cases} \tag{3.3}$$

For u_2

$$g'(u_2 = -2 + \sqrt{4+v}) = 2v + 8 - 4\sqrt{4+v} \begin{cases} > 0 \quad if \qquad\quad v > 0 \rightarrow unstable \\ < 0 \quad if \quad -4 < v < 0 \rightarrow stable \end{cases} \tag{3.4}$$

For u_3

$$g'(u_3 = -2 - \sqrt{4+v}) = 2v + 8 + 4\sqrt{v+4} > 0 \quad if \quad v > -4 \rightarrow unstable \tag{3.5}$$

From (3.3)-(3.4) we can see that the equilibrium points u_1 and u_2 exchanges their stability at $v = 0$. So TB occurs at (0,0). (Bifurcation Diagram in Fig.4)

Now Let's find out the solution of the system. We can first try to solve it analytically as the following.

$$u' = \frac{du}{dt} = u(u - (-2 + \sqrt{v+4}))(u - (-2 - \sqrt{v+4}))$$

$$\frac{du}{u(u - (-2 + \sqrt{v+4}))(u - (-2 - \sqrt{v+4}))} = dt$$

$$\frac{1}{u(u - (-2 + \sqrt{v+4}))(u - (-2 - \sqrt{v+4}))} = \frac{A}{u} + \frac{B}{u - (-2 + \sqrt{v-4})} + \frac{C}{u - (-2 - \sqrt{v-4})}$$

Solving the above equation gives us,

$$A = -\frac{1}{v}, \quad B = \frac{0.5}{v} + \frac{1}{v\sqrt{v+4}}, \quad C = \frac{0.5}{v} - \frac{1}{v\sqrt{v+4}}$$

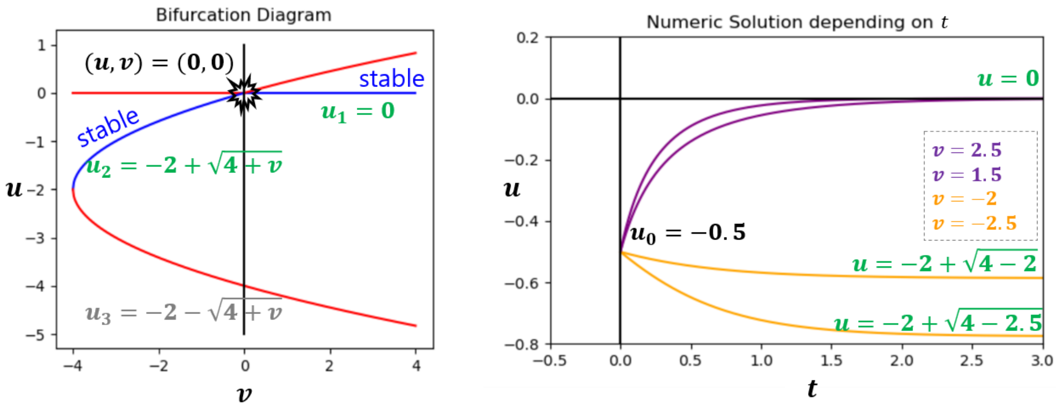
So,

$$A \int \frac{1}{u} du + B \int \frac{1}{u - (-2 + \sqrt{v+4})} du + C \int \frac{1}{u - (-2 - \sqrt{v+4})} du = \int 1 dt$$

However, this is too complicated to solve directly. So, we have decided to use Explicit Euler's method to find the numerics solution instead.

$$u_{k+1} = u_k + hg(t_k, u_k, v)$$

we assume that $u_0 = -0.5$, $h = \frac{5}{499} \approx 0.01$, $t \in [0, 5]$, $t_k = kh$ for $(k = 0, 1, \ldots, 499)$ Then, we can get this graph below:



(Figure 4)

We can also see that the solution tends to converge to the stable equilibrium point. Depending on the parameter v, the solution might be having different forms.

