1) Show that $E(c)=c$ and $var(c)=0$ where c is a constant.

Pf What do ~~we~~ we mean by expectation & variance of constants. Wasn't it for random variables only? Well, you can define a random variable X

(*) $Pr[X=c]=1$. then $E(X)$ can be essentially written as $E(c)$

$$E(X) = \sum_x x \, Pr[X=x] \qquad x \in \{\text{all realizations of } X\}$$

But x can only be c, as X only takes the value c with prob. 1.

$$E(X) = \underbrace{c.\, Pr[X=c]}_{\text{from } (*)} = c.1 = c. \quad \because [Pr[X=c]=1]$$

$$var(X) = \sum_x (x - E(X))^2 \, Pr[X=x]$$

$$= \sum_x (x-c)^2 \, Pr[X=x]$$

$$= \underbrace{(c-c)^2}_{=0} \underbrace{Pr[X=c]}_{=1} = 0. \text{ [Ans}$$

) Show that $E(aX) = a\,E(X)$ and $var(aX) = a^2 Var(X)$??

Pf $$E(aX) = \sum_x ax \, Pr[X=x] \qquad x \text{ being realization of } X$$

$$= a. \sum_x x \, Pr[X=x] = a\,E(X)$$

$$\text{var}(aX) = \sum_x (ax - E(aX))^2 \Pr[X=x]$$

$$= \sum_x (ax - aE(X))^2 \Pr[X=x]$$

$$= a^2 \sum_x (x - E(X))^2 \Pr[X=x]$$

$$= a^2 \text{var}(X) \qquad [\text{Ans}].$$

$$\Rightarrow \boxed{\text{var}(-X) = \text{var}(X)}$$

3) Show that $\underline{\text{var}(X) = E(X^2) - E^2(X)}$

$$\text{var}(X) = \sum_x (x - E(X))^2 \Pr[X=x]$$

$$= \sum_x \left(x^2 - 2E(X)x + E^2(X)\right) \Pr[X=x]$$

$$= \sum_x \Big[x^2 \Pr[X=x] - 2E(X)x\Pr[X=x] + E^2(X)\Pr[X=x]\Big]$$

$$= \sum_x x^2 \Pr[X=x] - 2E(X)\sum_x x\Pr[X=x] + E^2(X)$$

$$= \sum_x x^2 \Pr[X=x] - 2E(X)E(X) + E^2(X)$$

$$= E(X^2) - 2E^2(X) + E^2(X)$$

$$= E(X^2) - E^2(X)$$

4) $E(aX+bY) = aE(X) + bE(Y)$ where X, Y are random variables and a, b are constants

Check page 41 of 3rdweek.pdf slides.

5) $E(X+Y) = E(X) + E(Y)$ and

$$E(X_1 + X_2 + \dots + X_n) = E(X_1) + E(X_2) + \dots + E(X_n).$$

Pf

To show $E(X+Y) = E(X) + E(Y)$, put $a=1$ and $b=1$ in 4).

$$E(X+Y+Z) = E[\underbrace{(X+Y)}_{} + Z] = E[\omega + Z]$$

$$= E[\omega] + E[Z]$$

$$= E[X+Y] + E[Z]$$

$$= E(X) + E(Y) + E(Z).$$

$$E[(X_1 + X_2 + \dots + X_n)] = E[(X_1 + X_2 + \dots + X_{n-1}) + X_n]$$

$$= E[(X_1 + X_2 + \dots + X_{n-1})] + E(X_n)$$

$$\vdots$$

continue splitting like this

$$= E(X_1) + E(X_2) + \dots + E(X_n).$$

6) Show that $\mathrm{cov}(X,Y) = E(XY) - E(X)E(Y)$.

