STAT234: Lecture 4 - Sums of random Variables

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Binomial Distribution and Normal approx.

More generally if $X_1, X_2, \cdots, X_n$ be independent identically distributed (iid) $Ber(p)$ random variables,
then

$$Y_n = \sum_{i=1}^{n} X_i \sim Bin\,(n, p)$$

So,

$$E(Y_n) = np \qquad var(Y_n) = np(1-p)$$

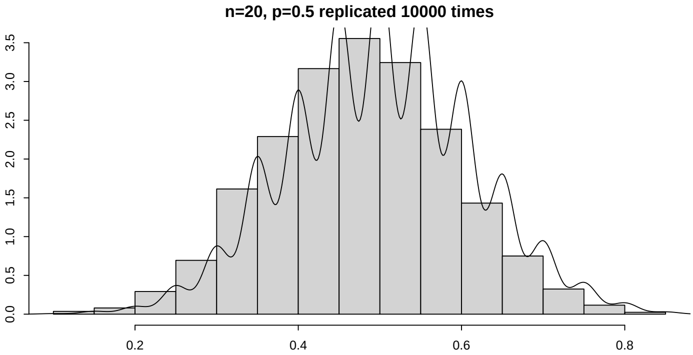
As $n \to \infty$

$$Y_n \approx N\,(np, np(1-p))$$

Lets look at variables generated at $Bin(20, 0.5)$ and repeat the process $10,000$ times.

```
Y1 <- rbinom(10000, 20, p=0.5)/ 20;
summary(Y1)
```

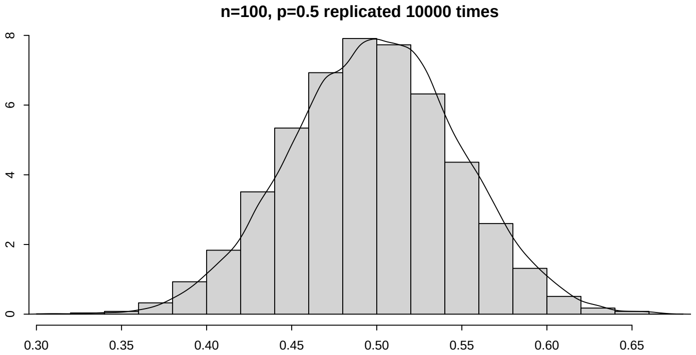
```
   Min. 1st Qu.  Median    Mean 3rd Qu.    Max.
  0.100   0.400   0.500   0.501   0.600   0.850
```


n=20, p=0.5 replicated 10000 times
0.0
0.5
1.0
1.5
2.0
2.5
3.0
3.5
0.2
0.4
0.6
0.8

Now lets look at variables generated at $Bin(100, 0.5)$ and repeat the process $10,000$ times.

```
Y2 <- rbinom(10000, 100, p=0.5)/ 100;
summary(Y2)
```

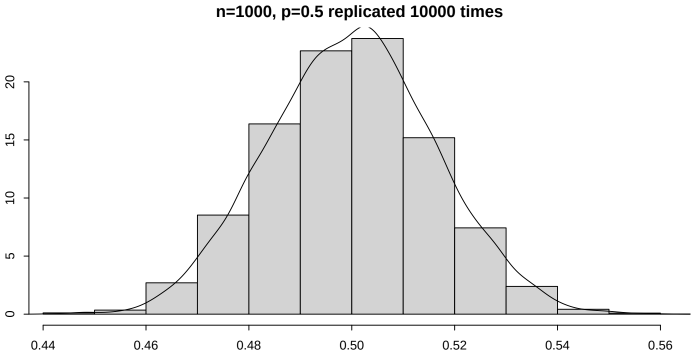
```
   Min. 1st Qu.  Median    Mean 3rd Qu.    Max.
   0.31    0.47    0.50    0.50    0.53    0.67
```


n=100, p=0.5 replicated 10000 times
0
2
4
6
8
0.30
0.35
0.40
0.45
0.50
0.55
0.60
0.65

Now lets look at variables generated at $Bin(1000, 0.5)$ and repeat the process $10,000$ times.

```
Y3 <- rbinom(10000, 1000, p=0.5)/ 1000;
summary(Y3)
```

```
   Min. 1st Qu.  Median    Mean 3rd Qu.    Max.
  0.442   0.489   0.500   0.500   0.511   0.561
```


n=1000, p=0.5 replicated 10000 times
0
5
10
15
20
0.44
0.46
0.48
0.50
0.52
0.54
0.56

Normal approx. of Binomial

http://digitalfirst.bfwpub.com/stats_applet/stats_applet_2_cltbinom.html

We define $\hat{p} = \frac{1}{n} \sum_{i=1}^{n} X_i$.

$$\hat{p} \approx N\left(p, \frac{p(1-p)}{n}\right)$$

This approximation is better when p is not to close to 0 or 1.