Remark Here, $u(t)$ is the solution of the transformed system (3.2). If we would like to find the solution $x(t)$ of the original system (3.1), then $x(t) = u(t) + 1$.

3.2 Example with a logarithm term

Consider the function $x' = r \ln x + x - 1$ and suppose it undergoes a TB at the point (x_0, r_0). To find this exact point, we will use the conditions stated in Theorem 2.1.

$$f(x_0, r_0) = r_0 \ln x_0 + x_0 - 1 = 0$$

$$f_x(x_0, r_0) = \frac{r_0}{x_0} + 1 = 0$$

$$f_r(x_0, r_0) = \ln x_0 = 0$$

$$f_{xx}(x_0, r_0) = -\frac{r_0}{x_0^2} \neq 0$$

$$f_{xr}(x_0, r_0) = \frac{1}{x_0} \neq 0$$

when we solve for these equations, we get $(x_0, r_0) = (1, -1)$. So the TB of this system takes place at $(1, -1)$. However, we can move this point to $(0, 0)$ by replacing $x = u + 1$ and $r = v - 1$ (Use Theorem2.2). Then the new equation after the substitution is

$$x' = f(x, r) = (v - 1) \ln (u + 1) + u = g(u, v) = u' \tag{3.6}$$

$$u' = g(u, v) = (v - 1) \ln (u + 1) + u \tag{3.7}$$

Now we should check if the system $u' = g(u, v) = (v - 1) \ln (u + 1) + u$ satisfies the conditions that would allow it to undergo a TB (Use Theorem 2.1).

$$g(0, 0) = 0 \tag{3.8}$$

$$g_u(u, v) = \frac{v - 1}{u + 1} + 1 \quad \Longrightarrow \quad g_u(0, 0) = 0 \tag{3.9}$$

$$g_v(u, v) = \ln (u + 1) \quad \Longrightarrow \quad g_v(0, 0) = 0 \tag{3.10}$$

$$g_{uu}(u, v) = -\frac{v - 1}{(u + 1)^2} \quad \Longrightarrow \quad g_{uu}(0, 0) = 1 \neq 0 \tag{3.11}$$

$$g_{uv}(u, v) = \frac{1}{u + 1} \quad \Longrightarrow \quad g_{uv}(0, 0) = 1 \neq 0 \tag{3.12}$$

Since all the conditions are satisfied, it will undergo a TB at $(0, 0)$.

To find the equilibrium points, we need to solve $g(u, v) = 0$ then we get,

$$g(u, v) = (v - 1) \ln (u + 1) + u = 0 \tag{3.13}$$

Here, we can only find one equilibrium point $u = 0$ analytically. Although we can't find another equilibrium point directly, we can draw the graph by using discrete values of u. The way of getting the graph of $u = h(v)$ is mentioned in the remark below.

Remark Here, we will provide the steps performed to obtain the second curve of equilibrium points $u = h(v)$ shown on Fig.5.

