Lecture 3: Probability

Kushal K Dey

Toss a coin

```
TossCoin <- function(n=1){
    return(replicate(n, sample(c("H","T"),1)))
}
TossCoin(1)

[1] "T"

TossCoin(n=2)

[1] "H" "T"

TossCoin(n=10)

 [1] "H" "T" "H" "T" "T" "T" "H" "T" "H" "T"
```

Experimental States

All possible states of an experiment.

1 toss of coin: $\{H, T\}$

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1 toss of coin: $\{H, T\}$

2 tosses : $\{H, H\}, \{H, T\}, \{T, H\}, \{T, T\}$

10 tosses: $\{x_1 x_2 \cdots x_n : x_i \in \{H, T\}\}$

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Ex: A coin is tossed 2 times. The random variable X is the number of tails that are noted.

state	X
HH	0
HT	1
TH	1
TT	2

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Are all the states of an experiment equally likely?

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$X = 0$ is also an event, comprises of $\{TT\}$.

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Repeat the experiment many many times (as many as you can imagine- may be 10^7).

The proportion of times you observe each experimental state after all these many repititions gives you the probability of that state.

Example of Probability

We cite an example of how to measure probability for one coin toss with states H and T.

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For a fair toss of coin, we saw

state	Prob
H	$\Pr(\text{H}) = 0.5$
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state	Prob
H	$\Pr(\text{H}) = 0.5$
T	$\Pr(\text{T}) = 0.5$

Let X be 1 if state is H and 0 if T.

X	Prob
1	$\Pr(\text{X}{=}1) := \Pr(\text{H}) = 0.5$
0	$\Pr(\text{X}{=}0) := \Pr(\text{T}) = 0.5$

Probability of an Event

Probability of an event is the sum of probability of all states making up the event.

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state	Prob
HH	Pr(HH) = Pr(H in toss 1) * Pr(H in toss 2) = 0.25
HT	Pr(HT) = Pr(H in toss 1) * Pr(T in toss 2) = 0.25
TH	Pr(TH) = Pr(T in toss 1) * Pr(H in toss 2) = 0.25
TT	Pr(HT) = Pr(T in toss 1) * Pr(T in toss 2) = 0.25

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Let X be 1 if we observe at least one H in two coin tosses.

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Then the probability table for the random variable X.

X	states	Prob
0	TT	Pr(X=0) = Pr(TT) = 0.25
1	{ HH, HT, TH }	Pr(X=1) = Pr(HH) + Pr(HT) + Pr(TH) = 0.7

Note that the sum of the probabilities for any probability table would be 1 (sum across all possible states).

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Let X be number of Ts observed in two tosses of a coin

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X	states	Prob
0	HH	Pr(X=0) = Pr(HH) = 0.25
1	{ HT, TH }	Pr(X=1) = Pr(HT) + Pr(TH) = 0.50
2	TT	Pr(X=2) = Pr(TT) = 0.25

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What type of interpreting?

Measures of central tendency and spread

Probability distribution of random variable

Suppose X is a random variable that takes values $x_1, x_2, \cdots .x_n$ and let us define the probability

$$p(x_i) = Pr(X = x_i)$$

Then we have

$$0 \leq p(x_i) \leq 1$$

$$\sum_{i=1}^{n} p(x_i) = 1$$

Cumulative probability

$$F(x) = Pr(X \leq x) = \sum_{x_i \leq x} p(x_i)$$

Expectation of random variable

If X is a discrete random variable with possible values

$x_1, x_2, x_3, ..., x_n$

$$E(X) := \sum_{i=1}^{n} x_i Pr(X = x_i)$$

For one coin toss, let X be 1 if H and 0 if T.

$$E(X) = 0.Pr(X = 0) + 1.Pr(X = 1) = 0.5$$

If X is number of Ts in 2 independent tosses of a coin

$$E(X) = 0.Pr(X = 0) + 1.Pr(X = 1) + 2.Pr(X = 2) = 0 + 0.5 + 2 * 0.25 = 1$$

If I am asked to choose ONE value of the variable X, this is the value we may want to choose.

