

1) Show that $E(c) = c$ and $\text{var}(c) = 0$ where c is a constant.

Pf What do we mean by expectation & variance of constants. Wasn't it for random variables only?
Well, you can define a random variable X
(*) $\Pr[X=c] = 1$. Then $E(X)$ can be essentially written as $E(c)$

$$E(X) = \sum_x x \Pr[X=x] \quad x \in \{\text{all realizations of } X\}$$

But x can only be c , as X only takes the value c with prob. 1.

$$E(X) = \underbrace{c \cdot \Pr[X=c]}_{\text{from (*).}} = c \cdot 1 = c. \quad \because [\Pr[X=c]=1]$$

$$\begin{aligned} \text{Var}(X) &= \sum_x (x - E(X))^2 \Pr[X=x] \\ &= \sum_x (x - c)^2 \Pr[X=x] \\ &= (\underbrace{c - c}_{=0})^2 \underbrace{\Pr[X=c]}_{=1} = 0. \quad [\text{Ans}] \end{aligned}$$

2) Show that $E(ax) = aE(x)$ and $\text{var}(ax) = a^2 \text{var}(x) ??$

Pf

$$\begin{aligned} E(ax) &= \sum_x ax \Pr[X=x] \quad x \text{ being realization of } X \\ &= a \cdot \sum_x x \Pr[X=x] \\ &= aE(x) \end{aligned}$$

$$\begin{aligned}
 \text{var}(aX) &= \sum_x (ax - E(ax))^2 \Pr[x=x] \\
 &= \sum_x (ax - aE(x))^2 \Pr[x=x] \\
 &= a^2 \sum_x (x - E(x))^2 \Pr[x=x] \\
 &= a^2 \text{var}(x) \quad [\text{Ans}]
 \end{aligned}$$

3) Show that $\text{var}(X) = E(X^2) - E^2(X)$

$$\begin{aligned}
 \text{var}(X) &= \sum_x (x - E(x))^2 \Pr[x=x] \\
 &= \sum_x (x^2 - 2E(x)x + E^2(x)) \Pr[x=x] \\
 &= \sum_x [x^2 \Pr[x=x] - 2E(x)x \Pr[x=x] \\
 &\quad + E^2(x) \Pr[x=x]] \\
 &= \sum_x x^2 \Pr[x=x] - 2E(x) \sum_x x \Pr[x=x] \\
 &\quad + E^2(x) \\
 &= \sum_x x^2 \Pr[x=x] - 2E(x)E(x) \\
 &\quad + E^2(x) \\
 &- E(X^2) + 2E^2(x) - E^2(x) \\
 &= E(X^2) - E^2(X)
 \end{aligned}$$

4) $E(aX+bY) = aE(X)+bE(Y)$ where X, Y
are random variables and a, b are constants

Check page 41 of 3rd week.pdf slides.

5) $E(X+Y) = E(X) + E(Y)$ and

Pf $E(X_1+X_2+\dots+X_n) = E(X_1)+E(X_2)+\dots+E(X_n)$

To show $E(X+Y) = E(X) + E(Y)$, put $a=1$ and $b=1$ in 4).

$$\begin{aligned} E(X+Y+Z) &= E[(\underbrace{X+Y}_\text{in } X+Y+Z) + Z] = E[\omega + Z] \\ &= E[\omega] + E[Z] \\ &= E[X+Y] + E[Z] \\ &= E(X) + E(Y) + E(Z) \end{aligned}$$

$$\begin{aligned} E[(X_1+X_2+\dots+X_n)] &= E[(X_1+X_2+\dots+X_{n-1}) + X_n] \\ &= E[(X_1+X_2+\dots+X_{n-1})] + E(X_n) \\ &\quad \vdots \text{ continue splitting like} \\ &= E(X_1) + E(X_2) + \dots + E(X_n). \end{aligned}$$

6) Show that $\text{cov}(X, Y) = E(XY) - E(X)E(Y)$.

By definition

$$\begin{aligned}\text{cov}(X, Y) &= E[(X - E(X))(Y - E(Y))] \\ &= E[XY - E(X)Y - XE(Y) \\ &\quad + E(X)E(Y)] \\ &= E[XY] + E[-E(X)Y] + E[-XE(Y)] \\ &\quad + E[E(X)E(Y)]\end{aligned}$$

[using the fact that
 $E(Z_1 + Z_2 + Z_3 + Z_4) = E(Z_1) + E(Z_2) + E(Z_3) + E(Z_4)$
Proof in 5].

$$\begin{aligned}&= E[XY] - E(X)E(Y) - E(Y)E(X) \\ &\quad + E[E(X)E(Y)] \\ &= E(XY) - 2E(X)E(Y) + E(X)E(Y) \\ &= E(XY) - E(X)E(Y)\end{aligned}$$

[using $E(c) = c$
Proof in 1].