By definition

$$\mathrm{cov}(X,Y) = E\left[(X-E(X))(Y-E(Y))\right]$$

$$= E\left[XY - E(X).Y - X.E(Y) + E(X).E(Y)\right]$$

$$= E\left[XY\right] + E\left[-E(X).Y\right] + E\left[-XE(Y)\right] + E\left[E(X)E(Y)\right]$$

[using the fact that $E(Z_1+Z_2+Z_3+Z_4) = E(Z_1)+E(Z_2)+E(Z_3)+E(Z_4)$ Proof in 5).]

$$= E[XY) - E(X)E(Y) - E(Y)E(X) + E\left[E(X)E(Y)\right]$$

[using $E(aZ) = aE(Z)$ proof in 2)]

$$= E(XY) - 2E(X)E(Y) + E(X)E(Y)$$

[using $E(c) = c$ Proof in 1)]

$$= E(XY) - E(X)E(Y)$$

7) Show
$\operatorname{var}(X+Y) = \operatorname{var}(X) + \operatorname{var}(Y) + 2\operatorname{cov}(X,Y)$

Pf

$$\operatorname{var}(X+Y) = \sum_x\sum_y (x+y-E(X+Y))^2 \Pr[X=x, Y=y]$$

$$= \sum_{x,y}\sum (x+y-E(X)-E(Y))^2 p_{XY}(x,y)$$

$[E(X+Y)=E(X)+E(Y)]$

$$= \sum_x\sum_y \left[(x-E(X)) + (y-E(Y))\right]^2 p_{XY}(x,y)$$

$$= \sum_x\sum_y \left\{(x-E(X))^2 + 2(x-E(X))(y-E(Y)) + (y-E(Y))^2\right\} p_{XY}(x,y)$$

$$= \sum_x\sum_y (x-E(X))^2 p_{XY}(x,y) \quad \left[\begin{array}{l}\text{using}\\ (a+b)^2\\ = a^2+2ab+b^2\end{array}\right]$$

$$+ \sum_x\sum_y (y-E(Y))^2 p_{XY}(x,y)$$

$$+ 2\sum_x\sum_y (x-E(X))(y-E(Y)) p_{XY}(x,y)$$

$$= \sum_x (x-E(X))^2 \sum_y p_{XY}(x,y) = p_X(x)$$

$$+ \sum_y (y-E(Y))^2 \sum_x p_{XY}(x,y) = p_Y(y)$$

$$+ 2\sum_x\sum_y (x-E(X))(y-E(Y)) p_{XY}(x,y)$$

$$= \sum_x (x-E(X))^2 p_X(x) + \sum_y (y-E(Y))^2 p_Y(y)$$
$$+ 2\sum_x \sum_y (x-E(X))(y-E(Y)) p_{XY}(x,y)$$
$$= var(X) + var(Y) + 2\,cov(X,Y) \quad [Ans]$$

8) $\left.\begin{aligned} var(X+Y) &= var(X) + var(Y) \\ var(X-Y) &= var(X) + var(Y) \end{aligned}\right\}$ when $cov(X,Y) = 0$

$$var(X-Y) = var(X) + var(-Y) + 2\,cov(X,-Y)$$
$$= var(X) + var(-Y) + 2\left[E(X(-Y)) - E(X)E(-Y)\right]$$
$$= var(X) + var(Y) + 2\left[-E(XY) + E(X)E(Y)\right]$$
$$= var(X) + var(Y) - 2\,cov(X,Y)$$

$$var(X+Y) = var(X) + var(Y) + 2\,cov(X,Y)$$

So, when $cov(X,Y) = 0$

$\left.\begin{aligned} var(X+Y) &= var(X) + var(Y) \\ var(X-Y) &= var(X) + var(Y) \end{aligned}\right\}$

9) When X and Y are independent

$$E(XY) = E(X)E(Y) \quad \text{or} \quad cov(X,Y) = 0$$

Pf X and Y independent means

$$\begin{aligned} p_{XY}(x,y) &= Pr[X=x, Y=y] \\ &= Pr[X=x]\, Pr[Y=y] \\ &= p_X(x)\, p_Y(y) \end{aligned}$$

$$\begin{aligned} E(XY) &= \sum_x \sum_y xy\, Pr[X=x, Y=y] \\ &= \sum_x \sum_y xy\, \underbrace{Pr[X=x]\, Pr[Y=y]}_{\text{(under independence)}} \\ &= \sum_x x\, Pr[X=x] \left[\sum_y y\, Pr[Y=y]\right] \\ &= \sum_x x\, Pr[X=x] \cdot E(Y) \\ &= E(Y) \cdot \sum_x x\, Pr[X=x] \\ &= E(Y)\, E(X). \end{aligned}$$

$$cov(X,Y) = E(XY) - E(X)E(Y) = 0$$

10) X, Y independent random variables, then show that $E(e^{t(X+Y)}) = E(e^{tX})E(e^{tY})$.