Expectation and $\bar{x}$

We saw that if we observe values $x_1, x_2, \cdots, x_n$ with frequencies $f_1, f_2, \cdots, f_n$, then

$$\bar{x} := \frac{1}{n}\sum_{i=1}^{n} x_i f_i = \frac{1}{n}\sum_{i=1}^{n} x_i \left(\frac{f_i}{n}\right)$$

Now we can argue that as $n \to \infty$, $\frac{f_i}{n} \approx Pr(X = x_i)$, then we get the expectation

$$E(X) = \sum_{i=1}^{n} x_i Pr(X = x_i)$$

Law of Large Numbers

Seems like as n becomes larger, the sample average $\bar{X}_n$ becomes closer and closer to the population average $\mu = E(X)$.

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Can we mathematically formulate it.

$$Pr(\left|\bar{X}_n - \mu\right| > \delta) \to 0$$

for any $\delta > 0$.

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If you tossed it many more times than 100, say 10,000, you would expect to get even closer to a 1:1 ratio of heads and tails (closer and closer to 50% heads and 50% tails).

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The more times you repeat the experiment, the closer you should get to the true probability (which remember is proportion for infinite many trials).

Chebyshev Inequality

General Chebyshev Inequality: For any random variable X, for any fixed $\epsilon > 0$ and a function $g(X) \geq 0$,

$$Pr[g(X) \geq \epsilon] \leq \frac{E(g(X))}{\epsilon}$$

Chebyshev Inequality:

$$Pr[\left|\bar{X}_n - \mu\right| \geq \epsilon] \leq \frac{E(\bar{X}_n - \mu)^2}{\epsilon^2}$$

Variance

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Define it as

$$
\begin{aligned}
Var(X) &:= \sum_{i=1}^{n} (x_i - E(X))^2 Pr(X = x_i) && (1) \\
&= E(X - E(X))^2 && (2) \\
&= E(X^2) - E(X)^2 && (3) \\
&= \sum_{i=1}^{n} x_i^2 Pr(X = x_i) - \left(\sum_{i=1}^{n} x_i Pr(X = x_i) \right)^2 && (4) \\
& && (5)
\end{aligned}
$$

Our first Probability Distribution

Let X be a random variable taking values 0 and 1 for T and H in a coin toss experiment.

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$$
\begin{aligned}
X &= 1 \qquad \textit{prob} \;\; p & (6)\\
&= 0 \qquad \textit{prob} \;\; (1-p) & (7)\\
& & (8)
\end{aligned}
$$

where p is the probability of head (0.5 in fair coin toss case).

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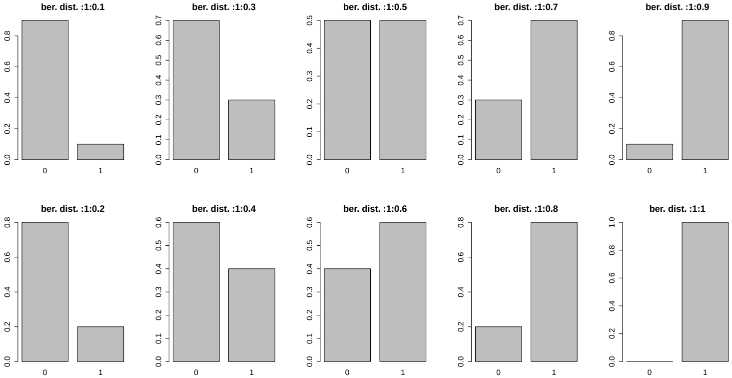
where p is the probability of head (0.5 in fair coin toss case).

This distribution is called a **Bernoulli distribution** and we term $X \sim Ber(p)$

$$E(X) = p \qquad var(X) = E(X^2) - E(X)^2 = p - p^2 = p(1-p)$$

Bernoulli distribution graph

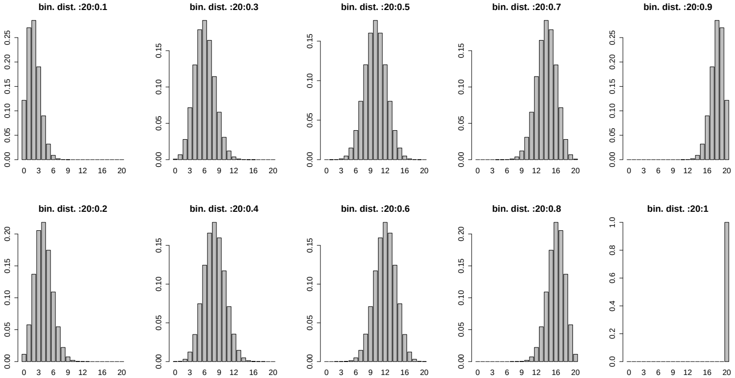


Binomial Distribution

Here X denotes number of heads observed in n tosses. Let probability of observing head in one toss be p