Continuity Correction

Discrete	Continuous
$X = 3$	$2.5 < X < 3.5$
$X > 3$	$X > 3.5$
$X \geq 3$	$X > 2.5$
$X < 3$	$X < 2.5$
$X \leq 3$	$X < 3.5$

Continuity Correction

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$$P(X \leq 3) = P(\frac{X - 9.6}{1.959} \leq \frac{3 - 9.6}{1.959})$$

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- $$P(X < 3.5) = P(\frac{X - 9.6}{1.959} \leq \frac{3.5 - 9.6}{1.959}) = P(z \leq -3.114) = 0.00092$$
- To find $P(X > 10)$ Should we use 10.5 or 9.5 ?

Continuous distribution

We would want to define $Pr(X = x)$ for a continuous random variable X, but unfortunately it is 0. We will soon see why

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So how do we proceed to build something like a probability table for discrete variable?
We define **cumulative density function** (cdf)

$$F(x) = Pr(X \leq x)$$

If we differentiate this function, we get **probability density function** (pdf)

$$\frac{d}{dx}F(x) = f(x)$$

$$f(x) \geq 0 \qquad \int f(x)dx = 1$$

Normal Distribution

For normal distribution with mean μ and variance σ^2,

$$\phi(x) := \int \frac{1}{\sqrt{2\pi}} exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)$$

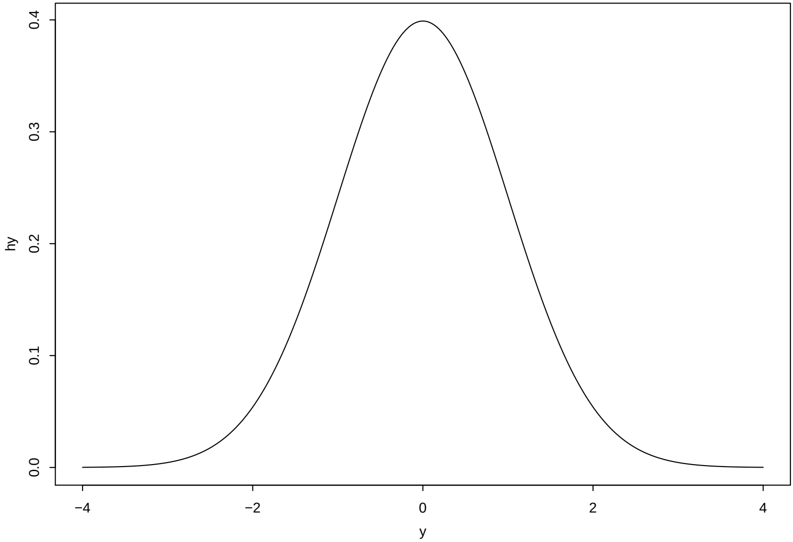
Cumulative distribution

$$\Phi(x) := \int_{-\infty}^{x} \phi(x)dx$$

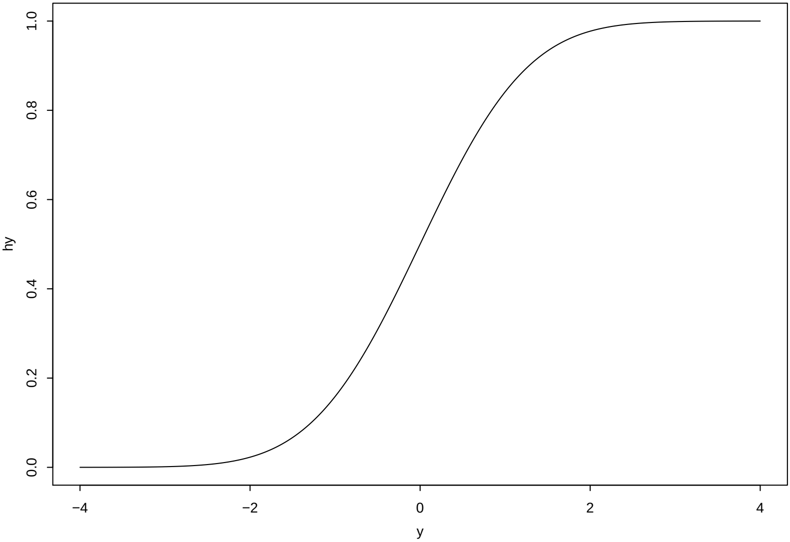
The mean and variance

$$E(X) = \mu \qquad var(X) = \sigma^2$$

probability density graph



cumulative density graph



Sum of normal variables

Suppose we consider two random variables X and Y which follow normal distribution.

