STAT234: Lecture 7 - Type I, Type II and Power

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We want to minimize Type I error and maximize Power

Two Types of Errors Revisited

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Reality	Our Decision H_0	Our Decision H_a
H_0	$\surd$ (Prob $= 1 - \alpha$)	Type I Error (Prob $= \alpha$)
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Hypothesis testing framework

Last time we considered the example,
$X_1, X_2, \cdots, X_n$ be data coming from

$$X_i \sim N(\mu, \sigma^2)$$

and we assumed that we know σ^2. In our model,

$$\sigma = 2$$

Hypothesis to test

$$H_0 : \mu = 0 \;\; H_1 : \mu > 0$$

We want to choose smallest c such that $\bar{X} > c$ happens with probability greater than or equal to the Type I error (0.05)

$$\textit{find smallest } c \;\; : Pr\left[\bar{X} > c | \mu = 0\right] < 0.05$$

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$$Type\ I\ error : Pr\left[Z > \frac{\sqrt{n}}{\sigma}c|\mu = 0\right] = 0.05 \qquad Z \sim N(0,1)$$

From this, we know

$$\frac{\sqrt{n}}{\sigma}c = 1.644$$

Our case: $\sigma = 2$ and $n = 25$ resulted in $c = 0.658$. Power needs a μ under alternate hypothesis. If we assume $\mu = 1$,

$$Power : Pr\left[\bar{X} > 0.658|\mu = 1\right]$$

$$Power(1) : Pr\left[Z > \frac{\sqrt{n}}{\sigma}(c-1)\right]\ Z \sim N(0,1)$$

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The right hand side quantity

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So, we can decide on how many samples to choose by deciding on the Type I and Type II errors we want.

Hypothesis testing framework

But $\mu = 1$ is only one possible alternative for the hypothesis. So, what we would want to do is get a formula of power for each value of μ in the alternate space.

$$Power(\mu) : Pr\left[Z > \frac{\sqrt{n}}{\sigma}(c - \mu)\right] \; Z \sim N(0,1)$$

Since from Type I error argument, we showed that

$$\frac{\sqrt{n}}{\sigma}c = 1.644$$

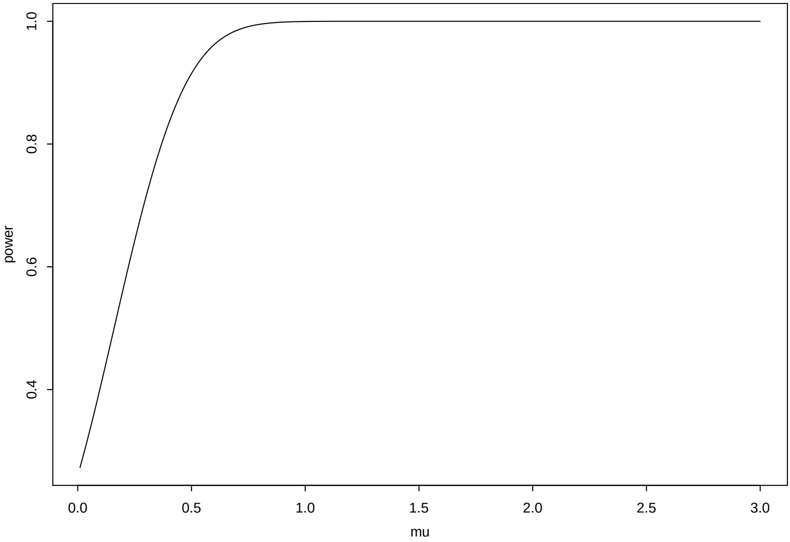
$$Power(\mu) : Pr\left[Z > 1.644 - \frac{\sqrt{n}}{\sigma}\mu\right] \; Z \sim N(0,1)$$

This can be written as

$$Power(\mu) := \Phi\left(\frac{\sqrt{n}}{\sigma}\mu - 0.644\right)$$

Hypothesis testing framework

We plot $Power(\mu)$ against μ. Say $n = 65$ and we know $\sigma = 2$.



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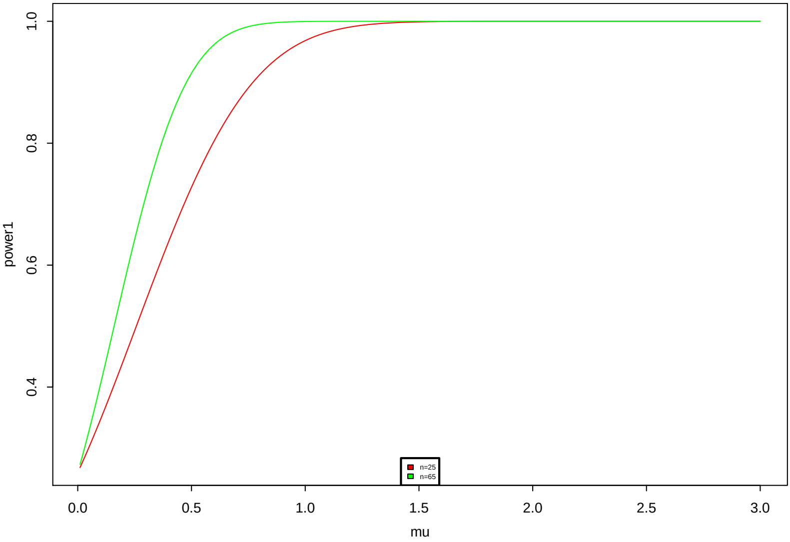