$$
\begin{aligned}
Pr(X = 0) &= (1-p)^n && (9)\\
Pr(X = 1) &= np(1-p)^{n-1} && (10)\\
Pr(X = 2) &= \binom{n}{k} p^2 (1-p)^{n-2} && (11)\\
&....... && (12)\\
Pr(X = x) &= \binom{n}{k} p^x (1-p)^{n-x} && (13)\\
&........ && (14)\\
Pr(X = n) &= p^n && (15)\\
& && (16)
\end{aligned}
$$

Binomial Distribution graph (n=20)



Moment generating function

The moment generating function (mgf) is given by

$$
\begin{aligned}
mgf(t) &:= E(e^{tX}) && (17)\\
&= E\left(\left\{\sum_{k=1}^{\infty}\frac{(tX)^k}{k!}\right\} Pr(X=k)\right) && (18)\\
&= \sum_{k=1}^{K}\frac{t^k}{k!}E\left(X^k\right) && (19)\\
&= \sum_{k=1}^{K}\frac{t^k}{k!}\mu_k && (20)
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Why do we care about $mgf(t)$?

The problem

Where is the connection to the last 4 classes?

How to connect to previous classes

Suppose I want to find out what proportion of Males/Females study in UChicago.

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Each record I take of a person is one run of the experiment. The states of the space for one run is $\{M, F\}$.

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Say, if I observe M, I am noting down 1 and if F, I am noting down 0.

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For the first record I note down, define

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\begin{aligned}
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&= 0 \;\; if \;\; F && (22)\\
& && (23)
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$$

X is a random variable.

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Assume the actual proportion of males in UChicago is p. We do not know p (otherwise we would not have sampled). Then

$$X \sim Ber(p)$$

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Now I can define n random variables $X_1, X_2, \cdots, X_n$, so that

$$\begin{aligned} X_1 &= 1 \;\; if \;\; (M x_2 x_3 \cdots x_n) && (24) \\ &= 0 \;\; if \;\; (F x_2 x_3 \cdots x_n) && (25) \\ & && (26) \end{aligned}$$

and so on.

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The point of sampling is if by just recording 100/1000 students can give you a somewhat close approximation to actual parameter p, why bother recording all students and waste your time and energy and resources?

That is the main reason why we do Statistics!!

A fun matching problem

http://www.rossmanchance.com/applets/randomBabies/RandomBabies.html

Joint Distribution

Consider two independent tosses of a coin.

Let X_1 be the number of Ts observed in the 2 tosses.

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The table

states	X_1	X_2	Prob
HH	0	0	0.25
HT	1	1	0.25
TH	1	1	0.25
TT	2	1	0.25

Joint Probability Table

$X1 \| X2$	0	1
0	0.25	0
1	0	0.50
2	0	0.25

Joint Probability Table

$X1 \vert X2$	0	1
0	0.25	0
1	0	0.50
2	0	0.25

$X1 \vert X2$	0	1	Total
0	0.25	0	0.25
1	0	0.50	0.50
2	0	0.25	0.25
Total	0.25	0.75	1

Joint Probability Table

Marginal distribution of $X1$

$X1 \mid X2$	0	1	Total
0	0.25	0	0.25
1	0	0.50	0.50
2	0	0.25	0.25
Total	0.25	0.75	1

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1	0	0.50	0.50
2	0	0.25	0.25
Total	0.25	0.75	1

Marginal distribution of $X2$

Joint Probability Table

Marginal distribution of $X1$

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0	0.25	0	0.25
1	0	0.50	0.50
2	0	0.25	0.25
Total	0.25	0.75	1

Marginal distribution of $X2$

$X1 \mid X2$	0	1	Total
0	0.25	0	0.25
1	0	0.50	0.50
2	0	0.25	0.25
Total	0.25	0.75	1

Joint and Marginal Probabilities

For two random variables X and Y, the joint probability

$$p(x, y) = Pr(X = x, Y = y)$$

where x and y are possible relaizations of the random variables X and Y.