7) Show
 $\text{Var}(X+Y) = \text{Var}(X) + \text{Var}(Y) + 2\text{Cov}(X, Y)$

$$\begin{aligned}
 \text{Pf} \quad \text{Var}(X+Y) &= \sum_{x,y} (x+y - E(X+Y))^2 p_{XY}(x,y) \\
 &= \sum_{x,y} (x+y - E(X) - E(Y))^2 p_{XY}(x,y) \\
 &= \sum_x \sum_y [(x - E(X)) + (y - E(Y))]^2 p_{XY}(x,y) \\
 &= \sum_x \sum_y \left\{ (x - E(X))^2 + 2(x - E(X))(y - E(Y)) + (y - E(Y))^2 \right\} p_{XY}(x,y) \\
 &= \sum_x \sum_y (x - E(X))^2 p_{XY}(x,y) + \sum_x \sum_y (y - E(Y))^2 p_{XY}(x,y) \\
 &\quad + 2 \sum_x \sum_y (x - E(X))(y - E(Y)) p_{XY}(x,y) \\
 &= \sum_x (x - E(X))^2 \left(\sum_y p_{XY}(x,y) \right) = p_X(x) \\
 &\quad + \sum_y (y - E(Y))^2 \left(\sum_x p_{XY}(x,y) \right) = p_Y(y) \\
 &\quad + 2 \sum_x \sum_y (x - E(X))(y - E(Y)) p_{XY}(x,y)
 \end{aligned}$$

$$\begin{aligned}
 &= \sum_x (x - E(x))^2 p_x(x) + \sum_y (y - E(y))^2 k_y(y) \\
 &\quad + 2 \sum_x \sum_y (x - E(x))(y - E(y)) p_x(x) k_y(y) \\
 &= \text{var}(X) + \text{var}(Y) + 2 \text{cov}(X, Y) \quad [\text{Ans}]
 \end{aligned}$$

8) $\text{var}(X+Y) = \text{var}(X) + \text{var}(Y)$ } when $\text{cov}(X, Y) = 0$
 $\text{var}(X-Y) = \text{var}(X) + \text{var}(Y)$

$$\begin{aligned}
 \text{var}(X-Y) &= \text{var}(X) + \text{var}(-Y) \\
 &\quad + 2 \text{cov}(X, -Y) \\
 &= \text{var}(X) + \text{var}(-Y) \\
 &\quad + 2 [E(X(-Y)) - E(X)E(-Y)] \\
 &= \text{var}(X) + \text{var}(Y) \\
 &\quad + 2 [-E(XY) + E(X)E(Y)]
 \end{aligned}$$

$$\text{var}(X+Y) = \text{var}(X) + \text{var}(Y) - 2 \text{cov}(XY)$$

when $\text{cov}(X, Y) = 0$

$\text{var}(X+Y) = \text{var}(X) + \text{var}(Y) + 2 \text{cov}(X, Y)$

$\text{var}(X-Y) = \text{var}(X) + \text{var}(Y)$

9) When X and Y are independent

$$E(XY) = E(X)E(Y) \quad \text{or} \quad \text{cov}(X, Y) = 0$$

PF X and Y independent means

$$\begin{aligned} p_{XY}(x,y) &= \Pr[X=x, Y=y] \\ &= \Pr[X=x] \Pr[Y=y] \\ &= p_X(x) p_Y(y) \end{aligned}$$

$$\begin{aligned} E(XY) &= \sum_x \sum_y xy \cancel{\Pr[X=x, Y=y]} \\ &= \sum_x \sum_y xy \underbrace{\Pr[X=x] \Pr[Y=y]}_{\substack{\text{under} \\ \text{independence}}} \\ &= \sum_x x \Pr[X=x] \left[\sum_y y \Pr[Y=y] \right] \\ &= \sum_x x \Pr[X=x] \cdot E(Y) \\ &= E(Y) \cdot \sum_x x \Pr[X=x] \\ &= \cancel{E(Y) E(X)}. \end{aligned}$$

$$\text{cov}(X, Y) = E(XY) - E(X)E(Y) = 0$$

10) X, Y independent random variables, then show that $E(e^{t(X+Y)}) = E(e^{tX}) E(e^{tY})$.