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What is the distribution of $X + Y$

$$E(X + Y) = E(X) + E(Y) = \mu + \mu = 2\mu$$

$$var(X + Y) = var(X) + var(Y) = \sigma^2 + \sigma^2 = 2\sigma^2$$

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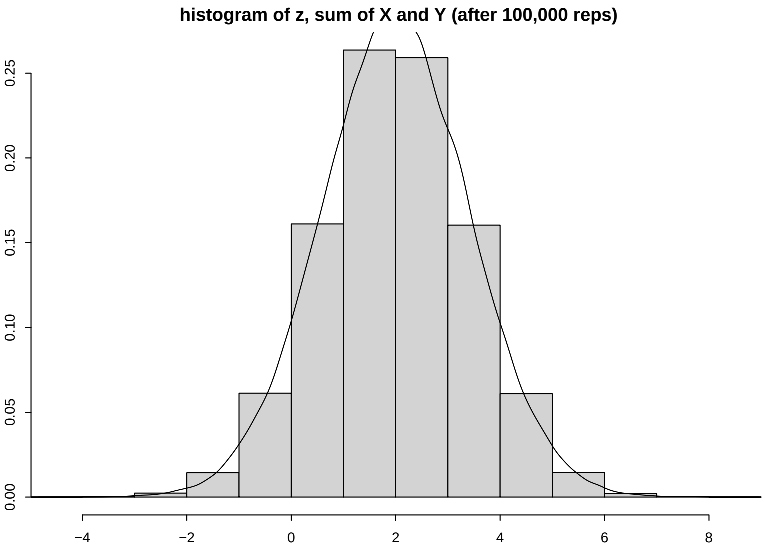
```
x <- rnorm(100000, 1, 1);
y <- rnorm(100000, 1, 1);
z <- x+y;
length(z)
```

```
[1] 100000
```

Sum of normal variables



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Assume now three random variables X, Y and W and consider their sum

$$Z = X + Y + W$$

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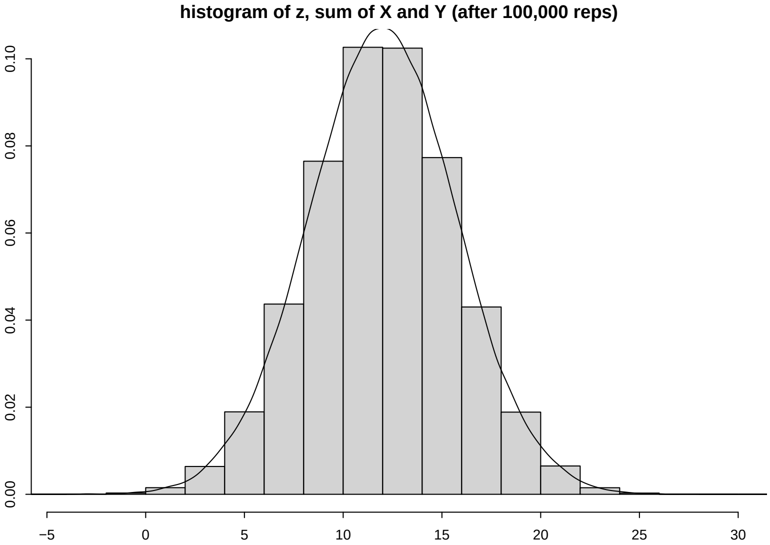
```
x <- rnorm(100000, 1, 1);
y <- rnorm(100000, 1, 3);
w <- rnorm(100000, 10, 2);
z <- x+y+w;
length(z)
```

```
[1] 100000
```

