Law of Large Numbers

Remember that <u>prior to collecting data</u>

from a popn, there are many possible samples of size n and so many possible $\bar{x}_n$'s

So, there is a distn of $\bar{x}_n$'s called the sampling distn of $\bar{X}_n$

Notice uppercase r.v.

Claims not yet proven

In fact, $E(\bar{X}_n) = \mu$ (unbiased for popn mean)

popn variance

& $var(\bar{X}_n) = \sigma^2/n$ precision improves with sample size

LLN: For large n, $\bar{X}_n$ is likely to be close to μ.

For any $\varepsilon > 0$, $\lim_{n \to \infty} P(|\bar{X}_n - \mu| < \varepsilon) = 1$

true for as small an ε you choose

LLN (proof using Chebyshev's inequality)

Claim: For any r.v. X, function $g(x) \geq 0$, $\varepsilon > 0$

$$P[g(X) \geq \varepsilon] \leq \frac{E[g(X)]}{\varepsilon}$$

↑ general form of Chebyshev's inequality

Proof: $E[g(X)] = \sum_{\text{all } x} g(x)\, p(x)$

$$\geq \sum_{A=\{x : g(x) \geq \varepsilon\}} g(x)\, p(x) \quad \leftarrow \text{subset of possible } x$$

$$\geq \sum_{A} \varepsilon\, p(x) \quad \text{since } g(x) \geq \varepsilon \text{ for } x \in A$$

$$= \varepsilon \sum_{A} p(x)$$

$$= \varepsilon P(A) = \varepsilon P[g(X) \geq \varepsilon]$$

So, $E[g(X)] \geq \varepsilon\, P[g(X) \geq \varepsilon]$

$$\Rightarrow P[g(X) \geq \varepsilon] \leq \frac{E[g(X)]}{\varepsilon}$$

Chebyshev's inequality for $\bar{X}_n$ in particular

$$P(|\bar{X}_n - \mu| \geq \varepsilon) = P[\underbrace{(\bar{X}_n - \mu)^2}_{g(\bar{X}_n) \geq 0} \geq \varepsilon^2]$$

$$\leq \frac{E[(\bar{X}_n - \mu)^2]}{\varepsilon^2} = \frac{\mathrm{var}(\bar{X}_n)}{\varepsilon^2} = \frac{\sigma^2}{n\varepsilon^2}$$

Claim: $\mathrm{var}(\bar{X}_n) = \frac{\sigma^2}{n}$

LLN (proof using Chebyshev)

So, $P(|\bar{X}_n - \mu| \geq \varepsilon) \leq \frac{\sigma^2}{n\varepsilon^2}$ for any $\varepsilon > 0$

$$\Rightarrow \lim_{n\to\infty} P(|\bar{X}_n - \mu| \geq \varepsilon) \leq \lim_{n\to\infty} \frac{\sigma^2}{n\varepsilon^2} = 0$$

since

$$\underbrace{1 - P(|\bar{X}_n - \mu| \geq \varepsilon)}_{\downarrow\, n\to\infty \quad 1 - 0} = \underbrace{P(|\bar{X}_n - \mu| \leq \varepsilon)}_{\downarrow\, n\to\infty}$$

So, $\lim_{n\to\infty} P(|\bar{X}_n - \mu| \leq \varepsilon) = 1$

LLN: For large n, $\bar{X}_n$ is <u>likely</u> to be close to μ.

Take-home message