This would mean $MgS_{X+Y}(t) = MgS_X(t)MgS_Y(t)$

$$\begin{aligned}
 \stackrel{\text{Pf}}{=} E(e^{t(X+Y)}) &= \sum_x \sum_y e^{t(x+y)} \Pr[X=x, Y=y] \\
 &= \sum_x \sum_y e^{t(x+y)} \Pr[X=x] \Pr[Y=y] \quad (\text{under independence}) \\
 &= \sum_x \sum_y e^{t(x+y)} p_x(x) p_y(y) \\
 &= \sum_x \sum_y e^{tx} e^{ty} p_x(x) p_y(y) \\
 &= \left[\sum_x e^{tx} p_x(x) \right] \left[\sum_y e^{ty} p_y(y) \right] \\
 &\quad \text{[} \cancel{\sum_x \sum_y} \text{]} \\
 &= E(e^{tX}) E(e^{tY}) = MgS_X(t)MgS_Y(t) \\
 &= \cancel{\text{[Ans]}}
 \end{aligned}$$

If $X \& Y$ have same distribution

$$MgS_X(t) = MgS_Y(t) = MgS(t)$$

$$MgS_{X+Y}(t) = [MgS(t)]^2 \quad \text{This for 2 random variables}$$

$$MgS_{X_1+X_2+\dots+X_n}(t) = [MgS(t)]^n \quad \text{for } n \text{ random variables}$$

$$11) \text{ cov}(x, x) = \text{var}(x). \underline{\text{Show}}$$

PS $\text{cov}(x, Y) = E(XY) - E(X)E(Y)$

$$\text{cov}(X, X) = E(X^2) - [E(X)]^2 = \text{var}(X) \quad [\text{Ans}]$$

$$12) \text{ cov}(c, X) = 0 \text{ where } c \text{ is a constant}$$

Define $Y \equiv c$ with prob. 1.

$$\text{cov}(c, X) = E(cx) - E(c)E(x)$$

$$= c \cdot E(x) - E(c)E(x) \quad [\because E(cx) = cE(x)]$$

$$= c \cdot E(x) - c \cdot E(x) \quad [\text{Ans}] \quad [\begin{matrix} E(c) = c \\ \text{from (1)} \end{matrix}]$$

$$13) \text{ cov}(ax, Y) = a \text{ cov}(x, Y) \quad \underline{\text{Show}}$$

$$\text{cov}(ax, Y) = E(axY) - E(ax)E(Y)$$

$$= aE(XY) - aE(X)E(Y)$$

$$= a \{ E(XY) - E(X)E(Y) \}$$

$$= a \text{ cov}(x, Y). \quad [\text{Ans}]$$

$$+). \text{ cov}(ax, bY) = ab \text{ cov}(x, Y) \quad \underline{\underline{\text{Show}}}$$

a, b constants

$$\text{cov}(ax, bY) = E(axbY) - E(ax)E(bY)$$

$$= abE(XY) - \{aE(X), bE(Y)\}$$

$$= abE(XY) - abE(X)E(Y)$$

$$= ab \{ E(XY) - E(X)E(Y) \}$$

$$= ab \operatorname{cov}(X, Y).$$

15)

$$\operatorname{cov}(X+Y, Z) = \operatorname{cov}(X, Z) + \operatorname{cov}(Y, Z): \underline{\text{Show}}$$

PF

$$\begin{aligned}
 \operatorname{cov}(X+Y, Z) &= E[(X+Y)Z] - E[X+Y]E[Z] \\
 &= E[XZ + YZ] - E(X+Y)E(Z) \\
 &= E(XZ) + E(YZ) \\
 &\quad - \{E(X) + E(Y)\}E(Z) \\
 &= E(XZ) - E(X)E(Z) \\
 &\quad + E(YZ) - E(Y)E(Z) \\
 &= \operatorname{cov}(X, Z) + \operatorname{cov}(Y, Z)
 \end{aligned}$$

[Ans]

16) Derive the moment generating function for
 $U(0,1)$ distribution

$$f(x) \equiv 1 \quad 0 \leq x \leq 1$$

$$\begin{aligned}
 M(t) &= \text{mgf of } X \sim U(0,1) \\
 E(e^{tx}) &= \int_0^1 e^{tx} f(x) dx = \int_0^1 e^{tx} dx \\
 &= \left[\frac{e^{tx}}{t} \right]_0^1 = \frac{e^t}{t} - \frac{1}{t} \\
 &= \frac{(e^t - 1)}{t}
 \end{aligned}$$

[Ans]

17) Moment generating function and mean and variance for Binomial distribution.