Sum of normal variables



Result

If X_1, X_2, $\cdots$, X_n are *independent* normal random variables such that

$$X_i \sim N\left(\mu_i, \sigma_i^2\right)$$

Then if we define

$$Z = X_1 + X_2 + \cdots + X_n$$

then

$$Z \sim N\left(\sum_{i=1}^{n} \mu_i, \sum_{i=1}^{n} \sigma_i^2\right)$$

Scaling of normal variables

Let X be a random variable that follows a distribution

$$X \sim N(\mu, \sigma^2)$$

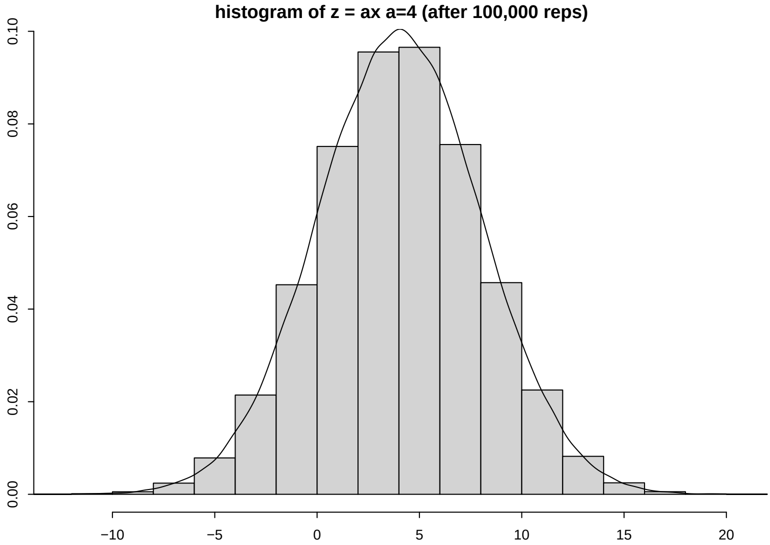
Let a be a constant.

What is the distribution of aX.

```
x <- rnorm(100000, 1, 1);
a <- 4;
z <- a*x
length(z)
```

```
[1] 100000
```

Sum of normal variables



Result

If X be a normal random variables such that

$$X \sim N\left(\mu, \sigma^2\right)$$

Let a be a constant,

$$Z = aX$$

$$E(Z) = aE(X) = a\mu$$

$$var(Z) = var(aX) = a^2 var(X) = a^2\sigma^2$$

and

$$Z \sim N\left(a\mu, a^2\sigma^2\right)$$

Result

If X_1, X_2, $\cdots$, X_n are *independent* normal random variables such that

$$X_i \sim N\left(\mu_i, \sigma_i^2\right)$$

Then if we define

$$Z = c_1 X_1 + c_2 X_2 + \cdots + c_n X_n$$

Check

$$E(Z) = \sum_{i=1}^{n} c_i \mu_i \qquad var(Z) = \sum_{i=1}^{n} c_i^2 \sigma_i^2$$

then

$$Z \sim N\left(\sum_{i=1}^{n} c_i \mu_i, \sum_{i=1}^{n} c_i^2 \sigma_i^2\right)$$

Corollary of Previous Result

If X_1, X_2, $\cdots$, X_n are *independent* normal random variables such that

$$X_i \sim N\left(\mu, \sigma^2\right)$$

then if we define

$$Z = \frac{1}{n}\sum_{i=1}^{n} X_i$$

and

$$Z \sim N\left(\mu, \frac{\sigma^2}{n}\right)$$

Conclusions

We showed that sum of independent normal random variables is normal.

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Central Limit Theorem

All the results we discussed today are true for sum of any n independent variables, where n can be small or large. What about large n?

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All the results we discussed today are true for sum of any n independent variables, where n can be small or large. What about large n?

if $X_1, X_2, \cdots, X_n$ be independent identically distributed (iid) random variables coming from a distribution with mean μ and variance σ^2,
then

$$\sum_{i=1}^{n} X_i \approx N\left(n\mu, n\sigma^2\right) \qquad n \ \ large$$

and

$$\frac{1}{n}\sum_{i=1}^{n} X_i \approx N\left(\mu, \frac{\sigma^2}{n}\right) \qquad n \ \ large$$

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CLT claims sum of a large number of random variables coming from any distribution (well behaved) is approximately normal.

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As a special case, Sum of a large number of Bernoulli or Binomial random variables is approximately normal.

When the underlying distribution is discrete, remember *continuity correction*.

Moment generating function

We define a moment generating function (mgf) as a function of t

$$mgf(t) := E\left(e^{tX}\right) = \int e^{tx} f(x) dx$$

where $f(x)$ is the probability density function (pdf) observed at point x.

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Normal Moment generating function

For a normal random variable X, it can be shown that the moment generating function has the following form

$$mgf(t) := exp\left(\mu t + \frac{1}{2}t^2\sigma^2\right)$$

Using the characterizing property of mgf, if suppose we have

$$mgf(t) : exp\left(2t + \frac{1}{2}6t^2\right)$$

then this is the mgf of

$$X \sim N(2, 6)$$

Moment generating function for sums

If X_1, X_2, $\cdots$, X_n are independent random variables following a distribution with pdf $f(x)$, then the moment generating function of X

$$mgf_X(t) : E(e^{tX})$$

If we define

$$Y = X_1 + X_2 + \cdots X_n$$

$$mgf_Y(t) : E(e^{tY}) = E(e^{tX_1+tX_2+\cdots+tX_n})$$

We can show that (check Canvas for proof)

$$mgf_Y(t) = [mgf_X(t)]^n$$

How to use mgf

We use the mgf properties to show that sum of normal random variables is normal, or sum of linear transformation of normal random variables is normal.

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