Step 1: Rewrite the system (3.13) in terms of u and get

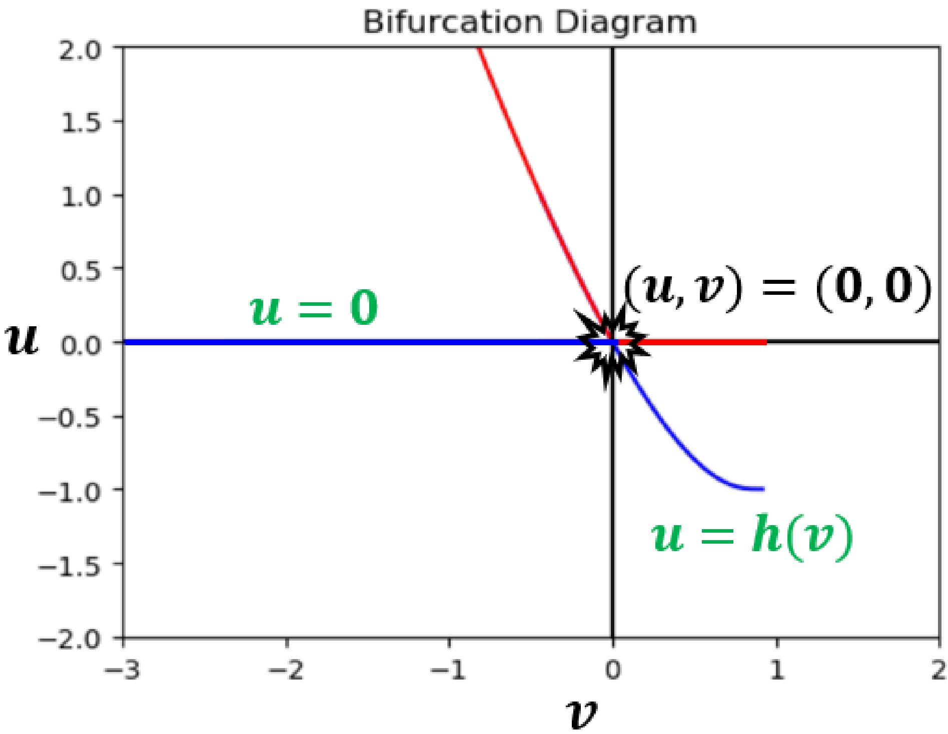
$$v = \frac{\ln(u+1) - u}{\ln(u+1)} \tag{3.14}$$

Step 2: Discretize u with the following settings.

$$u \in (-1, 7], \quad h = \frac{7-(-1)}{500000-0} \approx 0.000016, \quad u_k = -1 + kh$$

Here, $(k = 1, 2, \ldots, 500000)$ so we have $u = [u_1, \ldots, u_{500000}]$. If we put it into the equation (3.14), then we can also get discrete values for $v = [v_1, \ldots, v_{500000}]$.

Step 3: Plot the discretized u-values $u = [u_1, \ldots, u_{500000}]$ on the vertical axis and $v = [v_1, \ldots, v_{500000}]$ on the horizontal axis. ■



(Figure 5)

On Fig 5, the transcritical bifurcation occurs at $(0, 0)$. We can analyze the stability of $u = 0$ by putting it into the equation $g_u(u, v)$ in (3.9). The result is $g_u(u = 0, v) = v$. Clearly, $u = 0$ is stable when $v < 0$ and is unstable when $v > 0$. And it's also certain that another equilibrium point $u = h(v)$ has the opposite dynamics in terms of its stability compared to $u = 0$ because the system undergoes a transcritical bifurcation. That's why we can confirm their stability like on Fig.5 (Blue color is stable equilibrium, red color is unstable equilibrium). However, there is also another way of finding their stabilities by getting the approximated equilibrium point $u = h(v)$. It would be more mathematical so we will introduce the process.

We will use Taylor expansion of the second degree for $u' = g(u, v)$ and ignore the high order term. Setting them as zero to find approximated equilibrium point gives,

$$g(u,v) = g(0,0) + ug_u(0,0) + vg_v(0,0) + \frac{1}{2}(u^2 g_{uu}(0,0) + 2vug_{uv}(0,0) + v^2 g_{vv}(0,0)) + \mathcal{O}(3) = 0$$

we already know that $g(0,0) = g_u(0,0) = g_v(0,0) = g_{vv}(0,0) = 0$, $g_{uu}(0,0) = 1$, and $g_{uv}(0,0) = 1$. Therefore,

$$g(u,v) = \frac{1}{2}(u^2 + 2vu) = 0 \quad\implies\quad u^* = 0 \quad or \quad u^* = -2v$$