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For two random variables X and Y, the joint probability

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From what we observed in the last slide, marginal distribution of X

$$p_X(x) = Pr(X = x) = \sum_y p(x, y)$$

where the $\sum_y$ is over all realizations of Y.

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where the $\sum_y$ is over all realizations of Y.

Similarly for Y

$$p_Y(y) = Pr(Y = y) = \sum_x p(x, y)$$

Conditional Probabilities

we define a conditional probability as

$$Pr(X = x | Y = y) = \frac{Pr(X = x, Y = y)}{Pr(Y = y)} = \frac{p_{XY}(x, y)}{p_Y(y)}$$

Intuition: Given we have observed realization y of Y, what is the probability distribution of X.

Conditional Probabilities

we define a conditional probability as

$$Pr(X = x | Y = y) = \frac{Pr(X = x, Y = y)}{Pr(Y = y)} = \frac{p_{XY}(x, y)}{p_Y(y)}$$

Intuition: Given we have observed realization y of Y, what is the probability distribution of X.

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$X1 \vert X2$	0	1
0	1	0
1	0	0.67
2	0	0.33
Total	1	1

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2	0	1	1

Sum of random variables

Consider two independent tosses of a coin.

Let X_1 be the number of Ts observed in the 2 tosses.

Let X_2 be 1 if at least one T observed, 0 otherwise.

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The table

states	X_1	X_2	$Y = X_1 + X_2$	Prob
HH	0	0	0	0.25
HT	1	1	2	0.25
TH	1	1	2	0.25
TT	2	1	3	0.25

Sum of random variables

$$
\begin{aligned}
E(Y) &= E(X_1 + X_2) && (27)\\
&= 0 \times 0.25 + 2 \times 0.50 + 3 \times 0.25 && (28)\\
&= 1.75 && (29)\\
& && (30)
\end{aligned}
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Is that always true?

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Is that always true? YES!!! Lets derive it.

Similar Results

- $E(c) = c$ where c is constant
- $E(aX) = aE(X)$
- $E(X + Y) = E(X) + E(Y)$
- $E(aX + bY) = aE(X) + bE(Y)$ where a, b are constants.

Expected Value of a Linear Combination

Show that $E(aX + bY) = aE(X) + bE(Y)$.

$$E(aX + bY) = \sum_x \sum_y (ax + by)p(x, y)$$

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$$= \sum_x \sum_y axp(x, y) + \sum_x \sum_y byp(x, y)$$

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$$= \sum_x ax \underbrace{\left[\sum_y p(x, y)\right]} + \sum_y by \underbrace{\left[\sum_x p(x, y)\right]} \quad *$$

* See HW0 for more details about the "double sum."

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&= a \sum_x x p_X(x) + b \sum_y y p_Y(y) \qquad = aE(X) + bE(Y)
\end{aligned}
$$

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Independence of two random variables

- The variable X is **independent** of variable Y if its chances are not affected by the variable Y,

$$Pr[X = x|Y = y] = Pr[X = x] \quad \forall\, x \quad each \;\; y$$

- The following 3 definitions of independence are equivalent.
  - $Pr[X = x|Y = y] = Pr[X = x]$
  - $Pr[Y = y|X = x] = Pr[Y = y]$
  - $Pr[X = x, Y = y] = p_{XY}(x, y) = Pr[X = x]Pr[Y = y] = p_X(x)p_Y(y)$

Independence for Random Variables

$$p(x, y) = p_X(x)p_Y(y).$$

Note: Independence implies that

$$p(y|X = x) = \frac{p(x, y)}{p_X(x)} = \frac{p_X(x)p_Y(y)}{p_X(x)} = p_Y(y).$$

Y \ X	0	1	2	$p(y)$
0	2/6	1/6	0	3/6
1	0	1/6	2/6	3/6
$p(x)$	2/6	2/6	2/6	

Consider the pair $(x, y,) = (0, 0)$.

$$p(0, 0) = 2/6 \neq p_X(0)p_Y(0) = (1/3)(1/2) = 1/6.$$

X and Y are not independent.

