Moment Generating Functions

Pruim 3.3 or "Mean Getting Functions" $\underbrace{\text{Mean Getting}}_{\text{my words!}}$

mgf's: A mathematical tool that

1. Sometimes provides an easy way to find mean, $E(X)$, and variance, $V(X)$.
2. Provides another way to uniquely characterize a probability model (in addition to cdf)

$\longrightarrow$ (for technical reasons pdf not a unique identifier)
also, mean & variance are not unique identifiers)

<u>Definition of a mgf</u> for a r.v. X (a <u>function</u> of t for values of t <u>near ϕ</u>)

$$M_X(t) = E[e^{tX}] = \begin{cases} \sum_{\text{all } x} e^{tx} f(x) & (\text{discrete } X) \\ \int_{-\infty}^{\infty} e^{tx} f(x)\,dx & (\text{continuous } X) \end{cases} = E(e^{tX})$$

r.v. $\uparrow$ a #, not random

This may seem an arbitrary choice of functions...
<u>How</u> does the mgf generate <u>"moments"</u>?

First, a <u>"moment" is a mean</u>, an expected value

For example, $E(X)$ is the first moment of X

$\mu_1 = E(X^1)$

and $\mu_2 = E(X^2)$ is the second moment of X

and $\mu_3 = E(X^3)$, $\mu_4 = E(X^4)$, and so on.

The k^{th} moment of X is $\mu_k = E(X^k)$

Also, the variance, $E[(X-\mu)^2]$ is called the 2^{nd} moment about (around) the mean

$\mu_2' = E[(X-\mu)^2] = Var(X)$

Of course, we know that $Var(X) = E(X^2) - [E(X)]^2$

So, $\mu_2' = \mu_2 - (\mu_1)^2$

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So, if we can "generate" the 1st two moments μ_1 & μ_2, then we can find the mean & variance:

$$E(X) = \mu_1 \qquad Var(X) = \mu_2 - [\mu_1]^2$$

<u>How</u> does the mgf "<u>generate</u>" moments?

Recall the Taylor series expansion for e^x around <u>x near zero</u>.

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} \qquad \text{Similarly, } e^{tx} = \sum_{k=0}^{\infty} \frac{(tx)^k}{k!}$$

Then $M_X(t) = E[e^{tX}] = \int_{-\infty}^{\infty} e^{tx} f(x)\,dx$ ← for continuous X (use $\sum_x$ for discrete)

$$= \int_{-\infty}^{\infty} \sum_{k=0}^{\infty} \frac{(tx)^k}{k!} f(x)\,dx = \sum_{k=0}^{\infty} \int_{-\infty}^{\infty} \frac{(tx)^k}{k!} f(x)\,dx$$

limit switch ok if $E[e^{tX}] < \infty$, which is true for all r.v. X we will study in this course.

$$= \sum_{k=0}^{\infty} \frac{t^k}{k!} \underbrace{\int_{-\infty}^{\infty} x^k f(x)\,dx}_{= E(X^k) = \mu_k = k^{th} \text{ moment of } X}$$

So, $M_X(t) = \sum_{k=0}^{\infty} \frac{t^k}{k!} \mu_k$ for <u>all</u> r.v. we will study in this course (discrete & continuous)

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$$M_X(t) = \sum_{k=0}^{\infty} \frac{t^k}{k!}\mu_k$$

$$= \frac{t^0}{0!}\mu_0 + \frac{t^1}{1!}\mu_1 + \frac{t^2}{2!}\mu_2 + \frac{t^3}{3!}\mu_3 + \frac{t^4}{4!}\mu_4 + \cdots$$

$\mu_0 = E(X^0) = E(1) = 1$

& $0! \equiv 1$

$$= 1 + t\mu_1 + \frac{t^2}{2!}\mu_2 + \frac{t^3}{3!}\mu_3 + \frac{t^4}{4!}\mu_4 + \cdots$$

To "generate" 1st moment,
find 1st derivative of $M_X(t)$ (wrt t)
& then set $t=0$.

$$\frac{d}{dt}M_X(t) = 0 + \mu_1 + \frac{2t}{2!}\mu_2 + \frac{3t^2}{3!}\mu_3 + \frac{4t^3}{4!}\mu_4 + \cdots$$

set $t=0$:

$$= \mu_1 + \underbrace{\frac{2(0)}{2!}\mu_2 + \frac{3(0)^2}{3!}\mu_3 + \frac{4(0)^3}{4!}\mu_4 + 0 + 0 + \cdots}_{=0}$$

$$= \mu_1$$

$$= E(X^1) = E(X)$$

To "generate" the 2nd moment,
find 2nd derivative of $M_X(t)$ & set $t=0$

$$\frac{d}{dt}\left[\frac{d}{dt}M_X(t)\right] = 0 + \overbrace{\frac{2}{2!}}^{=1}\mu_2 + \overbrace{\frac{3\cdot 2\,t}{3!}}^{=1}\mu_3 + \frac{4\cdot 3}{4!}t^2\mu_4 + \cdots$$

$$= \mu_2 + t\mu_3 + \frac{t^2}{2}\mu_4 + \cdots$$

set $t=0$:

$$= \mu_2 + (0)\mu_3 + \frac{(0)^2}{2}\mu_4 + 0 + 0 + \cdots$$

$$= \mu_2 = E(X^2)$$

Then, can find $var(X) = \mu_2 - [\mu_1]^2$.