Here, $u^* = -2v$ is an approximated equilibrium point of $u = h(v)$, which works well near $(0,0)$. Putting these two equilibrium points into $g_u(u,v) = g'(u) = \frac{v-1}{u+1} + 1$ gives,

$$g'(u^* = 0) = v$$

and

$$g'(u^* = -2v) = \frac{v-1}{1-2v} + \frac{1-2v}{1-2v} = -\frac{v}{1-2v}$$

when the parameter v is replaced with real numbers like $v = 0.1$ and $v = -0.1$ which are near $v = 0$, then

$$v = 0.1 \implies \begin{cases} g'(u=0) = 0.1 > 0, & \text{unstable.} \\ g'(u=-2v) = -0.125 < 0, & \text{stable.} \end{cases} \tag{3.15}$$

$$v = -0.1 \implies \begin{cases} g'(u=0) = -0.1 < 0, & \text{stable.} \\ g'(u=-2v) = 0.083 > 0, & \text{unstable.} \end{cases} \tag{3.16}$$

Similar to the bifurcation diagram in Fig. 5, we can confirm $u = 0$ is stable, $u = h(v) \approx -2v$ is unstable when $v < 0$, $u = 0$ is unstable and $u = h(v) \approx -2v$ is stable when $v > 0$.

Remark What if we were to use a third degree Taylor expansion instead of a second degree expansion of $g(u,v)$ at $(0,0)$? Then, we would get a more precise approximation of the equilibrium point of $u = h(v)$. A third degree Taylor expansion gives,

$$g(u,v) = \frac{1}{2}(u^2 + 2vu) + \frac{1}{3!}(u^3 g_{uuu} + 3uv^2 g_{uvv} + 3u^2 v g_{uuv} + g_{vvv} v^3) = 0$$

$$g_{uuu}(0,0) = -2, \quad g_{uuv}(0,0) = -1, \quad g_{uvv}(0,0) = g_{vvv}(0,0) = 0$$

combining them gives,

$$g(u,v) = -\frac{1}{6}u(2u^2 + 3(v-1)u - 6v) = 0$$

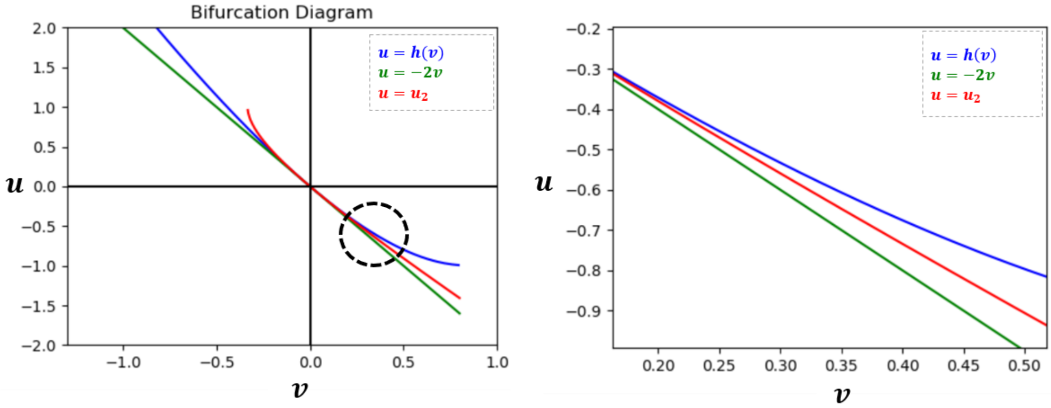
solving the above equation gives us

$$u_1 = 0, \quad u_2 = \frac{-3v + 3 - \sqrt{9v^2 + 30v + 9}}{4}, \quad u_3 = \frac{-3v + 3 + \sqrt{9v^2 + 30v + 9}}{4}$$

u_1 and u_2 both pass through $(0,0)$, meanwhile u_3 doesn't. So, TB related equilibrium points are only u_1 and u_2. Here, u_2 is a better approximated equilibrium point than $u = -2v$. Also, $g'(u_1 = 0) = v$ and $g'(u_2) = \frac{v+3-\sqrt{9v^2+30v+9}}{-3v+7-\sqrt{9v^2+30v+9}}$. Let's now check if it works well near $v = 0$ by replacing $v = 0.1$ and $v = -0.1$.