Covariance between Random Variables

The covariance between random variables X and Y is defined as

$$\begin{aligned}
cov(X, Y) &= E(X - E(X)(Y - E(Y))) && (31)\\
&= E(XY - E(X)Y - XE(Y) + E(X)E(Y)) && (32)\\
&= E(XY) - E(E(X)Y) - E(XE(Y)) + E(E(X)E(Y)) && (33)\\
&= E(XY) - E(X)E(Y) - E(X)E(Y) + E(X)E(Y) && (34)\\
&= E(XY) - E(X)E(Y) && (35)\\
&= \sum_{x,y} xyPr(X = x, Y = y) - E(X)E(Y) && (36)\\
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Example to disprove the converse

Let X be a random variable

$$X = 1 \;\; prob \;\; 0.5 \qquad\qquad = -1 \;\; prob \;\; 0.5 \tag{38}$$

Let Y be another random variable

$$\begin{aligned} Y &= 0 \;\; if \; X = 1 && (39)\\ &= +1 \;\; if \; X = -1 \; prob \, 0.5 && (40)\\ &= -1 \;\; if \; X = -1 \; prob \, 0.5 && (41)\\ & && (42) \end{aligned}$$

Then check that $E(XY) = E(X) = E(Y) = 0$ and so $cov(X, Y) = 0$

Show X and Y are not independent

Correlation between Random Variables

Correlation between two random variables X and Y is given by

$$cor(X, Y) = \frac{cov(X, Y)}{\sqrt{var(X)}\sqrt{var(Y)}}$$

It can be shown that (check Cauchy Schwartz inequality)

$$-1 \leq cor(X, Y) \leq +1$$

Correlation (and covariance) is a measure of *linear* dependency. Correaltion of -1 means negative linear dependency, $+1$ means positive linear dependency, 0 means no linear dependency.

Recap

- States of an experiment, Events, Random Variables
- Probability (Joint, Marginal and Conditional)
- Expectation, Variance, Moments, MGF
- Law of Large Numbers, Chebyshev Inequality
- Independence and Covariance
- Sums of random variables
- Connect with Statistical Data !!

Is there something I very conveniently skipped?

Continuous Random Variables

Position yourself again at the middle of the Quad. Suppose now you record the height of each person crossing the Quad.

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In this case each person whose height is measured is a run of the experiment. The state is the height (in ft/inches). Since the state itself is numeric, we can directly define a random variable X same as the state.

Now we record the states for n individuals. Then the state space

$$S = \{(h_1 h_2 \cdots h_n)\}$$

and define n random variables $X_1, X_2, \cdots, X_n$ such that

$$X_1 = h_1 \quad X_2 = h_2 \quad \cdots X_n = h_n$$

Continuous Random Variables

If we had recorded the heights of all students, the assume the histogram would have been approximated by a normal bell-shaped curve with center μ and variance σ^2 very accurately. Then

$$X_1, X_2, \cdots, X_n \overset{iid}{\sim} N(\mu, \sigma)$$

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The sample mean $\bar{X}_n$ and sample standard deviation s_n^2 will also get closer and closer to μ and σ respectively as $n \to \infty$.

But again, we save our time and energy by looking at only 100/1000 samples and then approximate the normal model by approximately determining the population or model parameters μ and σ from the n data points and corresponding histogram.

Continuous Probabilities

We would want to define $Pr(X = x)$ for a continuous random variable X, but unfortunately it is 0. We will soon see why

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So how do we proceed to build something like a probability table for discrete variable?
We define **cumulative density function** (cdf)

$$F(x) = Pr(X \leq x)$$

If we differentiate this function, we get **probability density function** (pdf)

$$\frac{d}{dx}F(x) = f(x)$$

$$f(x) \geq 0 \qquad \int f(x)dx = 1$$

Continuous Probabilities

For normal distribution

$$f(x)/\phi(x) := \int \frac{1}{\sqrt{2\pi}} exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)$$