Let $X_1, X_2, \dots, X_n \sim \text{Ber}(p)$ independent and same distribution [iid]

$$Y = X_1 + X_2 + \dots + X_n$$

$$\text{Mgf}_Y(t) = E(e^{tY}) = [\text{Mgf}_{X_i}(t)]^n$$

$$\text{Mgf}_{X_1} = \text{Mgf}_{X_2} = \dots = \text{Mgf}_{X_n}$$

$$X \sim \text{Ber}(p) \quad \text{Mgf}_X(t) = E(e^{tX})$$

$$= \sum_x e^{tx} \Pr[X=x]$$

$$= e^{t0} \Pr[X=0] + e^{t1} \Pr[X=1]$$

$$= 1 \cdot \Pr[X=0] + et \Pr[X=1]$$

$$= (1-p) + pet$$

$$\text{Mgf}_X(t) = (1-p) + pet$$

$$\text{Mgf}_Y(t) = [(1-p) + pet]^n$$

To calculate mean and second moment

~~E(Y)~~ & ~~E(Y^2)~~ we compute $\frac{d}{dt} \text{Mgf}_Y(t) \Big|_{t=0}$
 $E(Y) \quad E(Y^2) \quad \frac{d^2}{dt^2} \text{Mgf}_Y(t) \Big|_{t=0}$

$$\frac{d}{dt} \text{Mgf}_Y(t) = \frac{d}{dt} \left\{ [(1-p) + pet]^n \right\}$$

$$= n [(1-p) + pet]^{n-1} \frac{d}{dt} \{ 1-p + pe$$

$$\begin{aligned}
 &= n [(1-p) + p e^t]^{n-1} p \left\{ \frac{d}{dt} (1-p) + \frac{d}{dt} p e^t \right\} \\
 &= n [(1-p) + p e^t]^{n-1} \{ 0 + p e^t \} \\
 &= n p e^t [(1-p) + p e^t]^{n-1} \quad (**)
 \end{aligned}$$

$$\frac{d}{dt} M_{gS_Y}(t) \Big|_{t=0} = n p \cdot 1 [(1-p) + p]^{n-1} = n p. \quad \rightarrow (I)$$

$$\begin{aligned}
 \frac{d^2}{dt^2} M_{gS_Y}(t) &= \frac{d}{dt} \left[\frac{d}{dt} M_{gS_Y}(t) \right] = \frac{d}{dt} \left[n p e^t \left\{ (1-p) + p e^t \right\}^n \right] \\
 &= \frac{d}{dt} \left[n p e^t \left\{ (1-p) + p e^t \right\}^{n-1} \right] \quad (**)
 \end{aligned}$$

$$\begin{aligned}
 &= \left[\frac{d}{dt} (n p e^t) \right] [(1-p) + p e^t]^{n-1} \\
 &\quad + \frac{d}{dt} [(1-p) + p e^t]^{n-1} (n p e^t) \\
 &= n p e^t [(1-p) + p e^t]^{n-1} \quad [\text{chain rule}]
 \end{aligned}$$

$$+ (n-1) [(1-p) + p e^t]^{n-2} \left[\frac{d}{dt} \left\{ (1-p) + p e^t \right\} \right]$$

$$\begin{aligned}
 &= n p e^t [(1-p) + p e^t]^{n-1} \\
 &\quad + n(n-1) p e^t [(1-p) + p e^t]^{n-2}
 \end{aligned}$$

$$\left[\frac{d}{dt} (1-p) + p \frac{d}{dt} e^t \right]$$

$$\begin{aligned}
 &= n p e^t [(1-p) + p e^t]^{n-1} \\
 &\quad + n(n-1) p e^t [(1-p) + p e^t]^{n-2} \\
 &\quad [0 + p e^t]
 \end{aligned}$$

$$= n p e^t \left[(1-p) + p e^t \right]^{n-1} + n(n-1)(p e^t)^2 \left[\underbrace{(1-p)}_{+ p e^t} \right]^{n-2}$$

~~= - n p e^t~~

$$\begin{aligned} \frac{d^2}{dt^2} M_{g f_Y}(t) \Big|_{t=0} &= n p \cdot 1 \cdot [1]^{n-1} \\ &\quad + n(n-1) p^2 \left[\underbrace{(1-p)}_{=1} + p e^t \right]^{n-2} \\ &= n p + n(n-1)p^2 \\ &= n p + n^2 p^2 - n p^2 \quad - (\text{iii}) \end{aligned}$$

$$\begin{aligned} \cancel{\mathbb{E}(Y^2)} &= n p + n^2 p^2 - n p^2 \quad \text{from II} \\ \cancel{\mathbb{E}(Y)} &= n p \quad \text{from I} \end{aligned}$$

$$\text{var}(Y) = E(Y^2) - E^2(Y)$$

$$\begin{aligned} \text{var}(X) &= E(X^2) - E^2(X) \\ &= n p + n^2 p^2 - n p^2 - n^2 p^2 \\ &= n p - n p^2 \\ &= n p [1-p] \quad (\underline{\text{Ans}}) \end{aligned}$$

So, $Y \sim \text{Bin}(n, p)$ has $E(Y) = np$
 $\text{var}(Y) = np(1-p)$.