$$v = 0.1 \implies \begin{cases} g'(u_1 = 0) = 0.1 > 0, & \text{unstable.} \\ g'(u_2) = -0.117 < 0, & \text{stable.} \end{cases} \tag{3.17}$$

$$v = -0.1 \implies \begin{cases} g'(u_1 = 0) = -0.1 < 0, & \text{stable} \\ g'(u_2) = 0.09 > 0, & \text{unstable} \end{cases} \tag{3.18}$$



(Figure 6)

The graph of exact equilibrium point $u = h(v)$ and its approximated solutions are shown in Fig 6. $u = u_2$(red color,which was obtained by 3rd Taylor Expansion) is closer to $u = h(v)$(blue color) than $u = -2v$(green color,which was obtained by 2nd Taylor Expansion). Both approximated equilibrium points work well near $(0, 0)$. The conclusion is, a higher order of Taylor expansion will give us a better approximated equilibrium point to the exact equilibrium point near $v = 0$. ■

Now we will find the solution to the system and analyze the relationship with the stability of the equilibrium points. Since the exact solution can not be found analytically, we will solve the problem with numerical methods(Explicit Euler Method).

Note: Regarding initial value u_0, we set $t \in [0, 8]$, $h = \frac{8-0}{999} \approx 0.008$, $t_k = kh$ for $(k = 0, 1, \ldots, 999)$ and get,

$$u_{k+1} = u_k + hg(u_k, t_k, v)$$

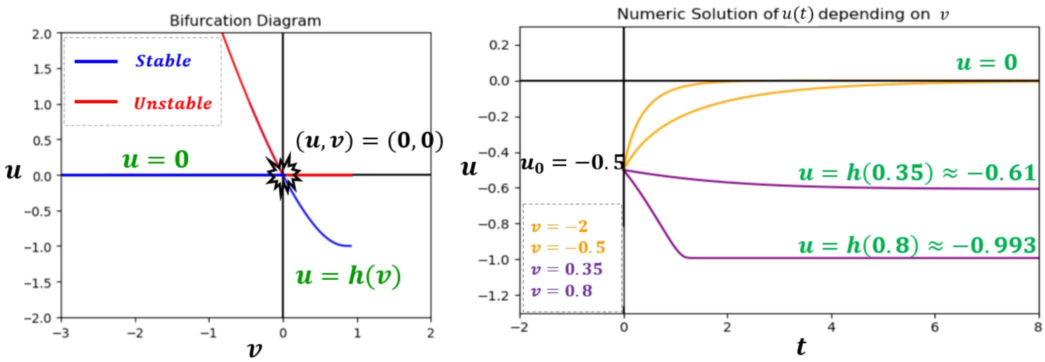
when $k = 0$

$$u_1 = u_0 + h((v-1)\ln(u_0 + 1) + u_0)$$

the logarithm term should be greater than zero. So,

$$u_0 > -1$$

Considering the initial value condition, we put the value $u_0 = -0.5$. Then we use explicit Euler method by setting $t \in [0, 8]$, $t_k = kh$, and $h = \frac{8}{999} \approx 0.008$. The graph of the numerical solution is below:



(Figure 7)

As shown on Fig.7, the solution tends to converge to a stable equilibrium point. When $v < 0$, the solution converges to $u = 0$, when $v > 0$, the solution converges to $u = h(v)$.

Remark Here, similar to the previous example, $u(t)$ is the solution of the transformed system (3.7). If we would like to find the solution $x(t)$ of the original system, then use $x(t) = u(t) + 1$.

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