STAT234: Lecture 8 - Analyzing Tables

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yawn.data <- matrix(c(10,4,24,12), nrow=2, byrow=TRUE,
                dimnames = list(c("Yawned", "No Yawn"),
                                c("Yawn Seed", "No Seed")))
thetable <-  addmargins(yawn.data)
thetable

        Yawn Seed No Seed Sum
Yawned         10       4  14
No Yawn        24      12  36
Sum            34      16  50
```

Is there any effect of planting yawn seed on yawning?

Tables

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How about using a theoretical model?

Our focus today - model based inference of tables.

Tables

For the 34 cases where yawn seed is planted, define random variables $X_1, X_2, \cdots, X_{34}$ such that

$$X_i = 1 \;\; \textit{if person yawns} \quad (1)$$
$$= 0 \;\; \textit{if person does not yawn} \quad (2)$$
$$(3)$$

Let p_1 be the probability

$$p_1 = Pr[\textit{a person yawns} | \textit{yawn seed is planted}]$$

then,

$$X_i \sim Bin(34, p_1)$$

The conditionality of yawn seed planted is there because all the X_i's are generated conditional on yawn seed being planted.

Tables

For the 16 cases where yawn seed is not planted, define random variables $Y_1, Y_2, \cdots, Y_{16}$ such that

$$
\begin{aligned}
Y_i &= 1 \;\; \textit{if person yawns} && (4)\\
&= 0 \;\; \textit{if person does not yawn} && (5)\\
& && (6)
\end{aligned}
$$

Let p_2 be the probability

$$p_2 = Pr[\textit{a person yawns}|\textit{yawn seed is not planted}]$$

then,

$$Y_i \sim Bin(16, p_2)$$

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Tables

We reduce a contingency table to a 2-sample problem

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$$\hat{p}_1 = \frac{10}{34} \quad \hat{p}_2 = \frac{4}{16}$$

$$\hat{p}_1 - \hat{p}_2 = \frac{10}{34} - \frac{4}{16} = 0.044$$

Tables

You know

$$E(\hat{p}_1 - \hat{p}_2) = E(\hat{p}_1) - E(\hat{p}_2) = p_1 - p_2$$

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$$E(\hat{p}_1 - \hat{p}_2) = E(\hat{p}_1) - E(\hat{p}_2) = p_1 - p_2$$

But to make inference on $p_1 - p_2$, we need to know the distribution of $\hat{p}_1 - \hat{p}_2$.

We know under **large sample assumption**

$$\hat{p}_1 \sim N(p_1, \frac{p_1(1-p_1)}{34})$$

$$\hat{p}_2 \sim N(p_2, \frac{p_2(1-p_2)}{16})$$

Note that X_i and Y_j are all independent, hence so are $\hat{p}_1$ and $\hat{p}_2$.

$$\hat{p}_1 - \hat{p}_2 \sim N\left(p_1 - p_2, \frac{p_1(1-p_1)}{34} + \frac{p_2(1-p_2)}{16}\right)$$

Tables

Under H_0, we have

$$p_1 = p_2 = p(say)$$

$$\hat{p}_1 - \hat{p}_2 \sim N\left(0, \frac{p(1-p)}{34} + \frac{p(1-p)}{16}\right)$$

$$\frac{\hat{p}_1 - \hat{p}_2}{\sqrt{\frac{p(1-p)}{34} + \frac{p(1-p)}{16}}} \sim N(0,1)$$

We replace p by $\hat{p}$,

$$\hat{p} = \frac{10+4}{36+14} = \frac{14}{50}$$

Tables

under large sample assumption

$$T = \frac{\hat{p}_1 - \hat{p}_2}{\sqrt{\hat{p}(1-\hat{p})(\frac{1}{34} + \frac{1}{16})}} \sim N(0,1)$$

We find the realized value of this quantity from the sample after substituting the values of $\hat{p}_1 = \frac{10}{36}$, $\hat{p}_2 = \frac{4}{16}$ and $\hat{p} = \frac{14}{50}$,

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We reject researchers' claim !!

Revisit Bootstrapping/ Resampling

We generate $N\hat{p} = 50 \times \frac{14}{50} = 14$ cards which are red (yawners) and $N \times (1 - \hat{p}) = 36$ cards which are black (non-yawners).

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Repeat this 1000 times and record the number of times we get the value of $\hat{p}_1 - \hat{p}_2$ greater than 0.044. Thats my p-value

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This is equivalent to construct the histogram of $\hat{p}_1 - \hat{p}_2$ values from the 10000 draws and counting the frequencies above 0.044

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Chi-square table

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What would be expected counts under the null, if we assume $p_1 = p_2 = p$, we should have

	Yawned	No Yawn	Total
Yawn Seed	$34 \times p$	$16 \times p$	$50 \times p$
No seed	$34 \times (1-p)$	$16 \times (1-p)$	$50 \times (1-p)$
Total	34	16	50

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The table under the null expected then is

```
yawn.data <- matrix(c(9.52,4.48,24.48,11.52), nrow=2, byrow=TRUE
                dimnames = list(c("Yawned", "No Yawn"),
                                c("Yawn Seed", "No Seed")))
E <-  addmargins(yawn.data)
E
```

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        Yawn Seed No Seed Sum
Yawned       9.52    4.48  14
No Yawn     24.48   11.52  36
Sum         34.00   16.00  50
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Take the observed table O and the expected table E. If the null hypothesis is not true, we will expect the squared differences

$$(O_{ij} - E_{ij})^2$$

to be high (i=1,2 , j=1,2).

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Lets consider the expression

$$X = \sum_{i,j} \frac{(O_{ij} - E_{ij})^2}{E_{ij}}$$

We need distribution for X. This distribution has a fancy name - **Chi-square distribution** with degrees of freedom $(r-1) \times (c-1)$ where r is the number of rows in table and c is the number of columns.

$$X \sim \chi^2_{(2-1)(2-1))} = \chi^2_1$$

Chi-Square Distribution

The chi-square distributions: χ_k^2

$$f(x) = \frac{x^{(k/2)-1} e^{-x/2}}{2^{k/2}\Gamma(k/2)} \qquad x > 0,\ k \geq 1$$

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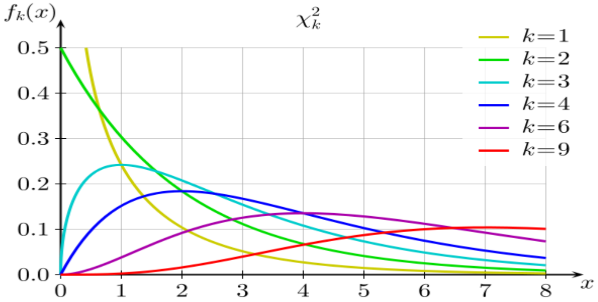

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Origin of the Chi-square distribution:
Let $Z \sim N(0,1)$ and define $Y = Z^2$. Then, $Y \sim \chi_1^2$.

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Let $Z \sim N(0,1)$ and define $Y = Z^2$. Then, $Y \sim \chi_1^2$.

In general, χ_n^2 is a sum of n i.i.d. standard normal distributions.