Cumulative distribution

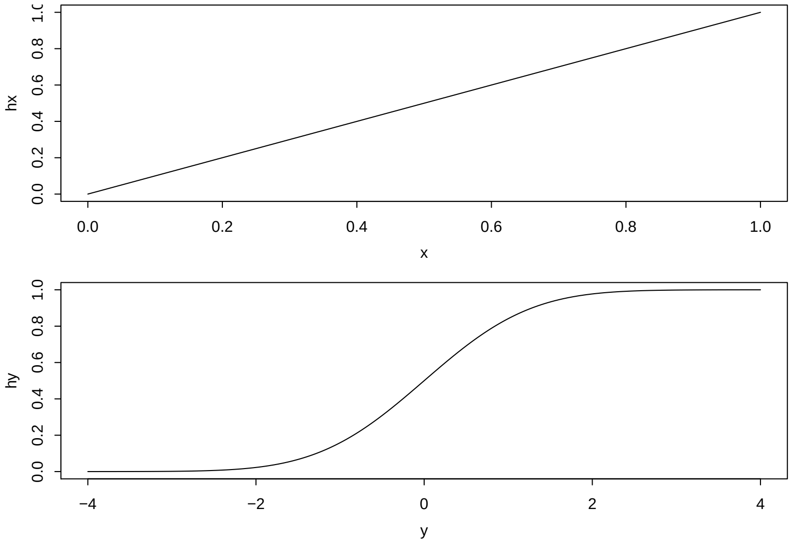
$$\Phi(x) := \int_{-\infty}^{x} \phi(x)dx$$

Uniform distribution

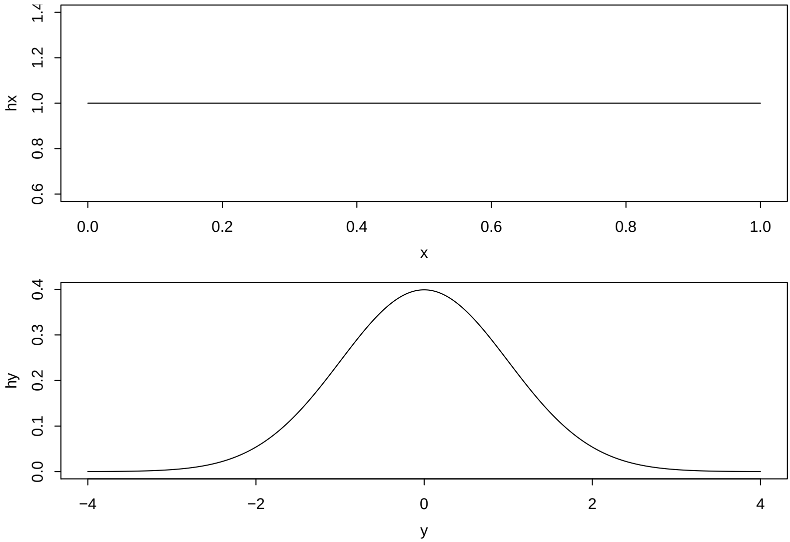
$$F(x) = Pr(X \leq x) = x$$

which implies $f(x) = 1$ for all x.

cumulative density graphs (Uniform and Normal)



probability density graphs (Uniform and Normal)



Properties of Continous Random Variable

We define the expectation analogous to the discrete random variable.

$$E(X) := \int xf(x)dx$$

$$var(X) := E(X - E(X))^2 = \int x^2 f(x)dx - E^2(X)$$

$$MGF(t) := E(e^{tX}) = \int e^{tx} f(x)dx$$

For normal distribution, one can easily show that

$$E(X) = \mu \quad var(X) = \sigma^2 \quad MGF(t) = exp(\mu t + \frac{\sigma^2 t^2}{2})$$

what about uniform? HW

Properties of Continous Random Variable

For two continuous random variables X and Y, we can define the joint cumulative density

$$F(x, y) = Pr(X \leq x, Y \leq y)$$

Then we define joint density

$$f(x, y) = \frac{d^2}{dxdy} F(x, y) = f_{XY}(x, y)$$

Define marginal

$$\int_X f_{XY}(x, y) = f_Y(y) \qquad \int_Y f_{XY}(x, y) = f_X(x)$$

Conditional

$$f_{X|Y}(x|y) := \frac{f_{XY}(x, y)}{f_Y(y)}$$

Properties of Continous Random Variable

We say X and Y are independent when

$$f_{X|Y}(x|y) = f_X(x)$$

or

$$f_{XY}(x, y) = f_X(x)f_Y(y)$$

As for discrete variables, independence implies uncorrelated but converse is not true.
Similarly results related to sums and linear transform of two or more variables stay true even for continuous variables.