This would mean $Mgf_{X+Y}(t) = Mgf_X(t) Mgf_Y(t)$

<u>Pf</u> $E(e^{t(X+Y)}) = \sum_x \sum_y e^{t(x+y)} \Pr[X=x, Y=y]$

$= \sum_x \sum_y e^{t(x+y)} \Pr[X=x] \Pr[Y=y]$

(under independence)

$= \sum_x \sum_y e^{t(x+y)} p_X(x) p_Y(y)$

$= \sum_x \sum_y e^{tx} e^{ty} p_X(x) p_Y(y)$

$= \left[\sum_x e^{tx} p_X(x)\right]\left[\sum_y e^{ty} p_Y(y)\right]$

$= E(e^{tX}) E(e^{tY}) = Mgf_X(t) Mgf_Y(t)$

[Ans].

If X & Y have same distribution

$Mgf_X(t) = Mgf_Y(t) = Mgf(t)$

$Mgf_{X+Y}(t) = [Mgf(t)]^2$. This for 2 random variables

$Mgf_{X_1+X_2+\cdots+X_n}(t) = [Mgf(t)]^n$ for n random variables

11) $\operatorname{cov}(X,X) = \operatorname{var}(X)$. Show

Pf $\operatorname{cov}(X,Y) = E(XY) - E(X)E(Y)$

$\operatorname{cov}(X,X) = E(X^2) - [E(X)]^2 = \operatorname{var}(X)$

[Ans].

12) $\operatorname{cov}(c,X) = 0$ where c is a constant

Define $Y \equiv c$ with prob. 1.

$$\begin{aligned}\operatorname{cov}(c,X) &= E(cX) - E(c)E(X) \\ &= c.E(X) - E(c)E(X) \quad [\because E(aX) = aE(X)] \\ &= c.E(X) - c.E(X) \quad [E(c) = c \text{ from (1)}] \\ &= 0\end{aligned}$$

[Ans]

13) $\operatorname{cov}(aX,Y) = a\operatorname{cov}(X,Y)$ Show!!

$$\begin{aligned}\operatorname{cov}(aX,Y) &= E(aX.Y) - E(aX)E(Y) \\ &= aE(XY) - aE(X)E(Y) \\ &= a\{E(XY) - E(X)E(Y)\} \\ &= a\operatorname{cov}(X,Y).\end{aligned}$$

[Ans]

4). $\operatorname{cov}(aX,bY) = ab\operatorname{cov}(X,Y)$ Show!!

a,b constants

$$\begin{aligned}\operatorname{cov}(aX,bY) &= E(aX.bY) - E(aX)E(bY) \\ &= abE(XY) - \{aE(X).bE(Y)\} \\ &= abE(XY) - abE(X)E(Y) \\ &= ab\{E(XY) - E(X)E(Y)\}\end{aligned}$$

$= ab \operatorname{cov}(X,Y).$

15)

$\operatorname{cov}(X+Y, Z) = \operatorname{cov}(X, Z) + \operatorname{cov}(Y,Z)$: Show

Pf $\operatorname{cov}(X+Y, Z) = E[(X+Y).Z] - E[X+Y]E[Z]$

$= E[XZ + YZ] - E(X+Y)E(Z)$

$= E(XZ) + E(YZ) - \{E(X) + E(Y)\} E(Z)$

$= E(XZ) - E(X)E(Z) + E(YZ) - E(Y)E(Z)$

$= \operatorname{cov}(X,Z) + \operatorname{cov}(Y,Z)$

[Ans]