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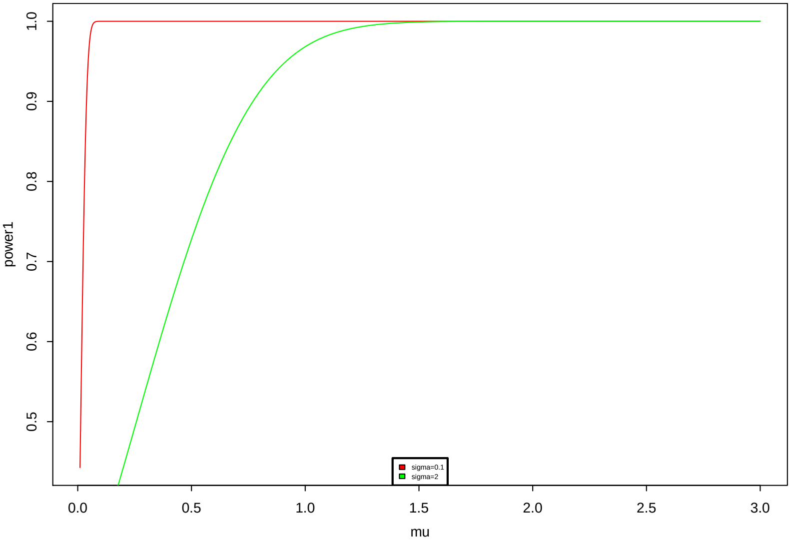

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- for fixed σ, the power for higher n remains higher at each μ. For fixed n, the power for lower σ remains higher at each μ

Another important continuous distribution

Chi-Square Distribution

The chi-square distributions: χ^2_k

$$f(x) = \frac{x^{(k/2)-1} e^{-x/2}}{2^{k/2}\Gamma(k/2)} \qquad x > 0,\ k \geq 1$$

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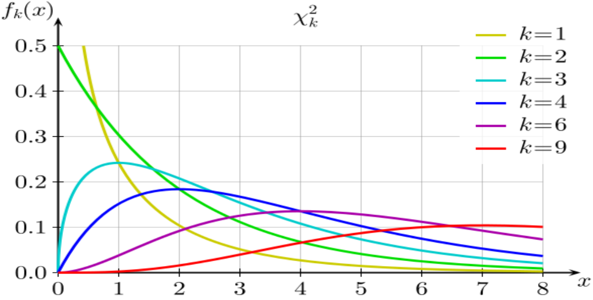

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In general, χ_n^2 is a sum of n i.i.d. standard normal distributions.

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- If $X_i \sim N()$, then $\bar{X}$ and S^2 are independent.
  Reason: $Cov(\bar{X}, X_i - \bar{X}) = 0 \Rightarrow \bar{X}$ and $X_i - \bar{X}$ are uncorrelated
  $\Rightarrow \bar{X}$ and $X_i - \bar{X}$ are independent.

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$$\frac{(n-1)S^2}{\sigma^2} \sim \chi^2_{(n-1)}$$

Mean of Sampling Distribution of s^2

Since we have,

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It follows that,

$$n-1 = E\left[\frac{(n-1)S^2}{\sigma^2}\right] = \frac{(n-1)}{\sigma^2}E(S^2) \Rightarrow E(s^2) = \sigma^2.$$

So, when sampling from a normal population,
s^2 is on target (σ^2) on average.

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$$f(t) = \frac{\Gamma(\frac{\nu+1}{2})}{\sqrt{\pi\nu}\Gamma(\frac{\nu}{2})}(1 + \frac{t^2}{\nu})^{-\frac{\nu+1}{2}}, -\infty < t < \infty$$

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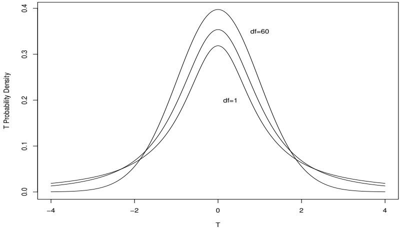

The "Student's" T Probability Distribution

Let $Z \sim N(0,1)$ and $Y \sim \chi^2_\nu$ with Z, Y **independent**. Then, $T = \frac{Z}{\sqrt{Y/\nu}}$ is said to have the $T-$distribution with ν degrees of freedom (df): $T \sim t_n$. p.d.f:

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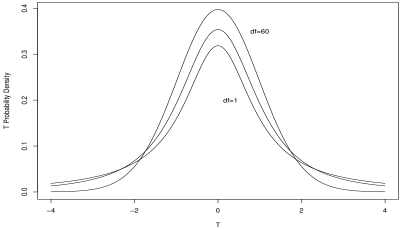

Question: What's the T distribution when $\nu \to \infty$?

How is the $T-$distribution Related to Estimation?

Let $X_1, X_2, \ldots, X_n$ be a random sample from a $N(\mu, \sigma^2)$ popn.

We know that $Z = \dfrac{\bar{X} - \mu}{\sigma/\sqrt{n}} \sim N(0, 1)$.

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Consider

$$\begin{aligned} T &= \frac{\bar{X} - \mu}{s/\sqrt{n}} = \frac{\bar{X} - \mu}{\sigma/\sqrt{n}} \cdot \frac{\sigma}{s} = Z\frac{\sigma}{s} \quad \text{(where } Z \sim N(0,1)\text{)} \\ &= \frac{Z}{s/\sigma} = \frac{Z}{\sqrt{s^2/\sigma^2}} \\ &= \frac{Z}{\sqrt{\frac{(n-1)s^2}{\sigma^2(n-1)}}} = \frac{Z}{\sqrt{\chi^2_{(n-1)}/(n-1)}} \sim t_{(n-1)} \end{aligned}$$

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