16) Derive the moment generating function for $U(0,1)$ distribution

$f(x) \equiv 1 \quad 0 \le x \le 1$

$M_X(t) =$ mgf of $X \sim U(0,1)$

$= E(e^{tX}) = \int_0^1 e^{tx} f(x)\,dx = \int_0^1 e^{tx}\,dx$

$= \left[\frac{e^{tx}}{t}\right]_0^1 = \frac{e^t}{t} - \frac{1}{t}$

$= \frac{(e^t - 1)}{t}$

[Ans]

17) Moment generating function and mean and variance for Binomial distribution.

Let $X_1, X_2 \ldots, X_n \sim Ber(p)$ independent and same distribution [iid]

$Y = X_1 + X_2 + \cdots + X_n$

$Mgf_Y(t) = E(e^{tY}) = [Mgf_X(t)]^n.$

$Mgf_{X_1} \equiv Mgf_{X_2} \equiv \cdots \equiv Mgf_{X_n}$

$X \sim Ber(p)$ $\quad Mgf_X(t) = E(e^{tX})$

$$= \sum_x e^{tx} P_r[X=x]$$

$$= e^{t0} P_r[X=0] + e^{t1} P_r[X=1]$$

$$= 1. P_r[X=0] + e^t P_r[X=1]$$

$$= (1-p) + pe^t$$

$Mgf_X(t) = (1-p) + pe^t$

$Mgf_Y(t) = [(1-p) + pe^t]^n.$

To calculate mean and second moment $E(Y)$ & $E(Y^2)$ we compute $\frac{d}{dt} Mgf_Y(t)\Big|_{t=0}$

$\frac{d^2}{dt^2} Mgf_Y(t)\Big|_{t=0}$

$$\frac{d}{dt} Mgf_Y(t) = \frac{d}{dt}\left\{[(1-p)+pe^t]^n\right\}$$

$$= n[(1-p)+pe^t]^{n-1} \frac{d}{dt}\{1-p+pe^t\}$$

$$= n\left[(1-p)+pe^t\right]^{n-1} p\left\{\frac{d}{dt}(1-p)+\frac{d}{dt}pe^t\right\}$$
$$= n\left[(1-p)+pe^t\right]^{n-1}\left\{0+pe^t\right\}$$
$$= npe^t\left[(1-p)+pe^t\right]^{n-1} \qquad (**)$$

$$\frac{d}{dt} Mgf_Y(t)\Big|_{t=0} = np.1\left[(1-p)+p\right]^{n-1} = np. \qquad \text{— (I)}$$

$$\frac{d^2}{dt^2} Mgf_Y(t) = \frac{d}{dt}\left[\frac{d}{dt} Mgf_Y(t)\right] = \frac{d}{dt}\left[npe^t\{(1-p)+pe^t\}^{n-1}\right]$$
$$(**)$$
$$= \frac{d}{dt}\left[npe^t\{(1-p)+pe^t\}^{n-1}\right]$$
$$= \left[\frac{d}{dt}(npe^t)\right]\left[(1-p)+pe^t\right]^{n-1} + \frac{d}{dt}\left[(1-p)+pe^t\right]^{n-1}(npe^t)$$
$$= npe^t\left[(1-p)+pe^t\right]^{n-1} \qquad [\text{chain rule}]$$
$$+ (n-1)\left[(1-p)+pe^t\right]^{n-2}\left[\frac{d}{dt}\{(1-p)+pe^t\} . npe^t\right]$$
$$= npe^t\left[(1-p)+pe^t\right]^{n-1}$$
$$+ n(n-1)pe^t\left[(1-p)+pe^t\right]^{n-2}\left[\frac{d}{dt}(1-p)+p\frac{d}{dt}e^t\right]$$
$$= npe^t\left[(1-p)+pe^t\right]^{n-1}$$
$$+ n(n-1)pe^t\left[(1-p)+pe^t\right]^{n-2}\left[0+pe^t\right]$$